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Stable Networks and Convex Payoffs

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Abstract

Recently a variety of link-based stability concepts have emerged in the literature on game theoretic models of social network formation. We investigate two basic formation properties that establish equivalence between some well known types of stable networks and their natural extensions. These properties can be identified as convexity conditions on the network payoff structures.

Keywords: Social networks; network formation; pairwise stability.

JEL classification: C72, C79.

1   Of stability and efficiency in network formation

Link-based stability is founded on the actions undertaken by individual decision makers—usually called "players"—with regard to the creation and breaking of links. Under the requirement of consent in link formation (Jackson and Wolinsky 1996, Gilles, Chakrabarti, and Sarangi 2005), a link can only be created if both players involved agree. On the other hand, links can be deleted by decisions to do so by individual players, i.e., deletion of links is accomplished without consent. Link-based stability requires that no *player* wants to delete one or more of her links to other players, and there is no *pair of players* that would like to establish a new link between them. Variations in the precise formalization of link addition and link deletion has resulted in a variety of link-based stability concepts in the literature.

Jackson and Wolinsky (1996) introduced this link-based stability approach. They developed the seminal notions of link addition and link deletion. They consider the creation (under consent) and deletion of a *single* link at a time. This resulted in their notion of *pairwise stability*. Since this foundational contribution there have been developed numerous alternative stability concepts. We refer to, e.g., Belleflamme and Bloch (2004), Bloch and Jackson (2004), Goyal and Joshi (2003), Gilles and Sarangi (2005) and Gilles, Chakrabarti, and Sarangi (2005) for extensions of this initial concept of pairwise stability. However, there is almost no work that examines the relationship between these different link-based stability concepts themselves. Our investigation makes a first step in this direction.

We limit our analysis to two notions of link addition and two notions of deletion. As mentioned, a network is *link deletion proof* if there is no player who wants to delete exactly one of her links. We also consider *strong* link deletion proof networks in which there is no individual who wants to deletion *one or more* of her links. Strong link deletion proofness is equivalent to imposing Nash equilibrium conditions in a non-cooperative game theoretic model of network formation as shown by Gilles, Chakrabarti, and Sarangi (2005). We show here that link deletion proofness results in exactly the same networks as the requirement of strong link deletion proofness if and only if the network payoffs satisfy a convexity property. This result has already been hinted at in Calvó-Armengol and Ilkiliç (2004), but has not been properly developed until now.

Further, we consider the *link addition proofness* condition introduced seminally by Jackson and Wolinsky (1996), which states that there is no link which will make both participating constituents better off. In Gilles and Sarangi (2005) we show that the implementation of belief systems in every player's behavior results in networks with stronger properties. We introduce there the notion of *strict* link addition proofness, which requires that both constituent players of a new link become strictly worse off if this link would be formed. Here we show that these two link addition proofness conditions result into the same class of networks if and only if the payoff function satisfies a sign-uniformity condition on the marginal payoffs.

Through these results we are able to clearly delineate the various link-based stability concepts clearly. These insights might be useful for future investigations of different equilibrium and stability concepts in these game theoretic models of network formation.

2 Preliminaries

Next we formally introduce the main tools of our analysis. Throughout we let N be some finite set of players or individuals.

2.1 Social networks

In our subsequent discussion we use established mathematical notation from Jackson and Wolinsky (1996), Dutta and Jackson (2003), and Jackson (2005). The reader may refer to these sources for a more elaborate discussion.

We limit our discussion to *non-directed networks* on N. Formally, if two players $i, j \in N$ with $i \neq j$ are related we say that there exists a *link* between players i and j. Now, if players i and j make up a single link, both players are equally essential; links have a bi-directional nature. Formally such a link can be expressed as a binary set $\{i, j\}$. We use the shorthand notation ij to describe the link $\{i, j\}$. We define

$$g_N = \{ij \mid i, j \in N, \ i \neq j\} \tag{1}$$

as the set of all potential links.

A *network* g on N is any set of links $g \subset g_N$. Particularly, g_N denotes the *complete network* and $g_0 = \varnothing$ is known as the *empty network*. The collection of all networks is defined as $\mathbb{G}^N = \{g \mid g \subset g_N\}$.

The set of (direct) *neighbors* of a player $i \in N$ in the network $g \in \mathbb{G}^N$ is given by

$$N_i(g) = \{j \in N \mid ij \in g\} \subset N. \tag{2}$$

Similarly we introduce

$$L_i(g) = \{ij \in g_N \mid j \in N_i(g)\} \subset g \tag{3}$$

as the *link set* of player i in the network g. These are exactly the links with i's direct neighbors in g.

For every pair of players $i, j \in N$ with $i \neq j$ we denote by $g + ij = g \cup \{ij\}$ the network that results from adding the link ij to the network g. Similarly, $g - ij = g \setminus \{ij\}$ denotes the network resulting from removing the link ij from network g.

2.2 Link-based stability concepts

We complete the preliminaries on network theory with the definition and discussion of the link-based stability conditions already mentioned in the introduction to this paper.

Within a network $g \in \mathbb{G}^N$, benefits for the players are generated depending on how they are connected to each other. Formally, for each player $i \in N$ the function $\varphi_i \colon \mathbb{G}^N \to \mathbb{R}$ denotes her *network payoff function* which assigns to every network $g \subset g_N$ a value $\varphi_i(g)$ that is obtained by player i when she participates in g. The composite network payoff function is now given by $\varphi = (\varphi_1, \ldots, \varphi_n) \colon \mathbb{G}^N \to \mathbb{R}^N$. We emphasize that these payoffs can be zero, positive, or negative and that the empty network $g_0 = \varnothing$ generates (reservation) values $\varphi(g_0) \in \mathbb{R}^N$ that might also be non-zero.[1]

For a given network $g \in \mathbb{G}^N$ we now define the following concepts:

(a) For every $ij \in g_N$, the *marginal benefit* of the link ij to player i in the network g for payoff function φ is given by

$$D_i(g, ij) = \varphi_i(g) - \varphi_i(g - ij) \in \mathbb{R}. \tag{4}$$

(b) For every player $i \in N$ and link set $h \subset L_i(g)$ the *marginal benefit* of link set h to player i in the network g for payoff function φ is given by

$$D_i(g, h) = \varphi_i(g) - \varphi_i(g - h) \in \mathbb{R}. \tag{5}$$

Using these additional tools we can give a precise description of the various link-based stability concepts:

(a) A network $g \subset g_N$ is **link deletion proof** for φ if for every player $i \in N$ and every neighbor $j \in N_i(g)$ it holds that $D_i(g, ij) \geqslant 0$.
Denote by $\mathcal{D}(\varphi) \subset \mathbb{G}^N$ the set of link deletion proof networks for φ.

(b) A network $g \subset g_N$ is **strong link deletion proof** for φ if for every player $i \in N$ and every $h \subset L_i(g)$ it holds that $D_i(g, h) \geqslant 0$.
Denote by $\mathcal{D}_s(\varphi) \subset \mathbb{G}^N$ the set of strong link deletion proof networks for φ.

(c) A network $g \subset g_N$ is **link addition proof** if for all players $i, j \in N$: $\varphi_i(g + ij) > \varphi_i(g)$ implies $\varphi_j(g + ij) < \varphi_j(g)$.
Denote by $\mathcal{A}(\varphi) \subset \mathbb{G}^N$ the set of link addition proof networks for φ.

(d) A network $g \in \mathbb{G}^N$ is **strictly link addition proof** for $\phi \colon \mathbb{G}^N \to \mathbb{R}$ if for all $i, j \in N$ it holds that $ij \notin g$ implies that $\varphi_i(g + ij) < \varphi_i(g)$.
Denote by $\mathcal{A}_s(\varphi) \subset \mathbb{G}^N$ the set of strict link addition proof networks for φ.

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[1] We remark that φ can be viewed as gross benefits $\overline{\varphi}$ minus the link maintenance costs c^m. Hence, we can reformulate $\varphi_i(g) = \overline{\varphi}_i(g) - \sum_{j \in N : ij \in g} c_{ij}^m$. However, explicit modeling of these maintenance costs is only essential in a dynamic model of network formation processes.

The two link deletion proofness notions are based on the severance of links in a network by individual players. The notion of link deletion proofness considers the stability of a network with regard to the deletion of a *single* link while strong link deletion proofness considers the possibility that a player deletes any subset of her existing links. Clearly, strong link deletion proofness implies link deletion proofness.

Adding a link on the other hand is considered for a pair at a time and also requires consent. A network is link addition proof if every pair of non-linked players as a whole does not have the incentive to add this link. Strict link addition proofness requires that every individual player in the pair being considered has a loss from adding a link. This formulation makes the consent requirement somewhat moot since neither player has an incentive to add the link. This is a significant strengthening of the link addition proofness requirement.

We conclude our discussion with an example which delineates the different link-wise stability concepts.

Example 1 Consider the network payoffs given in the following table:

Network	$\varphi_1(g)$	$\varphi_2(g)$	$\varphi_3(g)$	Stability
$g_0 = \varnothing$	0	0	0	D_s
$g_1 = \{12\}$	-1	-1	-1	
$g_2 = \{13\}$	-1	-1	-1	
$g_3 = \{23\}$	5	3	3	D_s, A
$g_4 = \{12, 13\}$	1	1	1	D_s
$g_5 = \{12, 23\}$	0	4	0	
$g_6 = \{13, 23\}$	0	0	4	
$g_7 = g_N$	1	5	5	D, A_s

In the table D stands for link deletion proofness and D_s for strong link deletion proofness. Similarly, A stands for link addition proofness and A_s for strict link addition proofness.

The main features here are that the complete network g_7 is link deletion proof, but not strong link deletion proof and that network g_3 is link addition proof, but not strict link addition proof. To make the differences between the various possibilities more clear we provide an overview of the marginal benefits:

Network	$D(g,12)$	$D(g,13)$	$D(g,23)$
$g_0 = \varnothing$	—	—	—
$g_1 = \{12\}$	$-1,-1,-1$	—	—
$g_2 = \{13\}$	—	$-1,-1,-1$	—
$g_3 = \{23\}$	—	—	$5,3,3$
$g_4 = \{12,13\}$	$2,2,2$	$2,2,2$	—
$g_5 = \{12,23\}$	$-5,1,-3$	—	$1,5,1$
$g_6 = \{13,23\}$	—	$-5,-3,1$	$1,1,5$
$g_7 = g_N$	$1,5,1$	$1,1,5$	$0,4,4$

In g_7 player 1 is stuck with bad company if she could delete only a single link at the time; she would like to break links with *both* players 2 and 3 and improve her payoff from 1 unit to 5 units. However, deleting either of these two links separately would make her only worse off. In this regard network convexity requires that no player is in such a bad company situation. ♦

Using the basic components of link formation we now define two more link-based stability concepts.

(a) A network $g \in \mathbb{G}^N$ is **pairwise stable** for φ if g is link deletion proof as well as link addition proof.
Denote by $\mathcal{P}(\varphi) = \mathcal{D}(\varphi) \cap \mathcal{A}(\varphi) \subset \mathbb{G}^N$ the family of pairwise stable networks for φ.

(b) A network $g \in \mathbb{G}^N$ is **strictly pairwise stable** for φ if g is strong link deletion proof as well as strict link addition proof.
Denote by $\mathcal{P}^\star(\varphi) = \mathcal{D}_s(\varphi) \cap \mathcal{A}_s(\varphi) \subset \mathbb{G}^N$ the family of strict pairwise stable networks for φ.

Jackson and Wolinsky (1996) seminally introduced the notion of pairwise stability. This requirement combines link deletion proofness and link addition proofness. Given that these two proofness conditions can be strengthened in various ways it is possible to define a variety of modifications depending on the context. In Gilles and Sarangi (2005) we introduce two modifications: (1) one where an agent can delete any subset of their links and (2) a variation of link addition proof where no agent in the pair wishes to add the link. The notion of strict pairwise stability combines these two features making it a natural link-based stability concept.

3 Equivalence and convexity

In this section we state the conditions that establish equivalence between the different stability concepts. We identify two related conditions under which the main proofness conditions result in the same networks.[2]

Equivalence Theorem

(a) *It holds that $\mathcal{D}_s(\varphi) = \mathcal{D}(\varphi)$ if and only if φ is **network convex** in the sense that for every network g, every player $i \in N$ and every link set $h \subset L_i(g)$:*

$$\sum_{ij \in h} D_i(g, ij) \geqslant 0 \text{ implies } D_i(g, h) \geqslant 0. \tag{6}$$

(b) *It holds that $\mathcal{A}_s(\varphi) = \mathcal{A}(\varphi)$ if and only if φ is **link uniform** on $\mathcal{A}(\varphi)$ in the sense that for every network $g \in \mathcal{A}(\varphi)$ and all players $i, j \in N$ with $ij \notin g$:*

$$D_i(g + ij, ij) \geqslant 0 \text{ implies } D_j(g + ij, ij) \geqslant 0. \tag{7}$$

(c) *It holds that $\mathcal{P}^\star(\varphi) = \mathcal{P}(\varphi)$ if and only if φ is network convex as well as link uniform on $\mathcal{A}(\varphi)$.*

A proof of this equivalence theorem is relegated to Section 4 of this paper.

To illustrate the two payoff conditions, network convexity and link uniformity, we return to the example discussed previously.

Example 2 Consider $N = \{1, 2, 3\}$ and the payoff structure φ described in Example 1. We show that this payoff structure is neither network convex nor link uniform. Indeed, first note that $D(g_7, 12) + D(g_7, 13) = (2, 6, 6)$ and that $D(g_7, \{12, 13\}) = (-4, 2, 2)$. Hence, the case of the removal of the links 12 and 13 from network g_7 shows that φ is not network convex.

With regard to link uniformity of φ we remark that for network $g_6 = \{13, 23\}$ we have that $D_1(g_6, 13) = -5 < 0$, while $D_3(g_6, 13) = 1 > 0$. This violates link uniformity of the marginal payoffs with regard to adding the link 13 to the network $g_3 = \{23\}$. That is also the reason why g_3 is link addition proof, but not strict link addition proof. ◆

Further, from the Equivalence Theorem it follows immediately that

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[2]Calvó-Armengol and Ilkiliç (2004) introduced the concept of α-convexity on a network payoff structure. This concept requires that the marginal benefits of link formation are related in a supermodular fashion. Note that the notion of "network convexity" defined here is weaker than their α-convexity condition.

Corollary: *If φ is (link) monotone in the sense that $D(g, h) \geqslant 0$ for all networks $g \subset g_N$ and link sets $h \subset g$, then $\mathcal{P}^{\star}(\varphi) = \mathcal{P}(\varphi)$.*

This equivalence result in fact follows from the observation that monotonicity of the value function φ implies that φ is both network convex as well as link uniform.

4 Proof of the equivalence theorem

PROOF OF ASSERTION (A):
Obviously from the definitions it follows that in general $\mathcal{D}_s(\varphi) \subset \mathcal{D}(\varphi)$.

Only if: Suppose that $g \in \mathcal{D}(\varphi)$ and that φ_i is not network convex on g for some $i \in N$ and some link set $h \subset L_i(g)$. We show that $g \notin \mathcal{D}_s(\varphi)$.
Indeed, from the hypothesis that g is link deletion proof, we know that $D_i(g, ij) \geqslant 0$ for every $ij \in L_i(g)$. Then for h it has to be true that since $\sum_h D_i(g, ij) \geqslant 0$, $D_i(g, h) < 0$. But then this implies that player i would prefer to sever all links in h. Hence, g cannot be strong link deletion proof, i.e., $g \notin \mathcal{D}_s(\varphi)$.

If: Let $g \in \mathcal{D}(\varphi)$ and assume that φ is network convex on g. Then for every player $i \in N$ and link $ij \in L_i(g)$ it has to hold that $D_i(g, ij) \geqslant 0$ due to link deletion proofness of g. In particular, for any link set $h \subset L_i(g)$: $\sum_h D_i(g, \cdot) \geqslant 0$. Now by network convexity this implies that $D_i(g, h) \geqslant 0$ for every link set $h \subset L_i(g)$. In other words, g is strong link deletion proof, i.e., $g \in \mathcal{D}_s(\varphi)$.

This completes the proof of the assertion.

PROOF OF ASSERTION (B):
First suppose that φ is link uniform on $\mathcal{A}(\varphi)$. Since $\mathcal{A}_s(\varphi) \subset \mathcal{A}(\varphi)$ we only have to show that $\mathcal{A}(\varphi) \subset \mathcal{A}_s(\varphi)$.
Let $g \in \mathcal{A}(\varphi)$. Then it follows by definition of link uniformity that for all $i, j \in N$ with $ij \notin g$:

$$\varphi_i(g) \leqq \varphi_i(g + ij) \text{ implies } \varphi_j(g) \leqq \varphi_j(g + ij)$$

since $D_i(g + ij, ij) = \varphi_i(g + ij) - \varphi_i(g)$. Hence, equivalently,

$$\varphi_i(g) > \varphi_i(g + ij) \text{ implies } \varphi_j(g) > \varphi_j(g + ij).$$

Now since g is link addition proof it holds that

$$\varphi_i(g + ij) > \varphi_i(g) \text{ implies } \varphi_j(g) > \varphi_j(g + ij),$$

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which in turn implies that $\varphi_i(g) > \varphi_i(g + ij)$.

This is a contradiction. Therefore, we conclude that $\varphi_i(g + ij) \leqq \varphi_i(g)$ as well as $\varphi_j(g + ij) \leqq \varphi_j(g)$. Hence, $g \in \mathcal{A}_s(\varphi)$.

Next suppose that $\mathcal{A}(\varphi) = \mathcal{A}_s(\varphi)$. Let $g \in A(\varphi)$ be such that there are $i, j \in N$ with $ij \notin g$. Then by strict link addition proofness of g—resulting from the assumed equivalence—it follows that

$$\varphi_i(g + ij) \leqq \varphi_i(g) \text{ as well as } \varphi_j(g + ij) \leqq \varphi_j(g).$$

This in turn implies that φ is link uniform for g.

PROOF OF ASSERTION (C):

This is a direct consequence of assertions (a) and (b) of the Equivalence Theorem proven above.

This completes the proof of the Equivalence Theorem stated in Section 3.

5 Conclusion

We have identified conditions under which strictly pairwise stable networks coincide with pairwise stable networks. This has useful applications since we can now apply the more natural notion of strict pairwise stability to the connections model of Jackson and Wolinsky (1996). However this equivalence does not hold for the co-author model also developed in Jackson and Wolinsky (1996).[3]

More importantly, we also conjecture that the network convexity condition can be extended to coalitions by identifying one such condition for each player. This will enable us to build establish the equivalence of pairwise stability and Jackson and van den Nouweland (2005)'s notion of strong stability.

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[3]For more on this we refer to Gilles, Chakrabarti, and Sarangi (2005).

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