Longevity and the Life Cycle

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Abstract

The life-cycle model is extended to analyze the causes of longer lifespans. In contrast to previous work, the extension delivers an explicit interior solution for consumption and longevity, where consumption is essential for living and longevity is financed with excess wealth. The solution is consistent with cross-country life-expectancy trends and explains gender differences in consumption and longevity. Also, a central testable hypothesis of the model is strongly supported by regressions using OECD panel data.

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The last enemy that shall be destroyed is death. (Corinthians 15:26)

Thus then the progress of knowledge, and in particular of medical science, the ever-growing activity and wisdom of Government in all matters relating to health, and the increase of material wealth, all tend to lessen mortality and to increase health and strength, and to lengthen life. (Alfred Marshall, *Principles - Book IV, Chapter V,* 1920)

Prolonging life is a universal human desire, yet it is costly in terms of resources and lifestyle adjustments. While the benefit-cost calculus involved in lifetime extension puts longevity squarely into the realm of economics, economists have largely been indifferent to the issue of endogenous longevity.[1] By contrast, historians, sociologists, demographers, biologists and medical scientists have examined the causes of the unprecedented rise of longevity over recent centuries and are hotly debating what the future holds.[2]

There is much that speaks for the idea that longevity is endogenous. For instance, one can not ignore that many individuals adopt healthy behaviors that are costly in terms of money, time, and perhaps current utility. The sacrifice is thought to increase the quality and quantity of the remaining years of one's life. Such thoughts are not irrational. Medical scientists have identified behaviors that lead to longer lives. Subjective longevity expectations tend to be accurate as shown by Smith et al. [18], which suggests that individuals are fully aware of the consequences of their actions on longevity. Also, Fogel [7] argues that the explosion in human life expectancies over recent centuries is a result of our increasing control over our environment. Thus, modern times require models where longevity is a result of economic behavior, in contrast to times prior to the 18th century when life expectancies were stable.

To examine the causes of longer lifespans, we extend the life-cycle model of Modigliani and Brumberg [13] and Yaari [20]. The extension has two novel features. Rather than choosing longevity indirectly through life-prolonging expenditures, rational individuals invert

[1] Notable exceptions are Grossman [10], Ehrlich and Chuma [5], Fogel [7], and Philipson and Becker [16].

[2] Expected lifespans have approximately tripled over the past three hundred years, after remaining stable at around 25 years for most of human history. See Wilmoth [19] and Fogel and Costa [8] for references.

the mapping from health expenditures to longevity and choose longevity directly. However, a solution to the model is not yet guaranteed. As we show in section 1, simple variations of the life-cycle model usually are not compatible with an interior maximum for longevity. One reason is that the simple life-cycle model in the steady state implies convex budget constraints in (consumption, longevity)-space with tangency occurring at a lifespan of zero. Thus, we make the critical assumption that energy-at-work is inversely related to longevity. In the spirit of Becker [2] and Fogel [7], this assumption formalizes the notion that energy is conserved and smoothed across an individual's lifetime and is supported by numerous findings that hard-working or fast-living Type A personalities also tend to die young (see Nanjundappa et al. [14] and references cited therein).

Under these circumstances, an interior optimal stationary solution for consumption and longevity is shown to exist in section 2. We also provide conditions that lead to a closed-form solution, an innovation compared to previous work by Grossman [10], Ehrlich and Chuma [5], and Philipson and Becker [16]. In steady state, consumption is pegged to recurring income, which is simply the minimum annual income earned in adulthood, as opposed to permanent income, which is the average of lifetime earnings. Recurring income appears late in life and broadens Philipson and Becker's [16] concept of mortality-contingent claims to environments without modern pension arrangements. In contrast to consumption, longevity is positively related to wealth in excess of recurring income and negatively related to recurring income. Thus, in terms of how wealth is allocated, consumption is essential for living and is financed with recurring income, while any excess wealth is allocated to life extension. This result is consistent with the historical view in Fogel and Costa [8] that reductions in chronic malnourishment set the basis for a mortality revolution.

The solution is also consistent with previous time series and cross-section evidence. In section 3, we show that the graph of the reduced-form longevity function resembles cross-country longevity profiles fitted by Preston [17]. In particular, we show that longevity is a concave function of the ratio of excess wealth to recurring income that shifts with advances in health technology. Thus, the model can explain historical trends for a variety of countries as well as evidence at the individual level, as long as one carefully distinguishes between the contributions to longevity of various sources of wealth. For example, the model explains the phenomenon that women tend to consume less and live longer than men. The model suggests that this difference is primarily due to lower recurring incomes of women as evidenced for example by the pension gap in advanced economies, while the wage gap between men and women tends to attenuate longevity differences.

Finally, in section 4, a central testable hypothesis of the model is strongly supported in a panel study of OECD countries. We argue that the replacement ratio of public pension systems adjusted for the old-age dependency ratio is an excellent measure of the ratio of recurring income to excess wealth. The variable has high explanatory power for average life expectancies and the estimated effect is negative, as theory demands, and significant in GMM regressions that correct for possible endogeneity bias. Traditional income proxies mismeasure the theoretical ratio, and we correct the resulting bias in a second GMM regression. The regressions underscore the importance of excess wealth for longevity.

1 Endogenous Longevity and the Life Cycle Model

In this section we explain why simple extensions of the life-cycle model fail to solve the longevity problem. The standard life-cycle problem for individuals is to choose a path for consumption that maximizes lifetime utility subject to a lifetime resource constraint. We broaden the scope of the model by allowing individuals to choose how long they will live.

Specifically, we consider an individual with time-separable preferences for consumption over a lifetime of length T :

$$\int_0^T e^{-(\rho+\delta)t} u\left(c\left(t\right)\right) dt$$

where $u\left(c\right)$ evaluates the utility of consumption c at time t and is assumed to be continuously differentiable with $u'(c) > 0 > u''(c)$ and $u'\left(0\right) = \infty$ and $u'\left(\infty\right) = 0$. The constant rate of time preference is ρ, while the constant hazard rate of death is δ. Thus, the probability of premature death is $e^{-\delta t}$ and the expected lifespan is $E(T) \equiv \int_0^T \delta t e^{-\delta t} dt$. As in Yaari [20], the effective discount rate of $\rho + \delta$ rises with the force of mortality and makes a person act more impatiently. When there is no risk of dying prematurely, we have $\delta = 0$.[3]

Individuals are endowed with initial wealth of y, have a fixed investment opportunity that yields a return of r, and operate in a perfect annuities market. They either work or are retired, and life spent outside of these two activities is ignored. During a working career that spans R periods, individuals inelastically supply a unit of labor and earn a wage rate of w in each year. In retirement and for the remainder of their lives, individuals earn a fixed annual income of a. We call a recurring income and assume $w \geq a > 0$. Recurring income is a more general concept than Philipson and Becker's mortality contingent claims. The idea is that we want to describe consumption and longevity choices in all sorts of economies, including those without modern pension arrangements. In traditional economies old age is sometimes financed by family networks. Sometimes though, individuals manage to grow old even without relatives and public transfers by engaging in home production. Thus, recurring income also includes alternative (usually non-market) means of subsistence.

With this preamble we can formally define lifetime wealth as

$$W(T) \equiv y + \int_0^R e^{-rt} w dt + \int_R^T e^{-rt} a dt$$

so that the individual's lifetime budget constraint is:

$$W(T) = \int_0^T e^{-rt} c(t)\, dt$$

We focus on $R < T$, but the mandatory age of retirement may be greater or smaller than the individual's expected lifespan $E(T)$. This focus presumes that individuals do not work until death. They may retire before or become terminally ill. In either case, they subsist on some form of recurring income.

The solution to the consumption problem of maximizing lifetime utility subject to the intertemporal revenue constraint and a fixed R and T is well known. In steady state where

[3] Life-cycle models based on Modigliani and Brumberg [13] fix the individual's maximum lifespan and allow no uncertainty about the time of death. Yaari [20] introduces the risk of premature death, which determines expected lifespans when maximum lifespans are infinite, but both lifespan measures remain exogenous.

the effective discount rate equals the interest rate, individuals perfectly smooth consumption in the sense of equalizing consumption in all periods. Thus, wealth is split evenly across years to give the annual consumption level of $c = \frac{W(T)}{A(T)}$, where the annuity value of a security paying one dollar every year until T is denoted $A(T) \equiv \int_0^T e^{-(\rho+\delta)t} dt = \frac{1}{\rho+\delta}\left(1 - e^{-(\rho+\delta)T}\right)$.

What if individuals also choose longevity? Specifically, suppose individuals choose longevity and consumption to maximize lifetime utility subject to the feasibility constraint.[4] If we continue with the simplifying assumption that $\rho+\delta = r$, consumption will be stationary. For the stationary problem we can write lifetime utility more compactly as

$$A(T)u(c) \tag{1}$$

and the lifetime budget constraint as

$$A(T)\,a + Y \equiv W(T) = A(T)c \tag{2}$$

where wealth from recurring income is $A(T)\,a$ and wealth net of recurring income is $Y \equiv y + A(R)(w - a)$. Because $w \geq a$, Y measures wealth in excess of lifetime minimum income $A(T)\,a$. The distinction between the two types of wealth is crucial for what follows.

Before solving the problem, we consider that a person's lifespan may also increase as a result of actions that have direct economic costs. For instance, time and money spent on healthcare and exercise are thought to have a positive effect on lifespans. With this motivation, Philipson and Becker [16] propose the following amendment to the simple life-cycle model. They aggregate the total resources spent that affect longevity into a lump-sum payment M that is subtracted from wealth, or $W(T) - M$. These resources affect longevity according to a concave production function $T(M)$, where $T' \geq 0 > T''$. Philipson and Becker also assume $a = 0$ and analyze the undiscounted case where $A(T)$ is replaced with T.

In our approach to the optimization problem, we invert the health production relationship and choose longevity directly, rather than choosing longevity indirectly by optimizing health

[4] Choosing longevity T is equivalent to choosing life expectancy $E(T)$. The maximum lifespan T is also the expected lifespan of a lucky individual that survives the risks of premature death. We assume that T is unconstrained, though it is natural to suppose that there is some biological upper bound. Medical science is sharply divided on the question of an upper bound on lifespans (see Wilmoth [19] for references).

expenditures. Inverting $T(M)$ yields $M(T)$, which is convex with $M' \geq 0$ and $M'' > 0$, and the budget constraint becomes

$$A(T)\,a + Y \equiv W(T) = A(T)c + M(T) \tag{3}$$

Intuitively, inverting $T(M)$ means that rational individuals see past the intermediate target of M to the ultimate target T. This is supported by evidence from Smith et al. [18] that individuals have rational subjective expectations about their lifespan.

With this modification of the lifetime budget constraint, we can write the LaGrangean for the longevity problem as:

$$Z = A(T)u(c) + \lambda\left(A(T)\,a + Y - A(T)c - M(T)\right)$$

The first-order conditions for an interior solution are then given by

$$\frac{\partial Z}{\partial \lambda} = A(T)\,a + Y - A(T)c - M(T) = 0 \tag{4}$$

$$\frac{\partial Z}{\partial c} = A(T)u'(c) - \lambda A(T) = 0 \tag{5}$$

$$\frac{\partial Z}{\partial T} = A'(T)u(c) + \lambda\left[A'(T)\,(a-c) - M'(T)\right] = 0 \tag{6}$$

The second order condition requires that the bordered Hessian is negative semi-definite:

$$\begin{vmatrix} 0 & -A & A'(a-c) - M' \\ -A & Au'' & 0 \\ A'(a-c) - M' & 0 & A''u + u'\left[A''(a-c) - M''\right] \end{vmatrix}$$

$$= -\left(A'(a-c) - M'\right)^2 Au'' + A^2u'M'\left(\frac{A''}{A'} - \frac{M''}{M'}\right) \geq 0 \tag{7}$$

Conditions (4) and (5) are familiar from the standard life-cycle problem and give decision rules for consumption conditional on a given lifespan. Once these decision rules are determined they could be used to find the optimal lifespan in the last condition (6). This last condition equates the marginal benefit of living longer to the marginal cost.[5]

[5] In the non-stationary version of the problem, (6) is the transversality condition for the optimal control problem with a free endpoint. Härtl et al. [11] explain that the non-stationary problem has never been completely characterized and a maximum is usually not guaranteed. It is sensible to suppose that if the stationary problem has a solution, then it is more likely that the non-stationary problem will have one, too.

Alternatively, one could combine equations (5) and (6) and obtain

$$\frac{u'(c)c}{u(c)} = \frac{A'(T)c}{A'(T)(c-a) + M'(T)} \tag{8}$$

Intuitively, individuals equate the marginal rate of substitution between T and c, $\frac{dT}{dc}\mid_{dU=0}= \frac{A(T)u'}{A'(T)u}$, and the relative price of consumption $\frac{p_c}{p_T}$, which because of the non-linearity of the budget constraint equals $\frac{dT}{dc}\mid_{dY=0}= \frac{A(T)}{A'(T)(c-a)+M'(T)}$. Thus, (8) indicates that individuals want to equate the elasticity of utility to $\frac{p_c}{p_T}\frac{A'(T)c}{A(T)}$. However, this may not be possible under reasonable assumptions and hence there may not exist an interior solution for longevity.

To see this, note that with $M(T) = 0$ and $a > 0$, equation (8) implies $\frac{u'(c)c}{u(c)} > 1$. However, this would violate strict concavity of the utility function which requires that the average utility of consumption exceeds marginal utility. When $a = 0$, (8) implies a linear utility function which still implies a bang-bang solution of instantaneous consumption and death, as noted by Ehrlich and Chuma [5]. Alternatively, when $M(T) > 0$, we need $A'(T)a = \frac{A'(T)}{A(T)}(W(T) - Y) < M'(T)$ in order to satisfy $0 \leq \frac{u'(c)c}{u(c)} < 1$. This condition restricts the admissible health production functions required for an interior solution. Even if this condition holds, a maximum is not guaranteed because $A'' < 0$ and $\frac{A''}{A'} - \frac{M''}{M'} < 0$ in (7). Philipson and Becker's assumptions of $a = 0$, convex $M(T)$, and $A(T) \to T$ so that $A' = 1$ and $A'' = 0$ do not guarantee an interior maximum without further restrictions on $M(T)$. To summarize:

Proposition 1. (*Non-Existence*). Choosing c and T to maximize (1) subject to (3) does not yield an interior maximum, unless $M'(T)\frac{A(T)}{A'(T)} > W(T) - Y$ and (7) holds.

We have just shown that simple extensions of the life-cycle model usually do not produce an interior solution for longevity. Instantaneous consumption and death is the natural outcome of these models even when utility is strictly concave. Intuitively, individuals face a convex budget constraint in $(c, A(T))$-space and convex indifference curves. Convex budget constraints normally make finding a unique solution difficult, but here they imply non-existence. To see why, note that because $A(T)$ is a monotonic transformation of T, we can draw budget constraints and indifference curves in (c, T)-space, as in Figure 1. When $M(T) = 0$, the budget constraint asymptotes at $c = a$, as can be seen by referring to (3). Because of this asymptote, the tangency of budget constraint and indifference curves occurs at the far right corner or as $T^* \to 0$. Adding a convex $M(T)$ twists the budget constraint toward the origin but does not easily create an interior tangency.

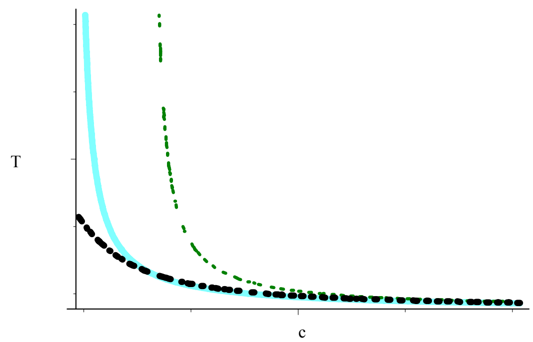

Figure 1: Indifference Curve (solid) and Budget Constraints

Because simple modifications to the life-cycle model (such as $M(T) \neq 0$) do not easily yield an interior tangency, we must consider other changes to the shape of the indifference curve or the budget constraint. In the next section we reshape the budget constraint but not the indifference curves. It seems more appropriate to alter the conditions individuals operate under rather than how they value these conditions.

2 The Longevity Problem Solved

In this section we extend the standard life cycle model and solve the longevity problem. Parallel to the exposition of the previous section, we posit that longevity is a negative and convex function of energy consumption, or $T(E)$, and that an individual's energy production increases market earnings, or wE. Again we suppose that longevity is chosen directly and invert $T(E)$ to yield $E(T)$ where $E'(T) < 0 < E''(T)$. After comparing the resulting wealth constraint with that of the simple life-cycle model, we develop the solution to the model.

We start with the following budget constraint:

$$(A\,(T) - A\,(R))\,a + A\,(R)\,E(T)w + y = A\,(T)\,c \tag{9}$$

where effort or energy production $E(T)$ is assumed to decrease with T. This formalizes the folk wisdom that "a short candle burns brightest" and builds on evidence that hard-working or fast-living Type A personalities die young [14]. Another interpretation follows Becker [2], who discusses effort smoothing across alternative uses of time, while here we emphasize energy smoothing over a whole lifetime. To conserve lifetime energy some individuals try to smooth their energy usage across activities and over time. Individuals with shorter horizons expend energy to a higher degree at work than long-lived individuals who try to conserve energy. The idea that individuals' capacity for energy production affects their productivity was also put forth by Fogel [7].

To emphasize similarities with the lifetime budget constraint of the standard life-cycle model, we would like to rewrite equation (9) to more closely resemble (3). Specifically, we require that the energy function is consistent with the following lifetime resource constraint:

$$G(T)A\,(T)\,a + Y \equiv W\,(T) = G(T)A\,(T)\,c \tag{10}$$

The two functions $E\,(.)$ and $G\,(.)$ can be related by equating the last two equations and rearranging to yield the following expression for the energy function:

$$E\,(T,.) = \left[\frac{1}{G\,(T)} - 1\right]\frac{Y}{A\,(R)\,w} + 1 \tag{11}$$

Differentiating both sides of (11) implies $G'(T) > 0$ and $G''(T) > 0$ unless $E''\,(T) \gg 0$. Thus, while (3) has a convex additive component $M(T)$, we see that (10) has a convex multiplicative component $G(T)$. This difference reshapes the budget constraint to create an interior tangency with an indifference curve.

An immediate implication of the energy function (11) is that effort rises for a given T as excess wealth rises relative to human wealth, where $\frac{Y}{A(R)w}$ is proportional to $\frac{y-A(R)a}{A(R)w}$. To understand the effect of the individual components on energy expenditure, suppose first that recurring income rises. In this case, falling energy consumption mimics an intertemporal substitution effect with energy being conserved in order to live longer and capture higher future incomes. Alternatively, the effect of wages on energy expenditure depends on the sign of $y - A(R)\,a$. The wage effect on energy expenditure is uncertain a priori because of conflicting income and substitution effects. On the one hand, individuals want to conserve energy when faced with higher rewards; on the other hand, higher incomes mean that individuals do not have to conserve on energy. Finally, the wealth effect of the endowment is to increase energy expenditure over a given horizon.

Under these assumptions, we can evaluate a new LaGrangean:

$$Z = A(T)u(c) + \lambda\left(G(T)A\left(T\right)a + Y - G(T)A(T)c\right)$$

The first-order conditions for an interior solution are then given by

$$\frac{\partial Z}{\partial \lambda} = G(T)A\left(T\right)a + Y - G(T)A(T)c = 0 \tag{12}$$

$$\frac{\partial Z}{\partial c} = A(T)u'(c) - \lambda G(T)A(T) = 0 \tag{13}$$

$$\frac{\partial Z}{\partial T} = A'(T)u(c) + \lambda\left[\left\{G'(T)A(T) + G(T)A'(T)\right\}(a-c)\right] = 0 \tag{14}$$

The second-order condition for this model reduces to:

$$\left[-\frac{G''}{G} - 2\frac{G'A'}{GA} + \frac{A''}{A'}\frac{G'}{G}\right](c-a) - \left[\left(\frac{G'}{G} + \frac{A'}{A}\right)(c-a)\right]^2 \frac{u''}{u'} \geq 0 \tag{15}$$

If we solve the last two first-order conditions for λ and then equate the results and rearrange terms, we find a condition analogous to (8):

$$\frac{u'(c)c}{u(c)} = \frac{A'(T)G(T)}{\left(A'(T)G(T) + A(T)G'(T)\right)}\frac{c}{(c-a)} \tag{16}$$

Thus, we have an interior solution if $\frac{A'G}{AG'} + 1 < \frac{c}{a}$. This condition simplifies to $\frac{A'a}{Y} < \frac{G'}{G^2}$ after substituting for $A(T)\,(c-a)$ from the wealth constraint. Finally, substituting for $W\left(T\right)$ implies $\frac{G'}{G}\,Y\frac{A}{A'} > W\left(T\right) - Y$. Compared to the previous restriction on $M(T)$, which required

$M'\frac{A}{A'} > W(T) - Y$ for an interior solution, the current restriction on $G(T)$ is much more forgiving, particularly if a or $W(T) - Y$ is large.

We proceed by making two functional form assumptions. First, define $G(T) = \Lambda A(T)^{\psi}$. Under this definition ψ acts as an efficiency parameter and Λ summarizes the effects of exogenous influences on individual energy production. A reduction in ψ or Λ raises the efficiency of energy production so that individuals need not work as hard to achieve a given level of earnings. In other words, individuals are more productive and earn higher wages for a given amount of time spent working. Second, assume that utility is CRRA or $u(c) = \frac{c^{\gamma}}{\gamma}$ with $\gamma \in (0, 1)$. This stringent form is widely used in empirical and theoretical work on consumption and helps clarify some of the theoretical results below. The two assumptions together have the virtue of yielding a closed form solution that looks very similar to that of the undiscounted case discussed in later sections.[6]

Applying both assumptions to (16) implies a closed-form solution for consumption:

$$c^* = \frac{\gamma(1+\psi)}{\gamma(1+\psi) - 1} a \tag{17}$$

When the solution for consumption is substituted into the wealth constraint, we can derive an implicit solution for longevity:

$$A(T^*) = \left[\frac{\gamma(1+\psi) - 1}{\Lambda} \frac{Y}{a}\right]^{\frac{1}{1+\psi}} \tag{18}$$

Thus, we need to constrain ψ so that $\gamma(1+\psi) > 1$ in order to guarantee an interior solution for consumption and longevity. With this solution it is a simple matter to compute T^* from $A(T^*)$, and then once longevity has been determined one can move to $E(T^*)$. The second-order condition for a maximum is also satisfied because (15) simplifies to

$$A^{1+2\psi}(\Lambda A')^2 \frac{c^{\gamma}}{\gamma}\left[3\psi + \gamma(1-\gamma)(1+\psi)^2\right] > 0$$

An immediate implication of equation (18) is that longevity depends negatively on recurring income and positively on all other sources of wealth. Also, from (17) we see that

[6] Perhaps more realistically, on-the-job energy could depend on the expected lifespan, or $G(T) = \Lambda E(T)^{\psi}$; however this functional form is less tractable without adding fundamental insight.

optimal consumption depends exclusively on recurring income. We discuss the solution in more depth in the next section. For now, we summarize this section in the following:

Proposition 2. (*Existence*). Choosing c and T to maximize (1) subject to (10) yields an interior maximum when $\frac{G'}{G} Y \frac{A}{A'} > W(T) - Y$ and (15) holds. The problem yields a maximum if $u(c) = \frac{c^\gamma}{\gamma}$ and $G(T) = \Lambda A(T)^\psi$ and an explicit solution given by (17) and (18).

The rest of the paper is concerned with developing the empirical implications of this theory. We start by showing how the model explains demographic trends and cross-sectional differences. We end by implementing a simple test of the central insight that longevity is positively related to the ratio of excess wealth to recurring income. To simplify the presentation we follow Modigliani and Brumberg [13] and Philipson and Becker [16] and analyze the model in the undiscounted stationary state. The solution is exactly the same except that the annuity function $A(T)$ simplifies to T and the discussion would not be much changed but would be slightly more cumbersome if this were not assumed.

3 Explaining Past Empirical Findings

In this section we look at the undiscounted steady-state of the model and show that the solution easily explains the main patterns found in previous empirical work on longevity. We argue that the model captures historical trends across countries and predicts previously unexplained differences across demographic groups. Critical to the explanations is a careful distinction between the effects of various components of wealth.

The solutions for the undiscounted case are

$$c^* = \frac{\gamma(1+\psi)}{\gamma(1+\psi)-1} a \tag{19}$$

$$T^* = \left[\frac{\gamma(1+\psi)-1}{\Lambda}\frac{Y}{a}\right]^{\frac{1}{1+\psi}} \tag{20}$$

As before, an increase in recurring income increases consumption and reduces longevity, while an increase in excess wealth increases longevity only.

An important aspect of the solution is that there must be some source of income in the later stages of life to have an interior solution, otherwise if $a = 0$ consumption is zero and lifespans infinite. It is reasonable to suppose that economies with public pension arrangements have $a > 0$, but what about economies without formal pensions? Because individuals lacking modern pension instruments still manage to live somehow, recurring income is broadly defined as including traditional sources of retirement income such as support from relatives and non-market production. With this definition we can talk about longevity choices in pre-industrial and industrial economies and we can distinguish between market wages and recurring income that comes from non-market sources. Recurring income may also be thought of as a minimum or subsistence income that is used to anchor consumption, while any excess wealth goes toward longevity enhancement.

We begin the discussion of equations (19) and (20) with a brief historical review. According to Fogel and Costa [8] and Wilmoth [19], the mortality decline began in the early 1700s with what has been termed the Second Agricultural Revolution. The mortality decline accelerated with the Industrial Revolution in the early 1800s and then continued with the onset of the Health Revolution around the turn of the 20th century.

The current model is useful for understanding the historical record. First, the Second Agricultural Revolution may have accomplished two things. It undoubtedly increased subsistence income from non-wage production, or a, and it also increased the yield of the largely agricultural endowment, y. Because consumption rose noticeably in comparison to more modest increases in longevity, one can conclude that $\frac{Y}{a}$ rose despite increases in a, implying $\frac{dy}{y} > (1 + \frac{Rw}{y})\frac{da}{a} > 0$.[7] Second, the Industrial Revolution primarily increased market productivity and wages and thus further increased longevity. Consumption continued to increase with further diffusion of agricultural technology and also as the technology revolution increased non-market productivity. In other words, $\frac{Y}{a}$ rose because $\frac{dw}{w} > \left(1 + \frac{y}{Rw}\right)\frac{da}{a} > 0$. Finally, according to Fogel and Costa [8], these early revolutions served to reduce chronic malnourishment and increased the population's fitness for market work. By contrast, the

[7] This discussion relies on the result that $\frac{dT^*}{T} = \frac{a}{(1+\psi)Y} d\left(\frac{Y}{a}\right)$ whereby $d\left(\frac{Y}{a}\right) = \frac{y}{a}\frac{dy}{y} + \frac{Rw}{a}\frac{dw}{w} - \frac{y+Rw}{a}\frac{da}{a}$.

subsequent Health Revolution served to raise the efficiency of individual energy production but did not affect consumption as much. Thus, we interpret the health revolution as a shock to Λ that primarily caused longevity to increase.

Before we consider the cross-sectional empirical evidence, it is instructive to graph the longevity function. In Figure 2, T^* appears as a concave function of $\frac{Y}{a}$.[8] The longevity function is more concave when ψ rises or as individuals' energy production becomes less efficient. There are two aspects of the longevity function that require comment. First, longevity is not bounded below by some number greater than zero. This should not be troubling because an individual without economic resources early in life is unlikely to live very long. Second, there is no upper limit to longevity. The model stresses economic factors not biological ones; and given recent material progress and technological innovation, perhaps having no upper bound is sensible at least locally.

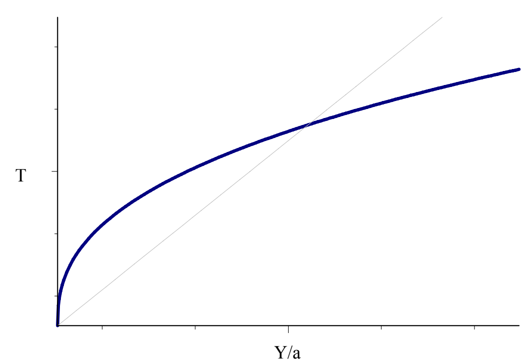

Figure 2: The Optimal Longevity Function

The graph bears a surprising resemblance to a well-known scattergram introduced by Preston [17]. Preston investigates how the average life expectancies for a variety of countries

[8]To draw the graph, we chose individual parameters so that the longevity function crosses the 45 degree line at age 60. Specifically, we assumed that $\Lambda = \frac{1}{600}, \gamma = \frac{3}{4}$, and $\psi = \frac{3}{2}$. This implies $\frac{Y}{a} \simeq 53$ at $T = 60$. Average life expectancies in the richest countries almost doubled in the 20th century, from roughly 40 to 75. With stable parameters, this implies that $\frac{Y}{a}$ more than quadrupled from around 19 to 93 during this period.

are related to their standard of living as measured by per capita GDP. He finds high positive correlation between the two variables and fits a logistic function to longevity and national income that resembles the above graph. However, a regression using a sample of 30 countries for which data were available in the 1930s and 1960s yields the conclusion that "factors exogenous to a country's current level of income are identified as being responsible for some 84 per cent of the increase in life expectancy during the period." While we revisit this issue empirically in the next section, the model suggests that the poor explanatory power may be because national incomes imprecisely capture excess wealth. Also, because non-market production or transfers are not counted as part of GDP, the recurring income effect is not measured and estimates will be biased. The unexplained residual is also partly explained by more productive health technologies. According to our simple model, shifts over time of the longevity function are due to changes in health technologies Λ or changes in the taste or energy efficiency parameters γ and ψ.

Empirical work at the individual level has established that longevity is positively related to income, wealth, and education. According to Deaton and Paxson [4], research has not yet settled why mortality is so strongly related to socioeconomic status and why the effect on mortality "is not the same for men and women, ...it is different over long periods and over the business cycle, and it is different in the cross-section from over time." On the basis of their evidence they speculate that "mortality is positively related to transitory income..., but is negatively related to permanent income." The longevity model suggests these puzzles may be explained by distinguishing whether the primary effect of an explanatory variable is through excess wealth, through recurring income, or through energy production by altering the efficiency by which energy is produced or conserved. For example, cross-sectional differences may be explained by such reasoning.

Specifically, the model provides an explanation for the fact that in most parts of the world females live longer than males. One might suppose that there is biological explanation for such gender differences. However, Carey [3] concludes that "unfortunately the scientific literature on sex longevity differences is ambiguous" and that one must carefully distinguish between influences on longevity through constitutional endowments (frailty and susceptibility to disease), productive biology, and behavior. Historical patterns suggest that a behavioral explanation may be warranted. A widespread gender gap in survival rates appears to be of recent origin with the gap growing wider in the 20th century in contrast to earlier times when many countries did not exhibit such an imbalance (Nathanson [15]).

The model links the longevity gap in advanced economies to wage and pension gaps that typically favor men. The pension gap acts through recurring income and serves to widen disparities in longevity between women and men, while the wage gap acts through excess wealth and tends to narrow such disparities. The model, also, implies that differences in recurring incomes create differences in consumption, whereas differences in excess wealth have no effect. Ghokale, Kotlikoff, and Sabelhaus [9] provide U.S. data for human wealth, pension wealth and consumption for men and women over the period of 1960 to 1990. Human wealth of men and women for the 20-89 age group grew at roughly the same rate over the period, but the percent increase of men for the 65-89 age group exceeded that of women. Also, pension wealth of men in both age groups increased by a larger percent than women. Finally, consumption of men and women aged 20 to 89 grew roughly by the same percent. Thus, the consumption gap remained stable with men's consumption exceeding that of women by an average of 1.35 percent. For the 65-80 age group, men's consumption was on average 4.15 percent higher than women's with the difference growing over time. Thus, the data are consistent with the model, which ties the widening of pre-existing gender gaps in longevity and consumption to a widening of the pre-existing pension gap.

4 Testing the Longevity Model

This section presents panel data estimates of the response of average life expectancy to the ratio of excess wealth to recurring income. We compare the performance of two measures of the theoretical ratio after correcting for mismeasurement and endogeneity bias. Using a sample of OECD economies, we show that GMM regression estimates of the longevity function have high explanatory power and yield correct and significant signs for both measures.

The solution for longevity in equation (18) is the starting point for specifying the form of the empirical model and what variables are important. The critical variable for longevity is $\frac{Y}{a}$ or the ratio of excess wealth to recurring income. This key variable has an excellent empirical counterpart, namely the replacement ratio of pension systems. The replacement ratio, denoted RR, is usually calculated as the ratio of retirement benefits to earnings and can be related to the ratio of wage earnings to recurring income according to $RR = \frac{(T-R)a}{Rw}$. Because the old-age dependency ratio is the ratio of retirees to workers, or $DR \equiv \frac{T-R}{R}$, we find that $\frac{a}{w} = \frac{RR}{DR}$. We can easily relate this formula to the ratio of excess wealth to recurring income if we abstract from the initial endowment by assuming $y = 0$. In this case, $\frac{Y}{a} = R\left(\frac{DR}{RR} - 1\right)$.

If we estimate the effects of $\frac{RR}{DR}$ on T, the model predicts a negative effect. One potential problem that we address is that DR may impart endogeneity bias to the regressions. This is because any calculation of average life expectancy is usually based on the current age distribution. Endogeneity of DR creates negative correlation with the error term of a least-squares regression and downward bias of the estimated effect of $\frac{RR}{DR}$ on T. Also, if the measure for R is taken to be the mandatory age of retirement, then the model predicts that R has a positive effect on T. However, there is the potential problem that the mandatory age of retirement is not binding. Many retirement systems specify a minimum age when benefits can be drawn and the effective retirement age varies with foregone pension income. Thus, the predicted positive effect assumes that retirement is exogenous, something that does not happen in practise and may affect the empirical results.

We are also interested in the performance of real per capita GDP, denoted I. The variable I has a tradition in the empirical mortality literature and was the key explanatory variable in Preston [17]. As argued previously, I only captures the excess wealth effect on longevity but not the recurring income effect of non-market production and old-age transfers. Using I to measure $\frac{Y}{a}$ will produce downwardly biased coefficient estimates because of mismeasurement. This is because least-squares regressions would allocate the recurring income effect to the error term, which forces the coefficient that estimates the effect of the theoretical ratio down. The bias resulting from mismeasurement error caused by using I affects least-squares estimates in the same way endogeneity bias caused by using $\frac{RR}{DR}$ affects least-squares estimates. We exploit the similarity by creating a common set of instruments.

Estimating equation (18) directly takes a particularly simple form:

$$T_{it} = \alpha + \beta D_t + X_{it}^{\prime}\Gamma + \varepsilon_{it}$$

where i indexes countries and t corresponds to time. The variable T_{it} is measured by a country's average life expectancy (for men and women) for the period 1960-65 and for 1995-2000. The variable D_t is a time dummy for the early 1960s period and the late 1990s. The variable X_{it} is a vector of the central explanatory variables and Γ is a vector of coefficients. We distinguish between $X_{it} = \left(\ln \frac{RR_{it}}{DR_{it}}, \ln R_{it}\right)$ and the traditional measure $X_{it} = (\ln I_{it})$. In the first case, RR is measured by a country's expected old-age replacement rate and is available for 1961 and 1995. Also, DR is the old-age dependency ratio and is available for 1960 and 2000, and R is measured by the standard age of entitlement to public old-age pensions for males and is available for 1961 and 1999. The data for the variables T, RR, DR, and R are taken from tables in the appendix of Feldstein and Siebert [6].[9] In the second case, I is real per capita GDP adjusted for purchasing power and is available for 1955 and 1990 from the Penn World Tables. These years were chosen because the latest available year was 1993 and we wanted preserve symmetry as much as possible.

We use three variables CR_{it}, S_{it}, and H_{it}, to construct two instrumental variables $\ln \frac{CR}{S}$ and HS. The two instruments were created to isolate the excess wealth effect on longevity and the recurring income effect. The variable CR is a country's pension contribution rate measured as a share of average earnings and is only available for 1967 and 1995 in [6]. The variable S is defined as the average number of years of schooling for the population 25 years

[9]The variable T is from Table 10, RR is from Table 21, DR is calculated from Tables 4 and 6, and R is from Table 23. The sample includes two observations for each of the variables for 18 countries: Australia, Belgium, Canada, Denmark, Finland, France, Germany, Ireland, Italy, Japan, Netherlands, New Zealand, Norway, Portugal, Sweden, Switzerland, United Kingdom, United States. The number of countries in the sample was limited by the availability of data for both periods for the core variables T, RR, DR, and R.

and older and is available from Barro and Lee [1] for 1960 and 1995. Finally, H is defined as total expenditures on health measured as a share of GDP. The variable is from the OECD and is available for the complete set of countries only for 1972 and 1995.

The regression results are reported in Table 1. The Table presents GLS regressions in equations (I) through (III). For comparison, GMM regressions are presented in equations (IV) and (V). All regressions correct for the possibility of heteroskedasticity.

Table 1: Pooled Longevity Regressions - Dependent Variable T

	(I) GLS	(II) GLS	(III) GLS	(IV) GMM	(V) GMM
Intercept	72.156	74.089	43.954	75.773	35.406
	(33.20)	(1.069)	(8.609)	(1.148)	(10.366)
D	6.887	6.884	3.721	7.257	1.978
	(0.466)	(0.462)	(1.029)	(0.483)	(1.489)
$\ln \frac{RR}{DR}$	-3.578	-3.597		-5.575	
	(1.411)	(1.221)		(1.146)	
$\ln R$	0.458				
	(7.766)				
$\ln I$			3.140		4.184
			(1.003)		(1.224)
$\overline{R}^2$	0.869	0.881	0.880	0.847	0.862
S.E.	1.399	1.378	1.339	1.513	1.438
Observations	36	36	36	36	36

Notes: White heteroskedasticity-adjusted standard errors are in parentheses. Sample means [and standard deviations] for the variables are: 74.1 [3.87] for T, 2.69 [0.93] for RR/DR, 65.2 [2.2] for R, and 9651 [4823] for I.

GLS estimates of the longevity equation do well for both sets of explanatory variables. The regressions have good explanatory power for longevity as measured by adjusted R^2s. Also, the coefficients for the replacement ratio adjusted for the dependency ratio have the correct sign and are highly significant.[10] The coefficient for the mandatory retirement age has the correct sign, but it is insignificant because of failure to measure the effective retirement age.[11] In contrast to Preston's [17] negative conclusion, the traditional regression (III) fairs well based on explanatory power and correct and significant signs of the regressor.

We use a GMM regression procedure to control for the biases that result when using $\frac{RR}{DR}$ or I as a regressor. The procedure uses $\ln(\frac{CR}{S})$ and HS as instrumental variables. Equations (IV) and (V) indicate that controlling for bias affects coefficients in the right direction. Compared to the GLS regressions, coefficients increase. There is not much change in the significance of the coefficients. The major difference between the GLS and GMM regressions is that the time dummy becomes insignificant in the traditional regression. Finally, both regressions do admirably based on adjusted R^2s.

We conclude that the theoretical model is strongly supported by the data. Because of the nature of the instrumental variables, we interpret the coefficient on $\frac{RR}{DR}$ in equation (IV) as estimating the recurring income effect, while the time dummy estimates the effect of excess wealth. By contrast, we interpret the coefficient on I in equation (V) as estimating the excess wealth effect, while the time dummy estimates the recurring income effect. For the countries in the sample, T rose on average by 6.85 years over the sample period, $\frac{RR}{DR}$ fell by 0.15, and I rose by 8394. The estimates suggest that the change in longevity over the sample period was largely due to excess wealth.

5 Conclusion

Longevity is perhaps the most important of all economic decision variables, but it has rarely been analyzed by economists. We bridge this gap and model the choice of the optimal quantity of life by extending the life-cycle model. After finding and characterizing the

[10]If we were to estimate separate effects for RR and DR on T, we would expect that RR has a negative effect and DR has a positive effect. We found that such unrestricted regressions behaved as predicted, with highly significant coefficients and slightly more explanatory power than the restricted regressions.

[11]Because R was insignificant in the GLS regression (I), we dropped it to conserve on instruments for the GMM regression and report GLS and GMM regressions without R in (II) and (IV).

optimal interior solution for consumption and longevity, we provide conditions that lead to a closed-form solution for longevity. We show that the model explains past evidence and that the solution is strongly supported by OECD panel data estimates. The framework should be useful for future analyses of longevity-related choices and policies.

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6 Appendix: Second Order Conditions Derived

6.1 Second LaGrangean

$$Z = A(T)u(c) + \lambda \left(G(T)A\left(T\right)a + Y - G(T)A(T)c\right)$$

The first-order conditions for an interior solution are then given by

$$\frac{\partial Z}{\partial \lambda} = G(T)A\left(T\right)a + Y - G(T)A(T)c = 0 \Longrightarrow G(T)A\left(T\right)a + Y = G(T)A(T)c$$

$$\frac{\partial Z}{\partial c} = A(T)u'(c) - \lambda G(T)A(T) = 0 \Longrightarrow \frac{u'(c)}{G(T)} = \lambda$$

$$\frac{\partial Z}{\partial T} = A'(T)u(c) + \lambda\left[X'(T)\left(a-c\right)\right] = 0 \Longrightarrow \frac{A'(T)u(c)}{X'(T)\left(c-a\right)} = \lambda$$

where $X(T) \equiv G(T)A(T)$. The last two conditions imply

$$\frac{A'(T)G(T)c}{X'(T)\left(c-a\right)} = \frac{u'(c)c}{u(c)} \Leftrightarrow \frac{u(c)}{u'(c)} = \left(1 + \frac{A(T)G'(T)}{A'(T)G(T)}\right)(c-a)$$

The **second order conditions** require a negative semi-definite Hessian, or

$$\begin{aligned}
&\left|\begin{array}{ccc} 0 & -GA & X'\left(a-c\right) \\ -GA & Au'' & A'u' - \lambda X' = u'\left(A' - \frac{X'}{G}\right) = -\frac{u'}{G}G'A \\ X'\left(a-c\right) & -\frac{u'}{G}G'A & A''u + \frac{u'}{G}X''\left(a-c\right) \end{array}\right| \\
= \;& GA\left|\begin{array}{cc} -GA & -\frac{u'}{G}G'A \\ X'\left(a-c\right) & A''u + \frac{u'}{G}X''\left(a-c\right) \end{array}\right| + X'\left(a-c\right)\left|\begin{array}{cc} -GA & Au'' \\ X'\left(a-c\right) & -\frac{u'}{G}G'A \end{array}\right| \\
= \;& (GA)^2\left[A''u + \frac{u'}{G}X''\left(a-c\right)\right] - \left[X'\left(a-c\right)\right]^2 Au'' \\
\propto \;& (GA)^2\left[A''G\frac{u}{u'} + X''\left(a-c\right)\right] - \left[X'\left(a-c\right)\right]^2 GA\frac{u''}{u'} \\
= \;& A''G\frac{u}{u'} + X''\left(a-c\right) - \left[\frac{\left(G'A + GA'\right)\left(a-c\right)}{GA}\right]^2 GA\frac{u''}{u'} \\
= \;& GA''\frac{u}{u'} + \left[G''A + 2G'A' + GA''\right]\left(a-c\right) - \left[\left(\frac{G'}{G} + \frac{A'}{A}\right)\left(a-c\right)\right]^2 GA\frac{u''}{u'} \\
= \;& GA''\frac{u}{u'} + \left[G''A + 2G'A' + GA''\right]\left(a-c\right) - \left[\left(\frac{G'}{G} + \frac{A'}{A}\right)\left(a-c\right)\right]^2 GA\frac{u''}{u'}
\end{aligned}$$

$$
\begin{aligned}
&= GA''\left(1+\frac{AG'}{A'G}\right)(c-a)+\left[-G''A-2G'A'-GA''\right](c-a) \\
&\quad -\left[\left(\frac{G'}{G}+\frac{A'}{A}\right)(c-a)\right]^2 GA\frac{u''}{u'} \\
&= \left[-\frac{G''A}{GA}-2\frac{G'A'}{GA}+\frac{A''}{A}\frac{G'A}{GA'}\right](c-a)-\left[\left(\frac{G'}{G}+\frac{A'}{A}\right)(c-a)\right]^2\frac{u''}{u'}
\end{aligned}
$$

6.2 LaGrangean for the Closed-Form Solution

$$
Z = A(T)\frac{c^{\gamma}}{\gamma}+\lambda\left(\Lambda A\left(T\right)^{1+\psi}a+Y-\Lambda A\left(T\right)^{1+\psi}c\right)
$$

The first-order conditions for an interior solution are then given by

$$
\frac{\partial Z}{\partial \lambda} = \Lambda A\left(T\right)^{1+\psi}a+Y-\Lambda A\left(T\right)^{1+\psi}c=0
$$

$$
\frac{\partial Z}{\partial c} = A(T)c^{\gamma-1}-\lambda\Lambda A\left(T\right)^{1+\psi}=0
$$

$$
\frac{\partial Z}{\partial T} = A'(T)\frac{c^{\gamma}}{\gamma}+\lambda\left[(1+\psi)\Lambda A\left(T\right)^{\psi}A'(T)\right](a-c)=0
$$

The **second order conditions** require a negative semi-definite Hessian for a maximum:

$$
\begin{vmatrix}
0 & -\Lambda A^{1+\psi} & (1+\psi)\Lambda A^{\psi}A'(a-c) \\
-\Lambda A^{1+\psi} & Ac^{\gamma-2}(\gamma-1) & Z_{cT} \\
(1+\psi)\Lambda A^{\psi}A'(a-c) & Z_{Tc} & Z_{TT}
\end{vmatrix}
$$

where

$$
\begin{aligned}
Z_{cT} &= Z_{Tc} = A'c^{\gamma-1}-\lambda(1+\psi)\Lambda A^{\psi}A'\frac{\partial Z}{\partial c}=0 \\
&\Rightarrow A'\left[c^{\gamma-1}-\lambda\Lambda A^{\psi}\right]=0 \\
&\Rightarrow Z_{Tc}=-\lambda\psi\Lambda A^{\psi}A'=-\frac{c^{\gamma-1}}{\Lambda A^{\psi}}\psi\Lambda A^{\psi}A'=-c^{\gamma-1}\psi A'
\end{aligned}
$$

and

$$
\begin{aligned}
Z_{TT} &= A''\frac{c^{\gamma}}{\gamma}+\lambda(1+\psi)\Lambda\left[\psi A^{\psi-1}\left(A'\right)^2+A^{\psi}A''\right](a-c) \\
&= A''\frac{c^{\gamma}}{\gamma}+\frac{c^{\gamma-1}}{A^{\psi}}(1+\psi)\left[\psi A^{\psi-1}\left(A'\right)^2+A^{\psi}A''\right](a-c)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{c^{\gamma-1}}{\gamma}\left\{A''c-\gamma(1+\psi)\left[\psi A^{-1}(A')^2+A''\right](c-a)\right\}\\
&= \frac{c^{\gamma-1}}{\gamma}\left\{A''\frac{\gamma(1+\psi)}{\gamma(1+\psi)-1}a-\gamma(1+\psi)\left[\psi A^{-1}(A')^2+A''\right]\frac{1}{\gamma(1+\psi)-1}a\right\}\\
&= \frac{c^{\gamma-1}}{\gamma}\frac{\gamma(1+\psi)a}{\gamma(1+\psi)-1}\left\{A''-\left[\psi A^{-1}(A')^2+A''\right]\right\}\\
&= -\frac{c^{\gamma}}{\gamma}\left\{\psi A^{-1}(A')^2\right\}
\end{aligned}
$$

Thus,

$$
\left|\begin{array}{ccc}
0 & -\Lambda A^{1+\psi} & (1+\psi)\Lambda A^{\psi}A'(a-c)\\
-\Lambda A^{1+\psi} & Ac^{\gamma-2}(\gamma-1) & -c^{\gamma-1}\psi A'\\
(1+\psi)\Lambda A^{\psi}A'(a-c) & -c^{\gamma-1}\psi A' & -\frac{c^{\gamma}}{\gamma}\psi A^{-1}(A')^2
\end{array}\right|
$$

$$
\begin{aligned}
=\;& \begin{array}{c}
\left\{\Lambda A^{1+\psi}\left|\begin{array}{cc} -\Lambda A^{1+\psi} & (1+\psi)\Lambda A^{\psi}A'(a-c)\\ -c^{\gamma-1}\psi A' & -\frac{c^{\gamma}}{\gamma}\psi A^{-1}(A')^2\end{array}\right|\right\}\\
\left\{+(1+\psi)\Lambda A^{\psi}A'(a-c)\left|\begin{array}{cc} -\Lambda A^{1+\psi} & (1+\psi)\Lambda A^{\psi}A'(a-c)\\ Ac^{\gamma-2}(\gamma-1) & -c^{\gamma-1}\psi A'\end{array}\right|\right\}
\end{array}\\
=\;& \begin{array}{c}
\Lambda A^{1+\psi}\left\{\begin{array}{c}\Lambda A^{1+\psi}\frac{c^{\gamma}}{\gamma}\psi A^{-1}(A')^2\\ +c^{\gamma-1}\psi A'(1+\psi)\Lambda A^{\psi}A'(a-c)\end{array}\right\}\\
+(1+\psi)\Lambda A^{\psi}A'(a-c)\left\{\begin{array}{c}\Lambda A^{1+\psi}c^{\gamma-1}\psi A'\\ -Ac^{\gamma-2}(\gamma-1)(1+\psi)\Lambda A^{\psi}A'(a-c)\end{array}\right\}
\end{array}\\
=\;& \begin{array}{c}
\left\{\Lambda A^{1+\psi}\Lambda A^{\psi}(A')^2\frac{c^{\gamma-1}}{\gamma}\left\{\psi c+\gamma\psi(1+\psi)(a-c)\right\}\right\}\\
+\left\{(1+\psi)\Lambda A^{\psi}(A')^2(a-c)\Lambda A^{1+\psi}c^{\gamma-2}\left\{c\psi-(\gamma-1)(1+\psi)(a-c)\right\}\right\}
\end{array}\\
=\;& \begin{array}{c}
\Lambda A^{1+\psi}\Lambda A^{\psi}(A')^2\frac{c^{\gamma-2}}{\gamma}\left\{c\left\{\psi c+\gamma\psi(1+\psi)(a-c)\right\}\right\}\\
+\Lambda A^{1+\psi}\Lambda A^{\psi}(A')^2\frac{c^{\gamma-2}}{\gamma}\left\{\gamma(1+\psi)(a-c)\left\{c\psi-(\gamma-1)(1+\psi)(a-c)\right\}\right\}
\end{array}\\
=\;& \begin{array}{c}
A^{1+2\psi}(\Lambda A')^2\frac{c^{\gamma-2}}{\gamma}*\\
\left\{c\left\{\psi c+\gamma\psi(1+\psi)(a-c)\right\}+\gamma(1+\psi)(a-c)\left\{c\psi-(\gamma-1)(1+\psi)(a-c)\right\}\right\}
\end{array}\\
\propto\;& c\left\{\psi c+\gamma\psi(1+\psi)(a-c)\right\}+\gamma(1+\psi)(a-c)\left\{c\psi-(\gamma-1)(1+\psi)(a-c)\right\}\\
=\;& \begin{array}{c}
\psi c^2+\gamma\psi(1+\psi)(a-c)c\\
+\gamma(1+\psi)(a-c)c\psi-\gamma(1+\psi)(a-c)(\gamma-1)(1+\psi)(a-c)
\end{array}\\
=\;& \psi c^2+2\gamma\psi(1+\psi)(a-c)c-\gamma(\gamma-1)(1+\psi)^2(a-c)^2\\
=\;& \left[\frac{\gamma(1+\psi)a}{\gamma(1+\psi)-1}\right]^2\left\{\frac{3\gamma\psi(1+\psi)}{\gamma(1+\psi)}+\gamma(1-\gamma)(1+\psi)^2\right\}>0
\end{aligned}
$$