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# Existence of Nash Networks in One-Way Flow Models

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**Abstract**

This paper addresses the existence of Nash networks for the one-way flow model of Bala and Goyal (2000) in a number of different settings. First, we provide conditions for the existence of Nash networks in models where costs and values of links are heterogenous and players obtain resources from others only through the directed path between them. We find that costs of establishing links play a vital role in the existence of Nash networks. Next we examine the existence of Nash networks when there are congestion effects in the model. Then, we provide conditions for the existence of Nash networks in a model where a player's payoff depends on the number of links she has established as well as on the number of links that other players in the population have created. More precisely, we show that convexity and increasing (decreasing) differences allow for the existence of Nash networks.

*JEL Classification: C72, D85*

*Key Words: Network Formation, Non-cooperative Games*

# Introduction

The importance of networks in economic and social activities has led to the emergence of a growing literature seeking to understand the formation of these networks. This literature in economics has focused on three main questions: Given that decisions are made by self-interested players (i) What is the set of stable networks? (ii) What networks are efficient? and (iii) Is there a conflict between the set of stable and efficient networks? We can discern two distinct strands in the literature differentiated by the type of stability concept used.

The first type employs the notion of pairwise stability and its variants and is inspired by Jackson and Wolinsky's (1996, [9]) work. These authors assume that a link is formed if both players involved in a link agree to form that link, though link deletion occurs unilaterally. While benefits depend on the overall graph, the cost of setting up a relationship is shared equally between the two participating players. In a pairwise stable network no pair of players has an incentive to form a link and no player has an incentive to delete a link. Necessary and sufficient conditions for the existence of pairwise stable networks can be found in Jackson and Watts (2001, [8]) .

The second literature develops a non-cooperative version of network formation. This literature was initiated by Bala and Goyal (BG, 2000, [1]) and assumes that a player can establish a link with another without the latter's consent, as long as she incurs the cost of forming the link. The authors present two versions of their model: the two-way flow model and the one-way flow model. The two-way flow models and consent or pairwise stability type models are the two extremes with the one-way flow model being in between the two.

Observe that in the pairwise stability type of models, in order to have two way flow of information, both players must pay a cost and agree to the link between them. In the two-way flow model, both players have access to each others information regardless of who initiates the link. Of course, as mentioned above, the initiating player bears the link formation cost. Consent issues are completely absent in this model. In the one-way flow model, only the (link) initiating player has access to the other player's information. Thus the importance of this model stems from the fact that it lies somewhere between the consent and no-consent models. While it does not explicitly require the other player's acquiescence, player $i$ has to incur link formation costs to access player $j$'s resources. To permit two-way information flow in this model both players have to incur the costs of a link. For both one-way and two-way flow models, the corresponding static stable networks are called Nash networks since Nash equilibrium is used to determine stability.

Most of the existing studies have explored the characterization of Nash networks, either in the two-way flow model (Galeotti, Goyal and Kamphorst (2005, [5]), Haller and Sarangi (2005,[7])) or in the one-way flow model (Galeotti (2004, [4]), Billand and Bravard (2005, [3])). The existence of Nash networks however has not been studied in great detail. Although BG (2000) provide a constructive proof of the existence of Nash networks in their original paper this is done in a very restrictive setting – assuming that all costs and benefits are homogeneous across players. In a recent paper Haller, Kamphorst and Sarangi (2005, [6]) study the existence of Nash networks in two-way flow models by incorporating value, cost and link heterogeneity. The existence issue had remained unexplored in the one-way flow setting.

In this paper, we investigate the existence of Nash networks in BG's one-way flow model. In the existing literature, there are two types of formulations based on the one-way flow model.

- In the first formulation, a player obtains resources from another player if and only if there exists a directed path between the two as in BG (2000). In the rest of the paper we refer to this framework as the (one-way flow) model with transitive spillovers (MTS).

- In the second set of models, a player's payoff depends on the number of links she has formed as well as the number of links the other players in the population have established. This framework has been used by Billand and Bravard (2004, [2]). In what follows, we call this framework the (one-way flow) model with global spillovers (MGS).

The existence of Nash networks in the MTS framework has been studied by BG (2000), when costs and values of links are homogeneous. But the existence of Nash networks has not been examined when costs and values of links are heterogeneous. The question of existence of equilibria under heterogeneity is important for several reasons. Firstly, the model with heterogeneity provides a robustness check for the results obtained from the model with homogeneous parameters. Secondly and more importantly, ex-ante asymmetries across players arise quite naturally in reality. For instance, in the context of information networks, it is often the case that some individuals are better informed, which makes them more valuable contacts. Similarly, as individuals differ, it seems natural that forming links is cheaper for some individuals as compared to others. Thirdly, our results complement the existing literature. Galeotti, (2004, [4]) characterizes the (strict) Nash networks when cost and values of links are heterogeneous. Yet we do not know under what conditions such equilibria exist. Finally, the existence of Nash networks in the MTS framework has never been studied, when there are congestion effects. Indeed, Billand and Bravard (2005, [3]) extend the model of BG (2000) and introduce the possibility of congestion effects. These effects exist in several instances where getting too many resources can actually prove an hindrance to agents. For instance, when researchers are seeking to get some information about a part of their field which they are unsure about, they often read a literature survey written by another scholar. This activity is costly in terms of time and effort, for instance, to identify relevant information sources. The reading effort can be expensive and tedious if they are too many sources. In extreme cases, if a survey is too exhaustive, it might have little or no value to the scholarly reader. Billand and Bravard (2005, [3]) characterize Nash networks when this assumption arises. However, they do not address the issue of existence of Nash networks.

The existence of Nash networks in MGS has never been studied. However, this framework can be useful as well, particularly in industrial organization applications. Indeed, prior to competing on the market, firms often have the opportunity to pick up externalities of other firms via economic intelligence activities (Prescott, Gibbons, 1993, [10]). These activities, which can be interpreted as directed links, include among others reading of industry trade press or patent literature, talking with technology vendors,

sales representative or industy experts, visiting the commercial trade fairs and analyzing the competitors' product. In an oligopoly market, the competitive strength of a firm depends both on the number of links she has formed and on the number of links the competitors have formed.
This framework has been explored by Billand and Bravard (2004, [2]) which characterize the Nash networks in that kind of frameworks. Thus our paper contributes to the literature by resolving the existence question for such networks.

We now provide a quick overview of the results for both types of models.

- *MTS models*: We show that there does not always exist a Nash network in MTS models when costs and values are heterogeneous. More precisely, we show that, as in the two-way flow model, heterogeneity of cost in forming links plays a great role in the non existence of Nash network. We then provide conditions on costs of setting links to allow for the existence of Nash networks. We also show that if costs are homogeneous, then there always exist Nash networks. Finally, we show that if costs and values are homogeneous, but congestion effects can occur, then a Nash network does not always exist.

- *MGS models*: We provide economically appealing conditions for the existence of Nash networks in MGS models. We show that in the MGS framework there always exists a Nash network when the players' utility functions satisfy the decreasing difference property or when each player's utility function is discretely convex with respect to the total number of links this player has established. Moreover, we give a general characterization of Nash networks when the players' utility functions satisfy the decreasing difference property and discrete convexity with respect to the total number of links that each player initiates.

The remainder of the paper is organized as follows. In Section 1 we set the basic model and study existence in the MTS model under various heterogeneity conditions for costs and values. We conclude this section by examining the model with congestion effects. Section 2 is devoted to the MGS model. The first part examines existence in the presence of increasing and decreasing differences. The second part focuses on the characterization and existence of Nash networks under discrete convexity. Section 3 concludes.

# 1 One-way Flow Model with Transitivity

In this section, we describe the model of one-way flow networks. This is followed by an examination of existence in the model without congestion effects. Next, we re-examine the issue incorporating congestion effects.

## 1.1 Model Setup

Let $N = \{1, \ldots, n\}$ be the set of players. The network relations among these players are formally represented by directed graphs whose nodes are identified with the players. A network $\boldsymbol{g} = (N, E)$ is a pair of sets: the set $N$ of players and the edges set $E(\boldsymbol{g}) \subset N \times N$ of directed links. A link initiated by player $i$ to player $j$ is denoted by $i, j$. Pictorially this is depicted as link from $j$ to $i$ to show the direction of information flow.[1] Each player

[1] Throughout the paper we refer to this as link from $j$ to $i$. The same is true for other network components like paths.

$i$ chooses a strategy $\boldsymbol{g}_i = (\boldsymbol{g}_{i,1}, \ldots, \boldsymbol{g}_{i,i-1}, \boldsymbol{g}_{i,i+1}, \ldots, \boldsymbol{g}_{i,n})$, $\boldsymbol{g}_{i,j} \in \{0, 1\}$ for all $j \in N \setminus \{i\}$, which describes the act of establishing links. More precisely, $\boldsymbol{g}_{i,j} = 1$ if and only if $i, j \in E(\boldsymbol{g})$. The interpretation of $\boldsymbol{g}_{i,j} = 1$ is that player $i$ forms a link with player $j \neq i$, and the interpretation of $\boldsymbol{g}_{i,j} = 0$ is that $i$ does not form a link with player $j$. We only use pure strategies. Note that $\boldsymbol{g}_{i,j} = 1$ does not necessarily imply that $\boldsymbol{g}_{j,i} = 1$. It can be that $i$ is linked to $j$, but $j$ is not linked to $i$. Let $\mathcal{G} = \times_{i=1}^{n} \mathcal{G}_i$ be the set of all possible networks where $\mathcal{G}_i$ is the set of all possible strategies of player $i \in N$.

We now provide some important graph theoretic definitions. For a directed graph, $\boldsymbol{g} \in \mathcal{G}$, a *path* $P(\boldsymbol{g})$ of length $m$ in $\boldsymbol{g}$ from player $j$ to $i$, $i \neq j$, is a finite sequence $i_0, i_1, \ldots, i_m$ of distinct players such that $i_0 = i$, $i_m = j$ and $\boldsymbol{g}_{i_k, i_{k+1}} = 1$ for $k = 0, \ldots, m-1$. If $i_0 = i_m$, then the path is a cycle. We denote the set of cycles in the network $\boldsymbol{g}$ by $\mathcal{C}(\boldsymbol{g})$. The complete network $\overline{\boldsymbol{g}}$, is a network such that for all $i \in N, j \in N$, we have $\overline{\boldsymbol{g}}_{i,j} = 1$. In the empty network, $\dot{\boldsymbol{g}}$, there are no links between any agents.

To sum up, a link from a player $j$ to a player $i$ ($\boldsymbol{g}_{i,j} = 1$) allows player $i$ to get resources from player $j$ and since we are in a one-way flow model, this link does not allow player $j$ to obtain resources from $i$. Moreover, a player $i$ may receive information from other players through a sequence of indirect links. To be precise, information flows from player $j$ to player $i$, if $i$ and $j$ are linked by a path of length $m$ in $\boldsymbol{g}$ from $j$ to $i$. Let

$$N_i(\boldsymbol{g}) = \{j \in N | \text{ there exists a path in } \boldsymbol{g} \text{ from } j \text{ to } i\},$$

be the set of players that player $i$ can access in the network $\boldsymbol{g}$. By definition, we assume that $i \in N_i(\boldsymbol{g})$ for all $i \in N$ and for all $\boldsymbol{g} \in \mathcal{G}$. Let $n_i(\boldsymbol{g})$ be the cardinality of the set $N_i(\boldsymbol{g})$. Information received from player $j$ is worth $V_{i,j}$ to player $i$. Moreover, $i$ incurs a cost $c_{i,j}$ when she initiates a direct link with $j$, i.e. when $\boldsymbol{g}_{i,j} = 1$. We can now define the payoff function of player $i \in N$:

$$\pi_i(\boldsymbol{g}) = \sum_{j \in N_i(\boldsymbol{g})} V_{i,j} - \sum_{j \in N} \boldsymbol{g}_{i,j} c_{i,j}.$$

We assume that $c_{i,j} > 0$ and $V_{i,j} > 0$ for all $i \in N$, $j \in N$, $i \neq j$. Moreover, we assume that, for all $i \in N$, $\pi_i(\boldsymbol{g}) = 0$ if $\boldsymbol{g}_{i,j} = 0$ for all $j \in N$, $j \neq i$. In other words, we assume that $V_{i,i} = 0$ for all $i \in N$. The next definition introduces the different notions of heterogeneity in our model.

**Definition 1** *Values (or costs) are said heterogeneous by pairs of players if there exist $i \in N$, $j \in N$, $k \in N$ such that $V_{i,j} \neq V_{i,k}$ $(c_{i,j} \neq c_{i,k})$ and there exist $i' \in N$, $j' \in N$, $k' \in N$ such that $V_{j',i'} \neq V_{k',i'}$. Values (or costs) are said heterogeneous by players if for all $i \in N$, $j \in N$, $k \in N$: $V_{i,j} = V_{i,k} = V_i$ $(c_{i,j} = c_{i,k} = c_i)$ but there exists $i \in N$, $i' \in N$ such that $V_i \neq V_{i'}$ $(c_i \neq c_{i'})$.*

We now provide some useful definitions for studying the existence of Nash networks. Given a network $\boldsymbol{g} \in \mathcal{G}$, let $\boldsymbol{g}_{-i}$ denote the network obtained when all of player $i$'s links are removed. Note that the network $\boldsymbol{g}_{-i}$ can be regarded as the strategy profile where $i$ chooses to form no links. The network $\boldsymbol{g}$ can be written as $\boldsymbol{g} = \boldsymbol{g}_{-i} \oplus \boldsymbol{g}_i$, where the operator $\oplus$ indicates that $\boldsymbol{g}$ is formed by the union of links in $\boldsymbol{g}_i$ and $\boldsymbol{g}_{-i}$. The strategy $\boldsymbol{g}_i$ is said to be a best response of player $i$ to $\boldsymbol{g}_{-i}$ if:

$$\pi_i(\boldsymbol{g}_i \oplus \boldsymbol{g}_{-i}) \geq \pi_i(\boldsymbol{g}'_i \oplus \boldsymbol{g}_{-i}), \text{for all } \boldsymbol{g}'_i \in \mathcal{G}_i.$$

The set of player $i$'s best responses to $\boldsymbol{g}_{-i}$ is denoted by $\mathcal{BR}_i(\boldsymbol{g}_{-i})$. Furthermore, a network $\boldsymbol{g} = (\boldsymbol{g}_1, \ldots, \boldsymbol{g}_i, \ldots, \boldsymbol{g}_n)$ is said to be a Nash network if $\boldsymbol{g}_i \in \mathcal{BR}_i(\boldsymbol{g}_{-i})$ for each $i \in N$.

**Definition 2** *We say that two networks $\boldsymbol{g}$ and $\boldsymbol{g}'$ are adjacent if there is a unique player $i$ such that $g_{i,j} \neq g'_{i,j}$ for at least one player $j \neq i$ and if for all player $k \neq i$, $g_{k,j} = g'_{k,j}$, for all $j \in N$.*

An improving path is a sequence of adjacent networks that results when players form or sever links based on payoff improvement the new network offers over the current network. More precisely, each network in the sequence differs from the previous one by the links formed by one unique player. If a player changes her links, then it must be that this player strictly benefits from such a change.

**Definition 3** *Formally, an improving path from a network $\boldsymbol{g}$ to a network $\boldsymbol{g}'$ is a finite sequence of networks $\boldsymbol{g}^1, \ldots, \boldsymbol{g}^k$, with $\boldsymbol{g}^1 = \boldsymbol{g}$ and $\boldsymbol{g}^k = \boldsymbol{g}'$, such that the two following conditions are verified :*

1. *for any $\ell \in \{1, \ldots, k\}$, there is a unique $i \in N$, such that: $\boldsymbol{g}^{\ell+1}_{-i} = \boldsymbol{g}^{\ell}_{-i}$, that is there is a unique player $i$ who has changed her strategy;*
2. *for this unique player $i$, we have $\boldsymbol{g}^{\ell+1}_i \in \mathcal{BR}_i(\boldsymbol{g}^{\ell}_{-i})$ and $\boldsymbol{g}^{\ell}_i \notin \mathcal{BR}_i(\boldsymbol{g}^{\ell}_{-i})$, that is $\boldsymbol{g}^{\ell+1}$ is a network where $i$ plays a best response while $\boldsymbol{g}^{\ell}$ is a network where $i$ does not play a best response.*

*Moreover, if $\boldsymbol{g}^1 = \boldsymbol{g}^k$, then the improving path is called an improving cycle.*

It is obvious that a network $\boldsymbol{g}$ is a Nash network if and only if it has no improving path emanating from it.

Finally, we define $\eta : \mathcal{G} \to \mathbb{R}$, $\eta(\boldsymbol{g}) = \sum_{i \in N} n_i(\boldsymbol{g})$ as a function.

## 1.2 Model with Heterogeneous Agents without Congestion Effect

Bala and Goyal (2000, [1]) outlines a constructive proof of the existence of Nash networks in the case of costs and values of links homogeneity. Here we begin by showing that in one-way flow models with cost and value heterogeneity by pairs of players (see Galeotti, [4] 2004) there always exists a Nash network if the number of players is $n = 3$. This result is no longer true if the number of players is $n \geqslant 4$. However, if values of links are heterogeneous by pairs of players and costs of links are heterogeneous by players, there always exists a Nash network.

**Proposition 1** *If the values and costs of links are heterogeneous by pairs and $n = 3$, then a Nash network exists.*

**Proof** Let $N = \{1, 2, 3\}$. We begin with the empty network $\dot{\boldsymbol{g}}$. Either $\dot{\boldsymbol{g}}$ is a Nash network and we are done, or $\dot{\boldsymbol{g}}$ is not a Nash network and there exists an improving path from $\dot{\boldsymbol{g}}$ to an adjacent network $\boldsymbol{g}^1$. That is, there exists a player, say without loss of generality player 1, such that $\dot{\boldsymbol{g}}_1 \notin \mathcal{BR}_1(\dot{\boldsymbol{g}}_{-1})$ and $\boldsymbol{g}_1^1 \in \mathcal{BR}_1(\dot{\boldsymbol{g}}_{-1})$. Since $1 \in N$ has no link in $\dot{\boldsymbol{g}}$ and forms links in $\boldsymbol{g}^1 = \boldsymbol{g}_1^1 \oplus \dot{\boldsymbol{g}}_{-1}$, we have $\eta(\dot{\boldsymbol{g}}) < \eta(\boldsymbol{g}^1)$. Now we will repeat the same step. Assume an improving path from a network $\boldsymbol{g}^1$ to a network $\boldsymbol{g}^k$ where for each player $i \in N$, we have $N_i(\boldsymbol{g}^{k-1}) \subseteq N_i(\boldsymbol{g}^k)$. We show that if there exists an improving path from $\boldsymbol{g}^k$ to $\boldsymbol{g}^{k+1}$, then for each player $i \in N$, $N_i(\boldsymbol{g}^k) \subseteq N_i(\boldsymbol{g}^{k+1})$. Let $i$ be a player such that $\boldsymbol{g}_i^{k+1} \in \mathcal{BR}_i(\boldsymbol{g}_{-i}^k)$ and $\boldsymbol{g}_i^k \notin \mathcal{BR}_i(\boldsymbol{g}_{-i}^k)$. We show that if $j \in N_i(\boldsymbol{g}^k)$, then $j \in N_i(\boldsymbol{g}^{k+1})$. Indeed there are two possibilities for $j \in N_i(\boldsymbol{g}^k)$.

1. Either $\boldsymbol{g}_{i,j}^k = 1$, that is $i$ directly obtains the resources of player $j$. Then there are

two possibilities.

- If $V_{i,j} - c_{i,j} > 0$ then $j \in N_i(\boldsymbol{g}^{k+1})$, otherwise $i$ does not play a best response in $\boldsymbol{g}^{k+1}$.
- If $V_{i,j} - c_{i,j} < 0$, then there is a network $\boldsymbol{g}^{k'}$, $k' < k$, such that $\ell \in N_j(\boldsymbol{g}^{k'})$ and $V_{i,j} + V_{i,\ell} - c_{i,j} > \max\{0, V_{i,\ell} - c_{i,\ell}\}$, else $\boldsymbol{g}^k_{i,j} = 0$. Since $N_j(\boldsymbol{g}^{k'}) \subseteq N_j(\boldsymbol{g}^k)$, for all $k' < k$ and for all $j \in N$, we have $\ell \in N_j(\boldsymbol{g}^k)$ and player $i$ deletes her link with $j$ only if $j \in N_\ell(\boldsymbol{g}^k)$ and $V_{i,j} + V_{i,\ell} - c_{i,j} < V_{i,j} + V_{i,\ell} - c_{i,\ell}$. In that case, $i$ forms a link with $\ell$ and $j \in N_i(\boldsymbol{g}^{k+1})$.

2. Or $\boldsymbol{g}^k_{i,j} = 0$, $\boldsymbol{g}^k_{i,\ell} = 1$ and $\boldsymbol{g}^k_{\ell,j} = 1$, that is $i$ indirectly obtains the resources of player $j$. Then, we use the same argument as above to show that player $i$ deletes her link with $\ell$ only if she has an incentive to form a link with $j$ and $j \in N_i(\boldsymbol{g}^{k+1})$.

We now show that there does not exist any cycle in an improving path $\mathcal{Q} = \{\dot{\boldsymbol{g}}, \boldsymbol{g}^1, \ldots, \boldsymbol{g}^t, \ldots, \boldsymbol{g}^{t+h}, \ldots, \boldsymbol{g}^{t+h'}, \ldots\}$, with $h' > h > 0$. We note that as $j \in N_i(\boldsymbol{g}^t)$ and $N_i(\boldsymbol{g}^t) \subseteq N_i(\boldsymbol{g}^{t+h})$, we have $j \in N_i(\boldsymbol{g}^{t+h})$. Also, as $\boldsymbol{g}^{t+h}_{i,j} = 0$, we have $\boldsymbol{g}^{t+h}_{i,k} = 1$ and $k \in N_i(\boldsymbol{g}^{t+h})$. Moreover, as $N_i(\boldsymbol{g}^{t+h}) \subseteq N_i(\boldsymbol{g}^{t+h'})$, we have $N_i(\boldsymbol{g}^{t+h'}) = \{j, k\}$.

Without loss of generality, we suppose that player $i$ deletes the link $i, j$ for the first time between $t$ and $t + h$. Likewise, we assume that player $i$ forms the link $i, j$ for the first time between $t + h$ and $t + h'$.

We have two cases.

1. Suppose we have $\boldsymbol{g}^t_{i,k} = 0$. To obtain a contradiction, assume that $k \in N_i(\boldsymbol{g}^t)$. It follows that $\boldsymbol{g}^{t+h}_{j,k} = 1$ since player $i$ does not form the link $i, k$ between $\boldsymbol{g}^t$ and $\boldsymbol{g}^{t+h}$ if $j$ preserves the link $j, k$. Also $j$ does not delete the link $j, k$ between $\boldsymbol{g}^t$

and $\boldsymbol{g}^{t+h}$ if $i$ does not form the link $i,k$ (recall that in our process only one player changes her strategy at each period). Since player $i$ chooses to delete the link $i,j$ in $\boldsymbol{g}^{t+h}$, then she must form the link $i,k$ and we must have $\boldsymbol{g}^{t+h}_{k,j} = 1$, since $k \in N_i(\boldsymbol{g}^t) \subseteq N_i(\boldsymbol{g}^{t+h})$. Moreover, we note that the substitution of the link $i,j$ by the link $i,k$ implies that $c_{i,j} > c_{i,k}$. Using same argument, player $k$ has not deleted the link $k,j$ between $\boldsymbol{g}^{t+h}$ and $\boldsymbol{g}^{t+h'}$. Therefore, if player $i$ forms the link $i,j$ in $\boldsymbol{g}^{t+h'}$ (and so deletes the link $i,k$), then we have $c_{i,j} < c_{i,k}$ and we obtain the desired contradiction.

2. Next, suppose that we have $\boldsymbol{g}^t_{i,k} = 1$. If player $i$ deletes the link $i,j$ in $\boldsymbol{g}^{t+h}$, then we obtain the situation in case 1 up to a permutation of players $j$ and $k$. Hence the proof follows.

□

We have shown that if values and costs of links are heterogeneous by pairs and $n = 3$, then there always exists a Nash network. Note that this result is not true for the model with directed links and two-way flow of resources (see Haller, Kamphorst and Sarangi 2005, [6] p. 7). We next show with an example that the above proposition is not valid for $n > 3$.

**Example 1** Let $N = \{1,2,3,4\}$ be the set of players and $V_{i,j} = V$ for all $i \in N$, $j \in N$. More precisely, we suppose that $c_{1,3} = V - V/16$ and $c_{1,2} = c_{1,4} = 4V$; $c_{2,1} = 2V - V/16$ and $c_{2,3} = c_{2,4} = 4V$; $c_{3,2} = 2V - V/8$, $c_{3,4} = 2V - V/6$ and $c_{3,1} = 4V$; $c_{4,1} = 3V - V/8$ and $c_{4,2} = c_{4,3} = 4V$.

1. In a best response, player 2 never forms any link with player 3 or player 4. Moreover, player 2 has an incentive to form a link with player 1 if the latter gets

resources from player 3 or player 4.

2. In a best response, player 4 never forms links with player 3 or player 2.

3. Then the unique best response of player 1 to any network in which she does not observe player 3 is to add a link with player 3 (since player 2 and player 4 never form a link with player 3). Moreover, we note that player 1 never has any incentive to form a link with player 2 or player 4.

4. In a best response, player 3 never forms any link with player 1.

Now let us take those best replies for granted and consider best responses regarding the remaining links $2,1$; $3,2$; $3,4$ and $4,1$. If player 2 initiates link $2,1$, then player 3's best response is to initiate link $3,2$. In that case player 4 must initiate the link $4,1$ and player 3 must replace the link $3,2$ by the link $3,4$. Then, player 4 must delete the link $4,1$ and the player 3 must replace the link $3,4$ by the link $3,2$. Hence there does not exist any mutual best response. Therefore, a Nash network does not exist. Finally, by appropriately adjusting costs it can be verified that this example holds even if we relax the assumption that $V_{i,j} = V$ for all $i \in N$, $j \in N$.

### 1.2.1 Existence of Nash networks and heterogeneity of values by pairs

We now prove the existence of Nash networks when values are heterogeneous by pairs and costs are heterogeneous by players. First, when values are heterogeneous by pairs and costs are heterogeneous by players, then we can write the profit function as follows:

$$\pi_i(\boldsymbol{g}) = \sum_{j \in N_i(\boldsymbol{g})} V_{i,j} - c_i \sum_{j \in N} \boldsymbol{g}_{i,j}.$$

Let $\pi_i^j(\boldsymbol{g})$ be the marginal payoff of player $i$ from player $j$ in the network $\boldsymbol{g}$. If $\boldsymbol{g}_{i,j} = 1$, then $\pi_i^j(\boldsymbol{g}) = \pi_i(\boldsymbol{g}) - \pi_i(\boldsymbol{g} \ominus i,j)$. Let $\mathcal{K}(\boldsymbol{g};i,j) = N_i(\boldsymbol{g} \ominus i,j) \bigcap N_i(\boldsymbol{g}_{-i} \oplus i,j)$, where $\boldsymbol{g} \ominus i,j$ denotes the network $\boldsymbol{g}$ without the link $i,j$. We can rewrite $\pi_i^j(\boldsymbol{g})$ as follows:

$$\pi_i^j(\boldsymbol{g}) = \sum_{k \in N_i(\boldsymbol{g}_{-i} \oplus i,j)} V_{i,k} - \sum_{k \in \mathcal{K}(\boldsymbol{g};i,j)} V_{i,k} - c_i. \tag{1}$$

**Proposition 2** *If values of links are heterogeneous by pairs and costs of links are heterogeneous by players, then a Nash network exists.*

The proof of Proposition 2 is long involving a number of lemmas. So we first provide a quick overview of the proof. It consists of constructing a sequence of networks, $\mathcal{Q} = (\boldsymbol{g}^0, \ldots, \boldsymbol{g}^{t-1}, \boldsymbol{g}^t, \ldots)$ beginning with the empty network. In each subsequent network, no player should have an incentive to decrease the amount of resources she obtains. Note that this sequence of networks is not an improving path. Indeed, we go from $\boldsymbol{g}^t$ to $\boldsymbol{g}^{t+1}$ in several operations. First, in $\boldsymbol{g}^t$ we let a player $i \in N$, who is not playing a best response in $\boldsymbol{g}^t$, to play a best response (if no such player exists, $\boldsymbol{g}^t$ is a Nash network) and obtain a network called $\mathrm{br}_i(\boldsymbol{g}^t)$. Second, we modify the network $\mathrm{br}_i(\boldsymbol{g}^t)$ as follows: we construct a cycle using all players $j \in N$ who obtain resources from a player $k$ who forms part of a cycle in $\mathrm{br}_i(\boldsymbol{g}^t)$, while preserving all links in $\mathrm{br}_i(\boldsymbol{g}^t)$ between a player $k \in N$ and a player $j$ who is not part of a cycle in $\mathrm{br}_i(\boldsymbol{g}^t)$. We obtain a network called $h(\mathrm{br}_i(\boldsymbol{g}^t))$. Thirdly, we delete all links $i,j$ which does not allow player $i$ to obtain additional resources in $h(\mathrm{br}_i(\boldsymbol{g}^t))$. We obtain a network called $m(h(\mathrm{br}_i(\boldsymbol{g}^t))) = \overline{\boldsymbol{g}}_i^t$, and in the sequence $\mathcal{Q}$, we have $\boldsymbol{g}^{t+1} = \overline{\boldsymbol{g}}_i^t$.

When a player $i$ receive an opportunity to revise her strategy, we go from a network $\boldsymbol{g}^{t-1}$ to a network $\boldsymbol{g}^t$, and we will show that $\eta(\boldsymbol{g}^{t-1}) > \eta(\boldsymbol{g}^t)$. Since the amount of resources that players can obtain in a network $\boldsymbol{g} \in \mathcal{Q}$ is finite, $\mathcal{Q}$ is finite and there exists a Nash network.

In the following paragraph, we define a class of networks $\mathcal{G}^3$ which contains all networks in the sequence $\mathcal{Q}$. Then, we provide a condition which implies that no player has an incentive to delete a link in a network $\boldsymbol{g} \in \mathcal{G}^3$ (Lemma 2). Finally, we show that all networks $\boldsymbol{g}^t \in \mathcal{Q}$ satisfy this condition since the empty network satisfies this condition (Lemma 6).

Let us formally define the set $\mathcal{G}^3$. Let $\mathcal{M} : \mathcal{G} \rightarrow \mathcal{P}(\mathcal{G})$, $\boldsymbol{g} \mapsto \mathcal{M}(\boldsymbol{g}) \subset \mathcal{G}$ be a correspondence. Let $m(\boldsymbol{g}) \in \mathcal{M}(\boldsymbol{g})$ be a minimal network associated to the network $\boldsymbol{g}$, $m(\boldsymbol{g})$ is a network such that, for all $i \in N$, $j \in N$, $N_i(\boldsymbol{g}) = N_i(m(\boldsymbol{g}))$ and if $m(\boldsymbol{g})_{i,j} = 1$, then $j \notin N_i(m(\boldsymbol{g}) \ominus i, j)$ and $\boldsymbol{g}_{i,j} = 1$. We note that in a network $m(\boldsymbol{g}) \in \mathcal{M}(\boldsymbol{g})$, there is at most one path from a player $i \in N$ to a player $j \in N$. In the following, we can take, without loss of generality, any element of $\mathcal{M}(\boldsymbol{g})$. Let $m(\boldsymbol{g})$ be a typical element of $\mathcal{M}(\boldsymbol{g})$. Obviously, we have $\eta(\boldsymbol{g}) = \eta(m(\boldsymbol{g}))$.

We say that $\boldsymbol{g}$ is a minimal network if $\boldsymbol{g} = m(\boldsymbol{g})$. We denote by $\mathcal{G}^m$ the set of minimal networks. Let $\mathcal{G}^1 = \{\boldsymbol{g} \in \mathcal{G}^m | i \in N_j(\boldsymbol{g}), j \notin N_i(\boldsymbol{g}), k \notin N_j(\boldsymbol{g}) \Rightarrow \boldsymbol{g}_{k,i} = 0\}$ be a subset of minimal networks. Essentially these are networks there there can be no more than one cycle involving any triad of players. Let $\mathcal{G}^2 \subset \mathcal{G}^1$ be the set of networks which belong to $\mathcal{G}^1$ and which contain at most one cycle. If $\boldsymbol{g} \in \mathcal{G}^2$ and $\boldsymbol{g}$ contains a cycle, then we denote by $C(\boldsymbol{g})$ the cycle in the network $\boldsymbol{g}$. We denote by $N^{C(\boldsymbol{g})}$ the set of players who belong to the cycle $C(\boldsymbol{g})$, and $E^{C(\boldsymbol{g})} \subset N^{C(\boldsymbol{g})} \times N^{C(\boldsymbol{g})}$ the set of links which belong to the cycle $C(\boldsymbol{g})$. Let $\mathcal{G}^3 = \{\boldsymbol{g} \in \mathcal{G}^2 | i \in C(\boldsymbol{g}), j \notin C(\boldsymbol{g}) \Rightarrow \boldsymbol{g}_{j,i} = 0\}$ be the set of networks which belong to $\mathcal{G}^2$ and where there does not exist any link from a player

$j \notin C(\boldsymbol{g})$ to a player $i \in C(\boldsymbol{g})$.

We now present some lemmas which allow us to prove Proposition 2. The first lemma presents some properties about links that cannot arise in the set $\mathcal{G}^3$.

**Lemma 1** *Suppose values of links are heterogeneous by pairs and costs of links are heterogeneous by players and* $\boldsymbol{g} \in \mathcal{G}^3$*.*

1. *If* $\boldsymbol{g}_{j,i} = 1$*, then there does not exist a player* $k$ *such that* $\boldsymbol{g}_{k,i} = 1$*.*
2. *If* $\boldsymbol{g}_{i,j} = 1$*, then* $\mathcal{K}(\boldsymbol{g}; i, j) = N_i(\boldsymbol{g} \ominus i, j) \bigcap N_i(\boldsymbol{g}_{-i} \oplus i, j)$ *is an empty set.*

**Proof** We successively prove both parts of the lemma.

1. To obtain a contradiction suppose that there exist two players $i$ and $j$ such that $\boldsymbol{g}_{j,i} = 1$ and $\boldsymbol{g}_{k,i} = 1$ in $\boldsymbol{g} \in \mathcal{G}^3$. Then there are two possibilities:

   Suppose $i \in N^{C(\boldsymbol{g})}$. Given that $i \in N^{C(\boldsymbol{g})}$ there can be at most one link to player $i$. Hence $j \notin N^{C(\boldsymbol{g})}$ and $k \notin N^{C(\boldsymbol{g})}$ simultaneously. Only one of them is in $N^{C(\boldsymbol{g})}$. Without loss of generality let $j \in N^{C(\boldsymbol{g})}$. Then $\boldsymbol{g}_{k,i} = 1$ violates the fact that $\boldsymbol{g} \in \mathcal{G}^3$.

   Suppose $i \notin N^{C(\boldsymbol{g})}$. Then we know that $\boldsymbol{g}_{i,j} = 0 = \boldsymbol{g}_{i,k}$ since $\boldsymbol{g} \in \mathcal{G}^3 \subseteq \mathcal{G}^1$. From the minimality of $\boldsymbol{g}$ we know that $j \notin N_k(\boldsymbol{g})$ and $k \notin N_j(\boldsymbol{g})$. Putting all this together we have $i \in N_j(\boldsymbol{g})$, $j \notin N_k(\boldsymbol{g})$, $k \notin N_j(\boldsymbol{g}) \Rightarrow \boldsymbol{g}_{k,i} = 0$. This is a contradiction.

2. Suppose there exists a player $k \in N_i(\boldsymbol{g} \ominus i, j) \bigcap N_i(\boldsymbol{g}_{-i} \oplus i, j)$. Then, there exist two different paths from player $k$ to player $i$ which is impossible by the minimality of $\boldsymbol{g}$.

□

It follows that if $\boldsymbol{g} \in \mathcal{G}^3$, then we can write $\pi_i^j(\boldsymbol{g})$ as follows:

$$\pi_i^j(\boldsymbol{g}) = \sum_{k \in N_i(\boldsymbol{g}_{-i} \oplus i,j)} V_{i,k} - c_i. \tag{2}$$

In the following lemma, we let $\boldsymbol{g}'_i \in \mathcal{G}_i$ be a strategy of player $i$, with $\boldsymbol{g}'_i \neq \boldsymbol{g}_i$. This lemma provides the best response properties of the networks $\boldsymbol{g} \in \mathcal{G}^3$.

**Lemma 2** *Suppose values of links are heterogeneous by pairs, costs of links are heterogeneous by players and* $\boldsymbol{g} \in \mathcal{G}^3$.

1. *Suppose players* $i \in N$, $j \in N$, $k \in N$ *are such that* $j \notin N_i(\boldsymbol{g})$, $i \in N_j(\boldsymbol{g})$, $k \notin N_j(\boldsymbol{g})$. *If* $\boldsymbol{g}'_{k,i} = 1$, *then* $\boldsymbol{g}'_k \notin \mathcal{BR}_k(\boldsymbol{g}_{-k})$.

2. *Suppose* $\boldsymbol{g}$ *contains a cycle* $C(\boldsymbol{g})$ *and for all* $i \in N^{C(\boldsymbol{g})}$, *and for all* $i, j \in E^{C(\boldsymbol{g})}$, *we have* $\pi_i^j(\boldsymbol{g}) > 0$. *If* $\boldsymbol{g}'_{i,j} = 0$, *then* $\boldsymbol{g}'_i \notin \mathcal{BR}_i(\boldsymbol{g}_{-i})$.

3. *Suppose* $i \in N$, $j \in N \setminus N^{C(\boldsymbol{g})}$ *and* $\boldsymbol{g}_{i,j} = 1 \Rightarrow \pi_i^j(\boldsymbol{g}) > 0$. *If* $\boldsymbol{g}'_{i,j} = 0$, *then* $\boldsymbol{g}'_i \notin \mathcal{BR}_i(\boldsymbol{g}_{-i})$.

**Proof** We now prove each part of the lemma.

1. Let players $i$, $j$ and $k$ be such that $j \notin N_i(\boldsymbol{g})$, $i \in N_j(\boldsymbol{g})$ and $k \notin N_j(\boldsymbol{g})$. By lemma 1.1, we know that $\boldsymbol{g}_{k,i} = 0$. Either already $i \in N_k(\boldsymbol{g})$ and the formation of the link $k, i$ is not a best response for player $k$, or $i \notin N_k(\boldsymbol{g})$. In the latter case, we have $j \notin N_k(\boldsymbol{g})$, $N_i(\boldsymbol{g}) \subset N_j(\boldsymbol{g})$, so $\pi_k(\boldsymbol{g} \oplus k, j) - \pi_k(\boldsymbol{g} \oplus k, i) \geq V_{k,j} > 0$. From this it follows that player $k$ does not play a best response if she forms a link with player $i$.

2. Without loss of generality, let $C(\boldsymbol{g})$ be such that $N^{C(\boldsymbol{g})} = \{1, 2, \ldots, p\}$ and $E^{C(\boldsymbol{g})} = \{1; 1, 2; 2, 3; \ldots; p-1, p; p, 1\}$. For simplicity now consider a player $i \neq p$.

   It is straightforward from $\pi_i^{i-1}(\boldsymbol{g}) > 0$ and the minimality of $\boldsymbol{g}$ that player $i$ does not play a best response if she deletes the link $i, i-1 \in E^{C(\boldsymbol{g})}$ and does not replace that link.

   We first show that player $i$ cannot play a best response if she replaces the link $i, i-1$ by a link $i, k$, with $k \neq i-1$. Indeed, if player $i$ replaces the link $i, i-1$ by a link $i, k$, $k \in N_i(\boldsymbol{g})$, then player $i$ is not playing a best response.

   We now show that if player $i$ replaces the link $i, i-1$ by a link $i, k$, $k \notin N_i(\boldsymbol{g})$, then player $i$ does not play a best response. Indeed, since $\boldsymbol{g} \in \mathcal{G}^3$, there does not exist a player $k \notin N_i(\boldsymbol{g})$, with $k \in N \setminus N^{C(\boldsymbol{g})}$, such that $\ell \in N_k(\boldsymbol{g})$ and $\ell \in N^{C(\boldsymbol{g})}$. Otherwise, there exist a player $k' \in N \setminus N^{C(\boldsymbol{g})}$, with $k \in N_{k'}(\boldsymbol{g})$, and a player $\ell' \in N^{C(\boldsymbol{g})}$ such that $\boldsymbol{g}_{k',\ell'} = 1$. In that case, $\boldsymbol{g} \notin \mathcal{G}^3$ and we obtain a contradiction. Likewise, there does not exist a player $k \notin N_i(\boldsymbol{g})$ such that $\ell \in N_k(\boldsymbol{g})$ and $\ell \in N_i(\boldsymbol{g}) \setminus N^{C(\boldsymbol{g})}$. Indeed, if $\ell \in N_k(\boldsymbol{g})$ and $\ell \in N_i(\boldsymbol{g}) \setminus N^{C(\boldsymbol{g})}$, then there exists a player $\ell'$ such that $\boldsymbol{g}_{\ell',\ell} = 1$, with $\ell' \in N_i(\boldsymbol{g})$ and a player $k'$ such that $\boldsymbol{g}_{k',\ell} = 1$, with $k' \in N_k(\boldsymbol{g})$ which is impossible by lemma 1.1. It follows that a player $i \in N^{C(\boldsymbol{g})}$ cannot obtain the resources of a player $\ell \in N_i(\boldsymbol{g}) \setminus N_i(\boldsymbol{g} \ominus i, i-1)$ from a player $k \notin N_i(\boldsymbol{g})$. Hence, if player $i$ replaces the link $i, i-1 \in E^{C(\boldsymbol{g})}$ by a link $i, k$ with $k \notin N_i(\boldsymbol{g})$, then player $i$ does not play a best response.

3. It is straightforward from $\pi_i^j(\boldsymbol{g}) > 0$ and the minimality of $\boldsymbol{g}$ that player $i$ has no incentive to delete the link $i, j$ if she does not replace that link.

   We now show that player $i$ has no incentive to replace the link $i, j$. In other words, we show that there does not exist a player $k$ who obtains a part of the resources of $j$ and allows $i$ to obtain more resources than $j$.

Let $k$ be such that $N_k(\boldsymbol{g}) \cap N_j(\boldsymbol{g}) = \emptyset$. Then player $i$ has no incentive to substitute the link $i, k$ to the link $i, j$. Hence $N_k(\boldsymbol{g}) \cap N_j(\boldsymbol{g}) \neq \emptyset$.

First, we must show that if $N_k(\boldsymbol{g}) \cap N_j(\boldsymbol{g}) \neq \emptyset$, then either $N_k(\boldsymbol{g}) \subset N_j(\boldsymbol{g})$ or $N_j(\boldsymbol{g}) \subset N_k(\boldsymbol{g})$. If the former is true the proof is obvious and we will only focus on the latter. Note that in $\boldsymbol{g}$, $N_k(\boldsymbol{g}) \neq N_j(\boldsymbol{g})$ since $j \notin N^{C(\boldsymbol{g})}$. To obtain a contradiction, suppose that $N_k(\boldsymbol{g}) \cap N_j(\boldsymbol{g}) \neq \emptyset$, $N_k(\boldsymbol{g}) \not\subseteq N_j(\boldsymbol{g})$ and $N_j(\boldsymbol{g}) \not\subseteq N_k(\boldsymbol{g})$. Then there exist players $\ell \in N_j(\boldsymbol{g}) \cap N_k(\boldsymbol{g})$, $\ell_j \in N_j(\boldsymbol{g})$ and $\ell_k \in N_k(\boldsymbol{g})$, such that $\boldsymbol{g}_{\ell_j,\ell} = \boldsymbol{g}_{\ell_k,\ell} = 1$, which is impossible by Lemma 1.1.

Second, we must show that there does not exist a player $k \in N$, such that $N_j(\boldsymbol{g}) \subset N_k(\boldsymbol{g})$ and $N_i(\boldsymbol{g}) \not\subseteq N_k(\boldsymbol{g})$, who obtains the resources of $j$ and allows $i$ additional resources. If $N_i(\boldsymbol{g}) = N_k(\boldsymbol{g})$, then $i \in N^{C(\boldsymbol{g})}$, $k \in N^{C(\boldsymbol{g})}$ and in that case player $i$ cannot obtain a part of the resources of player $j$ due to a link with player $k$, since $\boldsymbol{g}$ is a minimal network. Therefore, we just need to show that the above statement is true for strict set inclusion. To obtain a contradiction, suppose there exists a player $k \in N$ such that $N_j(\boldsymbol{g}) \subset N_k(\boldsymbol{g})$ and $N_i(\boldsymbol{g}) \not\subset N_k(\boldsymbol{g})$. Then there exists a player $\ell_k \in N_k(\boldsymbol{g})$ such that $\boldsymbol{g}_{\ell_k,j} = 1$. Therefore, we have $\boldsymbol{g}_{\ell_k,j} = 1$ and $\boldsymbol{g}_{i,j} = 1$ which is impossible by Lemma 1.1. Since $N_j(\boldsymbol{g}) \subset N_k(\boldsymbol{g})$, $N_i(\boldsymbol{g}) \subset N_k(\boldsymbol{g})$, and $\boldsymbol{g}_{i,j} = 1$, by Lemma 1.2, player $i$ cannot obtain a part of the resources of $j$ due to her link with player $k$. Consequently, if player $i$ deletes the link $i, j$ and replaces it by the link $i, k$, then she does not play a best response.

□

We now introduce some additional definitions that are required to complete the proof. Let $\mathcal{MBR}_i(\boldsymbol{g}_{-i})$ be a modified version of the best response function of player $i \in N$. More precisely, $\boldsymbol{g}'_i \in \mathcal{MBR}_i(\boldsymbol{g}_{-i})$ if $\boldsymbol{g}'_i$ is a best response of player $i$ against $\boldsymbol{g}_{-i}$ and if player $i$ does not form any links that yield zero marginal payoffs. Let $\mathrm{br}_i : \mathcal{G} \to \mathcal{G}$, $\boldsymbol{g} \mapsto \mathrm{br}_i(\boldsymbol{g})$ be a function. The network $\mathrm{br}_i(\boldsymbol{g}) = (\boldsymbol{g}'_i \oplus \boldsymbol{g}_{-i})$ is a network where $\boldsymbol{g}'_i \in \mathcal{MBR}_i(\boldsymbol{g}_{-i})$, and all other players $j \neq i$ having the same links as in the network $\boldsymbol{g}$. In other words, in $\mathrm{br}_i(\boldsymbol{g})$, we have $\mathrm{br}_i(\boldsymbol{g})_{i,j} = 1 \Rightarrow \pi_i^j(\mathrm{br}_i(\boldsymbol{g})) > 0$ and $\mathrm{br}_i(\boldsymbol{g})_{i,j} = 0 \Rightarrow \pi_i^j(\mathrm{br}_i(\boldsymbol{g})) \leq 0$.

Let $N^{\mathcal{C}(\boldsymbol{g})}$ be the set of players who belong to a cycle in $\boldsymbol{g}$. Let $\mathcal{H} : \mathcal{G} \to \mathcal{P}(\mathcal{G})$ be a correspondence. A network $h(\boldsymbol{g}) \in \mathcal{H}(\boldsymbol{g})$ is a network associated with $\boldsymbol{g}$ such that $h(\boldsymbol{g})$ contains at most one cycle, $C(h(\boldsymbol{g}))$. Moreover, if $k$ is such that $\ell \in N_k(\boldsymbol{g})$ and $\ell \in N^{\mathcal{C}(\boldsymbol{g})}$, then $k \in N^{C(h(\boldsymbol{g}))}$. If $k \notin N^{C(h(\boldsymbol{g}))}$, then for all $\ell \in N$, we have $g_{\ell,k} = h(\boldsymbol{g})_{\ell,k}$. This is different from the networks in $\mathcal{G}^2$ since there is no minimality restriction here. This operation creates one cycle leaving unchanged the strategies of those players that do not form a part of the cycle.

Observe that for all $\boldsymbol{g} \in \mathcal{G}$ and for all $k \in N$, we have, by construction, for all $\boldsymbol{g}' \in \mathcal{M} \circ \mathcal{H}(\boldsymbol{g})$, $N_k(\boldsymbol{g}) \subseteq N_k(\boldsymbol{g}')$.

Finally, we define

$$\overline{\boldsymbol{g}}^i \in \mathcal{M} \circ \mathcal{H} \circ \mathrm{br}_i(\boldsymbol{g}), \tag{3}$$

to be a network obtained from $\boldsymbol{g}$ after performing the three operations defined above. Note that the superscript in $\overline{\boldsymbol{g}}^i$ refers to the fact that in this network only player $i$ is playing her best response.

**Lemma 3** *If $\boldsymbol{g} \in \mathcal{G}^3$, then $\overline{\boldsymbol{g}}^i \in \mathcal{G}^3$.*

**Proof** We must show that $\overline{\boldsymbol{g}}^i$ has the following four properties: it is a minimal network, it contains at most one cycle, there does not exist a link from $j \notin N^{C(\overline{\boldsymbol{g}}^i)}$ to $k \in N^{C(\overline{\boldsymbol{g}}^i)}$ and if $\ell \in N_j(\overline{\boldsymbol{g}}^i), j \notin N_\ell(\overline{\boldsymbol{g}}^i), k \notin N_j(\overline{\boldsymbol{g}}^i)$ then $\overline{\boldsymbol{g}}^i_{k,\ell} = 0$. The first property follows from the correspondence $\mathcal{M}$ and the next two from the correspondence $\mathcal{H}$. We just need to verify that the last property is enjoyed.

First, we show that in $\mathrm{br}_i(\boldsymbol{g})$, we have $\ell \in N_j(\mathrm{br}_i(\boldsymbol{g}))$, $j \notin N_\ell(\mathrm{br}_i(\boldsymbol{g}))$, $i \notin N_j(\mathrm{br}_i(\boldsymbol{g}))$ $\Rightarrow \mathrm{br}_i\ (\boldsymbol{g})_{i,\ell} = 0$. We know that in $\boldsymbol{g}$ we have $\ell \in N_j(\boldsymbol{g}), j \notin N_\ell(\boldsymbol{g}), i \notin N_j(\boldsymbol{g}) \Rightarrow \boldsymbol{g}_{i,\ell} = 0$ since $\boldsymbol{g} \in \mathcal{G}^3$. By definition, we have $\mathrm{br}_i(\boldsymbol{g})_k = \boldsymbol{g}_k$, for all $k \in N \setminus \{i\}$. Hence, if we show that player $i \notin N_j(\mathrm{br}_i(\boldsymbol{g}))$ has not formed a link $i, \ell$ with a player $\ell$ such that $\ell \in N_j(\mathrm{br}_i(\boldsymbol{g}))$ and $j \notin N_\ell(\mathrm{br}_i(\boldsymbol{g}))$ in $\mathrm{br}_i(\boldsymbol{g})$, then we will have shown the conclusion for $\mathrm{br}_i(\boldsymbol{g})$. But, by Lemma 2.1, we know that if $i$ has formed a link with player $\ell$, then $i$ is not playing a best response which is a contradiction.

Second, by construction, if $\boldsymbol{g}$ is such that $\ell \in N_j(\boldsymbol{g}), j \notin N_\ell(\boldsymbol{g}), k \notin N_j(\boldsymbol{g}) \Rightarrow \boldsymbol{g}_{_{k,\ell}} = 0$, then $\boldsymbol{g}' \in \mathcal{M} \circ \mathcal{H}(\boldsymbol{g})$ is such that $\ell \in N_j(\boldsymbol{g}')$, $j \notin N_\ell(\boldsymbol{g}')$, $k \notin N_j(\boldsymbol{g}') \Rightarrow \boldsymbol{g}'_{k,\ell} = 0$. The conclusion follows. □

The next lemma covers properties of networks in $\overline{\boldsymbol{g}}^i$ and $\mathrm{br}_i(\boldsymbol{g})$.

**Lemma 4** *Suppose $\boldsymbol{g} \in \mathcal{G}^3$ and for all $k \in N$, $j \in N$, $\boldsymbol{g}_{k,j} = 1 \Rightarrow \pi^j_k(\boldsymbol{g}) > 0$.*

1. *If $k \in N_j(\boldsymbol{g})$, then $k \in N_j(\mathrm{br}_i(\boldsymbol{g}))$.*

2. *If $k \in N_j(\boldsymbol{g})$, then $k \in N_j(\overline{\boldsymbol{g}}^i)$.*

3. *If $\boldsymbol{g}_i \notin \mathcal{BR}_i(\boldsymbol{g}_{-i})$, then $\eta(\boldsymbol{g}) < \eta(\overline{\boldsymbol{g}}^i)$.*

**Proof** We successively prove each part of the Lemma.

1. Observe that for all $k \neq i$, and for all $j \in N$, we have $\boldsymbol{g}_{k,j} = \mathrm{br}_i(\boldsymbol{g})_{k,j}$. Hence, if $N_j(\boldsymbol{g}) \nsubseteq N_j(\mathrm{br}_i(\boldsymbol{g}))$, then there exists a player $k$ such that $k \in N_i(\boldsymbol{g})$ and $k \notin N_i(\mathrm{br}_i(\boldsymbol{g}))$. Since $\boldsymbol{g} \in \mathcal{G}^3$, we know from Lemma 2.2 and 2.3, that player $i$ will not be playing a best response if she deletes one of her links. Hence, if $k \in N_i(\boldsymbol{g})$, then $k \in N_i(\mathrm{br}_i(\boldsymbol{g}))$, and we obtain the desired conclusion.

2. We know from the first part of the lemma that $N_j(\boldsymbol{g}) \subseteq N_j(\mathrm{br}_i(\boldsymbol{g}))$, and we know that $N_j(\mathrm{br}_i(\boldsymbol{g})) \subseteq N_j(\boldsymbol{g}')$, for all $\boldsymbol{g}' \in \mathcal{M} \circ \mathcal{H}(\mathrm{br}_i(\boldsymbol{g}))$. The result follows.

3. From the second part of the lemma, we know that $N_j(\boldsymbol{g}) \subseteq N_j(\overline{\boldsymbol{g}}^i)$ for all $j \neq i$. We now show that if $\boldsymbol{g}_i \notin \mathcal{BR}_i(\boldsymbol{g}_{-i})$, then $N_i(\boldsymbol{g}) \subset N_i(\overline{\boldsymbol{g}}^i)$. By Lemma 2.2 and 2.3, we know that player $i$ cannot be playing a best response if she deletes links. Hence, if she is playing a best response, it must be that $N_i(\boldsymbol{g}) \subset N_i(\mathrm{br}_i(\boldsymbol{g}))$. Since, we know that, for all $\boldsymbol{g}' \in \mathcal{M} \circ \mathcal{H}(\mathrm{br}_i(\boldsymbol{g}))$, $N_i(\mathrm{br}_i(\boldsymbol{g})) \subseteq N_i(\boldsymbol{g}')$, we conclude that $N_i(\boldsymbol{g}) \subset N_i(\overline{\boldsymbol{g}}^i)$. Therefore, $\eta(\boldsymbol{g}) < \eta(\overline{\boldsymbol{g}}^i)$.

□

Let us denote by $\boldsymbol{g} \setminus \mathcal{MBR}_i(\boldsymbol{g}_{-i}) = \boldsymbol{gm}$. Then $\boldsymbol{gm} \oplus i, j$ is the network obtained from $\mathrm{br}_i(\boldsymbol{g})$ when player $i$ forms no link except the link $i, j$.

**Lemma 5** *Suppose* $\boldsymbol{g} \in \mathcal{G}^3$.

1. *If* $\overline{\boldsymbol{g}}^i_{i,j} = \mathrm{br}_i(\boldsymbol{g})_{i,j} = 1$, *then, for all* $j \in N \setminus \{i\}$, $N_j(\boldsymbol{gm} \oplus i, j) \subseteq N_j(\overline{\boldsymbol{g}}^i_{-i} \oplus i, j)$.

2. *Suppose for all* $i \in N$, $j \in N$, $\boldsymbol{g}_{i,j} = 1 \Rightarrow \pi_i^j(\boldsymbol{g}) > 0$. *If* $\overline{\boldsymbol{g}}^i_{k,\ell} = \boldsymbol{g}_{k,\ell} = 1$, *then* $N_\ell(\boldsymbol{g}_{-k} \oplus k, \ell) \subseteq N_\ell(\overline{\boldsymbol{g}}^i_{-k} \oplus k, \ell)$.

**Proof** We prove the two parts of the lemma successively.

1. If $j \notin N^{C(\overline{\boldsymbol{g}}^i)}$, then $N_j(\overline{\boldsymbol{g}}^i_{-i}) = N_j(\overline{\boldsymbol{g}}^i)$. Indeed, since $\overline{\boldsymbol{g}}^i \in \mathcal{G}^3$, $j \notin N^{C(\overline{\boldsymbol{g}}^i)}$, and $\overline{\boldsymbol{g}}^i_{i,j} = 1$, player $j$ does not obtain any resources from player $i$. Moreover, we have by construction, $N_j(\text{br}_i(\boldsymbol{g})) \subseteq N_j(\overline{\boldsymbol{g}}^i)$. It follows that $N_j(\boldsymbol{gm} \oplus i, j) \subseteq N_j(\text{br}_i(\boldsymbol{g})) \subseteq N_j(\overline{\boldsymbol{g}}^i) = N_j(\overline{\boldsymbol{g}}^i_{-i}) \subseteq N_j(\overline{\boldsymbol{g}}^i_{-i} \oplus i, j)$.

   Assume that $j \in N^{C(\overline{\boldsymbol{g}}^i)}$, $\overline{\boldsymbol{g}}^i_{i,j} = \text{br}_i(\boldsymbol{g})_{i,j} = 1$ and there exists a player $\ell$ such that $\ell \in N_j(\boldsymbol{gm} \oplus i, j)$ and $\ell \notin N_j(\overline{\boldsymbol{g}}^i_{-i} \oplus i, j)$. So in $\text{br}_i(\boldsymbol{g})$, player $i$ obtains resources from player $\ell$ through a path containing $j$, and in $\overline{\boldsymbol{g}}^i$ player $i$ obtains resources from player $\ell$ through a path which does not contain $j$, since for all $k \in N$, $N_k(\text{br}_i(\boldsymbol{g})) \subseteq N_k(\overline{\boldsymbol{g}}^i)$. Hence, there is a player $j'$ where $j' \in N_i(\overline{\boldsymbol{g}}^i)$, $j' \notin N^{C(\overline{\boldsymbol{g}}^i)}$ and $j' \in N_j(\overline{\boldsymbol{g}}^i)$ who has formed a link with player $\ell$ between $\text{br}_i(\boldsymbol{g})$ and $\overline{\boldsymbol{g}}^i$. This is not possible by construction.

2. If $\ell \notin N^{C(\overline{\boldsymbol{g}}^i)}$, then $N_\ell(\overline{\boldsymbol{g}}^i_{-k} \oplus k, \ell) = N_\ell(\overline{\boldsymbol{g}}^i)$ since player $\ell$ does not obtain any resources from player $k$. Moreover, we know by Lemma 4.1 and 4.2 that $N_\ell(\boldsymbol{g}) \subseteq N_\ell(\overline{\boldsymbol{g}}^i)$. It follows that $N_\ell(\boldsymbol{g}_{-k} \oplus k, \ell) \subseteq N_\ell(\boldsymbol{g}) \subseteq N_\ell(\overline{\boldsymbol{g}}^i) = N_\ell(\overline{\boldsymbol{g}}^i_{-k} \oplus k, \ell)$.

   Suppose now that $\ell \in N^{C(\overline{\boldsymbol{g}}^i)}$. Note that $k \in N^{C(\overline{\boldsymbol{g}}^i)}$ since $k$ has formed a link with $\ell$. For a contradiction assume that $\ell \in N^{C(\overline{\boldsymbol{g}}^i)}$ and $N_\ell(\boldsymbol{g}_{-k} \oplus k, \ell) \nsubseteq N_\ell(\overline{\boldsymbol{g}}^i_{-k} \oplus k, \ell)$. Then there is a player $j$ such that $j \in N_\ell(\boldsymbol{g}_{-k} \oplus k, \ell)$ and $j \notin N_\ell(\overline{\boldsymbol{g}}^i_{-k} \oplus k, \ell)$. Also note that $j \notin N^{C(\overline{\boldsymbol{g}}^i)}$, otherwise $j \in N_\ell(\overline{\boldsymbol{g}}^i_{-k} \oplus k, \ell)$. Moreover, if $j \in N_\ell(\boldsymbol{g}_{-k} \oplus k, \ell)$ and $j \notin N_\ell(\overline{\boldsymbol{g}}^i_{-k} \oplus k, \ell)$, then $j \notin N_k(\boldsymbol{g} \ominus k, \ell)$ and $j \in N_k(\overline{\boldsymbol{g}}^i \ominus k, \ell)$ since $\boldsymbol{g} \in \mathcal{G}^3$, and $N_\ell(\boldsymbol{g}) \subseteq N_\ell(\overline{\boldsymbol{g}}^i)$ by Lemma 4.1 and 4.2. In other words, player $k$ obtains resources from player $j$ in $\boldsymbol{g}$ through a path which contains $\ell$, and in $\overline{\boldsymbol{g}}^i$ player $k$ obtains resources from player $j$ through a path which does not contain $\ell$. Hence, there exists a player who has formed a link with a player $\ell'$ where $\ell' \in N_k(\overline{\boldsymbol{g}}^i)$,

$j \in N_{\ell'}(\overline{\boldsymbol{g}}^i)$, and $k \notin N_{\ell'}(\overline{\boldsymbol{g}}^i)$ between $\boldsymbol{g}$ and $\overline{\boldsymbol{g}}^i$. This is not possible by construction of $\overline{\boldsymbol{g}}^i$.

□

**Lemma 6** *Let $\overline{\boldsymbol{g}}^i$ be defined as in equation (3).*

1. *If $\boldsymbol{g} \in \mathcal{G}^3$, then $\overline{\boldsymbol{g}}^i_{i,j} = 1 \Rightarrow \pi_i^j(\overline{\boldsymbol{g}}^i) > 0$.*
2. *If for all $i \in N$, $j \in N$, $\boldsymbol{g}_{i,j} = 1 \Rightarrow \pi_i^j(\boldsymbol{g}) > 0$, then for all $i \in N \setminus \{k\}$, $j \in N$, $\overline{\boldsymbol{g}}^k_{i,j} = 1 \Rightarrow \pi_i^j(\overline{\boldsymbol{g}}^k) > 0$.*

**Proof** We now prove successively the two parts of the lemma.

1. (a) First, we show that this property is true if $\overline{\boldsymbol{g}}^i_{i,j} = 1$ and $j \notin N^{C(\overline{\boldsymbol{g}}^i)}$. If $j \notin N^{C(\overline{\boldsymbol{g}}^i)}$, then by construction $\mathrm{br}_i(\boldsymbol{g})_{i,j} = 1$ and so $\pi_i^j(\mathrm{br}_i(\boldsymbol{g})) > 0$. Using Lemma 5.1, Lemma 3, and the marginal profit function defined in equation (2) we have:

$$\begin{aligned}
\pi_i^j(\overline{\boldsymbol{g}}^i) &= \textstyle\sum_{k \in N_j(\overline{\boldsymbol{g}}^i_{-i} \oplus i,j)} V_{i,k} - c_i \\
&\geq \textstyle\sum_{k \in N_j(\boldsymbol{g} \setminus \mathcal{MBR}_i(\boldsymbol{g}_{-i}) \oplus i,j)} V_{i,k} - \sum_{k \in \mathcal{K}(\mathrm{br}_i(\boldsymbol{g}); i,j)} V_{i,k} - c_i \\
&= \pi_i^j(\mathrm{br}_i(\boldsymbol{g})) > 0
\end{aligned}$$

   (b) Second, we show that this property is true if $\overline{\boldsymbol{g}}^i_{i,j} = 1$ and $j \in N^{C(\overline{\boldsymbol{g}}^i)}$. By construction if $\overline{\boldsymbol{g}}^i_{i,j} = 1$ and $j \in N^{C(\overline{\boldsymbol{g}}^i)}$, then $i \in N^{C(\overline{\boldsymbol{g}}^i)}$. If $i \in N^{C(\overline{\boldsymbol{g}}^i)}$, then by construction of $\overline{\boldsymbol{g}}^i$, there is at least one player $\ell \in N^{C(\overline{\boldsymbol{g}}^i)}$, such that $\pi_i^\ell(\mathrm{br}_i(\boldsymbol{g})) > 0$. So for all players $\ell' \in N^{C(\overline{\boldsymbol{g}}^i)}$, there exists a network $(\overline{\boldsymbol{g}}^i)' \in \mathcal{M} \circ \mathcal{H} \circ \mathrm{br}_i(\boldsymbol{g})$ where player $i$ forms a link with player $\ell'$, and by construction $\pi_i^j(\overline{\boldsymbol{g}}^i) = \pi_i^{\ell'}((\overline{\boldsymbol{g}}^i)')$. We know by Lemma 5.1, that $N_j(\boldsymbol{gm} \oplus i,j) \subseteq N_j(\overline{\boldsymbol{g}}^i_{-i} \oplus i,j)$. Finally, by Lemma 3, we know that $\overline{\boldsymbol{g}}^i \in \mathcal{G}^3$. Hence using the marginal profit function defined in equation (2) we have:

$$
\begin{aligned}
\pi_i^j(\overline{\boldsymbol{g}}^i) &= \textstyle\sum_{k \in N_j(\overline{\boldsymbol{g}}^i_{-i} \oplus i,j)} V_{i,k} - c_i = \sum_{k \in N_\ell((\overline{\boldsymbol{g}}^i_{-i})' \oplus i,\ell)} V_{i,k} - c_i \\
&\geq \textstyle\sum_{k \in N_\ell(\boldsymbol{gm} \oplus i,\ell)} V_{i,k} - \sum_{k \in \mathcal{K}(\boldsymbol{gm} \oplus i,\ell;i,\ell)} V_{i,k} - c_i \\
&= \pi_i^\ell(\mathrm{br}_i(\boldsymbol{g})) > 0.
\end{aligned}
$$

2. First, we show that for all $i \in N \setminus \{k\}$, and for all $j \notin N^{C(\overline{\boldsymbol{g}}^k)}$, if $\boldsymbol{g}_{i,j} = 1 \Rightarrow \pi_i^j(\boldsymbol{g}) > 0$, then $\overline{\boldsymbol{g}}^k_{i,j} = 1 \Rightarrow \pi_i^j(\overline{\boldsymbol{g}}^k) > 0$. Indeed, if player $i \in N \setminus \{k\}$ has a link with player $j \notin N^{C(\overline{\boldsymbol{g}}^k)}$ in $\overline{\boldsymbol{g}}^k$, then, by construction of $\overline{\boldsymbol{g}}^k$, player $i$ has a link with player $j$ in $\boldsymbol{g}$, so $\pi_i^j(\boldsymbol{g}) > 0$. We know, from Lemma 5.2, that for all $j \in N$, we have $N_j(\boldsymbol{g}_{-i} \oplus i,j) \subseteq N_j(\overline{\boldsymbol{g}}^k_{-i} \oplus i,j)$. Moreover, by Lemma 3, $\overline{\boldsymbol{g}}^k \in \mathcal{G}^3$. So using the marginal profit function defined in equation (2) we have:

$$
\begin{aligned}
\pi_i^j(\overline{\boldsymbol{g}}^k) &= \textstyle\sum_{\ell \in N_j(\overline{\boldsymbol{g}}^k_{-i} \oplus i,j)} V_{i,\ell} - c_i \\
&\geq \textstyle\sum_{\ell \in N_j(\boldsymbol{g}_{-i} \oplus i,j)} V_{i,\ell} - c_i \\
&= \pi_i^j(\boldsymbol{g}) > 0.
\end{aligned}
$$

Next, we show that for all $i \in N \setminus \{k\}$, and for all $j \in N^{C(\overline{\boldsymbol{g}}^k)}$, if $\boldsymbol{g}_{i,j} = 1 \Rightarrow \pi_i^j(\boldsymbol{g}) > 0$, then $\overline{\boldsymbol{g}}^k_{i,j} = 1 \Rightarrow \pi_i^j(\overline{\boldsymbol{g}}^k) > 0$. Since $\overline{\boldsymbol{g}}^k \in \mathcal{G}^3$ and there exists a link from player $j$ to player $i$, we have $i \in N^{C(\overline{\boldsymbol{g}}^k)}$. If $i \in N^{C(\overline{\boldsymbol{g}}^k)}$, then there are two possibilities: either $k \in N_i(\text{br}_k(\boldsymbol{g}))$ or $i \in N^{C(\boldsymbol{g})}$. We deal with these two possibilities successively.

(a) If $k \in N_i(\text{br}_k(\boldsymbol{g}))$, then there exists in $\text{br}_k(\boldsymbol{g})$ a link from player $i$ to a player $\ell$ such that $\text{br}_k(\boldsymbol{g})_{i,\ell} = \boldsymbol{g}_{i,\ell} = 1$ and $k \in N_\ell(\text{br}_k(\boldsymbol{g}))$. Since, $\boldsymbol{g}_{i,\ell} = 1$, we have $\pi_i^\ell(\boldsymbol{g}) > 0$. Furthermore, by construction, player $\ell \in N^{C(\overline{\boldsymbol{g}}^k)}$, since $k \in N_\ell(\text{br}_k(\boldsymbol{g}))$. We note that for all players $h' \in N^{C(\overline{\boldsymbol{g}}^k)}$, there exists a network $(\overline{\boldsymbol{g}}^k)' \in \mathcal{M} \circ \mathcal{H} \circ \text{br}_k(\boldsymbol{g})$ where player $i$ forms a link with player $h'$, and by construction $\pi_i^j(\overline{\boldsymbol{g}}^k) = \pi_i^{h'}((\overline{\boldsymbol{g}}^k)')$. We know from Lemma 5.2 that for all $j \in N$, we have $N_j(\boldsymbol{g}_{-i} \oplus i,j) \subseteq N_j(\overline{\boldsymbol{g}}^k_{-i} \oplus i,j)$. Finally, we know by Lemma 3 that $\overline{\boldsymbol{g}}^i \in \mathcal{G}^3$. Hence, using the marginal profit function defined by equation (2), we obtain:

$$\begin{aligned}
\pi_i^j(\overline{\boldsymbol{g}}^k) &= \textstyle\sum_{\ell' \in N_j(\overline{\boldsymbol{g}}^k_{-i} \oplus i,j)} V_{i,\ell'} - c_i = \textstyle\sum_{\ell' \in N_\ell((\overline{\boldsymbol{g}}^k_{-i})' \oplus i,\ell)} V_{i,\ell'} - c_i \\
&\geq \textstyle\sum_{\ell' \in N_\ell(\boldsymbol{g}_{-i} \oplus i,\ell)} V_{i,\ell'} - c_i \\
&= \pi_i^\ell(\boldsymbol{g}) > 0.
\end{aligned}$$

(b) If $i \in N^{C(\boldsymbol{g})}$, then we have $\pi_i^\ell(\boldsymbol{g}) > 0$ for $i, \ell \in E^{C(\boldsymbol{g})}$. We assume, without loss of generality, that player $i$ forms in $C(\overline{\boldsymbol{g}}^i)$ a link with a player $j$ such that $\pi_i^j(\text{br}_i(\boldsymbol{g})) > 0$. By construction of $\overline{\boldsymbol{g}}^k$ we have $N^{C(\boldsymbol{g})} \subseteq N^{C(\overline{\boldsymbol{g}}^k)}$ and by Lemma 5.2, we have $N_j(\boldsymbol{g}_{-i} \oplus i,j) \subseteq N_j(\overline{\boldsymbol{g}}^k_{-i} \oplus i,j)$ for all $j \in N$. Note that for all players $h' \in N^{C(\overline{\boldsymbol{g}}^k)}$, there exists a network $(\overline{\boldsymbol{g}}^k)' \in \mathcal{M} \circ \mathcal{H} \circ \text{br}_k(\boldsymbol{g})$ where player $i$ forms a link with player $h'$. Also by construction $\pi_i^j(\overline{\boldsymbol{g}}^k) = \pi_i^{h'}((\overline{\boldsymbol{g}}^k)')$.

We know by Lemma 3 that $\overline{\boldsymbol{g}}^i \in \mathcal{G}^3$. Again, using the marginal profit function defined by equation (2), we obtain:

$$\begin{aligned}
\pi_i^j(\overline{\boldsymbol{g}}^k) &= \textstyle\sum_{\ell' \in N_j(\overline{\boldsymbol{g}}^k_{-i} \oplus i,j)} V_{i,\ell'} - c_i = \sum_{\ell' \in N_\ell((\overline{\boldsymbol{g}}^k_{-i})' \oplus i,\ell)} V_{i,\ell'} - c_i \\
&\geq \textstyle\sum_{\ell' \in N_\ell(\boldsymbol{g}_{-i} \oplus i,\ell)} V_{i,\ell'} - c_i \\
&= \pi_i^\ell(\boldsymbol{g}) > 0.
\end{aligned}$$

□

**Proof of Proposition 2** We start with the empty network $\dot{\boldsymbol{g}} = \boldsymbol{g}^0$. It is straightforward to check that $\boldsymbol{g}^0 \in \mathcal{G}^3$. Either $\boldsymbol{g}^0$ is a Nash network, and we are done, or there exists a player, say $i$, who does not play a best response in $\boldsymbol{g}^0$. In that case, we construct the network $\boldsymbol{g}^1 \in \mathcal{M} \circ \mathcal{H} \circ \mathrm{br}_i(\boldsymbol{g}^0)$. We know from Lemma 4.3 that $\eta(\boldsymbol{g}^0) < \eta(\boldsymbol{g}^1)$. From Lemma 3, $\boldsymbol{g}^1 \in \mathcal{G}^3$ and from Lemma 6.1 and 6.2, we know that for all players $j \in N$ and $\ell \in N$, $\boldsymbol{g}^1_{j,\ell} = 1 \Rightarrow \pi_j^\ell(\boldsymbol{g}^1) > 0$. Either $\boldsymbol{g}^1$ is a Nash network, and we are done, or there exists a player, say $j$, who does not play a best response in $\boldsymbol{g}^1$. In that case, we construct the network $\boldsymbol{g}^2 \in \mathcal{M} \circ \mathcal{H} \circ \mathrm{br}_j(\boldsymbol{g}^1)$. We know from Lemma 4.3 that $\eta(\boldsymbol{g}^1) < \eta(\boldsymbol{g}^2)$. Again from Lemma 3, $\boldsymbol{g}^2 \in \mathcal{G}^3$ and from Lemma 6.1 and 6.2, we know that for all players $j \in N$ and $\ell \in N$, $\boldsymbol{g}^2_{j,\ell} = 1 \Rightarrow \pi_j^\ell(\boldsymbol{g}^2) > 0$. It follows that we can construct a sequence of networks $\{\boldsymbol{g}^0, \boldsymbol{g}^1 \ldots, \boldsymbol{g}^t, \ldots\}$ such that in $\boldsymbol{g}^{t-1}$, there exists a player, say $k$, who does not play a best response, and $\boldsymbol{g}^t \in \mathcal{M} \circ \mathcal{H} \circ \mathrm{br}_k(\boldsymbol{g}^{t-1})$, $\eta(\boldsymbol{g}^{t-1}) < \eta(\boldsymbol{g}^t)$, $\boldsymbol{g}^t \in \mathcal{G}^3$ and for all $j \in N$, $\boldsymbol{g}^t_{j,\ell} = 1 \Rightarrow \pi_j^\ell(\boldsymbol{g}^t) > 0$. This sequence is finite since $\eta(\boldsymbol{g}) \leq n^2$, for all $\boldsymbol{g} \in \mathcal{G}$ .

□

Proposition 2 establishes that if values of links are heterogeneous by pairs of players and costs of links are heterogeneous by players, then a Nash network always exists. This result is similar to the result of Haller et al. [6] in two-way flow models. We now study one-way flow models when values of links are heterogeneous by players and costs of links are heterogeneous by pairs of players.

### 1.2.2 Existence of Nash networks and heterogeneity of costs by pairs

In example 1 we have shown that a Nash network does not always exist when values of links are heterogeneous by players and costs of links are heterogeneous by pairs of players. We now state a condition which allows for the existence of Nash networks when costs of links are heterogeneous by pairs. In that case, we can write the payoff function as follows:

$$\pi_i(\boldsymbol{g}) = \sum_{j \in N_i(\boldsymbol{g})} V_i - \sum_{j \in N} g_{i,j} c_{i,j}.$$

Let $\pi_i^j(\boldsymbol{g})$ denote the marginal payoff of player $i$ from player $j$ in the network $\boldsymbol{g}$. If $\boldsymbol{g}_{i,j} = 1$, then $\pi_i^j(\boldsymbol{g}) = \pi_i(\boldsymbol{g}) - \pi_i(\boldsymbol{g} \ominus i, j)$. Let $\mathcal{K}(\boldsymbol{g}; i, j) = N_i(\boldsymbol{g} \ominus i, j) \bigcap N_i(\boldsymbol{g}_{-i} \oplus i, j)$. We can rewrite $\pi_i^j(\boldsymbol{g})$ as follows:

$$\pi_i^j(\boldsymbol{g}) = \sum_{k \in N_i(\boldsymbol{g}_{-i} \oplus i, j)} V_i - \sum_{k \in \mathcal{K}(\boldsymbol{g}; i, j)} V_i - c_{i,j}. \tag{4}$$

To prove the following proposition, we need an additional new definition. Let $\mathcal{H}_i : \mathcal{G} \to \mathcal{G}$ be a correspondence where $h_i(\boldsymbol{g}) \in \mathcal{H}_i(\boldsymbol{g})$ satisfies the following conditions.

- If $\boldsymbol{g}$ contains at most one cycle and there does not exist any link from a player $j \notin C(\boldsymbol{g})$ to a player $k \in C(\boldsymbol{g})$, then $\boldsymbol{g} = h_i(\boldsymbol{g})$.

- If player $i$ has formed a link with no player $j \in N^{\mathcal{C}(\boldsymbol{g})}$ or with at least two players $j \in N^{\mathcal{C}(\boldsymbol{g})}$ in $\boldsymbol{g}$, then

  1. if $k$ is such that $\ell \in N_k(\boldsymbol{g})$ and $\ell \in N^{\mathcal{C}(\boldsymbol{g})}$, then $k \in N^{C(h_i(\boldsymbol{g}))}$;
  2. if $k \notin N^{C(h_i(\boldsymbol{g}))}$, then for all $\ell \in N$, we have $g_{\ell,k} = h_i(\boldsymbol{g})_{\ell,k}$.

- If player $i$ has formed a link with one and only one player $j \in N^{\mathcal{C}(\boldsymbol{g})}$ in $\boldsymbol{g}$, then:

  1. if $k$ is such that $\ell \in N_k(\boldsymbol{g})$ and $\ell \in N^{\mathcal{C}(\boldsymbol{g})}$, then $k \in N^{C(h_i(\boldsymbol{g}))}$;
  2. if $k \notin N^{C(h_i(\boldsymbol{g}))}$, then for all $\ell \in N$, we have $g_{\ell,k} = h_i(\boldsymbol{g})_{\ell,k}$;
  3. player $i$ and player $j$ belong to $N^{C(h_i(\boldsymbol{g}))}$ and the link $i, j \in E(h_i(\boldsymbol{g}))$.

We now define $\hat{\boldsymbol{g}}^i$ as follows: $\hat{\boldsymbol{g}}^i \in \mathcal{M} \circ \mathcal{H}_i \circ br_i(\boldsymbol{g})$.

**Proposition 3** *Consider a game where values of links are heterogeneous by players and costs of links are heterogeneous by pairs. There always exists a Nash network if for all $i \in N$, $j \in N$, $j' \in N$: $|c_{i,j} - c_{i,j'}| < V_i$.*

**Proof** The proof of this proposition is similar to the proof of the proposition 2 with $\hat{\boldsymbol{g}}^i$ playing the same role as $\overline{\boldsymbol{g}}^i$). □

**Corollary 1** *Suppose a game where values and costs of links are heterogeneous by pairs. If for all $i \in N$, $j \in N$, $j' \in N$: $|c_{i,j} - c_{i,j'}| < \min_{k \in N}\{V_{i,k}\}$, then there is a Nash network.*

The importance of these results stems from the fact that they identify conditions under which Nash networks always exist under heterogeneity.

## 1.3 Model with Congestion Effect

In one-way flow models with homogeneous players BG [1] establish that Nash networks always exist. We show that this result is no longer true when the payoff function incorporates congestion effects – a phenomenon that frequently arises in many network settings. Billand and Bravard (2005, [3]) characterize Nash networks under congestion effects. In this section, we use their framework to show the non-existence of Nash networks.

Let us define $\phi : N \times \{0, \ldots, n-1\} \to I\!R$, $(x, y) \mapsto \phi_i(x, y)$ be such that:

$$\phi_i(x, y) > \phi_i(x, y + 1).$$

Let $c_i(\boldsymbol{g}) = \sum_{j \neq i} \boldsymbol{g}_{i,j}$ be the costs incurred by $i$ in the network $\boldsymbol{g}$. We now define the payoff function of player $i \in N$ as

$$\bar{\pi}_i(\boldsymbol{g}) = \phi_i(n_i(\boldsymbol{g}), c_i(\boldsymbol{g})).$$

As before we assume that player $i$ obtains her own resources. We now provide an example where a Nash network does not exist.

**Example 2** Let $N = \{1, 2, 3\}$, and $\phi_1(2, 1) > \phi_1(1, 0) > \phi_1(3, 1)$, $\max\{\phi_k\ (2, 1)\ , \phi_k(3, 2)\} < \phi_k(1, 0) < \phi_k(3, 1)$, for $k \in \{2, 3\}$.

First, networks in which a player forms two links are not Nash.

Second, the unique best response of player 2 (respectively 3) to any network $\boldsymbol{g}'$ in which player 1 and player 3 (respectively 2) have formed no link is to form no link. Moreover, the unique best response of player 1 to a network $\boldsymbol{g}$ in which player 2 and player 3 have formed no link is to form a link with player 2 or player 3. Therefore, the empty network is not a Nash network.

Third, a network $\boldsymbol{g}$ where $n_1(\boldsymbol{g}) \neq 2$ cannot be a Nash network. Indeed, it is obvious that $n_1(\boldsymbol{g}) = 3$ cannot be a Nash network since $\phi_1(1,0) > \phi_1(3,1) > \phi_1(3,2)$. Moreover, a network $g$ where $n_1(\boldsymbol{g}) = 1$ cannot be a Nash network. Indeed, in a Nash network where player 1 has formed no links, players 2 and 3 cannot have established any links, since at least one of these players gets the ressources of one player only and we have $\phi_k(2,1) < \phi_k(1,0)$, for $k \in \{2,3\}$. In that case, when players 2 and 3 create no links, player 1 has an incentive to establish a link with player 2 or player 3. To sum up if there exists a Nash network $\boldsymbol{g}$, then $n_1(\boldsymbol{g}) = 2$.

Without loss of generality, we consider networks $\boldsymbol{g}$ in which player 1 has formed a link with player 2. In these networks,

1. player 2 has not formed a link with player 3 because in that case $2,3 \in N_1(\boldsymbol{g})$ and player 1 would have an incentive to delete the link $1,2$.
2. Player 3 has an incentive to establish a link with player 1, since $\phi_3(1,0) < \phi_3(3,1)$.
3. The networks in which a player has formed two links are not Nash networks.

Hence a Nash network does not exist.

## 2 One-Way Flow Model with Global Spillovers

In this section, we modify the framework in order to describe new situations. More precisely, in the models of section 1, the payoff of a player $i$ from a link with player $j$ depends on the identities of both players. In this section, what matters is the number of links that player $i$ has formed as well as the total number of links that the other players have formed.

Recall that the number of links formed by $i \in N$ is $c_i(\boldsymbol{g}) = \sum_{j \neq i} \boldsymbol{g}_{i,j}$. Let $c_{-i}(\boldsymbol{g}) = \sum_{j \neq i} \sum_{k \neq j} \boldsymbol{g}_{j,k}$ denote the number of links formed by all players except $i \in N$, in the network $\boldsymbol{g}$. Define $A = \{0, \ldots, n-1\}$ and $B = \{0, \ldots, (n-1)^2\}$. The payoff function of each player $i \in N$ is given by $u_i : A \times B \to I\!R, (c_i(\boldsymbol{g}), c_{-i}(\boldsymbol{g})) \mapsto u_i(c_i(\boldsymbol{g}), c_{-i}(\boldsymbol{g}))$.

The following example illustrates that a Nash network does not always exist under this general payoff function.

**Example 3** Let $N = \{1, 2, 3\}$. We define the following payoff function for all $i \in N$:

$$u_i(2,1) > u_i(1,0) > u_i(0,0),$$

$$u_i(x,y) < u_i(0,0) \text{ for all } (x,y) \notin \{(2,1),(1,0)\}.$$

It is obvious that in this example there does not exist any Nash network.

Since Nash networks do not always exist in one-way flow models with global spillovers, we now provide two conditions which allow the existence of Nash networks under situations of interest in economics.

The first condition is the increasing and decreasing differences property. The second condition is the discrete convexity property.

## 2.1 Increasing and Decreasing Differences

**Definition 4** *The payoff function $u_i$ has strictly increasing (decreasing) differences in its two arguments $(c_i(\boldsymbol{g}), c_{-i}(\boldsymbol{g}))$ if $u_i(c_i(\boldsymbol{g}), c_{-i}(\boldsymbol{g})) - u_i(c_i(\boldsymbol{g}), c_{-i}(\boldsymbol{g}'))$ is strictly increasing (decreasing) in $c_i(\boldsymbol{g})$ for all $c_{-i}(\boldsymbol{g}) > c_{-i}(\boldsymbol{g}')$. (See Vives, 1999, [12]).*

Let $\delta(\boldsymbol{g}) = \sum_{i \in N} c_i(\boldsymbol{g})$, as the total number of links formed in the network $\boldsymbol{g}$.

**Proposition 4** *In a one-way flow model with global spillovers and increasing (or decreasing) differences a Nash network always exists.*

**Proof** Consider a one-way flow model with global spillovers and increasing differences. To prove the proposition, we begin with the empty network $\dot{\boldsymbol{g}}$. Either $\dot{\boldsymbol{g}}$ is Nash and we are done, or $\dot{\boldsymbol{g}}$ is not a Nash network and there exists an improving path from $\dot{\boldsymbol{g}}$ to an adjacent network $\boldsymbol{g}^1$. That is, there exists a player $i_1$ such that $\dot{\boldsymbol{g}}_{i_1} \notin \mathcal{BR}_{i_1}(\dot{\boldsymbol{g}}_{-i_1})$ and $\boldsymbol{g}^1_{i_1} \in \mathcal{BR}_{i_1}(\boldsymbol{g}^1_{-i_1})$. Since $i_1$ had formed no link in $\dot{\boldsymbol{g}}$ and forms links in $\boldsymbol{g}^1$, we have $\delta(\dot{\boldsymbol{g}}) < \delta(\boldsymbol{g}^1)$. Now, either $\boldsymbol{g}^1$ is Nash and we are done, or there is a player $i_2$ such that $\boldsymbol{g}^1_{i_2} \notin \mathcal{BR}_{i_2}(\boldsymbol{g}^1_{-i_2})$. In that case, there exists an improving path from $\boldsymbol{g}^1$ to an adjacent network $\boldsymbol{g}^2$ such that $\boldsymbol{g}^2_{i_2} \in \mathcal{BR}_{i_2}(\boldsymbol{g}^2_{-i_2})$. Hence we have $\delta(\boldsymbol{g}^1) < \delta(\boldsymbol{g}^2)$ since player $i_2$ had formed no links in $\boldsymbol{g}^1$. More generally, $\boldsymbol{g}^k$ is defined as follows: $\boldsymbol{g}^k$ is adjacent to $\boldsymbol{g}^{k-1}$, $\boldsymbol{g}^{k-1}_{i_k} \notin \mathcal{BR}_{i_k}(\boldsymbol{g}^{k-1}_{-i_k})$, $\boldsymbol{g}^k_{i_k} \in \mathcal{BR}_{i_k}(\boldsymbol{g}^{k-1}_{-i_k})$ and $i_k \notin \{i_1, \ldots, i_{k-1}\}$, that is we have $\boldsymbol{g}^k = \boldsymbol{g}^k_{i_k} \oplus \boldsymbol{g}^{k-1}_{i_k-1}$. By construction, we have: $\delta(\boldsymbol{g}^{k-1}) < \delta(\boldsymbol{g}^k)$. Let $\boldsymbol{g}^m$ denote the network after agents $i_1, \ldots, \ldots, i_k, \ldots, i_m$ have sequentially chosen a best response and there is no other player $i \notin \{i_1, \ldots, i_m\}$ who has an incentive to form links in $\boldsymbol{g}^m$. Let $\mathcal{C} = \boldsymbol{g}^1, \boldsymbol{g}^2, \ldots, \boldsymbol{g}^m$ be the improving path from $\boldsymbol{g}^1$ to $\boldsymbol{g}^m$. This path is finite and $m \leq n$. There are now two possible cases.

1. No player $i_k$, $k \in \{1, \ldots, m\}$, has an incentive to change her strategy. Then the proof is complete.

2. There exists a player $i_k$, $k \in \{1, \ldots, m\}$, who has an incentive to modify her strategy. Without loss of generality let this be player $i_1$. Let $\boldsymbol{g}^{(1)} = \boldsymbol{g}^{(1)}_{i_1} \oplus \boldsymbol{g}^m_{-i_1}$, where

$\boldsymbol{g}_{i_1}^{(1)} \in \mathcal{BR}_{i_1}(\boldsymbol{g}_{-i_1}^m)$. Clearly player $i_1$ has no incentive to reduce the total number of her links in $\boldsymbol{g}^m$. Indeed, we have $c_{-i_1}(\boldsymbol{g}^m) > c_{-i_1}(\boldsymbol{g}^1)$. Hence for all $c_{i_1}(\boldsymbol{g}) < c_{i_1}(\boldsymbol{g}^1)$, by the property of increasing difference we get that $0 < u_{i_1}(c_{i_1}(\boldsymbol{g}^1), c_{-i_1}(\boldsymbol{g}^1)) - u_{i_1}(c_{i_1}(\boldsymbol{g}), c_{-i_1}(\boldsymbol{g}^1)) < u_{i_1}(c_{i_1}(\boldsymbol{g}^1), c_{-i_1}(\boldsymbol{g}^m)) - u_{i_1}(c_{i_1}(\boldsymbol{g}), c_{-i_1}(\boldsymbol{g}^m))$. Since player $i_1$ changes her strategy we have $c_{i_1}(\boldsymbol{g}^m) \neq c_{i_1}(\boldsymbol{g}^{(1)})$. Consequently, we have $c_{i_1}(\boldsymbol{g}^m) < c_{i_1}(\boldsymbol{g}^{(1)})$ and $c_{-i_1}(\boldsymbol{g}^m) = c_{-i_1}(\boldsymbol{g}^{(1)})$ which implies that $\delta(\boldsymbol{g}^m) < \delta(\boldsymbol{g}^{(1)})$.

If $\boldsymbol{g}^{(1)}$ is not a Nash equilibrium there are two possibilities.

1. There exists a player $\ell \notin \{i_1, \ldots, i_m\}$ such that $\boldsymbol{g}_\ell^{(1)} \notin \mathcal{BR}_\ell(\boldsymbol{g}_{-\ell}^{(1)})$ and $\boldsymbol{g}_\ell^{(2)} \in \mathcal{BR}_\ell(\boldsymbol{g}_{-\ell}^{(1)})$. Then let $N^{\mathcal{BR}}(\boldsymbol{g}^{(2)}) = \{i_1, \ldots, i_m\} \cup \{\ell\}$ to be the set of players who have played a best response (and who have formed links). Also, we have $\boldsymbol{g}^{(2)} = \boldsymbol{g}_\ell^{(2)} \oplus \boldsymbol{g}_{-\ell}^{(1)}$, and $\delta(\boldsymbol{g}^{(1)}) < \delta(\boldsymbol{g}^{(2)})$, since by construction player $\ell$ has not formed any links in $\boldsymbol{g}^{(1)}$.

2. There does not exist a player $\ell \notin \{i_1, \ldots, i_m\}$ such that $\boldsymbol{g}_\ell^{(1)} \notin \mathcal{BR}_\ell(\boldsymbol{g}_{-\ell}^{(1)})$. In that case, we have $N^{\mathcal{BR}}(\boldsymbol{g}^{(2)}) = \{i_1, \ldots, i_m\}$ and there exists a player $j \in \{i_2, \ldots, i_m\}$ such that $\boldsymbol{g}_j^{(1)} \notin \mathcal{BR}_j(\boldsymbol{g}_{-j}^{(1)})$ and $\boldsymbol{g}_j^{(2)} \in \mathcal{BR}_j(\boldsymbol{g}_{-j}^{(1)})$. We have $\boldsymbol{g}^{(2)} = \boldsymbol{g}_j^{(2)} \oplus \boldsymbol{g}_{-j}^{(1)}$. Again using the property of increasing differences we obtain $\delta(\boldsymbol{g}^{(1)}) < \delta(\boldsymbol{g}^{(2)})$.

More generally, we define $\boldsymbol{g}^{(k)}$ as follows: $\boldsymbol{g}^{(k)}$ is adjacent to $\boldsymbol{g}^{(k-1)}$, $\boldsymbol{g}^{(k-1)}$ is not a Nash network, and:

1. if there exists a player, say $\ell' \notin N^{\mathcal{BR}}(\boldsymbol{g}^{(k-1)})$, such that $\boldsymbol{g}_{\ell'}^{(k-1)} \notin \mathcal{BR}_{\ell'}(\boldsymbol{g}_{-\ell'}^{(k-1)})$, then $\boldsymbol{g}^{(k)} = \boldsymbol{g}_{\ell'}^{(k)} \oplus \boldsymbol{g}_{-\ell'}^{(k-1)}$. In that case, we have $N^{\mathcal{BR}}(\boldsymbol{g}^{(k)}) = N^{\mathcal{BR}}(\boldsymbol{g}^{(k-1)}) \cup \{\ell'\}$;

2. otherwise, there exists a player, say $j' \in N^{\mathcal{BR}}(\boldsymbol{g}^{(k-1)})$, such that $\boldsymbol{g}_{j'}^{(k-1)} \notin \mathcal{BR}_{j'}$ $(\boldsymbol{g}_{-j'}^{(k-1)})$ and $\boldsymbol{g}_{j'}^{(k)} \in \mathcal{BR}_{j'}(\boldsymbol{g}_{-j'}^{(k-1)})$. In that case, $\boldsymbol{g}^{(k)} = \boldsymbol{g}_{j'}^{(k)} \oplus \boldsymbol{g}_{-j'}^{(k-1)}$ and $N^{\mathcal{BR}}(\boldsymbol{g}^{(k)}) =$

$N^{\mathcal{BR}}(\boldsymbol{g}^{(k-1)})$.

For case 1, $\delta(\boldsymbol{g}^{(k)}) > \delta(\boldsymbol{g}^{(k-1)})$ since player $\ell'$ has formed no links in $\boldsymbol{g}^{(k-1)}$ but has formed links in $\boldsymbol{g}^{(k)}$. For case 2, $\delta(\boldsymbol{g}^{(k)}) > \delta(\boldsymbol{g}^{(k-1)})$ by the property of increasing differences.

To summarize, if the empty network is not a Nash network, then there is an improving path, $\mathcal{C} = \boldsymbol{g}^0, \ldots, \boldsymbol{g}^\ell, \boldsymbol{g}^{\ell+1}, \ldots, \boldsymbol{g}^t$, from the network $\dot{\boldsymbol{g}} = \boldsymbol{g}^0$ to a network $\boldsymbol{g}' = \boldsymbol{g}^t$. Moreover, for all $\boldsymbol{g}^\ell \in \mathcal{C}$, $\boldsymbol{g}^\ell \neq \boldsymbol{g}^t$, $\delta(\boldsymbol{g}^{\ell+1}) > \delta(\boldsymbol{g}^\ell)$. Hence, there does not exist any improving cycle between $\dot{\boldsymbol{g}}$ and $\boldsymbol{g}'$.

Since the set $\mathcal{G}$ is finite and there does not exist any improving cycle, the improving path beginning from the empty network $\dot{\boldsymbol{g}}$ is finite. Hence a Nash network always exists.

The proof of the existence of Nash networks in one-way flow models with global spillovers and decreasing differences is similar, except that we need to start with the complete network. □

The next example illustrates the importance of this result in a Cournot model.

**Example 4** *Cost reducing collaborative activities in oligopoly.*[2] Consider an homogeneous product Cournot Oligopoly consisting of $n$ *ex ante* symmetric firms who face the linear inverse demand function $p = \alpha - \sum_{i \in N} q_i$, $\alpha > 0$. The firms initially have zero fixed costs and identical constant returns-to-scale cost functions. Establishing a link lowers marginal costs in a linear way: $C_i(\boldsymbol{g}) = \gamma_0 - \gamma c_i(\boldsymbol{g})$, where $\gamma_0$ is a positive parameter representing a firm $i$'s marginal cost if it has no link. Given any network $\boldsymbol{g}$, the Cournot equilibrium output can be written as:

$$q_i(\boldsymbol{g}) = \frac{(\alpha - \gamma_0) + n\gamma c_i(\boldsymbol{g}) - \gamma c_{-i}(\boldsymbol{g})}{(n+1)}, i \in N.$$

[2]This model is taken from Billand and Bravard (2004, [2]).

The Cournot prodits for firm $i \in N$ are given by $u_i(c_i(\boldsymbol{g}), c_{-i}(\boldsymbol{g})) = (q_i(\boldsymbol{g}))^2 - f c_i(\boldsymbol{g})$, where $f$ is the cost of establishing a link. Let us define $\boldsymbol{g}'$ in which there exists a player $j$ such that $\boldsymbol{g}'_{-j} = \boldsymbol{g}_{-j}$ and $\sum_{k \neq j} g_{j,k} = \sum_{k \neq j} g'_{j,k} + 1$. We get:

$$\begin{aligned}\Delta u_i &= u_i(c_i(\boldsymbol{g}), c_{-i}(\boldsymbol{g})) - u_i(c_i(\boldsymbol{g}'), c_{-i}(\boldsymbol{g}')) \\ &= -\frac{\gamma(2\gamma n c_i(\boldsymbol{g}) - 2\gamma c_{-i}(\boldsymbol{g}) - \gamma + 2(\alpha - \gamma_0))}{(n+1)^2}\end{aligned}$$

and $\Delta u_i$ decreases with $c_i(\boldsymbol{g})$. The profit function satisfies decreasing differences. Hence, by proposition 4, there always exists a Nash network.

## 2.2 One-Way Flow Model with Global Spillovers and Discrete Convexity

We now give a new condition allowing for the existence of Nash networks: the discrete convexity property. We begin by characterizing this property. Then, we examine the existence of Nash networks in this setting and characterize the architectures of these networks.

### 2.2.1 Discrete convexity

A function $f : \mathbb{R} \to \mathbb{R}$ satisfies strict midpoint convexity if for any $x, y \in \mathbb{R}$,

$$f\left(\frac{x+y}{2}\right) < \frac{f(x) + f(y)}{2}.$$

We consider a similar property for a function defined on the discrete space $X \subset \mathbb{Z}$, inspired by Ui (2005, [11]).[3] Let $|x| = \max\{-x, x\}$. We say that a function $f : X \to \mathbb{R}$

[3] Ui (2005, [11]) deals with discrete concavity and provides a more general definition of larger midpoint property.

satisfies the strict smaller midpoint property if, for any $x, y, z \in X$, with $|x - y| = 2$, and $|z - x| = |z - y| = 1$, there exists $t \in (0, 1)$, such that,

$$f(z) < tf(x) + (1 - t)f(y).$$

Note that, in defining the strict smaller midpoint property, we postulate that the midpoint of $x, y \in X$ is $z \in X$.

We assume that for all $i \in N$, the payoff function $u_i$ satisfies the strict smaller midpoint property in the first argument. That is, for any $x \in A$, $y \in A$, $z \in \{1, \ldots, n-2\}$, with $|x - y| = 2$, and $|z - x| = |z - y| = 1$, and for all $w \in B$, there exists $t \in (0, 1)$ such that:

$$u_i(z, w) < tu_i(x, w) + (1 - t)u_i(y, w), \forall i \in N.$$

For simplicity, we assume that $u_i(\cdot, \cdot) = u(\cdot, \cdot)$, for all $i \in N$. Dropping the subscript we can write this as:

$$u(z, w) < tu(x, w) + (1 - t)u(y, w). \tag{5}$$

We now give some results about functions which satisfy the strict smaller midpoint property.

**Lemma 7** *A function $u : A \times B \to \mathbb{R}$ satisfies the strict smaller midpoint property in the first argument if and only if, for any $z \in \{1, \ldots, n-2\}$ and for any $w \in B$, with $|x - z| = |z - y| = 1$, $x \neq y$,*

$$u(z, w) < \max\{u(x, w), u(y, w)\} \tag{6}$$

**Proof** Without loss of generality, suppose that $u(x, w) \geq u(y, w)$, and (5) is true. Then we have:

$$u(z, w) < tu(x, w) + (1 - t)u(y, w) \leq u(x, w),$$

and (6) holds.

Without loss of generality, let $u(x,w) \geq u(y,w)$, and (6) be true. Then we can choose $t$ sufficiently large ($t < 1$), such that $tu(x,w) + (1-t)u(y,w)$ is sufficiently close to $u(x,w)$ and thus (5) is true. Therefore (5) and (6) are equivalent. □

**Lemma 8** *Suppose that the payoff function $u : A \times B \to \mathbb{R}$ satisfies the strict smaller midpoint property in the first argument. If $u(n-1,w) \leq u(n-2,w)$ for all $w \in B$, then $u(z,w) > u(z+1,w)$ for all $z \in \{0,\ldots,n-3\}$.*

**Proof** Assume that $u(n-1,w) \leq u(n-2,w)$ for all $w \in B$. By Lemma 7, we know that $u(n-2,w) < \max\{u(n-3,w), u(n-1,w)\}$. Given that $u(n-2,w) < \max\{u(n-3,w), u(n-1,w)\}$ and $u(n-1,w) \leq u(n-2,w)$ for all $w \in B$, we have $u(n-2,w) < u(n-3,w)$. Suppose now that there exists $k \in \{1,\ldots,n-3\}$ such that $u(k,w) > u(k+1,w)$ for all $w \in B$. Then, by Lemma 7, we have $u(k,w) < u(k-1,w)$. □

**Lemma 9** *Suppose that the payoff function $u : A \times B \to \mathbb{R}$ satisfies the strict smaller midpoint property in the first argument. Then, for any $w \in B$, we have*

$$\max\{u(0,w), u(n-1,w)\} > u(z,w), \forall z \in \{1,\ldots,n-2\}.$$

**Proof** By Lemma 8 we know that if $u(n-1,w) \leq u(n-2,w)$, then $u(z,w) < u(z-1,w)$, for all $z \in \{1,\ldots,n-1\}$. Hence, $u(0,w) > u(z,w)$ for all $z \in \{1,\ldots,n-1\}$. Also, assume that $u(n-1,w) > u(n-2,w)$. There are now two cases.

1. Suppose $u(0,w) < u(1,w)$,. Then by Lemma 7, $u(1,w) < u(2,w)$, for all $w \in B$. Moreover, if there exists $k \in \{3,\ldots,n-2\}$ such that $u(k-1,w) < u(k,w)$,

then by Lemma 7, $u(k, w) < u(k+1, w)$. Hence, $u(n-1, w) > u(z, w)$ for all $z \in \{1, \ldots, n-2\}$.

2. Suppose $u(0, w) \geq u(1, w)$ for all $w \in B$. Then, we show that there exists a unique $d \in \{2, \ldots, n-2\}$ such that $u(d-1, w) > u(d, w) < u(d+1, u)$.

   - If $d$ does not exist, then we know that $u(\cdot, w)$ is decreasing in its first argument and we have a contradiction since $u(n-1, w) > u(n-2, w)$.

   - Suppose that there exist $d$ and $d'$, $d \neq d'$, such that $u(d-1, w) > u(d, w) < u(d+1, w)$ and $u(d'-1, w) > u(d', w) < u(d'+1, w)$. Without loss of generality let $d' > d$. Since $u(d, w) < u(d+1, w)$, we have $u(d+1, w) < u(d+2, w)$ and by induction $u(d+k, w) < u(d+k+1, w)$ for all $k \in \{1, \ldots, n-d-2\}$ and $w \in B$. Hence, there does not exist $d' \in \{d+2, \ldots, n-2\}$ such that $u(d'-1, w) > u(d', w) < u(d'+1, w)$ which yields a contradiction.

   Therefore, we have for all $z \in \{1, \ldots, d\}$, $u(0, w) > u(z, w)$, for all $w \in B$ and we have for all $z \in \{d, \ldots, n-2\}$, $u(n-1, w) > u(z, w)$, for all $w \in B$. This gives us the desired conclusion.

□

### 2.2.2 Existence of Nash Networks and Discrete Convexity

Let us define two strategies for player $i \in N$: $\boldsymbol{g}_i = \mathbf{0}$ with $c_i(\boldsymbol{g}) = 0$ (player $i$ forms no links) and $\boldsymbol{g}_i = \boldsymbol{n-1}$ with $c_i(\boldsymbol{g}) = n-1$ (player $i$ forms a link with each of the other players).

**Lemma 10** *Suppose that the payoff function $u(\cdot,\cdot)$ satisfies strict smaller midpoint property. Then, the best response of each player $i \in N$ is either $\mathbf{0}$ or $\boldsymbol{n-1}$.*

**Proof** To obtain a contradiction, assume that there exist a player $i \in N$ and a network $\boldsymbol{g} \in \mathcal{G}$ such that $\mathcal{B}R_i(\boldsymbol{g}) \notin \{\mathbf{0}, \boldsymbol{n-1}\}$. Then, there exists $\boldsymbol{g} \in \mathcal{G}$ such that $c_i(\boldsymbol{g}) \in \{2,\ldots,n-2\}$, $c_{-i}(\boldsymbol{g}) \in B$, $u(c_i(\boldsymbol{g}), c_{-i}(\boldsymbol{g})) \geq u(0, c_{-i}(\boldsymbol{g}))$ and $u(c_i(\boldsymbol{g}), c_{-i}(\boldsymbol{g})) \geq u(n-1, n_{-i}(\boldsymbol{g}))$. By Lemma 9, we have for any $c_i(\boldsymbol{g}) \in \{2,\ldots,n-2\}$,

$$\max\{u(0, c_{-i}(\boldsymbol{g})), u(n-1, c_{-i}(\boldsymbol{g}))\} > u(c_i(\boldsymbol{g}), c_{-i}(\boldsymbol{g})), \forall n_{-i}(\boldsymbol{g}) \in B,$$

which is a contradiction. □

**Proposition 5** *Suppose that the payoff function $u(\cdot,\cdot)$ satisfies the strict smaller midpoint property. Then the one-way flow model with global spillovers contains a Nash network.*

**Proof** We start from the empty network $\dot{\boldsymbol{g}}$, and show that we can reach a Nash network. In other words, there is no improving cycle originating from the empty network. If there is no improving path from $\dot{\boldsymbol{g}}$, we are done. Otherwise, there exists a player, say $i_1$, such that $\mathbf{0} \notin \mathcal{BR}_{i_1}(\dot{\boldsymbol{g}}_{-i_1})$. Hence, by Lemma 10, we have, $\mathcal{BR}_{i_1}(\dot{\boldsymbol{g}}_{-i_1}) = \boldsymbol{n-1}$. Let $\boldsymbol{g}^1$ be the network in which no player has formed links except player $i_1$ who has formed $n-1$ links. Either $\boldsymbol{g}^1$ is a Nash network and we are done, or there is a player say $i_2$ such that $\mathbf{0} \notin \mathcal{BR}_{i_2}(\boldsymbol{g}^1_{-i_2})$. In the latter case, by lemma 10, we have $\mathcal{BR}_{i_2}(\boldsymbol{g}^1_{-i_2}) = \boldsymbol{n-1}$. Let $\boldsymbol{g}^2$ be the network in which no player has formed links except players $i_1$ and $i_2$ who have formed $n-1$ links. We observe that player $i_1$ has no incentive to modify her strategy in $\boldsymbol{g}^2$. Indeed, we have $\mathcal{BR}_i = \mathcal{BR}_j$ for all $i \in N$, $j \in N$, and by construction

$\boldsymbol{g}^2_{-i_1} = \boldsymbol{g}^2_{-i_2}$. Therefore, if $\mathcal{BR}_{i_2}(\boldsymbol{g}^1_{-i_2}) = \boldsymbol{n-1}$, then $\mathcal{BR}_{i_1}(\boldsymbol{g}^2_{-i_1}) = \boldsymbol{n-1}$. More generally, we define $\boldsymbol{g}^k$ the network in which no player has formed links except players $i_1, i_2, \ldots, i_k$ who have formed $n-1$ links and $\mathcal{BR}_i(\boldsymbol{g}^k_{-i}) = \boldsymbol{n-1}$ for all $i \in \{i_1, \ldots, i_k\}$. Either $\boldsymbol{g}^k$ is a Nash network and we are done, or there exists a player, say $i_{k+1}$, such that $\boldsymbol{0} \notin \mathcal{BR}_{i_{k+1}}(\boldsymbol{g}^k_{-i_{k+1}})$. By Lemma 10, $\mathcal{BR}_{i_{k+1}}(\boldsymbol{g}^k_{-i_{k+1}}) = \boldsymbol{n-1}$. Let $\boldsymbol{g}^{k+1}$ be the network in which no player has formed links except players $i_\ell$, with $\ell \in \{1, \ldots, k+1\}$ who has formed $n-1$ links. We observe that players $i_\ell$ have no incentive to modify their strategy in $\boldsymbol{g}^{k+1}$ since $\mathcal{BR}_i(\cdot) = \mathcal{BR}_j(\cdot)$ for all $j, i \in N$ and $\boldsymbol{g}^{k+1}_{-i} = \boldsymbol{g}^{k+1}_{-i_k}$ for all $i \in \{i_1, \ldots, i_k\}$.

Hence, there does not exist any improving cycle starting from $\dot{\boldsymbol{g}}$ and, since the set of players $N$ is finite, a Nash network exists. □

### 2.2.3 Characterization of Nash Networks and Discrete Convexity

We define a class of networks that are important in what follows. A network $\boldsymbol{g}$ is a $k$-all-or-nothing network if $k$ firms have established links with all other firms while $n-k$ firms have formed no link.

**Proposition 6** *Suppose that the payoff function $u(\cdot,\cdot)$ satisfies strict smaller midpoint property. The Nash networks are $k$-all-or-nothing networks.*

**Proof** We know by Lemma 10 that in a Nash network each player forms either 0 or $n-1$ links. □

Now, we state sufficient conditions for the empty and complete networks to be Nash networks.

**Proposition 7** *Suppose $u(\cdot,\cdot)$ has strictly decreasing differences in its two arguments and satisfies the strict smaller midpoint property.*

1. *The complete network is the unique Nash network if and only if $u\left(n-1,(n-1)^2\right) > u\left(0,(n-1)^2\right)$.*

2. *The empty network is a Nash network if $u\left(0,0\right) > u\left(n-1,0\right)$.*

**Proof** Since the second part of the proposition is straightforward, we only prove the first part. First, it is obvious that if the complete network is the unique Nash network, then $u\left(n-1,(n-1)^2\right) > u\left(0,(n-1)^2\right)$. Second, we show that if $u\left(n-1,(n-1)^2\right) > u\left(0,(n-1)^2\right)$ then the complete network is the unique Nash network. Indeed, assume that there is a non complete network $\boldsymbol{g}^*$ which is a Nash network. We have $u\left(c_i(\boldsymbol{g}^*), c_{-i}(\boldsymbol{g}^*)\right) \geq u\left(c_i(\boldsymbol{g}), c_{-i}(\boldsymbol{g}^*)\right)$, for all $\boldsymbol{g} \in \mathcal{G}$ and for all $i \in N$. By decreasing difference property, we have:

$$u(n-1,(n-1)^2\big) > u(0,(n-1)^2\big) \Rightarrow u(n-1,c_{-i}(\boldsymbol{g})\big) > u(0,c_{-i}(\boldsymbol{g})\big)\,,$$

for all $c_{-i}(\boldsymbol{g}) \in \{0,\ldots,(n-1)^2-1\}$. By Lemma 10, and the strict smaller midpoint property, we have:

$$u(n-1,c_{-i}(\boldsymbol{g})) > u(0,c_{-i}(\boldsymbol{g})) \Rightarrow u(n-1,c_{-i}(\boldsymbol{g})) > u(c_i(\boldsymbol{g}),c_{-i}(\boldsymbol{g}))\,,$$

for all $c_i(\boldsymbol{g}) \in \{0,\ldots,n-2\}$. This tells us that each player $i$ always has an incentive to form $n-1$ links in $\boldsymbol{g}^*$. Hence, $\boldsymbol{g}^*$ cannot be a Nash network which gives us the necessary contradiction. □

**Example 5** Consider again the framework described in example 4. Let $\boldsymbol{g}'$ be the network such that $\boldsymbol{g}'_{-i} = \boldsymbol{g}_{-i}$ and $\sum_{j\neq i} g_{i,j} = \sum_{j\neq i} g'_{i,j} + 1$ ($\boldsymbol{g}$ is supposed to be a non empty network) and let $\boldsymbol{g}''$ be the network such that $\boldsymbol{g}''_{-i} = \boldsymbol{g}_{-i}$ and $\sum_{j \neq i} g_{i,j} = \sum_{j \neq i} g''_{i,j} - 1$. We show that the profit function satisfies the strict smaller midpoint property, that is $u(c_i(\boldsymbol{g}), c_{-i}(\boldsymbol{g})) < \max\{u(c_i(\boldsymbol{g}'), c_{-i}(\boldsymbol{g}')), u(c_i(\boldsymbol{g}''), c_{-i}(\boldsymbol{g}''))\}$. To obtain a contradiction, assume that $u(c_i(\boldsymbol{g}), c_{-i}(\boldsymbol{g})) \geq \max\{u(c_i(\boldsymbol{g}'), c_{-i}(\boldsymbol{g}')), u(c_i(\boldsymbol{g}''), c_{-i}(\boldsymbol{g}''))\}$. Then,

$$u(c_i(\boldsymbol{g}), c_{-i}(\boldsymbol{g})) - u(c_i(\boldsymbol{g}'), c_{-i}(\boldsymbol{g}')) \geq 0$$

$$u(c_i(\boldsymbol{g}''), c_{-i}(\boldsymbol{g}'')) - u(c_i(\boldsymbol{g}), c_{-i}(\boldsymbol{g})) \leq 0$$

But, straightforward calculations give us:

$$\Delta u = 2u(c_i(\boldsymbol{g}), c_{-i}(\boldsymbol{g})) - u(c_i(\boldsymbol{g}'), c_{-i}(\boldsymbol{g}')) - u(c_i(\boldsymbol{g}''), c_{-i}(\boldsymbol{g}'')) < 0,$$

which is a contradiction.

Since $u$ has strictly decreasing differences in its two arguments and satisfies the strict smaller midpoint property, we can conclude, by proposition 6, that Nash networks are $k$-all-or-nothing networks. Moreover, by proposition 7, we know that the complete network is the unique Nash network if and only if $u(n-1, (n-1)^2) > u(0, (n-1)^2)$ and the empty network is a Nash network if $u\,(0,0) > u\,(n-1, 0)$.

# Concluding remarks

Much of the existing literature on one-way flow models contains the assertion that for some parameters ranges, the models admit Nash networks with specific properties. This amounts to providing sufficient conditions for the existence of Nash networks. However, these conditions often do not cover the entire parameters space and are unable to answer if Nash networks always exist. Our paper fills this void in the literature.

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