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Abstract

A methodology for nonmarket goods is presented based on preference algebra and set theory that allows us to specify exactly when preference assumptions such as weak complementarity can be tested against revealed preference information. Revealed preference is insufficient for welfare analysis involving state preference variables such as nonmarket goods. The preference and set-theoretic structure presented here is specifically designed to characterize the minimal additional preference information necessary for exact welfare analysis, and also provides a common basis for specifying the many context-specific methods that have been proposed for closing the information gap (whether or not they provide this minimal information). The paper closes with examples demonstrating how this structure can be used as a methodology for working with assumptions about preference structure, focusing on when such assumptions can be tested against revealed preference. This includes an extended examination of weak complementarity and related issues, followed by five shorter examples including two types of repackaging for price indices and the new and disappearing goods problem.

Keywords: Identifying preference, Preference-theoretic, Methodology, Nonmarket goods, Testing preference restrictions, Weak complementarity, Existence value, Weak substitutability, Repackaging price index, Cross-product repackaging, New and disappearing goods.

Welfare analysis involving a nonmarket good, such as environmental quality, requires knowledge of the consumer's preference over distinctions where the consumer is not able to choose. For example, the individual consumer is not given a choice in the marketplace between clean and dirty air.[1] This is also true for some contexts that are not traditionally understood as nonmarket goods. I therefore use the term "state preference variable" defined as a nontrivial argument in the consumer's preference relation over which she has no economic control. The consumer may care about the state preference variable itself and also the variable may affect her commodity preferences. In addition to environmental variables such as global warming, examples of state preference variables include the existence and quality of public goods, the quality of some market goods such as monopoly goods, and perhaps even aspects of Behavioral Economics where for instance a person may prefer to be not clinically depressed (or not a drug addict), but is not able to obtain this state without the development of a drug or a government program providing the drug to the indigent.

Any welfare analysis concerned with variance or changes in state preference variable values requires knowledge of the consumer's joint preference over that variable and commodities. While this may only require very local knowledge such as to verify the efficiency of a solution's first order conditions, more extensive knowledge is required for more typical applications involving larger discrete changes in the state preference variables, such as with measures like Equivalent Variation. With any of these applications we have a problem: preference information can be recovered from the observable demand function only to the extent that is does not involve any distinctions in a state preference variable. We can thus only partially identify the joint preference over these variables and commodities.

In this paper I develop a general structure based on preference logic and basic set theory that enables us to characterize the missing preference information that is needed to fully specify the overall joint preference relation (when combined with the available revealed preference information). The structure allows us to precisely describe the minimal amount of missing information, but is also robust enough so that we deal with potential approaches for filling in this preference information gap that provide more than enough information.

Many context-specific methods have been proposed to supply the missing preference information, especially for settings involving product quality and environmental issues. These applications typically include assumptions or qualitative information that allow us to at least partially close the information gap. Each such application is thus defined by the additional preference information (API) it provides. What I might call an instance of "additional preference information" is typically referred to in the literature more narrowly as a "preference (or utility) restriction" or perhaps as a "maintained hypothesis."[2] This restrictive terminology emphasizes the assumptive nature of the additional information, while my more general terminology is explicitly open to the possibility of using real preference information.

The immediate application of the structure developed here is that we can state exactly when preference assumptions such as weak complementarity are sufficient and also when

[1] I realize that there may be plausible exceptions to this statement. However, please accept it in the spirit that it is provided, for the purpose of illustrating the nonmarket good concept.

[2] Such as in Smith and Banzhaf (2004) and Ebert (2001), respectively.

they can be tested against revealed preference. The structure can be used to determine whether any given API only partially fills the information gap, exactly fills the gap, or includes more than enough information. When an API includes excess preference information we can always test it against the demand function in the form of revealed preference information. In addition to one-way tests of specific individual preference restrictions such as weak complementarity, we can also sometimes obtain two-ways tests for the validity of whole classes of API's. Examples of using this structure for testing and other analysis with specific API's and classes of API's are provided in this paper after the structure is developed. Other potential applications of this structure are discussed in the conclusion.

The theoretical literature concerned with welfare analysis in the context of state preference variables such as nonmarket goods and product quality is dominated by applications of real analysis, and hence requires continuous state preference variables and other regularity conditions to enable various techniques from calculus. My major departure from this literature is that I instead rely on the algebraic properties of preference, such as transitivity, in combination with basic set theory. This has two advantages. The first is the obvious one, the results are more general as no restrictions are imposed on the state preference variables – they do not even need to be numbers. The second is that by stepping away from the calculus paradigm, and hence from the quasi-traditional microeconomic constructions that are the workhorse in this literature, we can obtain results that are not available with a methodology that is so narrowly focused on real analysis.[3] Combining these two approaches (preference algebra and real analysis) should be quite powerful but is beyond the scope of this paper.

The rest of this paper is organized into five sections. The modelling setup presented in the first allows me to specify the nature of the missing information problem in the second section. Then with the third, I develop the just discussed preference-theoretic structure. Weak complementarity is examined in the fourth section along with several other examples of how this general structure may be applied, followed by a conclusions section. All proofs are deferred to the Appendix.[4]

1 Modelling Setup

Given the wide range of potential application areas, modelling assumptions are kept to a minimum so as to keep the results as general as possible, while at the same time maintaining tractability by remaining close to our standard understanding of preference and demand. However, this paper only deals with the preference and demand of an individual consumer, so that any direct application of this work in the context of market demand would require the usual simple "representative consumer" assumption.

[3]For example, we are able to find that exact welfare analysis is possible in conditions where it is deemed impossible in that literature. See Note 61.

[4]Editorial note: All theorems are presented with the intention of either demonstrating the nature of the problem, developing the preference-theoretic structure, or illustrating its application. Presenting proofs in the main text would detract from that purpose. Furthermore, for the most part the proofs consist of multiple applications of simple preference logic and set theory which many readers would find rather tedious.

State preference variables, which may be a scalers, vectors or non-numbers, and may be discrete or continuous, are represented by lower case z. Upper case Z represents the set of admissible values of the state preference variable, such as $Z = \{\text{Global Warming, Not Global Warming}\}$. Superscripts are used to distinguish individual elements of Z, such as in $z^a, z^b \in Z$.[5] We shall assume that Z is non-trivial in that it has at least two elements. Let X be the commodity consumption set (typically $X = \Re^L_+$). The consumer has a preference relation over $Y = X \times Z$ represented by $\succsim_Y$, and given prices, wealth and z, she chooses $x \in X$ to achieve the highest affordable preference level,

$$\widehat{x}(p, z, w) = \{x \in X \mid p \cdot x \leq w, \text{ and } (x, z) \succsim_Y (\bar{x}, z) \text{ for all } \bar{x} \text{ such that } p \cdot \bar{x} \leq w\}, \tag{1}$$

with prices $p \in P = \Re^L_{++}$ and wealth $w \in W = \Re_{++}$ strictly positive. Demand is thus defined as an extended function of prices, state preference variables and wealth. The basic preference and choice assumptions are that the unobservable preference relation is rational on Y, as well as continuous and locally nonsatiated for any distinctions in X,[6] and that the observable demand function is single valued.[7] The distinction introduced here between the choice domain X with typical element x, and the preference domain Y with typical element (x, z), is the root source of the problem addressed in this paper.

For purposes of welfare analysis we are only concerned with elements of Y that might actually occur with market interaction, i.e., those can be obtained with the demand function. For each $z \in Z$, the obtainable set in X is $\{x \in X \mid x = \widehat{x}(p, z, w) \text{ for some } (p, w) \in \Re^{L+1}_{++}\}$.[8] To ease the presentation in this paper I assume that the obtainable set is the same for all $z \in Z$, denoted by $\widehat{X}$.[9] With $\widehat{X}$, the overall obtainable subset of Y is defined as $\widehat{Y} = \widehat{X} \times Z$. The sets $\widehat{X}$ and $\widehat{Y}$ are known to the analyst if demand is fully observable. The notation $\succsim_{\widehat{Y}}$ indicates the restriction of $\succsim_Y$ to the obtainable preference domain $\widehat{Y}$. Thus, for purposes of welfare analysis we are only interested in the preference information represented by $\succsim_{\widehat{Y}}$.

For each $z \in Z$, we can use $\succsim_{\widehat{Y}}$ to define a z-fixed preference relation on $\widehat{X}$, $\succsim_z$, such that $x^1 \succsim_z x^2 \iff (x^1, z) \succsim_{\widehat{Y}} (x^2, z)$ for all $x^1, x^2 \in \widehat{X}$. It then follows that each $\succsim_z$ is rational, continuous and locally nonsatiated. I assume that each $\succsim_z$ can be uniquely identified from the demand function.[10] This key assumption should be noncontroversial as it is a more careful

[5] Reserving subscripts for individual components in a z vector.

[6] Continuity on the Z portion of Y is a vacuous property for any discrete z and therefore meaningless without special restrictions on the nature of Z. Local nonsatiation is only useful if restricted to the choice set X, even if meaningful on the larger preference set.

[7] The preference relation is not necessarily monotone or convex. While strict convexity of preference is sufficient for the demand relationship to be single valued, even weak convexity is not necessary. If preference is monotone, then weak convexity is necessary, and strict convexity becomes necessary when preference is strongly monotone.

[8] This distinction between X and the obtainable subset is also used by Richter (1971). There is no "usual" obtainable set. For example with $X = \Re^2_+$, three different obtainable consumption sets are obtained with Cobb-Douglas, quasilinear and Stone-Geary preferences.

[9] However all of the results presented here except for Theorem A3 have been obtained for the more general case without this assumption.

[10] More formally, suppose that the same demand function was obtained with two preference relations on Y,

statement of the widely understood idea that there is a one-to-one relationship between ordinary preference relations and ordinary demand functions (both without state preference variables).[11] Working in the other direction, we can construct the extended demand function with only $\{\succsim_z \mid z \in Z\}$:

$$\widehat{x}(p, z, w) = \{x \in \widehat{X} \mid p \cdot x \leq w, \text{ and } x \succsim_z \bar{x} \text{ for all } \bar{x} \in \widehat{X} \text{ such that } p \cdot \bar{x} \leq w\}. \qquad (2)$$

Thus the information content of the demand function is identical with the set $\{\succsim_z \mid z \in Z\}$.[12] I shall refer to this information as revealed preference information, or more formally, as the identifiable z-fixed preference relations $\{\succsim_z \mid z \in Z\}$.

2 Problem: Missing Preference Information

The previous paragraph sets up our missing preference information problem: since the demand function can be constructed with only $\{\succsim_z \mid z \in Z\}$, any other preference distinctions specified by $\succsim_{\widehat{Y}}$ cannot recovered from demand. Thus for any $(x^a, z^a), (x^b, z^b) \in \widehat{Y}$ with $z^a \neq z^b$, we cannot determine whether or not $(x^a, z^a) \succsim_{\widehat{Y}} (x^b, z^b)$. This should not be surprising. Since the consumer never faces a choice between (x^a, z^a) and (x^b, z^b), it is not possible for this preference information to be reflected in consumer behavior and hence cannot be incorporated into the demand function – there can be no revealed preference that involve distinctions in z.

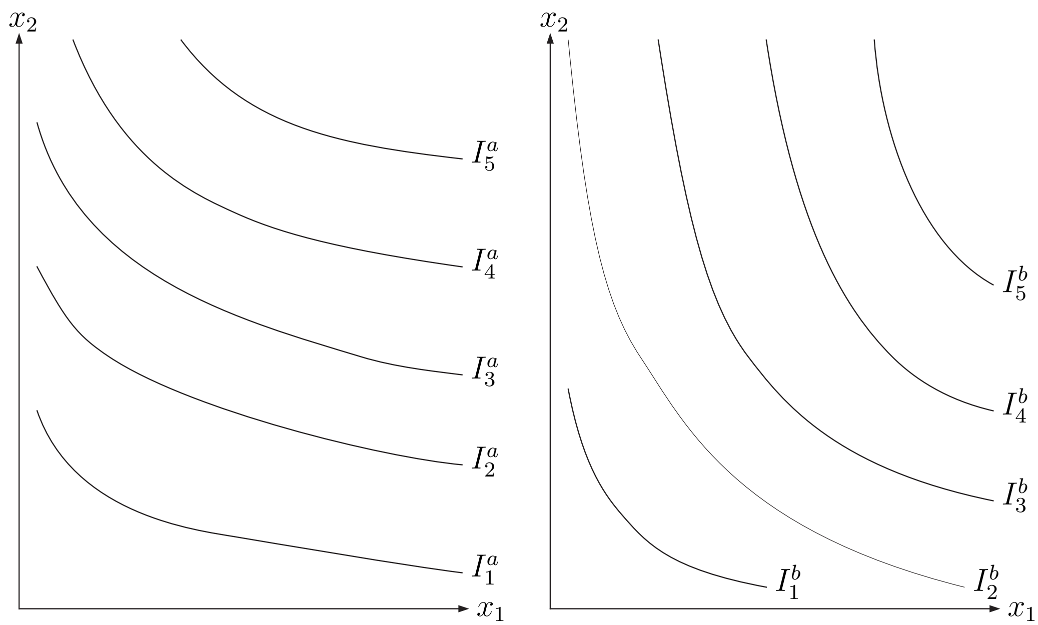
The missing preference information problem is illustrated by Figure 1, for some $z^a, z^b \in Z$ with $z^a \neq z^b$, $X = \Re^2_+$ and $\widehat{X} = \Re^2_{++}$. With fixed $z = z^a$, from the demand function $\widehat{x}(p, z, w)$, we can identify the preference relation $\succsim_{z^a}$ as represented in part (a) by the indifference curves I^a_j. Similarly, with $z = z^b$ we can identify $\succsim_{z^b}$ as represented by the indifference curves I^b_j in part (b). Then we know for example that all the points in I^a_5 are preferred to the points in I^a_3 and all the points in I^b_4 are preferred over the elements of I^b_1. However from revealed preference alone, we do not know whether or not the consumer prefers the points of I^a_5 (with $z = z^a$) over those of I^b_3 (with $z = z^b$). The problem is then recovering the remaining preference information that will enable us to compare the indifference curves in part (a) with those in part (b).

The information content of $\succsim_{\widehat{Y}}$ not available with $\{\succsim_z \mid z \in Z\}$ is necessary for exact welfare analysis involving distinctions in z. Furthermore, the whole of $\succsim_{\widehat{Y}}$ is sufficient

$\succsim_A$ and $\succsim_B$ (each replacing $\succsim_Y$ in equation (1)). Then for each $z \in Z$, I am assuming $(x^1, z) \succsim_A (x^2, z) \iff (x^1, z) \succsim_B (x^2, z)$ for all $x^1, x^2 \in \widehat{X}$.

[11] Proofs for this typically require additional conditions such as that the demand function satisfies the Lipschitz condition (Uzawa, 1971). Thus with this assumption I am implicitly assuming whatever regularity conditions are required to make it true in given a situation. It is only here, in the traditional context of X, that I tacitly take advantage of real analysis. I dispense with it in the context of Z.

[12] This implies an interpretation of the demand function such that it includes only quantitative information, with no qualitative information (such as about how the products are used and thereby provide utility, or descriptive information about the similarity or dissimilarity of different products). Product names are considered only nominal and by themselves provide no substantive information.



(a) Indifference curves with $z = z^a$ (b) Indifference curves with $z = z^b$

Figure 1: Preference indifference curves in $X = \Re^2_+$ with alternative z values.

for such analysis. The most basic welfare question in this context would ask which of some $(x^a, z^a), (x^b, z^b) \in \widehat{Y}$ is "better" or preferred. As already discussed, this can be answered with $\succsim_{\widehat{Y}}$, but not if $z^a \neq z^b$ and we only know $\{\succsim_z \mid z \in Z\}$. The state preference variable theoretical literature is predominantly concerned with specifying exact wealth-compensation-type welfare measures or price index constructs. Suppose that we are concerned with a change in price and z from (p^a, z^a) to (p^b, z^b) with wealth fixed at w^0. Then the equivalent compensating wealth may be measured by EV where $x^a = \widehat{x}(p^a, z^a, w^0 + EV)$, $x^b = \widehat{x}(p^b, z^b, w^0)$ and $(x^a, z^a) \sim_{\widehat{Y}} (x^b, z^b)$.[13] From this we can also obtain a price index, $\varphi = (w^0 + EV)/w^0$. Thus $\succsim_{\widehat{Y}}$ is sufficient for exact welfare analysis while $\{\succsim_z \mid z \in Z\}$ by itself is insufficient. Moreover, all of the preference information present in $\succsim_{\widehat{Y}}$ is necessary if we need to be in a position to consider any pair $(p^a, z^a), (p^b, z^b) \in P \times Z$. Since $\succsim_{\widehat{Y}}$ is sufficient without additional assumptions, it follows that continuity and other regularity conditions typically imposed on z are not necessary for exact welfare analysis, however useful they may be.

Given our inability to identify the "true" preference relation $\succsim_{\widehat{Y}}$, we will often want to deal with the set of feasible candidate preference relations that could be the "true" relation. This set can be characterized with a key demand-consistency concept,

Definition. Given an extended demand function $\widehat{x}(p, z, w)$, with all of the z-fixed relations identifiable from $\widehat{x}$, $\{\succsim_z \mid z \in Z\}$, a preference relation $\succsim$ defined on some $\widetilde{Y} \subseteq Y$ is said to be $\widehat{x}$-*consistent* if it is consistent with each $\succsim_z$, $z \in Z$.

[13] EV is an equivalent variation welfare measure; a compensating variation measure is also easily constructed. The property "unified preference" to be introduced later is a sufficient but not a necessary condition for the existence of x^a and hence also for EV.

Thus $\succsim$ is $\widehat{x}$-consistent if and only if $(x^a, z) \succsim (x^b, z) \Leftrightarrow x^a \succsim_z x^b$ for all $(x^a, z), (x^b, z) \in \widetilde{Y} \bigcap \widehat{Y}$. With $\widetilde{Y} = \widehat{Y}$, any $\widehat{x}$-consistent relation could be the true relation. Hence the set of feasible candidate preference relations is $\Phi(\widehat{x}) = \{\succsim \text{ defined on } \widehat{Y} \mid \succsim \text{ is } \widehat{x}\text{-consistent}\}$. Since the unknown "true" preference relation $\succsim_{\widehat{Y}}$ is an element, we know that $\Phi(\widehat{x})$ is not empty. With only revealed preference information we are able exclude any preference relation on $\widehat{Y}$ that is not an member of $\Phi(\widehat{x})$, but are not able exclude any element of $\Phi(\widehat{x})$.

To help motivate the rest of this paper, the remainder of this section focuses on the breadth of $\Phi(\widehat{x})$ membership. Most of the concepts in this article are more easily developed and presented purely in terms of preference relations. However representative utility functions can be used to provide a characterization of $\Phi(\widehat{x})$ that the reader may find more meaningful. I assume that all utility functions have the same range $\Re_u \subseteq \Re$,[14] and that $\succsim_Y$ can be represented by a utility function, $u_Y : Y \to \Re_u$ with notation $u_Y(x, z)$. The constrained utility maximization program is then,

$$\begin{aligned} \text{Program UY:} \quad \max_x \quad & u_Y(x, z) \\ s.t. \quad & p \cdot x \leq w, \\ & x \in X. \end{aligned}$$

We thus obtain the same extended demand function as with Equation (1), $\widehat{x}(p, z, w)$.

We can precisely characterize the membership of $\Phi(\widehat{x})$ in terms of utility functions that are related by a special class of transformations:

Theorem 1. *Let u_1 and u_2 be two utility functions representing preferences on Y.*

a. The utility functions u_1 and u_2 will yield the same demand function with Program UY if and only if there is some transformation $g : \Re_u \times Z \to \Re$, $g(u, z)$, such that $u_2(x, z) = g(u_1(x, z), z)$ for all $(x, z) \in \widehat{Y}$, with g strictly increasing in u. Such a transformation is called a "g-transform."[15]

b. u_1 and u_2 represent the same preference relation on $\widehat{Y}$ if and only if the g-transform of part (a) is actually a traditional monotonic transformation $f : \Re_u \to \Re$ such that $g(u, z) = f(u)$ for all $(u, z) \in \widehat{\Re}_1 \times Z$, where $\widehat{\Re}_1$ is the range of u_1 when restricted to the domain $\widehat{Y}$, $\widehat{\Re}_1 = \{u \in \Re_u \mid u = u_1(x, z) \text{ for some } (x, z) \in \widehat{Y}\}$.

If both u_1 and u_2 are also differentiable in x and related by a g-transform as described in part (a), then their respective Kuhn-Tucker Conditions associated with Program UY will be equivalent. If u_1 and u_2 represent the same preference order and if the g-transform is differentiable with respect to z, then part (b) implies that $\partial g / \partial z \equiv 0$.

From Theorem 1 we know that $\succsim_i \in \Phi(\widehat{x})$ if and only if there is a g-transform such that $u_i(x, z) = g(u_Y(x, z), z)$, where u_i represents an extension of $\succsim_i$ to Y.[16] With the wide

[14] This simplifying assumption only reduces the available utility representations for any given preference relation, and thereby enables the "if and only if" statements in Theorem 1. Typically, $\Re_u = [0, \infty)$.

[15] The ($\Leftarrow$) aspect of part a of this theorem has also been identified by Ebert (2001).

[16] Assuming that $\succsim_i$ is representable by a utility function. Henceforth I will gloss over the distinction between an $\succsim \in \Phi(\widehat{x})$ and an extension of $\succsim$ that may be represented by a utility function defined on Y.

variety of potential g-transforms, we can see how welfare analysis for differences in a state preference variable is impossible using only the preference information available from the demand function. For example, the maximal range of our previously defined welfare measure EV is $(-w^0, \infty)$, and as a consequence of Theorem 1, every EV value in this maximal range will be taken on by some element of $\Phi(\widehat{x})$.[17] The same is true for other traditional exact welfare measures such as compensating variation and assorted price indices with respect to their maximal ranges. Moreover, we do not even know if such traditional measures are well defined (i.e., exist). To see this I need to introduce a new concept.

For some purposes it useful to know that a preference level obtained with one $z \in Z$ can also be obtained with any other element of Z.

Definition. An indifference relation "$\sim$" defined on some $\widetilde{Y} \subseteq Y$.[18] is said to be *unified* if whenever $z^a, z^b \in Z$ and $(x^a, z^a) \in \widetilde{Y}$, there exists some $(x^b, z^b) \in \widetilde{Y}$ such that $(x^a, z^a) \sim (x^b, z^b)$. A preference relation defined on $\widetilde{Y}$ is said to be unified if its associated indifference relation is unified.

Economists typically assume that preference is unified. In particular, this property is locally necessary for the application of any of our standard compensating welfare measures such as EV. However, non-unified preferences may be useful in capturing the effect of dramatic state preference variables such as the loss of a child or some catastrophic environmental state variable. Theorem 1 tells us that $\Phi(\widehat{x})$ always includes both unified and non-unified preferences.[19] Thus we cannot determine from the demand function whether our standard welfare measures are globally well defined.

Finally, suppose that a traditional demand function $\widehat{x}(p, w)$ can be obtained with an ordinary utility function $u(x)$, but that there is also a state preference variable z that might affect preference (we do not know if it does or does not). Then any utility function of the form $g(u(x), z)$ would yield the same demand function. Thus the lack of demand sensitivity to

[17]For any fixed $(x^a, z^a), (x^b, z^b) \in \widehat{Y}$ with $z^a \neq z^b$.

[18]Formally, an indifference relation on a set B is based on a complete partition of the set into indifference sets, $\mathbf{I} = \{I_i \subseteq B\}$. The indifference relation is then a binary relation defined on elements of B represented by "$\sim$" such that for $a, b \in B$, $a \in I_a$, $b \in I_b$ with $I_a, I_b \in \mathbf{I}$, we have $a \sim b$ if and only if $I_a = I_b$. At the extreme, this definition allows for indifference relations where each I_i is a singleton and $a \sim b \Leftrightarrow a = b$. As the individual sets become larger (and fewer), an indifference relation includes more indifference information in that we are then able to say "$a \sim b$" for more pairs $a, b \in B$. Sometimes an indifference relation may be specified by some rule of the form "If $x, y \in B$ share some property A, then $x \sim y$." The actual indifference relation is then defined by the transitive closure with respect to this property, which thus determines a partition on B.

[19]For example if $\succsim_i \in \Phi(\widehat{x})$ is unified with representative utility function u_i, then for any fixed $z^0 \in Z$, the preference relation represented by

$$u_j(x, z) = \begin{cases} \tanh\left(u_i(x, z)\right), & \text{if } z = z^0, \\ u_i(x, z), & \text{if } z \neq z^0. \end{cases},$$

is a non-unified element of $\Phi(\widehat{x})$. The hyperbolic tangent function $\tanh(\cdot)$ maps $[0, \infty)$ to $[0, 1)$. For any non-unified preference it is also possible to specify a g-transform that will convert it into a unified preference.

potential state preference variables cannot by itself be used to preclude preference sensitivity, and offers no particular help in identifying the "true" underlying preference relation.[20] Again, what we need is additional preference information (API). In the rest of the paper, I first specify a structure for characterizing the missing preference information that allows us to know when some suggested API is sufficient and test the API when it is more than sufficient, and then close with several examples.

3 Reference Sets and Seed Relations

Reference sets and seed relations provide a way of specifying the additional information required to identify a unique complete preference relation on $\widehat{Y}$. As will be seen, they can also be used to provide precise characterizations of the various preference restrictions prevalent in the literature that allow us to determine precisely when these restrictions can be tested against revealed preference information. With a fixed reference set and seed preference relations we can uniquely specify each element of $\Phi(\widehat{x})$.

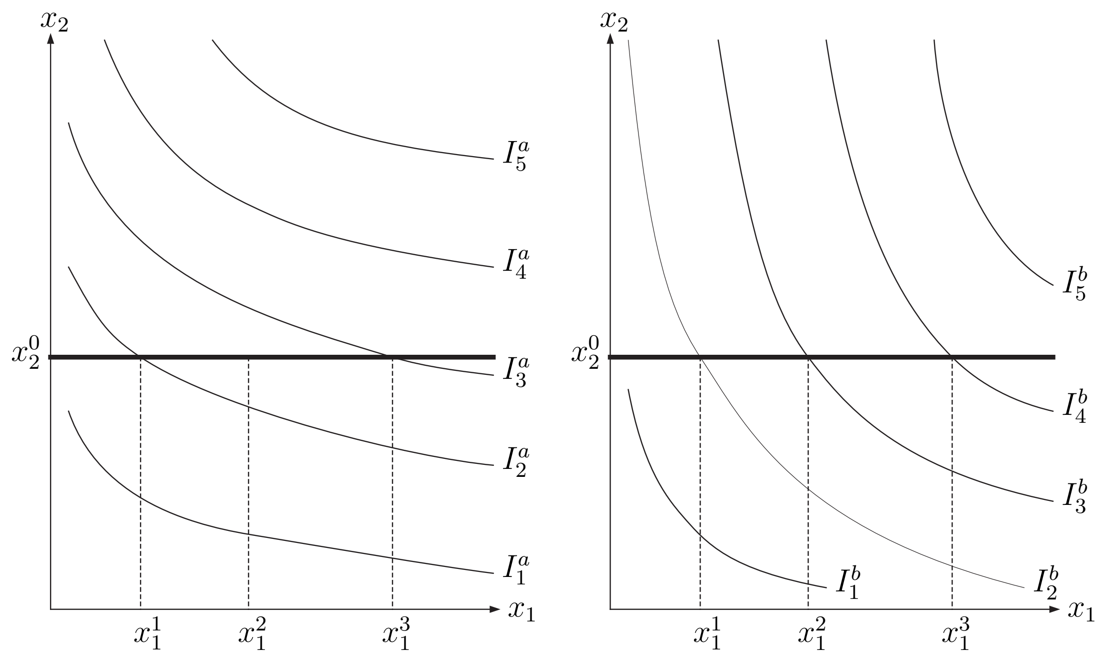
The reference set concept is illustrated with Figure 2 using the same indifference curves depicted in Figure 1. Recall that each of these curves drawn in X space actually represents an indifference set in $X \times Z$, with z fixed in parts (a) and (b) at either $z = z^a$ or $z = z^b$ respectively. A horizontal line has been added in both parts of the figure representing the set $X_R = \{(x_1, x_2) \in \Re^2_+ \mid x_2 = x^0_2\}$, for some fixed value x^0_2. As will be demonstrated shortly, X_R provides a common frame of reference between the two sets of indifference curves presented in the two parts of Figure 2, and is hence called a reference set. In general, a reference set is some subset of the choice set, $X_R \subseteq X$.

With this common frame of reference and some additional information we can compare the I^a and I^b indifference sets, and thereby establish a complete preference relation on $\widehat{Y}$. With Figure 2, suppose for example that we knew the consumer is indifferent between z^a and z^b for any fixed $x \in X_R$: $(x, z^a) \sim_Y (x, z^b)$. Then from $(x^1_1, x^0_2, z^a) \sim_Y (x^1_1, x^0_2, z^b)$, with $(x^1_1, x^0_2, z^a) \in I^a_2$ and $(x^1_1, x^0_2, z^b) \in I^b_2$, we know that the consumer does not perceive any preference distinction between the I^a_2 and I^b_2 indifference sets, and thus is indifferent between all the (x, z) points in $I^a_2 \bigcup I^b_2$. Similarly the consumer is also indifferent between all points in $I^a_3 \bigcup I^b_4$. Thus we can combine an understanding of preference on $X_R \times Z$ with revealed preference to establish a preference relation on all of $\widehat{Y}$, showing for example that each point in I^a_3 is preferred to each point in I^b_3. The preference relation on $X_R \times Z$ is called a seed relation. I first discuss some properties of reference sets, and then the role of seed relations.

3.1 Reference Set Properties

There are four desirable properties of reference sets and I begin with three of them. A reference set X_R is said to be *sufficient* if for any $(x, z) \in \widehat{Y}$ we can always find some $x_R \in X_R$ that is in the same $\succsim_z$ indifference set as x, that is $x \sim_z x_R$. A reference set is

[20] The lack of demand sensitivity only tells us that $\succsim_{z^a} = \succsim_{z^b}$ for all $z^a, z^b \in Z$.



(a) With $z = z^a$ (b) With $z = z^b$

Figure 2: A complete non-redundant reference set.

termed *non-redundant* if the just discussed x_R is always unique.[21] A reference set X_R is non-redundant if and only if the indifference sets of $X_R \bigcap \widehat{X}$ under each $\succsim_z$ are all singletons. For example, the $x_2 = x_2^0$ line in Figure 2 intersects each indifference curve at exactly one point. Finally, a subset of X is said to be *naturally ordered* if each element is either monotonically superior or monotonically inferior to any other element.[22] The first two properties depend on the demand function via revealed preference while the third does not. The reference set depicted in Figure 2 is sufficient, non-redundant and naturally ordered for $Z = \{z^a, z^b\}$.

There are also some useful relationships between these three properties. With the following theorem we know that any naturally ordered reference set is universally non-redundant for all demand functions.

Theorem 2. *A naturally ordered reference set $X_R \subseteq X$ is non-redundant for any feasible demand function $\widehat{x} : \Re^L_{++} \times Z \times \Re_{++} \to \Re^L$, for any possible Z.*

While a reference set that is non-redundant with a given demand function might not be naturally ordered, the next theorem shows that any universally non-redundant reference set must be naturally ordered.

[21]More formally, a reference set X_R is redundant if for some $(x, z) \in \widehat{Y}$, there are $x_R^1, x_R^2 \in X_R$ with $x_R^1 \neq x_R^2$ such that $x \sim_z x_R^1$ and $x \sim_z x_R^2$.

[22]Formally, a set $\widetilde{X} \subseteq X$ is naturally ordered if for any $x^1, x^2 \in \widetilde{X}$ with $x^1 \neq x^2$, we have either $x^1 \leq x^2$ or $x^2 \leq x^1$. Editorial note: This terminology indicates that we immediately know the complete preference ordering on X_R for all $\succsim_z$. Alternatively, we could say that X_R is monotone. However in practice that would create confusion with monotone preference, such as with Theorems P2 and P3 (next subsection).

Theorem 3. *Let $X_R \subseteq X$ be a non-redundant reference set for any feasible demand function $\widehat{x} : \Re^L_{++} \times Z \times \Re_{++} \to \Re^L$, for any possible Z. Then X_R must be naturally ordered.*

There is also a useful relationship between sufficiency and being naturally ordered.

Theorem 4. *Let $X_R \subseteq X$ such that X_R is naturally ordered and sufficient. For any fixed $z \in Z$, if $\succsim_z$ can be extended to a monotone relation on X, then $X_R \backslash \{0\} \subseteq \widehat{X}$.*

As discussed below in the context of weak complementarity, this last theorem can sometimes be used to substantially simplify testing for sufficiency.

The notion of universal sufficiency is not as easily captured as universal non-redundancy. However some reference sets are more likely to be sufficient than others. For example if $\succsim_Y$ is strictly convex and strongly monotone, so that $\Re^L_{++} \subseteq \widehat{X}$ (as is typical), then the diagonal reference set $X_D = \{x \in \Re^L_+ \,|\, x_i = x_j \,\forall\, i, j,\, 1 \leq i < j \leq L\}$ must be sufficient. On the other hand, the reference set depicted in Figure 2 may be insufficient in the same circumstance.[23] Unfortunately, in many situations the most natural or convenient reference set does not have this universal quality (as we see below with our main example, weak complementarity).

Our last reference set property builds upon sufficiency and redundancy. Sometimes a proper subset of a sufficient redundant reference set is also sufficient. In such a case, we may prefer to use the smaller set as our reference set. Any sufficient reference set whereby the deletion of any point will make the set insufficient is called *irreducible*. Obviously any sufficient and non-redundant reference set is irreducible. However, many sufficient redundant reference sets are also irreducible. Thus given sufficiency, non-redundancy is stronger that irreducibility, as it is also weaker than being naturally ordered.

3.2 Seed Preference Relations and Preference Generation

The desirability of sufficiency and non-redundancy becomes apparent in the context of seed preference relations. Given a reference set X_R, a seed preference relation, $\succsim_s$, is a complete and transitive preference relation defined on the set $\widehat{Y}_R = (X_R \times Z) \bigcap \widehat{Y}$. Let $\succsim_s$ be a $\widehat{x}$-consistent seed preference relation defined on $(X_R \times Z) \bigcap \widehat{Y}$, where X_R is a sufficient and non-redundant reference set (given the demand function $\widehat{x}$), and let $(x^a, z^a), (x^b, z^b) \in \widehat{Y}$. Then we know that there are unique $x^a_R, x^b_R \in X_R$ such that $x^a \sim_{z^a} x^a_R$ and $x^b \sim_{z^b} x^b_R$, and we can therefore define the relation $\succsim_S \in \Phi(\widehat{x})$ such that $(x^a, z^a) \succsim_S (x^b, z^b)$ if and only if $(x^a_R, z^a) \succsim_s (x^b_R, z^b)$. Here, we have used the seed preference relation $\succsim_s$ defined on $\widehat{Y}_R$ to generate the complete preference relation $\succsim_S$ on $\widehat{Y}$, where $\succsim_s$ and $\succsim_S$ are both $\widehat{x}$-consistent. If X_R were not sufficient, then x^a_R does not exist for some $(x^a, z^a) \in \widehat{Y}$, preventing this construction and thus leaving $\succsim_S$ incomplete. If X_R were redundant then x^a_R and x^b_R might not be unique. More importantly, with redundancy it becomes harder to specify $\widehat{x}$-consistent seed relations (as will be demonstrated in the applications section).

The purpose of a seed relation is to provide the missing preference information so that the "true" $\succsim_{\widehat{Y}} \in \Phi(\widehat{x})$ can be identified and recovered as the generated preference relation.

[23]For example, with CES preference, many indifference curves will not intersect with this reference set.

The following theorem specifies how individual seed relations uniquely identify elements of $\Phi(\widehat{x})$, as well as showing that any $\succsim \in \Phi(\widehat{x})$ can be generated by such a seed relation.[24]

Theorem P1. *Given the demand function $\widehat{x}$ with obtainable set $\widehat{Y}$, let X_R be a sufficient reference set and define $\widehat{Y}_R = (X_R \times Z) \bigcap \widehat{Y}$.*

a. Let $\succsim_s$ be a $\widehat{x}$-consistent seed preference relation defined on $\widehat{Y}_R$. Then there is a unique $\succsim_S \in \Phi(\widehat{x})$ that is consistent with $\succsim_s$ on $\widehat{Y}_R$. This $\succsim_S$ is said to be generated by $\succsim_s$.

b. Any $\succsim_S \in \Phi(\widehat{x})$ can be generated by a unique $\widehat{x}$-consistent seed preference relation on $\widehat{Y}_R$.

Part a tells us that with the information available from $\widehat{x}$ and $\succsim_s$, we can identify a unique element of $\Phi(\widehat{x})$. Together, parts a and b tell us that with X_R fixed there is a one-to-one relationship between the set of all $\widehat{x}$-consistent seed preference relations defined on $\widehat{Y}_R$ and the elements of $\Phi(\widehat{x})$. Thus with X_R fixed, identifying the unique "true" element of $\Phi(\widehat{x})$ is equivalent to identifying the unique "true" $\widehat{x}$-consistent seed preference relation on $\widehat{Y}_R$.

Theorem P1 does not explicitly deal with redundancy or reducibility. However these properties affect preference generation as described by the theorem. Suppose that X_R is reducible. Then for any given seed relation, the generated relation $\succsim_S \in \Phi(\widehat{x})$ can also be generated with an alternative seed relation defined on a smaller preference domain, $(\widetilde{X}_R \times Z) \bigcap \widehat{Y}$ where $\widetilde{X}_R \subset X_R$ is also sufficient. Thus we may generate each element of $\Phi(\widehat{x})$ with less additional preference information. Alternatively, if X_R is non-redundant, or at least irreducible, then in a sense each demand-consistent seed relation represents the minimal additional preference information required to generate individual elements of $\Phi(\widehat{x})$. Moreover, as an already sufficient reference set is augmented by additional points, the number of possible seed preference relations is vastly increased but the number of $\widehat{x}$-consistent seed relations remains constant. Consequently, it becomes increasingly difficult to specify a demand-consistent seed preference relation as the reference set becomes more redundant.

On the other hand, the $\widehat{x}$-consistency requirement is quite easy to satisfy if the reference set is naturally ordered (i.e., "super-non-redundant"). Any $\widehat{x}$-consistent preference relation defined on some $\widetilde{Y} \subseteq \widehat{Y}$ is strongly monotone.[25] However, strong monotonicity does not imply that a preference relation is $\widehat{x}$-consistent. As indicated by the following theorem, a strongly monotone seed preference relation is always demand-consistent if and only if $X_R \bigcap \widehat{X}$ is naturally ordered.

[24] Editorial note: Fourteen theorems are presented in this paper. To help the reader keep track of their individual roles, I use four distinct numbering sequences. With Theorems 1 through 4 we have already encountered most of the "regular" sequence. With "P" for "seed **P**reference," the three P-theorems presented in this subsection match up one-to-one with the three I-theorems in the next subsection ("I" for "seed **I**ndifference"). Finally, to distinguish between the general aspects of reference sets and seed relations, and properties that are specific to particular applications, we have the A-sequence in the applications section.

[25] See Lemma 4 in the Appendix. In the realm of Y, without special restrictions on the nature of Z, notions of monotonic preference can only be concerned with distinctions in the values of x, with z held fixed. For example, if the elements of Z are not numbers and do not otherwise have a natural order, such as with $Z = \{Rain, Snow, Sunshine\}$, then monotonic preference on Z is meaningless. Therefore, a preference relation $\succsim$ defined on some $\widetilde{Y} \subseteq \widehat{Y}$ is *strongly monotone* if $(x^a, z) \succ (x^b, z)$ whenever $x^a \geq x^b$ and $x^a \neq x^b$ for all such $(x^a, z), (x^b, z) \in \widetilde{Y}$ with $z \in Z$.

Theorem P2. *Given the demand function $\widehat{x}$ with obtainable set $\widehat{X}$, let X_R be a reference set and define $\widehat{Y}_R = (X_R \times Z) \bigcap \widehat{Y}$.*

a. Let $X_R \bigcap \widehat{X}$ be naturally ordered and let $\succsim_s$ be a strongly monotone seed preference relation on $\widehat{Y}_R$. Then $\succsim_s$ is $\widehat{x}$-consistent.

b. Let all strongly monotone seed preference relations on $\widehat{Y}_R$ be $\widehat{x}$-consistent. Then $X_R \bigcap \widehat{X}$ is naturally ordered.

With X_R naturally ordered, $X_R \bigcap \widehat{X}$ is also so that part a of this theorem follows. Thus, with a naturally ordered reference set, demand-consistency requirements are substantially weakened in the sense that being strongly monotone is generically a weaker requirement than being $\widehat{x}$-consistent, and typically easier to demonstrate. With Theorems 2, 3, 4 and P2, we can see that being naturally ordered has implications for non-redundancy, sufficiency, as well as demand-consistency.

Part b of Theorem P2 suggests that when the reference set is not naturally ordered it is possible for a strongly monotone seed relation and a demand function to be mutually inconsistent. In particular, we know that there must be some strongly monotone seed preference relation that is not x-consistent.[26] This following existence theorem is concerned with demand-inconsistency from the context of a fixed seed relation.

Theorem P3. *Let X_R be a reference set that is not naturally ordered and let $\succsim_s$ be a strongly monotone seed preference relation defined on $Y_R = X_R \times Z$. Then $\succsim_s$ is not demand-consistent for an infinite number of valid demand functions.*

Thus when the reference set is not naturally ordered, no seed relation is universally valid with all demand functions so that it is possible to test suggested seed relations against the demand function in the form of revealed preference. As demonstrated below, Theorems P2 and P3 are together quite useful in specifying when a preference restriction can be tested.

3.3 Seed Indifference Relations

Heretofore I have been concerned with seed preference relations to generate elements of $\Phi(\widehat{x})$. Sometimes we can get by with less seed information in the form of seed indifference relations. In the following, after developing some common aspects of indifference relations, I establish an important generic case where we can get by with just indifference information in the seed relation. I also show that we can get by with less than what may be called "complete" indifference information.

A preference relation $\succsim$ and an indifference relation $\sim$ defined on the same set A are said to be *associated* if $a \sim b \Leftrightarrow [a \succsim b$ and $b \succsim a]$ for all $a, b \in A$.[27] There is also a weaker condition whereby $\succsim$ and $\sim$ are said to be *consistent* if $a \sim b \Rightarrow [a \succsim b$ and $b \succsim a]$ for all

[26] From the proof we know that there are an infinite number of demand-inconsistent strongly monotone seed preference relations (one for each value of α).

[27] Recall that a indifference relation is based on a partition of the set. See note 18.

$a, b \in A$. Given a complete and transitive preference relation, there is a unique associated indifference relation and typically many merely consistent indifference relations.[28] However, for any nontrivial indifference relation there are is no unique ordering of the individual indifference sets and hence no unique associated preference relation. Thus, the information content of a indifference relation is generally less than that of a preference relation defined on the same preference domain, and the information content of an associated indifference relation is more than that of a "merely consistent" relation. The indifference relation associated with any generic $\succsim_i$ is represented by $\sim_i$, so that in particular, for any $z \in Z$, the indifference relation associated with $\succsim_z$ is denoted by $\sim_z$.

To be useful, our definition of demand-consistency in the case of indifference relations needs to be more complicated than our previous definition for preference relations.

Definition. Given an extended demand function $\widehat{x}(p, z, w)$, with identifiable z-fixed preference relations $\{\succsim_z \mid z \in Z\}$, an indifference relation $\sim$ defined on some $\widetilde{Y} \subseteq \widehat{Y}$ is $\widehat{x}$*-consistent* if two conditions hold: 1) For any $(x^a, z), (x^b, z) \in \widetilde{Y}$ with $z \in Z$, we have $(x^a, z) \sim (x^b, z) \Rightarrow x^a \sim_z x^b$; and 2) For any $(x^1, z^a), (x^2, z^b), (x^3, z^a), (x^4, z^b) \in \widetilde{Y}$ and $z^a, z^b \in Z$, such that $(x^1, z^a) \sim (x^2, z^b)$ and $(x^3, z^a) \sim (x^4, z^b)$, we have $x^1 \succsim_{z^a} x^3 \Leftrightarrow x^2 \succsim_{z^b} x^4$.

A stronger version of the first condition is also sometimes useful: 1′) For any $(x^a, z), (x^b, z) \in \widetilde{Y}$ with $z \in Z$, $(x^a, z) \sim (x^b, z) \Leftrightarrow x^a \sim_z x^b$.

Given a reference set X_R, a seed indifference relation $\sim_s$ is an indifference relation defined on the set $\widehat{Y}_R = (X_R \times Z) \bigcap \widehat{Y}$. With the help of a lemma we can transform the previous results concerning seed preference relations into similar conclusions about seed indifference relations. I begin with a result very similar to Theorem P1,

Theorem I1. *Given the demand function $\widehat{x}$ with obtainable set $\widehat{Y}$, let X_R be a sufficient reference set, and define $\widehat{Y}_R = (X_R \times Z) \bigcap \widehat{Y}$.*

a. Let $\sim_s$ be a unified $\widehat{x}$-consistent seed indifference relation on $\widehat{Y}_R$. Then there is a unique $\succsim_S \in \Phi(\widehat{x})$ that is consistent with $\sim_s$ on $\widehat{Y}_R$. This $\succsim_S$ generated by $\sim_s$ is also unified.

b. Any unified $\succsim_S \in \Phi(\widehat{x})$ can be generated by a unique unified $\widehat{x}$-consistent seed indifference relation defined on $\widehat{Y}_R$ that satisfies condition 1′, represented by $\sim_s$. Furthermore, where $\sim_t$ is a unified $\widehat{x}$-consistent seed indifference relation on $\widehat{Y}_R$, $\succsim_S$ can generated by $\sim_t$ if and only if $(x^a, z^a) \sim_t (x^b, z^b) \Rightarrow (x^a, z^a) \sim_s (x^b, z^b)$ for all $(x^a, z^a), (x^b, z^b) \in \widehat{Y}_R$.

As before with Theorem P1, this theorem allows us to identify a unique member of $\Phi(\widehat{x})$ with the information available from $\widehat{x}$ and a seed relation. Part a of Theorem I1 provides sufficiency conditions for when a seed indifference relation may generate a complete unified preference relation on $\widehat{Y}$, while part b tells us that any such unified element of $\Phi(\widehat{x})$ can

[28]Given $\succsim$, if $\sim_a$ is the unique associated indifference relation and $\sim_c$ another merely consistent relation ($\sim_c \neq \sim_a$) we have $r \sim_c s \Rightarrow r \sim_a s$ for all $r, s \in A$, but for some $t, v \in A$ we have $t \sim_a v$ and not $t \sim_c v$. The associated relation $\sim_a$ provides a complete account of the indifference relations implied by $\succsim$, where as $\sim_c$ does not. Thus the merely consistent indifference relation is in a sense incomplete. In allowing this kind of incompleteness, we have a relatively weak understanding of indifference.

be generated by possibly several seed indifference relations, of which exactly one satisfies condition $1'$. Thus, "$\succsim_S$ is unified" is a sufficient condition for $\succsim_S$ to be generated by an indifference seed relation on X_R.[29] With X_R fixed, from the uniqueness properties of both parts of Theorem I1 we have a one-to-one relationship between the set of all possible unified $\widehat{x}$-consistent seed indifference relations defined on $\widehat{Y}_R$ satisfying condition $1'$ and the unified elements of $\Phi(\widehat{x})$. (Recall that only with unified $\succsim \in \Phi(\widehat{x})$ can we be sure that any of our standard compensating welfare measures such as EV are well defined.)

From part b of Theorem I1, any unified member of $\Phi(\widehat{x})$ can be generated by a unique unified seed indifference relation that satisfies condition $1'$, and often by many relations that do not satisfy condition $1'$. If we use anyone of these latter relations in part a, then we are in effect using less seed indifference information to obtain the same outcome as compared with using the unique relation that satisfies condition $1'$. As before with seed preference relations, if X_R is reducible we can also lower the information content of each seed indifference relation by using a smaller reference set. Thus, the minimal amount of indifference information required to generate a unified $\succsim_S \in \Phi(\widehat{x})$ is a $\widehat{x}$-consistent unified seed relation that either does not satisfy condition $1'$, or is defined on a non-redundant reference set.[30]

We also have the following seed indifference relation equivalents of Theorems P2 and P3,

Theorem I2. *Given the demand function $\widehat{x}$ with obtainable set $\widehat{X}$, let X_R be a reference set and define $\widehat{Y}_R = (X_R \times Z) \bigcap \widehat{Y}$.*

a. Let $X_R \bigcap \widehat{X}$ be naturally ordered and let $\sim_s$ be a strongly monotone seed indifference relation on $\widehat{Y}_R$. Then $\sim_s$ is $\widehat{x}$-consistent and satisfies condition $1'$.[31]

b. Let all strongly monotone unified seed indifference relation on $\widehat{Y}_R$ be $\widehat{x}$-consistent. Then $X_R \bigcap \widehat{X}$ is naturally ordered.

Theorem I3. *Let X_R be a reference set that is not naturally ordered and let $\sim_s$ be a unified seed indifference relation defined on $Y_R = X_R \times Z$. Then $\sim_s$ is not demand-consistent for an infinite number of valid demand functions.*

With the two series of theorems, P1, P2 & P3 and I1, I2 & I3, we can respectively work with either seed preference relations or unified seed indifference relations. The first theorem of both series is the main preference generation theorem that allows us to identify unique members of $\Phi(\widehat{x})$, while the second and third theorems are concerned with the relationship between demand-consistency and whether the reference set is naturally ordered. These four

[29] It is not a necessary condition as there always exist non-unified elements of $\Phi(\widehat{x})$ that can be generated by seed indifference relations. However, it is a necessary screen in that there also always exist non-unified elements of $\Phi(\widehat{x})$ that cannot be generated by seed indifference relations.

[30] With a non-redundant reference set, any $\widehat{x}$-consistent seed indifference relation also satisfies condition $1'$.

[31] An indifference relation defined on some $\widetilde{Y} \subseteq \widehat{Y}$ is said to be strongly monotone if it is consistent with a strongly monotone preference relation defined on $\widetilde{Y}$. Then any $\widehat{x}$-consistent unified indifference relation defined on some $\widetilde{Y} \subseteq \widehat{Y}$ is strongly monotone. This follows from Lemma 5 and a previous observation. See note 25.

latter theorems are used extensively in the following examples as tools in helping us discern when possible API specifications may be tested against revealed preference. The relationships between reference set properties as specified by Theorems 2, 3 and 4 also support this work.

4 Application with Preference Assumptions

Applications with state preference variables in the literature typically invoke some assumption about preference that enables the analyst to sufficiently identify $\succsim_Y$ so that some welfare measure may be specified. Each such assumption is an instance of Additional Preference Information. These suggested API's can be specified with the just developed preference-theoretic structure in a systematic way that allows us to state whether they are sufficient to identify a unique $\widehat{x}$-consistent complete preference relation on $\widehat{Y}$, and also whether they can be tested against revealed preference information. To demonstrate this, I first present an in-depth analysis of weak complementarity and the related concept of existence value, followed by shorter looks at five other applications including two important API concepts from the price index literature for product quality, and also a treatment of the new and disappearing goods problem.

4.1 Weak Complementarity and Existence Value

Most state preference variable applications in the literature are concerned with environmental variables, product quality, or traditional public goods such as local roads. Therefore, it is particulary appropriate to start off with "weak complementarity," a specific API that has been used in all three areas. The notion of weak complementarity was introduced by Mäler as a methodology to estimate the benefits of improving an environmental quality variable such as the quality of a sport fishery stock or of lake water for swimming.[32] It requires that a given state preference variable be associated with one of the market goods in a manner such that it is reasonable to assume that the consumer is indifferent between values of the state preference variable when she is consuming a zero amount of the market good. With Mäler's first example, the state preference variable is the quality of a sport fishery and the market good is the use of that fishery. In this case weak complementarity applies in that we might reasonably assume that non-fishermen do not care about the quality of the fishery, i.e., fishing and the quality of the fishery are weak complements.

With weak complementarity we are assuming the consumer only cares about the state preference variable as it affects the benefit she derives from her personal consumption of the market good. That is to say, the state preference variable does not have "existence value" – I derive no benefit from the existence of a public good that is a quality of a private good unless I consume that private good. However reasonable this assumption may sound, forbidding existence value is an explicitly restrictive modelling assumption that precludes many valid preference relations. For example, I may prefer a strong sport fishery so that I will always

[32]The concept was first developed in Mäler (1971) and the terminology was introduced in Mäler (1974).

have the option of using it, or perhaps because my good friend enjoys fishing.

The first step in applying the structure developed here with a given API is specifying a reference set and seed relation that together capture the defining characteristics of the API. Suppose that z and the private good x_1 are weak complements. Then "no existence value" requires $(x, z^a) \sim_Y (x, z^b)$ for all $x \in X$ such that $x_1 = 0$, and all $z^a, z^b \in Z$.[33] We thus have a ready-made reference set $X_{WC} = \{x \in X \mid x_1 = 0\}$ and a unified seed indifference relation defined by $(x, z^a) \sim_{wc} (x, z^b)$ for all $x \in X_{WC}$ and all $z^a, z^b \in Z$.[34] With these we are ready for a precise consideration of weak complementarity as an API to supplement the revealed preference information in order to identify a unique element of $\Phi(\widehat{x})$.

Preference generation as specified by Theorem I1 part a requires that X_{WC} be sufficient and $\sim_{wc}$ be $\widehat{x}$-consistent. For every $(x^a, z) \in \widehat{Y}$, sufficiency of X_{WC} necessitates the existence of some $(0, x^b_{-1}) \in X_{WC}$ such that $x^a \sim_z (0, x^b_{-1})$.[35] In the words of Willig (1978), this means that with any fixed z "any bundle including good 1 can be matched in the [identifiable $\succsim_z$] preference ordering by some other bundle which excludes good 1" so that x_1 is "nonessential." Therefore, implementing weak complementarity requires two properties, no existence value and the nonessentiality of x_1. In the context of Theorem I1, the first property is associated with $\widehat{x}$-consistency of $\sim_{wc}$ and the second with the sufficiency of X_{WC}.

Testing for sufficiency and demand-consistency would be the next natural step in applying this methodology based on reference sets and seed relations. Given a complete demand function $\widehat{x} : P \times Z \times W \to X$, sufficiency can always be tested against revealed preference, and demand-consistency of a specific seed relation is only sometimes testable.[36] For both properties, testing is affected by whether or not the reference set is naturally ordered. With our current application, X_{WC} is naturally ordered if and only if $X = \Re^2_+$.

Sufficiency in the form of nonessentiality holds if and only if for every $(x^a, z) \in \widehat{Y}$ there exists some $(0, x^b_{-1}) \in \widehat{X}$ such that $x^a \sim_z (0, x^b_{-1})$. From $\widehat{x}$, we can identify the obtainable set $\widehat{X}$ and revealed preference in form of $\{\succsim_z \mid z \in Z\}$. Thus nonessentiality can always be fully verified with revealed preference. However, this may necessitate examining all $\succsim_z$ indifference sets in $\widehat{X}$ for all $z \in Z$. With X_{WC} naturally ordered we do have additional one-way tests of

[33] The no existence value aspect of weak complementarity is usually defined in the literature in terms of a partial directive and a utility function such as

$$\frac{\partial u(0, x_2, x_3, \ldots, x_L, z)}{\partial z} = 0,$$

for state preference variable z associated with good 1. However this definition only applies with continuous state preference variables for which the derivative is well defined.

[34] This is a valid indifference relation as it clearly defines a partition on $X_{WC} \times Z$; this partition includes exactly one set for each $x \in X_{WC}$. (See Note 18 for the relationship between indifference and partitions.)

[35] Here I adopt the notational convention $x_{-1} = (x_2, \ldots, x_L)$ so that $x = (x_1, x_{-1})$.

[36] This testability depends on our assumption that the given demand function is fully known. If instead, for example, the demand function was estimated from a more limited data set using a parametric form that assumed either nonessentiality or no existence value, then we have effectively assumed away our ability to test these issues by the methods described here.

nonessentiality that might save some labor. Theorem 4 allows us to reject sufficiency by only examining the membership of a naturally ordered reference set. In particular, with $X = \Re^2_+$ we can reject nonessentiality if for any $z \in Z$, $\succsim_z$ is consistent with a monotone relation on X and we find some $x \in X_{WC}$ where $x \neq 0$ and $x \notin \widehat{X}$. We thus have a separate one-way test of nonessentiality for each distinct $\succsim_z$, $z \in Z$. With each such test, all we have to do is linearly scan the reference set looking for a nonzero element that is not obtainable. For a very simple example, if any $\succsim_z$ is Cobb-Douglas then $\widehat{X} \bigcap X_{WC} = \emptyset$ so that we may reject nonessentiality on the basis of Theorem 4.

We can test the demand-consistency of $\sim_{wc}$ if and only if $L > 2$ (where $X = \Re^L_+$). When such testing is permitted, we can either affirm or reject so that we have a two-way test of $\widehat{x}$-consistency. However, we have at most only a one-way test concerning the "true" $\succsim_{\widehat{Y}}$. If $X = \Re^2_+$, then X_{WC} is naturally ordered and hence by Theorem I2 we know that $\sim_{wc}$ is automatically $\widehat{x}$-consistent,[37] and as a consequence we cannot use revealed preference to test the no existence value hypothesis. On the other hand, if $X = \Re^L_+$ with $L > 2$, X_{WC} is not naturally ordered and possibly redundant (recall Theorem 3) and therefore $\sim_{wc}$ might not be $\widehat{x}$-consistent. Moreover, Theorem I3 guarantees the existence of demand functions for which $\sim_{wc}$ is in fact not demand-consistent. Thus demand-consistency requires additional special conditions. Typically such conditions may be found by simply applying the definition of $\widehat{x}$-consistency with the seed relation. In this case, we thereby obtain the property of "single-preference." We say that a demand function is *single-preferenced* on some $\widetilde{X} \subseteq X$ if all the identifiable z-fixed preference relations are identical on this restricted set. More formally, for all possible pairs $z^a, z^b \in Z$ and all $x^1, x^2 \in \widetilde{X} \bigcap \widehat{X}$, single-preference requires $x^1 \succsim_{z^a} x^2 \Leftrightarrow x^1 \succsim_{z^b} x^2$. We then have our first applications theorem,

Theorem A1. *The weak complementarity seed indifference relation $\sim_{wc}$ is demand-consistent if and only if the demand function is single-preferenced on X_{WC}.*

Thus whenever we observe demand that is not single-preferenced on X_{WC}, we must reject "no existence value," so that there is no element of $\Phi(\widehat{x})$ that is consistent with $\sim_{wc}$. On the other hand when $\widehat{x}$ is single-preferenced X_{WC}, we know that there does exist some member of $\Phi(\widehat{x})$ that can be generated from $\sim_{wc}$ in the context of Theorem I1. However, we cannot affirm that this member is the "true" $\succsim_{\widehat{Y}}$. It is in this sense that Theorem A1 only provides a one-way test of no existence value, and hence of weak complementarity.

With $L = 2$, demand is trivially always single preferenced on X_{WC} and $\sim_{wc}$ is automatically $\widehat{x}$-consistent, so that Theorem A1 is not informative. However with $L > 2$, this theorem is meaningful since neither single preference nor demand-consistency is then automatic. From Theorems I2 and I3, we understand that this ability to test $\widehat{x}$-consistency is a consequence of X_{WC} not being naturally ordered. However, I believe our intuitive understanding of this should focus on the redundancy of the reference set. With this redundancy it is possible to obtain contradictions when we apply a seed relation with the reference set,

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[37] Application of Theorem I2 part a requires that $\sim_{wc}$ be strongly monotone. The seed preference relation defined by $(0, x_2^a, z^a) \succsim_{wc} (0, x_2^b, z^b) \Leftrightarrow x_2^a \geq x_2^b$ is strongly monotone and also associated with $\sim_{wc}$. Therefore $\sim_{wc}$ is also strongly monotone.

contradictions that refute demand-consistency.[38]

Weak complementarity is a good example for demonstrating the advantages of using condition 1 rather than $1'$ in defining demand-consistency for indifference relations. If I had instead used condition $1'$ then $\sim_{wc}$ as currently defined would be deficient since with $L > 2$ it would not satisfy the requirement that $x^a \sim_z x^b \Rightarrow (x^a, z) \sim_{wc} (x^b, z)$ for any $z \in Z$ and any $x^a, x^b \in \widehat{X} \bigcap X_{WC}$ with $x^a \neq x^b$. Satisfying this requirement would necessitate additional statements in the seed relation definition that simply replicate revealed preference information so that the definition is no longer fully specified independently of the demand function. Moreover this additional seed indifference information is not necessary for preference generation, and with it we would lose the simple intuitive clarity of our current definition. With weak complementarity I have reduced the seed relation information in two ways from what it might be otherwise, first by using a seed indifference relation instead of a seed preference relation and by second by availing ourselves of condition 1 instead of $1'$.

With our methodology it is possible to extend a result such as Theorem A1 so that we are able to characterize the entire membership of $\Phi(\widehat{x})$. In particular, we are able to reject "no existence value" (in the form of $\sim_{wc}$) as indicated by Theorem A1 if and only if we can also reject a specific kind of existence value. By definition, existence value occurs when the consumer has strict preference distinctions between z values when $x_1 = 0$. These preference distinctions generally depend on the quantities of the remaining commodities. We say that there is a *separable existence value* if the preference interaction between z and x_{-1} depends only on a simple aggregate metric of the commodity values. More formally, a preference relation on $\widehat{Y}$ has separable existence value if its restriction to $X_{WC} \times Z$ can be specified by some $\succsim_{sx}$ where,

$$(x^a, z^a) \succsim_{sx} (x^b, z^b) \quad \Leftrightarrow \quad (u(x^a), z^a) \succsim_{UZ} (u(x^b), z^b), \tag{3}$$

for all $(x^a, z^a), (x^b, z^b) \in \widehat{Y}_{WC} = (X_{WC} \times Z) \bigcap \widehat{Y}$, where u is some utility function on X_{WC} and $\succsim_{UZ}$ is a preference relation defined on $\Re \times Z$.[39] Equation (3) is also valid in the case of no existence value so that the latter may be regarded as a special case of separable existence value.[40] If existence value instead depends on the individual values of the x_{-1} vector in a manner that cannot be captured by equation (3) then it is nonseparable. We might expect this if the existence value has a particular relationship with specific consumption goods other than x_1. With $L = 2$ the distinction between separable and nonseparable existence value degenerates so that all preference relations on $\widehat{Y}$ trivially have separable existence value.

The following theorem gives us a strong relationship between the two types of existence value and whether the demand function is single-preferenced on X_{WC}.

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[38] Also recall the close link between naturally ordered and redundancy as specified by Theorems 2 and 3.

[39] Editorial note: The "sx" of $\succsim_{sx}$ is for "separable existence," and the "UZ" of $\succsim_{UZ}$ refers to the preference domain as the cross of utility values and Z.

[40] Therefore, for some purposes we may want a more strict notion of separable existence value that requires $\succsim_{UZ}$ to be active in z such that there exists some $x \in X_{WC}$ and $z^a, z^b \in Z$ so that $(u(x), z^a) \succ_{UZ} (u(x), z^b)$.

Theorem A2. *If $\widehat{x}$ is single-preferenced on X_{WC}, then all members of $\Phi(\widehat{x})$ have separable existence value.*[41] *Otherwise all members of $\Phi(\widehat{x})$ have nonseparable existence value.*

Each instance of $\succsim_{sx}$ as defined by equation (3) is a seed preference relation so that we can apply Theorem P1, just as we applied Theorem I1 with the seed indifference relation $\sim_{wc}$. However preference generation with either of these theorems requires the seed relations be $\widehat{x}$-consistent. Theorems A1 and A2 tell us that demand-consistency of these seed relations can be tested by determining if demand is single-preferenced. Theorem A2 generalizes Theorem A1.[42] Previously, with only Theorem A1, we could reject "no existence value" if demand is not single-preferenced, while now with Theorem A2, we can also reject all separable existence value seed relations. Moreover, Theorem A1 gave us only had a one-way test in that we could not affirm the nature of the "true" seed relation – we could not state that the seed relation must be $\sim_{wc}$. However with Theorem A2 we have a two-way test such that with single preferenced demand we can state that the true seed relation must have the form of $\succsim_{sx}$. The distinction is that $\sim_{wc}$ is a specific seed relation, while $\succsim_{sx}$ is a seed relation form that admits a whole class individual seed relations as specified by all the feasible $\succsim_{SX}$.[43]

Application of these theorems depends on Theorems P2 and I2 since they tell us that we can reject seed relations only when X_{WC} is not naturally ordered, i.e., when $L > 2$ so that single-preference is not a degenerate property. Theorem I3 provided the initial basis for developing this context-specific analytic structure. From it we knew that special conditions are required to guarantee demand-consistency of $\sim_{wc}$ when $L > 2$, giving us an impetus to find those conditions in the form of single-preferenced demand on X_{WC} as stated in Theorem A1. Then recognizing the general implications of single-preference, we extended our results with Theorem A2 to all separable existence value seed relations, thereby providing a characterization of all the entire $\Phi(\widehat{x})$ set that depends only on the observable property as to whether $\widehat{x}$ is single-preferenced.

In general, if a reference set X_R is complete but not naturally ordered, then the demand function and seed relation together provide an overabundance of preference information so that the generated relation is over-determined. Trusting the empirical demand information, I have developed a structure which uses this excess information to test the feasibility of proposed seed relations. However if the researcher is quite confident with a given seed relation, the same process of testing for demand-consistency could instead be used to test the accuracy of demand information. For example, if in a specific context the logical case for weak complementarity seems irrefutable, then an estimated demand function that is not single-preferenced on X_{WC} would be suspect.

On the other hand, if the researcher has no need to test either source of preference

[41] Where again "no existence value" is a special case of separable existence value.

[42] Ignoring the technical distinctions between indifference and preference seed relations.

[43] With $\widehat{x}$ single-preferenced on X_{WC}, it can be shown that for any $\succsim_{SX}$ relation that is strictly monotone on u, there is an element of $\Phi(\widehat{x})$ whose restriction to $(X_{WC} \times Z) \bigcap \widehat{Y}$ can be represented by a preference relation defined by equation (3) using this $\succsim_{SX}$. Thus $\Phi(\widehat{x})$ includes elements for all possible separable existence value structures (including "no existence value") as represented by all the feasible $\succsim_{SX}$ relations.

information, we can reduce the information requirements so that the generated relation is no longer over-identified, i.e., so that the seed relation is automatically demand-consistent. With weak complementarity, one simple but extreme example would be to treat all the other commodities as a single composite commodity so that we force $L = 2$, and thereby radically reduce the required preference information from both the demand function and seed relation. However, with our structure we can usually get automatic demand-consistency without changing L by simply selecting a new naturally ordered reference set that is a subset of the previous reference set. The seed relation is then redefined as the simple restriction of the old relation with respect to the new reference set. While this clearly reduces the amount of seed preference information, it can also be interpreted, perhaps more usefully, as a reduction in the required demand preference information.

In the context of weak complementarity, this process may be thought of as a strengthening of nonessentiality and a consequent weakening of the no existence value requirement. For example, with $L > 2$ suppose that $(0, x_{-1}) \in \widehat{X}$ for any $x_{-1} \in \Re^{L-1}_{++}$. Then X_{WC} is clearly reducible, so that we might consider an alternative sufficient reference set that is naturally ordered such as $X_{D1} \subset X_{WC}$ defined by $X_{D1} = \{x \in X \,|\, x_1 = 0, x_i = x_j \text{ for all } 1 < i, j \leq L\}$.[44] The sufficiency of X_{D1} may be formally interpreted as a strengthening of nonessentiality such that "any bundle including good 1 can be matched in the preference ordering by some other bundle which excludes good 1 *and where all other goods are of the same quantity.*" With X_{D1}, $\sim_{wc}$ is automatically $\widehat{x}$-consistent. This weakens the no existence value requirement so that now the consumer is allowed to care about differences in the state preference variable with $x_1 = 0$ if $x_i \neq x_j$ for some $1 < i, j \leq L$. This example with X_{D1} suggests a whole family of possible extensions to the basic weak complementarity assumptions, some of which may be intuitive in some contexts.

Changing the preference domain of $\sim_{wc}$ from $X_{WC} \times Z$ to $X_{D1} \times Z$ clearly reduces the amount of seed indifference information. On a practical level this can also be interpreted as a reduction in the required revealed preference information. Nonessentiality formally only requires that each z-fixed indifference surface include at least one point in X_{WC}. However, with $L > 2$ it is difficult to imagine indifference surfaces that would include only one such point. Instead, the nonempty intersection of any z-fixed indifference surface with X_{WC} would typically be a non-trivial curve or surface. Previously we required complete knowledge of each $\succsim_z$ on the preference domain $\widehat{X}$, including all such preference information in X_{WC}. This information is obtained as revealed preference from the demand function when $x_1 = 0$, and therefore may be particularly difficult to obtain: tracking the consumption of an item may be simpler than tracking non-consumption. However, with a weakened nonessentiality requirement defined with respect to a complete naturally ordered reference set such as X_{D1}, we need the preference information for only one point in X_{WC} on each z-fixed indifference surface in order to identify an almost complete $\widehat{x}$-consistent preference relation on $\widehat{Y}$.[45] Thus the required revealed preference information for points in X_{WC} is substantially reduced.

[44] This is the diagonal reference set in the $\Re^{L-1}_{+}$ space defined by $x_1 = 0$. When $L = 2$, $X_{D1} = X_{WC}$.

[45] It is "almost" complete because we may not be able to recover preferences between other points in X_{WC}.

4.2 Other Examples

The preceding analysis of weak complementarity and existence value illustrates the depth of analysis possible with this methodology. Five shorter applications presented here provide some indication of the possible breadth of analysis. These include two brief examples closely related to weak complementarity, two from the price index literature concerned with product quality, and finally a treatment of the problem of new and disappearing goods. These examples especially differ in the choice of a most convenient reference set.

The first two and last examples use a common seed relation definition that we have already seen. For any reference set X_R, one possible seed indifference relation is the "neutral relation" defined by $(x, z^a) \sim_n (x, z^b)$ for all $x \in X_R$ and all $z^a, z^b \in Z$.[46] The weak complementarity seed relation is an example of the neutral relation, as is the seed relation discussed at the beginning of the main section in the context of Figure 2. On the other hand, most instances of the $\succsim_{sx}$ seed relation form are "non-neutral" seed relations.

After substantial consideration of weak complementarity, Smith and Banzhaf (2004) in their closing section briefly consider two other potential APIs that can also be specified by the neutral seed relation. The first generalizes weak complementarity so that the consumer does not care about z as long as $x_1 \leq x_1^0$ for some fixed $x_1^0 \geq 0$. For the example given, z is the availability of campsites in a wilderness area and x_1 is the length of a hike in the wilderness, so that any $x_1 \leq x_1^0$ represents a day-hike and hence does not require camping facilities. In this case the natural reference set is clearly $X_{\leq x_1^0} = \{x \in X \mid x_1 \leq x_1^0\}$.[47]

Let $x_1^0 > 0$ so that this reference set is typically sufficient[48] but not naturally ordered (with $L > 1$). Then from Theorem I3 we know that special conditions are again required to ensure demand-consistency. These conditions imply a result very similar to Theorem A1 requiring demand to be single-preferenced on $X_{\leq x_1^0}$. We can also obtain a result similar to Theorem A2 as it relates to equation (3) but where the preference seed relation form is instead defined on $X_{\leq x_1^0} \times Z$; $\Phi(\widehat{x})$ either contains only preference relations that are consistent with this seed relation form, or contains no such member. Smith and Banzhaf's proposed API is only feasible in the first instance. The main difference from before is that these results now also have meaning when $L = 2$ (demand-consistency is not automatic).

Smith and Banzhaf's third API involves "weak substitution" as introduced by Feenberg and Mills.[49] Here the consumer does not care about z as long as $x_1 \geq x_1^0$, with x_1^0 again fixed. With the Smith and Banzhaf specification as applied to Feenberg and Mills' original education example, x_1 is the quantity of private education and z is the quality of public education. Increased consumption of private education is associated with lower consumption of public education so that at some point the consumer no longer cares about the quality of public education. This API is fully specified by applying the neutral seed indifference

[46] The consumer is "neutral" (or apathetic) about distinctions in z for any fixed $x \in X_R$.

[47] So that $X_{\leq x_1^0} = X_{WC}$ when $x_1^0 = 0$.

[48] However there are exceptions such as when one or more $\succsim_z$ have CES preference.

[49] Feenberg and Mills (1980), p.80.

relation with the reference set $X_{\geq x_1^0} = \{x \in X \,|\, x_1 \geq x_1^0\}$. All of the results with $X_{\leq x_1^0}$ discussed in the immediately preceding paragraph also apply here in an obvious way.

Our two examples with product quality in the context of price indexes are based on the seminal work by Fisher and Shell[50] with some subsequent development by Willig (1978). The first example concerns repackaging, so that with the classic shrinking candy bar example, we might expect preference (or utility) to depend on the total volume consumed of a given candy type (e.g. ounces) rather than on the number of bars irrespective of size. Fisher and Shell first provide a general formulation of repackaging such that

$$(x^a, z^a) \succsim_{rp} (x^b, z^b) \;\;\Leftrightarrow\;\; U(f(x_1^a, z^a), x_{-1}^a) \geq U(f(x_1^b, z^b), x_{-1}^b), \tag{4}$$

for all $(x^a, z^a), (x^b, z^b) \in \widehat{Y}$ where z is a quality vector for good x_1, and with appropriately defined real valued functions U and f.[51] The special case of "pure repackaging," such as with the candy bar example, is then captured by imposing a multiplicative restriction on the functional form, $f(x_1, z) = x_1 h(z)$, so that preference for a good with per unit satisfaction content $h(z)$ depends on the product of that satisfaction content and the number of items. For this example application with $\succsim_{pr}$ I shall assume that $\widehat{X} = \Re^L_{++}$ so that $\widehat{Y} = \Re^L_{++} \times Z$.

Repackaging is a universal property in the sense that it requires equation (4) to be true for all x vectors. Thus our reference set is naturally defined as the entirety of $X = \Re^L_+$ which is always sufficient, redundant and not naturally ordered (with $L > 1$). Thus we can apply Theorem P3 (as we have previously applied Theorem I3) requiring additional conditions for demand-consistency. However we typically do not start out with specific U and f functions, and therefore have a somewhat different demand-consistency question. Given the demand function $\widehat{x}$, we instead want to know if there are any U and f functions such that $\succsim_{rp}$ is $\widehat{x}$-consistent. Or, if we are concerned with pure repackaging, we wish to verify the existence of U and h functions. Thus we are not concerned with verifying the demand-consistency of any specific seed relation, but rather the feasibility of the form given by equation (4).

As is evident from Theorem A2, the methodology presented in this paper can be used to draw conclusions about the demand-consistency of general seed forms such as $\succsim_{sx}$ and $\succsim_{rp}$. We shall see that demand-consistency of the $\succsim_{rp}$ preference form requires a condition that bears some similarity to single-preference. Suppose there exists some U and f functions such that $\succsim_{rp}$ is $\widehat{x}$-consistent. For this application we shall adopt a simplifying assumption similar to unified preference: for any $x_1^a > 0$ and $z^a, z^b \in Z$ there exists some $x_1^b > 0$ so that $f(x_1^a, z^a) = f(x_1^b, z^b)$. (In the case of pure repackaging we only need $h > 0$ so that $x_1^b = x_1^a h(z^a)/h(z^b) > 0$.) Thus with x_{-1} fixed, any change in the "packaging" of good 1 can be offset by a change in the quantity consumed so that the consumer is left indifferent. This is represented by the function χ so that $x_1^b = \chi(x_1^a, z^a, z^b)$. Then for any $x_1^{\alpha a}, x_1^{\beta a} \in \Re_{++}$, $x_{-1}^{\alpha}, x_{-1}^{\beta} \in \Re^{L-1}_{++}$ and $z^a, z^b \in Z$, demand-consistency of $\succsim_{rp}$ implies

$$(x_1^{\alpha a}, x_{-1}^{\alpha}) \succsim_{z^a} (x_1^{\beta a}, x_{-1}^{\beta}) \;\;\Leftrightarrow\;\; (x_1^{\alpha b}, x_{-1}^{\alpha}) \succsim_{z^b} (x_1^{\beta b}, x_{-1}^{\beta}), \tag{5}$$

[50] Fisher and Shell (1971) as originally published in Griliches (1971) has omitted notation (especially brackets) with much of the displayed mathematical type and is consequently often difficult to read. These typesetting errors are corrected in the version subsequently published in Fisher and Shell (1972).

[51] See Fisher and Shell (1971) equation (5.15).

where $x_1^{\alpha b} = \chi(x_1^{\alpha a}, z^a, z^b)$ and $x_1^{\beta b} = \chi(x_1^{\beta a}, z^a, z^b)$.[52]

To more easily see the similarity with single-preference, let $x_1^{\alpha a} = x_1^{\beta a} \equiv x_1^a$ so that we also have $x_1^{\alpha b} = x_1^{\beta b} \equiv x_1^b$. Then equation (5) simplifies to

$$(x_1^a, x_{-1}^\alpha) \succsim_{z^a} (x_1^a, x_{-1}^\beta) \quad \Leftrightarrow \quad (x_1^b, x_{-1}^\alpha) \succsim_{z^b} (x_1^b, x_{-1}^\beta), \tag{6}$$

so that the restricted preference relation on all $x_{-1} \in \Re_{++}^{L-1}$ is the same with (x_1^a, z^a) and (x_1^b, z^b). This compares with single-preferenced demand such as on X_{WC} where the restricted preference relation on all $x_{-1} \in \Re_{++}^{L-1}$ is the same with for all $z^a, z^b \in Z$ when $x_1 = 0$. This simpler equation also suggests a one-way test for $\widehat{x}$-consistency of the $\succsim_{rp}$ form utilizing revealed preference in the form of $\{\succsim_z \ | z \in Z\}$. Simply choose some x_1^a, z^a and z^b, and systematically look for some x_1^b so that this simpler equation is valid, i.e., where we find the same preference relation on x_{-1} with (x_1^b, z^b) as we did with (x_1^a, z^a). Demand-consistency must be rejected if there is no such x_1^b. To be more precise, we must reject either $\widehat{x}$-consistency or the existence of χ. When $\succsim_{rp}$ is $\widehat{x}$-consistent, the existence of χ has a strong intuitive appeal that is allied with the compensating aspect of the price indexing concept. We will reject $\widehat{x}$-consistency if this intuition is stronger than that of demand-consistency itself. With pure repackaging there is an especially strong case for the existence of χ.

However, even if this test is passed for all possible combinations of x_1^a, z^a and z^b, demand-consistency is not guaranteed.[53] Instead, as indicated by the following theorem, we need the existence of a χ function whereby equation (5) is universally valid.

Theorem A3. *Given $\widehat{x}$, let a function χ exist such that for any $x_1^{\alpha a}, x_1^{\beta a} \in \Re_{++}$, $x_{-1}^\alpha, x_{-1}^\beta \in \Re_{++}^{L-1}$ and $z^a, z^b \in Z$, equation (5) is valid with $x_1^{\alpha b} = \chi(x_1^{\alpha a}, z^a, z^b)$ and $x_1^{\beta b} = \chi(x_1^{\beta a}, z^a, z^b)$. Then some functions U and f exist such that $\succsim_{rp}$ as defined by equation (4) is $\widehat{x}$-consistent.*

This gives us a two-way test of demand-consistency: the $\succsim_{rp}$ seed form is $\widehat{x}$-consistent if and only if equation 5 is satisfied for some χ function. However as just noted, rejection of demand-consistency depends on the intuitive appeal of χ.

If the conditions of Theorem A3 are satisfied then we know that there exists at least one repackaging preference relation that is demand-consistent. However, it will generally not be unique. In particular from Theorem 1, a $\succsim_{rp}$ preference relation will be uniquely demand-consistent only if there are no other functions U_{alt} and f_{alt} such that $U_{alt}(f_{alt}(x_1, z), x_{-1}) = g(U(f(x_1, z), x_{-1}), z)$ for some g-transform $g(U, z)$ that is an active function of z.[54] For example, it will not be unique if U is either additively or multiplicatively separable in its first argument.[55] Therefore we cannot typically use revealed preference information to identify a

[52]Equation (5) is obtained by directly applying the definition of $\widehat{x}$-consistent preference relations with equation (4) and the χ function.

[53]For example this test never fails when $L = 2$ even though there exist preference relations that cannot be specified with equation (4).

[54]Because the reference set is all of X and we have a utility representation, we are able to use Theorem 1 to directly characterize $\phi(\widehat{x})$. Our previous tool for such characterizations, Theorem P1 part b, does not tell us anything meaningful with this reference set since each seed relation is its own generated relation.

[55]That is if either $U(f, x_{-1}) = f + \bar{U}(x_{-1})$ or $U(f, x_{-1}) = f \cdot \bar{U}(x_{-1})$ for some real valued function $\bar{U}$.

unique repackaging preference relation. Thus again, like Theorem A2 we have a result concerning a preference relation form that provides necessary and sufficient conditions for when there exists demand-consistent representatives of this form, but which permits a large class of such demand-consistent representatives ($\succsim_{sx}$ is never unique and $\succsim_{rp}$ is rarely unique).

If we are specifically interested in the demand-consistency of the pure repackaging preference form, Theorem A3 needs to be strengthened such that the required χ function must have the form $x_1\bar{\chi}(z^a, z^b)$ for some function $\bar{\chi} : Z \times Z \to \Re$. Willig (1978) examines pure repackaging for the more specialized case where x_1 is nonessential, z has no existence value, and utility and demand functions are differentiable on z (so that z must be continuous).[56] He shows that under these conditions, whenever the component demand function for good one has the form $\widehat{x}_1(p, z, w) = H(p_1/h(z), p_{-1}, w)/h(z)$, the pure repackaging preference relation using this h function will be demand-consistent. However he does not show that pure repackaging (as I have defined it) must yield an $\widehat{x}_1$ component function of this form.[57]

Sometimes a quality change in a good primarily affects preference through its effect on one or more other goods. Fisher and Shell use the example of quality change of refrigerators enhancing the enjoyment of ice-cream. This could be the case with any good that primarily provides a functionality in support of other desired outcomes such as with most transportation. Following Fisher and Shell (1971) equation (5.26), this may be represented by the general seed relation form,

$$(x^a, z^a) \succsim_{cpr} (x^b, z^b) \Leftrightarrow U(x_1^a, f(x_1^a, x_2^a, z^a), x_3^a, \ldots, x_L^a) \geq U(x_1^b, f(x_1^b, x_2^b, z^b), x_3^b, \ldots, x_L^b),$$

for all $(x^a, z^a), (x^b, z^b) \in \widehat{Y}$ where z is again the quality of x_1. Here the quantity of good one can have a direct impact on preference, but the quality of that good only affects preference through the second good argument in the outside utility function U. Fisher and Shell again offer a more specific multiplicative form, $f(x_1, x_2, z) = x_2 h(x_1, z)$, while Willig (1978) provides an additive form $f(x_1, x_2, z) = x_2 + x_1 h(z)$ and the terminology "cross-product repackaging" (and hence the "cpr" of $\succsim_{cpr}$). Our analysis of repackaging can be also be applied to the more complicated cross-product repackaging, to include parallels to equations 5 and 6 and Theorem A3.

Our modelling structure is intended to be quite general to accommodate a large variety of applications. However, it may need to be modified for some applications, as with our last example, the new and disappearing goods problem, an important issue in the construction of price indices. Suppose that in going from one period to the next there are D disappearing goods $\{x_{1d}, x_{2d}, \ldots, x_{Dd}\}$, N new goods $\{x_{1n}, x_{2n}, \ldots, x_{Nn}\}$, and the usual L goods that exist in both periods. With $D > 0$ and $N > 0$ we have the problem of simultaneous new and disappearing goods. On the other hand with either $D = 0$ or $N = 0$ we would respectively have the new goods problem or the disappearing goods problem. With this application z indexes the consumption set: initially with $z = z^a$, we have $X_a = \Re_+^{L+D}$ with typical element $(x_1, \ldots, x_L, x_{1d}, \ldots, x_{Dd})$, and in the second period ($z = z^b$) we have $X_b = \Re_+^{L+N}$ with typical element $(x_1, \ldots, x_L, x_{1n}, \ldots, x_{Nn})$.

[56] With $\widehat{X} = \Re_{++}^L$ I have explicitly (and deliberately) not taken advantage of nonessentiality.

[57] Willig (1978) defines pure repackaging by this $\widehat{x}_1$ functional form.

Even though this violates our assumption that the consumption set is invariant with z, the modelling structure developed here is still applicable.[58] In particular, this situation naturally lends itself to an application of the neutral indifference seed relation with the reference set $X_{DN} = \Re^L_+$ and typical element $(x_1, \ldots, x_L)$. X_{DN} may be identified with the subset of X_a where $x_{1d} = \ldots = x_{Dd} = 0$, and also as the subset of X_b with $x_{1n} = \ldots = x_{Nn} = 0$, so that $X_a \bigcap X_b = X_{DN}$. Sufficiency of X_{DN} requires that $x_{1d}, \ldots, x_{Dd}$ are all simultaneously nonessential in the first period. That is, "any bundle including positive amounts of any the goods $x_{1d}, \ldots, x_{Dd}$ can be matched in the $\succsim_{z^a}$ preference ordering by some other bundle which excludes all of these goods." In this same way, sufficiency also requires that $x_{1n}, \ldots, x_{Nn}$ are all simultaneously nonessential in the second period.[59]

In this case, demand-consistency of the neutral indifference seed relation requires that $\succsim_{z^a}$ and $\succsim_{z^b}$ be identical on X_{DN} (i.e., demand is single-preferenced on X_{DN}). In implementing the neutral seed relation we are assuming that none of the goods $x_{1d}, \ldots, x_{Dd}, x_{1n}, \ldots, x_{Nn}$ has any existence value. As with the previous discussion with respect to Theorems A1 and A2, single-preference provides a one-way test of this assumption; it can be rejected but cannot be affirmed. With sufficiency and $\widehat{x}$-consistency, application of Theorem I1 part a guarantees in this context a unique generated complete preference relation on $\widehat{Y}$. Thus, any obtainable consumption bundle in the first period is fully comparable with any obtainable consumption bundle in the second.

In this section I have shown how the previously developed structure based on reference sets and seed relations can be applied with various API rationales. Some preference assumptions such as weak complementarity are most naturally stated as seed indifference relations. For these applications we have Theorems I1, I2 and I3. However other API rationales such as separable existence value and repackaging are best specified as seed preference relations, for which we have Theorems P1, P2 and P3.

Our two core requirements for preference generation – a sufficient reference set and a $\widehat{x}$-consistent seed relation – provide the basis for examining the efficacy of a given API. Typically the most important consideration in this context is our ability to test the demand-consistency of a given seed relation or seed relation form. In particular, with reference sets that are not naturally ordered, $\widehat{x}$-consistency requires additional conditions that we can specify on the identifiable z-fixed revealed preference relations $\{\succsim_z \mid z \in Z\}$. The original API rationale can thus be tested against revealed preference.

With a specific seed relation, demand-consistency testing provides only a one-way test of a rationale's validity. For example, we can reject weak complementarity if demand is not single-preferenced on X_{WC}, but with single-preference we are not able to affirm that the "true" relation on $\widehat{Y}$ conforms with weak complementarity. On the other hand, with seed forms such as $\succsim_{sx}$ and $\succsim_{rp}$ we typically lose uniqueness but it is sometimes possible to obtain a two-way test of the seed form's validity. For instance, we know that the "true" relation on

[58] Also, the simplifying assumption that $\widehat{X}$ is identical for all $z \in Z$ is clearly not viable with this application. However, none of our general results, such as the P and I theorems, depend on this assumption.

[59] With the pure new goods problem nonessentiality is trivially true for $\succsim_{z^a}$ since consumption of any x_{id} is not possible. Similarly, nonessentiality is trivially true for $\succsim_{z^b}$ with the pure disappearing goods problem.

$\widehat{Y}$ has separable existence value if and only if demand is single-preferenced on X_{WC}.

With a complete redundant reference set, between the seed relation and revealed preference information, there is a surplus of preference information so that the generated relation on $\widehat{Y}$ is over-identified. This excess preference information may be used to either test the seed relation against revealed preference, test an estimated demand function against the seed relation, or reduce the overall preference information requirements. As was illustrated with weak complementarity, that last option may be formally specified so that the seed relation includes less information (by using a smaller reference set), but in practice may be interpreted as reducing the required amount of revealed preference information from the demand function. This may have practical implications.

5 Conclusions

The problem addressed in this paper is the specification of the missing preference information in the context of state preference variables so that we can identify complete individual demand-consistent relations on $\widehat{Y}$, i.e., elements of $\Phi(\widehat{x})$. The key assumption underlying this work is that a complete demand function is available from which we can obtain revealed preference information for distinctions in commodity space. With Theorem 1 the significance of the problem becomes clear: without additional preference information, meaningful welfare analysis involving differences in state preference variables is impossible.

The core results of this paper are contained in Theorems P1 and I1 which show how elements of $\Phi(\widehat{x})$ can be uniquely identified by $\widehat{x}$-consistent seed relations defined with respect to fixed sufficient reference sets. Application of these theorems depends on knowing that a given seed relation is indeed demand-consistent. With a general (reducible) reference set, $\widehat{x}$-consistency among seed relations can be very rare. However, from Theorems P2 and I2 we know that strongly monotone seed relations are always $\widehat{x}$-consistent if (and only if) the reference set is naturally ordered. This result is strengthened with Theorems P3 and I3 so that when a reference set is not naturally ordered, there always exists a large class of demand functions for which a gvien seed relation is not demand-consistent.

The structure developed here can be applied in at least three distinct ways. First, we may customize both the reference set and seed relation to capture the intuition of a specific assumed API such as with weak complementarity, repackaging or any of the other examples of the preceding section. We can then know whether or not a given API is testable against the revealed preference information. If it is testable, we can construct one-way tests of specific seed relations such as weak complementarity and possibly two-way tests of seed relation forms such as with separable existence value. In this context, Theorems P3 and I3 are critical tools in telling us when demand-consistency requires additional conditions that can be verified with revealed preference. A more substantial summary of this approach is presented in the last four paragraphs of the applications section.

A second way of applying this structure is to focus on alternative seed relations that might be used with the same reference set. If a reference set is sufficient and non-redundant, then each seed relation defined with respect to that set represents the minimal information (preference or indifference) required to identify a unique demand-consistent relation on $\widehat{Y}$. We can compare all the members of $\Phi(\widehat{x})$ in complete detail by restricting our attention to differences in the individual seed relations. This works particularly well when the reference set is naturally ordered so that all monotonic seed relations are demand-consistent. Thus Theorems P2 and I2 may be a critical tools for this type of application.[60] Applications with the first approach, such as with $\sim_{wc}$ and $\succsim_{rp}$, focus on specifying and vetting potential API's that are based on specific rationales, whereas with this second approach we would typically only work with seed relations that are known to be valid without necessarily considering rationales. I approximated this second type of analysis with the characterization of $\Phi(\widehat{x})$ in terms of separable versus nonseparable existence value (Theorem A2).

The structure developed here could also be applied in a third way whereby it is used to provide a precise characterization of the minimal missing preference information that in turn can be used to design behavioral experiments or surveys to elicit the missing information. Such experiments and surveys would be designed to elicit a seed relation defined with respect to a specific non-redundant reference set. Alternatively, a redundant reference set may be carefully designed to strategically verify the internal consistency of this non-market preference data. An important distinction with this third type of application in comparison with the first type is that with experiments or surveys, API's would actually represent real "additional preference information" instead of mere supposition on the part of the investigators (however thoughtful that supposition might be).

Focusing on the first type of application, the system presented here is a general structure that can be used with the many varied methodologies (API specifications) that have been developed to deal with state preference variables in diverse areas such as the nonmarket goods and price index literatures. It is thus a meta-system for examining and applying these individual methodologies that brings to bear a level of precision and rigor for dealing with issues such as whether a given methodology allows us to specify a complete preference relation on $\widehat{Y}$, or whether the assumptions of another methodology are testable against revealed preference. At present there is a lack of clarity in the literature concerning these issues. For example, even thought weak complementarity is much discussed in the literature, it does not seem to be well understood that a complete preference relation on $\widehat{Y}$ sufficient for exact welfare analysis is fully determined by revealed preference information in the context of nonessentiality and the no existence value assumption.[61] There is also misunderstanding in the literature concerning when assumptions such as weak complementarity can be

[60]With the first approach Theorems P3 and I3 are important in telling us that not naturally ordered implies instances of not demand-consistent, while with the second approach Theorems P2 and I2 are important because they tells that naturally ordered implies demand-consistency.

[61]For example, Smith and Banzhaf (2004) perceive a need for an additional preference relation assumption in the form of the "Willig conditions" (Willig, 1978) in order to facilitate exact welfare analysis. This is discussed some in their footnote 11. However, with nonessentiality and no existence value, the state preference relation is fully determined on $\widehat{Y}$, thus enabling any of our standard exact measures such as EV. Moreover, with a complete state preference relation, the Willig conditions are fully testable so that introducing them as an assumption is questionable. This distinction does not seem to be a consequence of my key assumption that all the z-fixed relations $\{\succsim_z \mid z \in Z\}$ are fully known.

tested against revealed preference.[62] Application of this structure in a systematic way with these various methodologies may substantially increase our understanding of them, and thus increase their efficacy with real world applications.

All work to date known to this author with the various API's discussed here is specifically concerned with continuous state preference variables, where results are obtained by exploiting differential and integral calculus conditions involving these variables in relation to price and quantity.[63] By comparison, the findings presented here are not restricted to continuous state preference variables or other special conditions required for the application of calculus with these variables.[64] This generality is obtained by working instead directly with the logical properties of preference such as transitivity, in conjunction with basic set theory. Given the results obtained here which have escaped this calculus-based literature, it seems that methods of real analysis are not only overly specialized for some applications, but by themselves are also substantially incomplete as a general tool for understanding state preference variables for purposes of welfare analysis.

Some potential API specifications are defined in terms of the preference interaction between price and state preference variables. Examples may be found in Willig (1978) and Ebert (2001). This is often the most natural way to state price indexing constructs. The core concept of such a specification is most naturally captured with a reference set defined in the realm of prices, $P_R \subseteq P$, and a seed relation defined on $P_R \times Z$. It is possible to extend the structure presented here in terms of commodity preference to the realm of prices. That development includes results which allow us to combine the power of preference generation with commodities and prices respectively, so that we can identify unique elements of $\Phi(\widehat{x})$ from seed relations defined with respect to price reference sets.

Appendix: Proofs of Theorems

We begin with three lemmas. Lemma 1 is used in the proofs of Lemma 3 and Theorem 4, Lemma 2 in that of Lemma 3 and Theorem 1, and Lemma 3 in the proof of Theorem 1.

Lemma 1. *Let $\succsim_Y$ be a preference relation on Y that yields the demand function $\widehat{x}$, with obtainable preference set $\widehat{Y}$, and let $z \in Z$.*

[62]For example, in the introduction of Ebert (2001), he states that "one is unable to reject" preference assumptions such as weak complementarity with revealed preference. However, from the work presented here we know that we can reject (but not affirm) weak complementarity when $L > 2$. (Ebert (2001) is presented in the realm of general L.) Ebert is sufficiently aware of Theorem 1 (see Note 15) such that it is impossible to affirm a specific preference assumption such as weak complementarity using only revealed preference information. However, it does not follow that we cannot reject a specific preference assumption with only this information.

[63]Such as with Bradford and Hildebrandt (1977), Feenberg and Mills (1980), Fisher and Shell (1971), Mäler (1971), Mäler (1974), Smith and Banzhaf (2004), and Willig (1978).

[64]My treatment of the new and disappearing goods problem is an example of this whereby z is explicitly discrete with no continuous analog. Consequently, the usual differential definition of the neutral seed relation such as with weak complementarity (see Note 33) cannot be applied here.

a. Let $x^a \in X$ with $(x^a, z) \succ_Y (0, z)$. Then $(x^a, z) \sim_Y (x^b, z)$ for some $x^b \in \widehat{X}$.

b. Let $x^a \in X$ with $(0, z) \succsim_Y (x^a, z)$. Then $(x^b, z) \succ_Y (x^a, z)$ for all $x^b \in \widehat{X}$.

Proof. a. Let $x^a \in X$ with $(x^a, z) \succ_Y (0, z)$. If $x^a \in \widehat{X}$ we would be done since trivially $(x^a, z) \sim_Y (x^a, z)$. Suppose that $x^a \notin \widehat{X}$. For any $p \in \Re^L_{++}$ let $\hat{w} = p \cdot x^a$. Since x^a is in the budget set defined by p and $\hat{w}$, but $x^a \neq \widehat{x}(p, z, \hat{w})$, it must be that $(\widehat{x}(p, z, \hat{w}), z) \succ_Y (x^a, z)$. We have restricted the domain of our demand function to $w > 0$. However, a natural extension to include $w = 0$ would give us $\widehat{x}(p, z, 0) = 0$ while still maintaining the continuity of $\widehat{x}$. We thus have $(x^a, z) \succ_Y (\widehat{x}(p, z, 0), z)$. Then by the continuity of $\succsim_Y$ and $\widehat{x}$ (in p and w), there must be some $\bar{w} \in (0, \hat{w})$ such that where $x^b = \widehat{x}(p, z, \bar{w})$ we have $(x^a, z) \sim_Y (x^b, z)$.

b. Let $x^a \in X$ with $(0, z) \succsim_Y (x^a, z)$ and let $x^b \in \widehat{X}$. We know that for some $(p, w) \in \Re^{L+1}_{++}$, $\widehat{x}(p, z, w) = x^b$. Then since $0 \in X$ is in the budget set defined by (p, w), we have $(x^b, z) \succ_Y (0, z)$. Thus by the transitivity of $\succ_Y$ we have $(x^b, z) \succ_Y (x^a, z)$. □

Lemma 2. *Let u_1 and u_2 be two utility functions representing respectively the preference relations, $\succsim_1$ and $\succsim_2$, on some generic preference domain Q, with $u_i : Q \to \Re$, $i = 1, 2$. Let $R \subseteq \Re$ be the range of u_1. Then $\succsim_1 = \succsim_2$ if and only if there is some (strictly) increasing monotonic transformation $f : R \to \Re$ such that $u_2 = f(u_1)$.*

Proof. ($\Leftarrow$): Let $f : R \to \Re$ be an increasing monotonic transformation such that $f(u_1) = u_2$. Let $q^a, q^b \in Q$. Then, $u_1(q^a) \geq u_1(q^b) \Leftrightarrow f(u_1(q^a)) \geq f\left(u_1(q^b)\right) \Leftrightarrow u_2(q^a) \geq u_2(q^b)$. Thus, u_1 and u_2 represent the same preference relation.

($\Rightarrow$): Define the utility value sets $U_i = \{v \in \Re \,|\, v = u_i(q) \text{ for some } q \in Q\}$, for $i = 1, 2$. Then with the common preference relation $\succsim_Q \equiv \succsim_1 = \succsim_2$, the preference set Q can be partitioned into indifference sets such that if $I \subseteq Q$ is such an indifference set, then $q^1, q^2 \in I \Rightarrow q^1 \sim_Q q^2$, and if $q^1, q^2 \in Q$ and $q^1 \sim_Q q^2$ then q^1 and q^2 are elements of the same indifference set. The set of such indifference sets is $\mathbf{I} = \{I \subseteq Q \,|\, I \text{ is indifference set with respect to } \succsim_Q\}$.

For $i = 1, 2$, there is a one-to-one relationship, f_i, between the elements of $\mathbf{I}$ and U_i such that for $I \in \mathbf{I}$ and $v_i \in U_i$, then $I = f_i(v_i)$ if and only if $v_i = u_i(q)$ for all $q \in I$. With these we can construct a one-to-one relationship, f, between the elements of U_1 and U_2 such that for $v_i \in U_i$, $i = 1, 2$, $v_2 = f(v_1)$ if and only if $f_1(v_1) = f_2(v_2)$. Moreover, where f_2^{-1} is the inverse function of f_2, we have $v_2 = f_2^{-1}(f_1(v_1))$, so that $f = f_2^{-1} \circ f_1$.

Let $q \in Q$. Then there is some $I \in \mathbf{I}$, $v_1 \in U_1$ and $v_2 \in U_2$, such that $q \in I$, $u_1(q) = v_1$ and $u_2(q) = v_2$, with $f_1(v_1) = f_2(v_2) = I$. It then follows that $u_2(q) = f(u_1(q))$, so that $u_2 = f \circ u_1$, often written as $u_2 = f(u_1)$.

Let $v_1^a, v_1^b \in U_1$ such that $v_1^a > v_1^b$. Then there is some $q^a, q^b \in Q$ such that $u_1(q^a) = v_1^a$, $u_1(q^b) = v_1^b$ and $q^a \succsim_Q q^b$. Then where $u_2(q^a) = v_2^a$ and $u_2(q^b) = v_2^b$, it must be that $v_2^a > v_2^b$. Thus since $f(v_1^a) = v_2^a$ and $f(v_1^b) = v_2^b$, we have $f(v_1^a) > f(v_1^b)$, showing that f is a monotonically increasing transformation. The domain of f and f_1 is $U_1 = R$. □

Lemma 3. *Let u_1 and u_2 be two utility functions representing preferences on X with respec-*

tive demand functions $\widehat{x}_1$ and $\widehat{x}_2$ obtained from solving the optimization program,

$$\text{Program UX:} \quad \max_x \; u(x) \quad s.t. \; p \cdot x \leq w, \; x \in X,$$

so that they have the same obtainable set $\widehat{X} = \{x \in X \,|\, x = \widehat{x}_i(p,w)$ for some $(p,w) \in \Re^{L+1}_{++}\}$, $i = 1,2$. Then $\widehat{x}_1 = \widehat{x}_2$ if and only if there is some monotonic transformation $f : R \to \Re$ such that $u_2(x) = f(u_1(x))$ for all $x \in \widehat{X}$ (where R is the range of u_1).

Proof. ($\Leftarrow$): Let there be some monotonic transformation $f : R \to \Re$ such that $u_2(x) = f(u_1(x))$ for all $x \in \widehat{X}$. Then for any $(p^a, w^a) \in \Re^{L+1}_{++}$ with $x^a = \widehat{x}_1(p^a, w^a)$, we need to show that $x^a = \widehat{x}_2(p^a, w^a)$. We know that $\widehat{x}_2(p^a, w^a) \in \widehat{X}$. Let $x^b \neq x^a$ be any other feasible solution ($x^b \in \widehat{X}$ with $p^a x^b \leq w^a$). Then $u_1(x^b) < u_1(x^a) \Rightarrow f(u_1(x^b)) < f(u_1(x^a)) \Rightarrow u_2(x^b) < u_2(x^a)$. Thus $x^b \neq \widehat{x}_2(p^a, w^a)$ so that $x^a = \widehat{x}_2(p^a, w^a)$, and hence $\widehat{x}_1 = \widehat{x}_2$.

($\Rightarrow$): Let $\widehat{x}_1 = \widehat{x}_2$. Define the utility functions $\widehat{u}_1$ and $\widehat{u}_2$ to be the respective restrictions of u_1 and u_2 to $\widehat{X}$: $\widehat{u}_i(x) = u_i(x)$, $i = 1,2$ for all $x \in \widehat{X}$.

For each $i = 1,2$: Denote the range of u_i as $\Re_i$, the range of $\widehat{u}_i$ as $\widehat{\Re}_i \subseteq \Re_i$, and define $v_i^0 = u_i(0)$. Then from Lemma 1 part a, for any $v_i \in \Re_i$ with $v_i > v_i^0$ there is some $x^i \in \widehat{X}$ such that $u_i(x^i) = v_i$ and hence $v_i \in \widehat{\Re}_i$. From Lemma 1 part b, for any $v_i \in \Re_i$ such that $v_i \leq v_i^0$, there is no $x^i \in \widehat{X}$ such that $u_i(x^i) = v_i$. Thus $\widehat{\Re}_i = (v_i^0, \infty) \bigcap \Re_i$. (For these applications of Lemma 1, Z is a singleton so that preference on Y is equivalent to preference on X.)

By an assumption stated in the text of the paper, $\widehat{u}_1$ and $\widehat{u}_2$ represent the same preference relation on $\widehat{X}$, and hence by Lemma 2 there is some (strictly) increasing monotonic transformation $h : \widehat{\Re}_1 \to \Re$ such that $\widehat{u}_2 = h(\widehat{u}_1)$, or $\widehat{u}_2(x) = h(\widehat{u}_1(x))$ for all $x \in \widehat{X}$. Finally we can define the monotonic transformation $f : \Re_1 \to \Re$, by $f(v) = h(v)$ if $v \in \widehat{\Re}_1$, and $f(v) = v + v_2^0 - v_1^0$ otherwise, so that $u_2(x) = f(u_1(x))$ for all $x \in \widehat{X}$. ☐

Theorem 1

Proof. Let u_1 and u_2 be two utility functions representing preferences on Y.

a. Then for each $i \in \{1,2\}$ and $z \in Z$, we have a utility function on X, u_{iz}, such that $u_{iz}(x) = u_i(x,z)$. u_1 and u_2 will yield the same demand function $\widehat{x}(p,z,w)$ with Program UY if and only if for all $z \in Z$, u_{1z} and u_{2z} yield the same demand function $\widehat{x}_z(p,w)$ with Program UX of Lemma 3, where $\widehat{x}(p,z,w) = \widehat{x}_z(p,w)$.

From Lemma 3, for each $z \in Z$, u_{1z} and u_{2z} yield the same demand function $\widehat{x}_z(p,w)$ with Program UX if and only if there is some increasing monotonic transformation $f_z : \Re_u \to \Re$ such that $u_{2z}(x) = f_z(u_{1z}(x))$ for all $x \in \widehat{X}$.

When all the f_z exist we can define $g : \Re_u \times Z \to \Re$ by $g(u,z) = f_z(u)$. Since each f_z is increasing monotonic, it follows that g is strictly increasing in u. Also, if start with a g-transform, then we can similarly define a complete set of f_z transformations. Thus, having such a g-transform is equivalent to having all the f_z-transforms.

Therefore, from the three previous paragraphs, u_1 and u_2 will yield the same demand function $\widehat{x}(p,z,w)$ with Program UY if and only if there is some transformation $g : \Re_u \times Z \to \Re$ such that $u_2(x,z) = g\left(u_1(x,z), z\right)$ for all $(x,z) \in \widehat{Y}$, with g increasing in u.

b. Let g be a g-transform such that $u_2 = g\left(u_1, z\right)$. From Lemma 2, u_1 and u_2 represent the same preference relation on $\widehat{Y}$ if and only if there is some (strictly) increasing monotonic transformation $f : \Re_u \to \Re$ such that $u_2(x,z) = f(u_1(x,z))$ for all $(x,z) \in \widehat{Y}$. But we also have $u_2(x,z) = g\left(u_1(x,z), z\right)$, and hence $g\left(u_1(x,z), z\right) = f(u_1(x,z))$ for all $(x,z) \in \widehat{Y}$. Thus $g(u,z) = f(u)$ for all $u \in \widehat{\Re}_1$, where $\widehat{\Re}_1$ is the range of u_1 when restricted to the domain $\widehat{Y}$, $\widehat{\Re}_1 = \{u \in \Re_u \,|\, u = u_1(x,z) \text{ for some } (x,z) \in \widehat{Y}\}$. □

The following lemma, showing that each $\succsim_z$ is strongly monotone is used in the proofs of Theorems 2 and P2.

Lemma 4. *For each $z \in Z$, the preference relation $\succsim_z$ is strongly monotone: if $x^0, x^1 \in \widehat{X}$, with $x^1 \geq x^0$ ($x^1_\ell \geq x^0_\ell$ for all $\ell = 1, \ldots, L$) and $x^1 \neq x^0$, then $x^1 \succ_z x^0$.*

Proof. Let $z \in Z$, $x^0 \in X$ and $x^1 \in \widehat{X}$, with $x^1 \geq x^0$ ($x^1_\ell \geq x^0_\ell$ for all $\ell = 1, \ldots, L$) and $x^1 \neq x^0$. Then for some $p^1 \in \Re^L_+$, $\widehat{x}(p^1, z, 1) = x^1$ and hence $p^1 \cdot x^1 = 1$. Then $x^1 \geq x^0$ implies that $p^1 \cdot x^0 \leq 1$ so that bundle x^0 is affordable when x^1 is chosen. Since x^0 is not chosen, it must be that $x^1 \succ_z x^0$. □

Theorem 2

Proof. Let $X_R \subseteq X$ be a naturally ordered reference set, and for any possible Z let $\widehat{x} : \Re^L_{++} \times Z \times \Re_{++} \to \Re^L$ be a feasible demand function. Now suppose that X_R is redundant so that for some $(x,z) \in \widehat{Y}$, there are $x^1_R, x^2_R \in X_R$ with $x^1_R \neq x^2_R$ such that $x \sim_z x^1_R$ and $x \sim_z x^2_R$. Then we also have $x^1_R \sim_z x^2_R$ with both $x^1_R, x^2_R \in \widehat{X}$. Applying Lemma 4 we know that $\succsim_z$ is strongly monotone. Then $x^1_R \sim_z x^2_R$ precludes the possibility of either $x^1_R \leq x^2_R$ or $x^2_R \leq x^1_R$, and hence X_R cannot be naturally ordered. Thus by contradiction, X_R must be non-redundant. □

Theorem 3

Proof. Let $X_R \subseteq X$ be a reference set that is not naturally ordered, so that for some $x^1, x^2 \in X_R$ with $x^1 \neq x^2$ we have neither $x^1 \leq x^2$ nor $x^2 \leq x^1$. Then for some integers k and ℓ with $1 \leq k \leq L$ and $1 \leq \ell \leq L$, we have $x^1_k < x^2_k$ and $x^1_\ell > x^2_\ell$. Define $\alpha = (x^2_k - x^1_k) \Big/ \left(\sqrt{x^1_\ell} - \sqrt{x^2_\ell}\right)$. With the singleton set of feasible state preference variable values, $\widetilde{Z} = \{\widetilde{z}\}$, we define the z-fixed preference relation $x^a \succsim_{\widetilde{z}} x^b \Leftrightarrow u(x^a) \geq u(x^b)$ where $u(x)$ is the quasilinear utility function $u(x) = x_k + \alpha\sqrt{x_\ell}$. Then $\succsim_{\widetilde{z}}$ is strictly convex and strongly monotone over $\Re^L_+$ such that $\Re^L_{++} \subset \widehat{X}_{\widetilde{z}} \subset \Re^L_+$. The utility function was constructed so that $u(x^1) = u(x^2)$ and hence X_R is redundant with respect to the demand function obtained with $\succsim_{\widetilde{z}}$. Thus a reference set that is not naturally ordered cannot be universally non-redundant. □

Theorem 4

Proof. Let $X_R \subseteq X$ such that X_R is naturally ordered and sufficient. For any fixed $z \in Z$, let $\succsim_{\overline{z}}$ be a strongly monotone preference relation on X such that the identifiable preference relation $\succsim_z$ is the restriction of $\succsim_{\overline{z}}$ to $\widehat{X}$.

Let $x^1 \in X_R \backslash \{0\}$. Then $x^1 \succ_{\overline{z}} 0$, and hence by Lemma 1 part a, there exists some $x^2 \in \widehat{X}$ such that $x^1 \sim_{\overline{z}} x^2$. From sufficiency, there also must be some $x^3 \in X_R$ so that $x^2 \sim_z x^3$ and hence $x^3 \in \widehat{X}$. Assume that $x^1 \neq x^3$. Since X_R is naturally ordered, we must have either $x^1 \geq x^3$ or $x^3 \geq x^1$ (but not both). Then from the strong monotonicity of $\succsim_{\overline{z}}$, we have either $x^1 \succ_{\overline{z}} x^3$ or $x^3 \succ_{\overline{z}} x^1$, but not $x^1 \sim_{\overline{z}} x^3$. Thus by contradiction $x^1 = x^3$, and hence $x^1 \in \widehat{X}$. We have demonstrated that $X_R \backslash \{0\} \subseteq \widehat{X}$. □

Theorem P1

Proof. For the demand function $\widehat{x} : \Re^L_{++} \times Z \times \Re_{++} \rightarrow \Re^L$ with identifiable $\{\succsim_z \ | z \in Z\}$ and obtainable set $\widehat{X}$, let X_R be a sufficient reference set relative to $\widehat{x}$.

Part a:

Let $\succsim_s$ be a complete and transitive preference relation on $\widehat{Y}_R = (X_R \times Z) \bigcap \widehat{Y}$, such that for any $z \in Z$ and $x^a, x^b \in X_R \bigcap \widehat{X}$, $(x^a, z) \succsim_s (x^b, z) \Leftrightarrow x^a \succsim_z x^b$ (i.e., $\succsim_s$ is $\widehat{x}$-consistent).

Let $(x^a, z^a), (x^b, z^b) \in \widehat{Y}$. Then by the properties of a complete reference set we know that there are some $x^a_R, x^b_R \in X_R$ such that $x^a \sim_{z^a} x^a_R$ and $x^b \sim_{z^b} x^b_R$, and we can define the relation $\succsim_S$ on $\widehat{Y}$ such that $(x^a, z^a) \succsim_S (x^b, z^b)$ if and only if $(x^a_R, z^a) \succsim_s (x^b_R, z^b)$. It follows from this construction that $\succsim_S$ is complete on $\widehat{Y}$.

Suppose that we also had $\tilde{x}^a_R, \tilde{x}^b_R \in X_R$ such that $x^a \sim_{z^a} \tilde{x}^a_R$ and $x^b \sim_{z^b} \tilde{x}^b_R$, allowing us to define the alternative preference relation $\succsim_T$ on $\widehat{Y}$ such that $(x^a, z^a) \succsim_T (x^b, z^b)$ if and only if $(\tilde{x}^a_R, z^a) \succsim_s (\tilde{x}^b_R, z^b)$. Since $\succsim_{z^a}$ and $\succsim_{z^b}$ are both transitive, we have $x^a_R \sim_{z^a} \tilde{x}^a_R$ and $x^b_R \sim_{z^b} \tilde{x}^b_R$, so that by the consistency property of $\succsim_s$ we know that $(x^a_R, z^a) \sim_s (\tilde{x}^a_R, z^a)$ and $(x^b_R, z^b) \sim_s (\tilde{x}^b_R, z^b)$. Then $(x^a_R, z^a) \succsim_s (x^b_R, z^b) \Leftrightarrow (\tilde{x}^a_R, z^a) \succsim_s (\tilde{x}^b_R, z^b)$, and hence $(x^a, z^a) \succsim_S (x^b, z^b)$ if and only if $(x^a, z^a) \succsim_T (x^b, z^b)$. Thus given the seed relation $\succsim_s$, the generated relation $\succsim_S$ is well defined.

We still need to show that $\succsim_S$ is transitive; completely consistent with $\succsim_s$ on $\widehat{Y}_R$ and with all $\succsim_z$ as they are respectively defined on $\widehat{X}$; and unique.

Let $(x^a, z^a), (x^b, z^b), (x^c, z^c) \in \widehat{Y}$ such that $(x^a, z^a) \succsim_S (x^b, z^b)$ and $(x^b, z^b) \succsim_S (x^c, z^c)$. Then let $x^a_R, x^b_R, x^c_R \in X_R$ such that $x^a \sim_{z^a} x^a_R$, $x^b \sim_{z^b} x^b_R$ and $x^c \sim_{z^c} x^c_R$, so that $(x^a_R, z^a) \succsim_s (x^b_R, z^b)$ and $(x^b_R, z^b) \succsim_s (x^c_R, z^c)$. By the transitivity of $\succsim_s$, we then have $(x^a_R, z^a) \succsim_s (x^c_R, z^c)$, and hence $(x^a, z^a) \succsim_S (x^b, z^b)$ so that $\succsim_S$ is transitive.

For any $(x^a_R, z^a), (x^b_R, z^b) \in \widehat{Y}_R$ we have $(x^a_R, z^a) \succsim_S (x^b_R, z^b) \Leftrightarrow (x^a_R, z^a) \succsim_s (x^b_R, z^b)$, so that $\succsim_S$ completely consistent with $\succsim_s$.

For any $z \in Z$, let $(x^a, z), (x^b, z) \in \widehat{Y}$ and assume that $(x^a, z) \succsim_S (x^b, z)$. Then for some $x^a_R, x^b_R \in X_R$ such that $x^a \sim_z x^a_R$ and $x^b \sim_z x^b_R$, we have $(x^a_R, z) \succsim_s (x^b_R, z)$, and hence $x^a \succsim_z x^b$. Now working in the other direction, assume that $x^a \succsim_z x^b$ for some $z \in Z$. Then there are some $x_R^a, x_R^b \in X_R$ such that $x^a \sim_z x_R^a$ and $x^b \sim_z x_R^b$, giving us $x_R^a \succsim_z x_R^b$ and $(x_R^a, z) \succsim_s (x_R^b, z)$, and hence $(x^a, z) \succsim_S (x^b, z)$. Thus for any $z \in Z$, $\succsim_S \equiv \succsim_z$ when restricted to $\widehat{X}$ with $z^a = z^b = z$, and hence $\succsim_S$ is $\widehat{x}$-consistent.

Let $\succsim_T$ be any complete and transitive preference relation defined on $\widehat{Y}$ that is completely consistent with $\succsim_s$ and all $\succsim_z$ for $z \in Z$. Let $(x^a, z^a), (x^b, z^b) \in \widehat{Y}$. Then there is some $x_R^a, x_R^b \in X_R$ such that $x^a \sim_{z^a} x_R^a$, $x^b \sim_{z^b} x_R^b$, and $(x^a, z^a) \succsim_S (x^b, z^b) \Leftrightarrow (x_R^a, z^a) \succsim_s (x_R^b, z^b)$. Then we have $(x^a, z^a) \sim_T (x_R^a, z^a)$, $(x^b, z^b) \sim_T (x_R^b, z^b)$ and $(x_R^a, z^a) \succsim_s (x_R^b, z^b) \Leftrightarrow (x_R^a, z^a) \succsim_T (x_R^b, z^b)$, so that be the transitivity of $\succsim_T$, $(x_R^a, z^a) \succsim_s (x_R^b, z^b) \Leftrightarrow (x^a, z^a) \succsim_T (x^b, z^b)$. Thus $\succsim_T \equiv \succsim_S$, and hence $\succsim_S$ is unique.

Part b:

Let $\succsim_S$ be a complete and transitive preference relation on $\widehat{Y}$, such that for any $z \in Z$ and any $x^a, x^b \in \widehat{X}$, $(x^a, z) \succsim_S (x^b, z) \Leftrightarrow x^a \succsim_z x^b$ (i.e., $\succsim_S$ is $\widehat{x}$-consistent). Now define $\succsim_s$ as the restriction of $\succsim_S$ to $\widehat{Y}_R = (X_R \times Z) \bigcap \widehat{Y}$: for any $(x_R^a, z^a), (x_R^b, z^b) \in \widehat{Y}_R$, $(x_R^a, z^a) \succsim_s (x_R^b, z^b)$ if and only if $(x_R^a, z^a) \succsim_S (x_R^b, z^b)$. The completeness and $\widehat{x}$-consistency of $\succsim_S$ implies the completeness and $\widehat{x}$-consistency of $\succsim_s$ (with completeness now defined on a smaller preference domain). By part a just proven, there is a unique complete and transitive preference relation on $\widehat{Y}$ that is consistent with $\succsim_s$ and with $\succsim_z$ for all $z \in Z$. Clearly that preference relation on $\widehat{Y}$ must be $\succsim_S$.

It remains to show that $\succsim_s$ is unique. Let $\succsim_t$ be any complete and transitive preference relation on $\widehat{Y}_R$ that is consistent with $\succsim_S$. Then for any $(x_R^a, z^a), (x_R^b, z^b) \in \widehat{Y}_R$, $(x_R^a, z^a) \succsim_t (x_R^b, z^b) \Leftrightarrow (x_R^a, z^a) \succsim_S (x_R^b, z^b)$, so that $(x_R^a, z^a) \succsim_t (x_R^b, z^b) \Leftrightarrow (x_R^a, z^a) \succsim_s (x_R^b, z^b)$. Thus $\succsim_t \equiv \succsim_s$, and hence $\succsim_s$ is unique. □

Theorem P2

Proof. For the demand function $\widehat{x} : \Re_{++}^L \times Z \times \Re_{++} \to \Re^L$ with identifiable z-fixed preference relations $\{\succsim_z \mid z \in Z\}$ and obtainable set $\widehat{X}$, let X_R be a reference set.

a. Let X_R be a reference set such that $X_R \bigcap \widehat{X}$ is naturally ordered, and let $\succsim_s$ be a strongly monotone seed preference relation on $\widehat{Y}_R = (X_R \times Z) \bigcap \widehat{Y}$. Now consider any $z \in Z$ and $x^a, x^b \in X_R \bigcap \widehat{X}$. We need to show that $(x^a, z) \succsim_s (x^b, z) \Leftrightarrow x^a \succsim_z x^b$.

Since $X_R \bigcap \widehat{X}$ is naturally ordered and $\succsim_s$ is strongly monotone, we have either $x^a \geq x^b$ with $(x^a, z) \succsim_s (x^b, z)$, or $x^b \geq x^a$ and $x^b \neq x^a$ with $(x^b, z) \succ_s (x^a, z)$. Applying Lemma 4 we have $\succsim_z$ strongly monotone, and hence either $x^a \geq x^b \Rightarrow x^a \succsim_z x^b$, or $[x^b \geq x^a$ and $x^b \neq x^a] \Rightarrow x^b \succ_z x^a$. Thus $(x^a, z) \succsim_s (x^b, z) \Leftrightarrow x^a \succsim_z x^b$, so that $\succsim_s$ is $\widehat{x}$-consistent.

b. Let all strongly monotone seed preference relation on $\widehat{Y}_R = (X_R \times Z) \bigcap \widehat{Y}$ be $\widehat{x}$-consistent. Suppose that $X_R \bigcap \widehat{X}$ is not naturally ordered. Then for some $x^1, x^2 \in X_R \bigcap \widehat{X}$ we have neither $x^1 \geq x^2$ nor $x^2 \geq x^1$, so that for some integers k and ℓ with $1 \leq k \leq L$ and $1 \leq \ell \leq L$, we have $x_k^1 < x_k^2$ and $x_\ell^1 > x_\ell^2$. Without loss of generality we may assume that $x^1 \succsim_{z^0} x^2$ for some $z^0 \in Z$. For any fixed α, $0 < \alpha < x_k^2 - x_k^1$, define $\beta = (x_k^2 - x_k^1 - \alpha) \Big/ \left(\sqrt{x_\ell^1} - \sqrt{x_\ell^2} \right)$. We now consider the strongly monotone seed preference relation $\succsim_s$ defined on $\widehat{Y}_R$ such that $(x^a, z^a) \succsim_s (x^b, z^b)$ if and only if $u(x^a) \geq u(x^b)$ where $u(x)$ is the quasilinear utility function

$u(x) = x_k + \beta\sqrt{x_\ell}$. This utility function was constructed so that $u(x^1) < u(x^2)$, and hence $\succsim_s$ is not consistent with $\widehat{x}$. Thus by contradiction, $X_R \bigcap \widehat{X}$ is naturally ordered. Note that with the range of possible α values, we have an infinite number of preference seed relations that are not demand-consistent if $X_R \bigcap \widehat{X}$ is not naturally ordered. $\square$

Theorem P3

Proof. Let X_R be a reference set that is not naturally ordered and let $\succsim_s$ be a strongly monotone seed preference relation defined on on $Y_R = X_R \times Z$. We need to show that $\succsim_s$ is not demand-consistent for an infinite number of valid demand functions.

Let $\widehat{x} : \Re^L_{++} \times Z \times \Re_{++} \to \Re^L$ be a valid demand function with identifiable z-fixed preference relations $\{\succsim_z \ | z \in Z\}$ and obtainable set $\widehat{X}$. Let $z^0 \in Z$. Since X_R is not naturally ordered, for some $x^1, x^2 \in X_R$ we have neither $x^1 \geq x^2$ nor $x^2 \geq x^1$, so that for some integers k and ℓ with $1 \leq k \leq L$ and $1 \leq \ell \leq L$, we have $x^1_k < x^2_k$ and $x^1_\ell > x^2_\ell$. Without loss of generality we may assume that $(x^1, z^0) \succsim_s (x^2, z^0)$. For any fixed α, $0 < \alpha < x^2_k - x^1_k$, define $\beta = (x^2_k - x^1_k - \alpha) \Big/ \left(\sqrt{x^1_\ell} - \sqrt{x^2_\ell}\right)$. We now consider the new strongly monotone z-fixed preference relation $\dot{\succsim}_{z^0}$ defined on X such that $x^a \dot{\succsim}_{z^0} x^b$ if and only if $u(x^a) \geq u(x^b)$ where $u(x)$ is the quasilinear utility function $u(x) = x_k + \beta\sqrt{x_\ell}$. This utility function was constructed so that $u(x^1) < u(x^2)$, and hence $\dot{\succsim}_{z^0}$ is not consistent with $\succsim_s$. We now define a new demand function via equation (2) with the previous z-fixed relations $\{\succsim_z \ | z \in Z\}$, except that $\succsim_{z^0}$ is replaced by $\dot{\succsim}_{z^0}$. With the range of possible α values, we have an infinite number of such demand functions for which $\succsim_s$ is not demand-consistent. $\square$

The following lemma, establishing a connection between $\widehat{x}$-consistent preference and indifference relations, is used in the proofs of the subsequent three theorems.

Lemma 5. *Given the demand function $\widehat{x}$ with obtainable set $\widehat{Y}$ with subset $\widetilde{Y} \subseteq \widehat{Y}$:*

a. Let $\succsim$ be a $\widehat{x}$-consistent preference relation on $\widetilde{Y}$. Then any indifference relation that is consistent with $\succsim$ is $\widehat{x}$-consistent. Furthermore, the (unique) associated indifference relation of $\succsim$ also satisfies condition 1′.

b. Let $\sim$ be a unified $\widehat{x}$-consistent indifference relation on $\widetilde{Y}$. Then there is a unique $\widehat{x}$-consistent transitive complete preference relation on $\widetilde{Y}$ that is consistent with $\sim$. With $\sim$ also satisfying condition 1′, it is the unique indifference relation associated with the preference relation.[65]

Proof. For the demand function $\widehat{x} : \Re^L_{++} \times Z \times \Re_{++} \to \Re^L$ with identifiable z-fixed preference relations $\{\succsim_z \ | z \in Z\}$ and obtainable set $\widehat{Y}$, let $\widetilde{Y} \subseteq \widehat{Y}$.

a. Let $\succsim$ be a $\widehat{x}$-consistent preference relation on $\widetilde{Y}$ and let $\sim$ be any indifference relation on $\widetilde{Y}$ that is consistent with $\succsim$, so that $(x^a, z^a) \sim (x^b, z^b) \Rightarrow [(x^a, z^a) \succsim (x^b, z^b)$ and $(x^b, z^b) \succsim$

[65]Without the unified property in part b, there could be either multiple, one, or even no $\widehat{x}$-consistent transitive complete preference relations that are consistent with the given demand-consistent indifference relation (even with condition 1′).

$(x^a, z^a)]$ for all $(x^a, z^a), (x^b, z^b) \in \widetilde{Y}$. We need to show that $\sim$ is: 1) $\widehat{x}$-consistent; and 2) satisfies condition $1'$ if it is also the (unique) associated indifference relation of $\succsim$.

Let $(x^a, z), (x^b, z) \in \widetilde{Y}$ with $z \in Z$. Then we have $(x^a, z) \sim (x^b, z) \Rightarrow [(x^a, z) \succsim (x^b, z)$ and $(x^b, z) \succsim (x^a, z)] \Rightarrow [x^a \succsim_z x^b$ and $x^b \succsim_z x^a] \Rightarrow x^a \sim_z x^b$, so that $\sim$ satisfies the first property of $\widehat{x}$-consistent indifference relations.

Let $(x^1, z^a), (x^2, z^b), (x^3, z^a), (x^4, z^b) \in \widetilde{Y}$ and $z^a, z^b \in Z$ such that $(x^1, z^a) \sim (x^2, z^b)$ and $(x^3, z^a) \sim (x^4, z^b)$. Then by definition we have $(x^1, z^a) \succsim (x^2, z^b)$, $(x^2, z^b) \succsim (x^1, z^a)$, $(x^3, z^a) \succsim (x^4, z^b)$ and $(x^4, z^b) \succsim (x^3, z^a)$. If $(x^1, z^a) \succsim (x^3, z^a)$, then by transitivity we would also have $(x^2, z^b) \succsim (x^4, z^b)$. Similarly, $(x^2, z^b) \succsim (x^4, z^b) \Rightarrow (x^1, z^a) \succsim (x^3, z^a)$, and hence $(x^1, z^a) \succsim (x^3, z^a) \Leftrightarrow (x^2, z^b) \succsim (x^4, z^b)$. Since $\succsim$ is $\widehat{x}$-consistent, we have $x^1 \succsim_{z^a} x^3 \Leftrightarrow (x^1, z^a) \succsim (x^3, z^a)$ and $x^2 \succsim_{z^b} x^4 \Leftrightarrow (x^2, z^b) \succsim (x^4, z^b)$. Then applying the three "$\Leftrightarrow$" relationships, we obtain $x^1 \succsim_{z^a} x^3 \Leftrightarrow x^2 \succsim_{z^b} x^4$, and hence $\sim$ also satisfies the second property of $\widehat{x}$-consistent indifference relations.

Now suppose that $\sim$ is also the (unique) associated indifference relation of $\succsim$ so that $(x^a, z^a) \sim (x^b, z^b) \Leftrightarrow [(x^a, z^a) \succsim (x^b, z^b)$ and $(x^b, z^b) \succsim (x^a, z^a)]$ for all $(x^a, z^a), (x^b, z^b) \in \widetilde{Y}$. Then for any $(x^a, z), (x^b, z) \in \widetilde{Y}$ with $z \in Z$, we have $(x^a, z) \sim (x^b, z) \Leftrightarrow [(x^a, z) \succsim (x^b, z)$ and $(x^b, z) \succsim (x^a, z)] \Leftrightarrow [x^a \succsim_z x^b$ and $x^b \succsim_z x^a] \Leftrightarrow x^a \sim_z x^b$, so that condition $1'$ is satisfied.

b. Let $\sim$ be a unified $\widehat{x}$-consistent indifference relation on $\widetilde{Y}$. We will construct a unique well defined $\widehat{x}$-consistent transitive complete preference relation on $\widetilde{Y}$ that is consistent with $\sim$. When $\sim$ also satisfies condition $1'$, we must show that it is the unique indifference relation associated with the constructed preference relation.

Let $(x^a, z^a), (x^b, z^b) \in \widetilde{Y}$. Since $\sim$ is unified, there are some $(x^A, z^b), (x^B, z^a) \in \widetilde{Y}$ such that $(x^a, z^a) \sim (x^A, z^b)$ and $(x^b, z^b) \sim (x^B, z^a)$. Then from the second $\widehat{x}$-consistency property of $\sim$ we know that $x^a \succsim_{z^a} x^B$ if and only if $x^A \succsim_{z^b} x^b$. We define a preference relation $\succsim_i$ on $\widetilde{Y}$ such that $(x^a, z^a) \succsim_i (x^b, z^b)$ if and only if these two equivalent conditions are true. As constructed $\succsim_i$ is complete on $\widetilde{Y}$, but perhaps not well defined since it may depend on the selection of x^A and x^B.

Suppose that we also had $(\tilde{x}^A, z^b), (\tilde{x}^B, z^a) \in \widetilde{Y}$ such that $(x^a, z^a) \sim (\tilde{x}^A, z^b)$ and $(x^b, z^b) \sim (\tilde{x}^B, z^a)$, allowing us to define the alternative preference relation $\succsim_j$ on $\widetilde{Y}$ such that $(x^a, z^a) \succsim_j (x^b, z^b)$ if and only if $x^a \succsim_{z^a} \tilde{x}^B$, and equivalently if and only if $\tilde{x}^A \succsim_{z^b} x^b$. Then $(x^A, z^b) \sim (\tilde{x}^A, z^b)$ and $(x^B, z^a) \sim (\tilde{x}^B, z^a)$, so that by the first $\widehat{x}$-consistency property of $\sim$ we know that $x^A \sim_{z^b} \tilde{x}^A$ and $x^B \sim_{z^a} \tilde{x}^B$. Thus by the transitive properties of $\succsim_{z^a}$ and $\succsim_{z^b}$ we respectively have $x^a \succsim_{z^a} x^B \Leftrightarrow x^a \succsim_{z^a} \tilde{x}^B$ and $\tilde{x}^A \succsim_{z^b} x^b \Leftrightarrow x^A \succsim_{z^b} x^b$, and hence $(x^a, z^a) \succsim_i (x^b, z^b) \Leftrightarrow (x^a, z^a) \succsim_j (x^b, z^b)$. This shows that the selection of x^A and x^B has no effect on the definition of $\succsim_i$, and hence this preference relation is well defined.

Let $(x^a, z^a), (x^b, z^b) \in \widetilde{Y}$ with $(x^a, z^a) \sim (x^b, z^b)$. By defining $x^B = x^a$ (for the construction of $\succsim_i$) we have both $(x^b, z^b) \sim (x^B, z^a)$ and $(x^a, z^a) \sim (x^B, z^a)$. Then from the first $\widehat{x}$-consistency property of $\sim$ we get $x^a \succsim_{z^a} x^B$ and $x^B \succsim_{z^a} x^a$, so that by definition, $(x^a, z^a) \succsim_i (x^b, z^b)$ and $(x^b, z^b) \succsim_i (x^a, z^a)$. Thus $(x^a, z^a) \sim (x^b, z^b) \Rightarrow [(x^a, z^a) \succsim_i (x^b, z^b)$ and $(x^b, z^b) \succsim_i (x^a, z^a)]$ and hence $\succsim_i$ is consistent with $\sim$.

For this paragraph only, let $\sim$ also satisfy condition $1'$ and assume that for some (x^a, z^a), $(x^b, z^b) \in \widetilde{Y}$ we have $(x^a, z^a) \succsim_i (x^b, z^b)$ and $(x^b, z^b) \succsim_i (x^a, z^a)$. Then by definition there is some $(x^A, z^b) \in \widetilde{Y}$ such that $(x^a, z^a) \sim (x^A, z^b)$, $x^A \succsim_{z^b} x^b$ and $x^b \succsim_{z^b} x^A$, so that $x^A \sim_{z^b} x^b$. From condition $1'$ we then have $(x^A, z^b) \sim (x^b, z^b)$, and hence by transitivity of $\sim$, $(x^a, z^a) \sim (x^b, z^b)$. Combining this with the results of the previous paragraph, we have $(x^a, z^a) \sim (x^b, z^b) \Leftrightarrow [(x^a, z^a) \succsim_i (x^b, z^b)$ and $(x^b, z^b) \succsim_i (x^a, z^a)]$, so that $\sim$ is the unique indifference relation associated with $\succsim_i$.

Let $(x^a, z^a), (x^b, z^b), (x^c, z^c) \in \widetilde{Y}$ such that $(x^a, z^a) \succsim_i (x^b, z^b)$ and $(x^b, z^b) \succsim_i (x^c, z^c)$. Then there are some $(x^{Ab}, z^b), (x^{Cb}, z^b) \in \widetilde{Y}$ such that $(x^a, z^a) \sim (x^{Ab}, z^b)$, $(x^{Cb}, z^b) \sim (x^c, z^c)$, $x^{Ab} \succsim_{z^b} x^b$ and $x^b \succsim_{z^b} x^{Cb}$. By the transitivity of $\succsim_{z^b}$ we have $x^{Ab} \succsim_{z^b} x^{Cb}$. Since $\sim$ is unified there exists some $(x^{Ca}, z^a) \in \widetilde{Y}$ such that $(x^{Ca}, z^a) \sim (x^c, z^c)$, and by the transitivity of $\sim$ then have $(x^{Ca}, z^a) \sim (x^{Cb}, z^b)$. Combining this result with $(x^a, z^a) \sim (x^{Ab}, z^b)$ and $x^{Ab} \succsim_{z^b} x^{Cb}$, the second $\widehat{x}$-consistency property of $\sim$ gives us $x^a \succsim_{z^a} x^{Ca}$. Thus by definition $(x^a, z^a) \succsim_i (x^c, z^c)$, demonstrating that $\succsim_i$ is transitive.

For any $z \in Z$, let $(x^a, z), (x^b, z) \in \widetilde{Y}$. Then applying our definition of $\succsim_i$ with $x^b = x^B$ and $z = z^a = z^b$, we have $(x^a, z) \succsim_i (x^b, z) \Leftrightarrow x^a \succsim_z x^b$. Thus $\succsim_i$ is $\widehat{x}$-consistent.

Suppose that $\succsim_k$ is an $\widehat{x}$-consistent complete and transitive preference relation defined on $\widetilde{Y}$ that is consistent with $\sim$, and for some $(x^a, z^a), (x^b, z^b) \in \widetilde{Y}$ let $(x^a, z^a) \succsim_i (x^b, z^b)$. By the definition of $\succsim_i$ we have some $(x^A, z^b) \in \widetilde{Y}$ such that $(x^a, z^a) \sim (x^A, z^b)$ and $x^A \succsim_{z^b} x^b$. Since $\succsim_k$ is consistent with $\sim$ we have $(x^a, z^a) \succsim_k (x^A, z^b)$, and since $\succsim_k$ is $\widehat{x}$-consistent also have $(x^A, z^b) \succsim_k (x^b, z^b)$. Then by transitivity, $(x^a, z^a) \succsim_k (x^b, z^b)$ and hence $\succsim_k = \succsim_i$. Thus $\succsim_i$ is the unique $\widehat{x}$-consistent complete and transitive preference relation defined on $\widetilde{Y}$ consistent with $\sim$. □

Theorem I1

Proof. For the demand function $\widehat{x} : \Re^L_{++} \times Z \times \Re_{++} \to \Re^L$ with identifiable z-fixed preference relations $\{\succsim_z \mid z \in Z\}$ and obtainable set $\widehat{X}$, let X_R be a sufficient reference set relative to $\widehat{x}$ and define $\widehat{Y}_R = (X_R \times Z) \bigcap \widehat{Y}$.

a. Let $\sim_s$ be a unified $\widehat{x}$-consistent indifference relation defined on $\widehat{Y}_R$. From part b of Lemma 5, there is a unique $\widehat{x}$-consistent transitive complete preference relation on $\widehat{Y}_R$ consistent with $\sim_s$ denoted by $\succsim_s$. Then from part a of Theorem P1, there is a unique complete and transitive $\widehat{x}$-consistent generated preference relation on $\widehat{Y}$ that is completely consistent with $\succsim_s$ on $\widehat{Y}_R$, denoted by $\succsim_S$.

We need to show that $\succsim_S$ is consistent with $\sim_s$, unique and unified. For any (x^a_R, z^a), $(x^b_R, z^b) \in (X_R \times Z) \bigcap \widehat{Y}$ we have, $(x^a_R, z^a) \sim_s (x^b_R, z^b) \Rightarrow \big[(x^a_R, z^a) \succsim_s (x^b_R, z^b)$ and $(x^b_R, z^b) \succsim_s (x^a_R, z^a)\big] \Rightarrow \big[(x^a_R, z^a) \succsim_S (x^b_R, z^b)$ and $(x^b_R, z^b) \succsim_S (x^a_R, z^a)\big]$, so that $\succsim_S$ is consistent with $\sim_s$.

Let $\succsim_T$ be a complete and transitive $\widehat{x}$-consistent preference relation defined on $\widehat{Y}$ that is consistent with $\sim_s$ such that $\succsim_T \neq \succsim_S$. Where $\succsim_t$ is the restriction of $\succsim_T$ to $\widehat{Y}_R$, $\succsim_t$ must be consistent with $\sim_s$. From the unique relationship between $\succsim_s$ and $\succsim_S$ we know that $\succsim_t \neq \succsim_s$. Then from the unique relationship between $\succsim_s$ and $\sim_s$ we know that $\succsim_t$ cannot have all the properties of $\widehat{x}$-consistency, $\sim_s$-consistency, transitivity and completeness. Hence $\succsim_T$ also cannot have all three properties. Thus by contradiction $\succsim_S$ is unique.

Let $\sim_S$ be the (unique) indifference relation associated with $\succsim_S$ and let $z^a, z^b \in Z$ with $x^a \in \widehat{X}$. Since X_R is sufficient there is some $x_R^a \in X_R$ such that $x^a \sim_{z^a} x_R^a$ and hence $(x^a, z^a) \sim_S (x_R^a, z^a)$. Since $\sim_s$ is unified, there is some $x_R^b \in X_R$ such that $(x_R^a, z^a) \sim_s (x_R^b, z^b)$ and therefore $(x_R^a, z^a) \sim_S (x_R^b, z^b)$. We then have $(x^a, z^a) \sim_S (x_R^b, z^b)$ so that $\succsim_S$ is unified.

b. Let $\succsim_S$ be a complete, transitive and unified $\widehat{x}$-consistent preference relation on $\widehat{Y}$. From part b of Theorem P1, $\succsim_S$ is generated by (and hence consistent with) a unique $\widehat{x}$-consistent seed preference relation on $\widehat{Y}_R$, denoted by $\succsim_s$. From the proof of that Theorem we also know that $\succsim_s$ is the restriction of $\succsim_S$ to $\widehat{Y}_R$. Let $\sim_s$ be the unique indifference relation associated with $\succsim_s$. From part a of Lemma 5 we know $\sim_s$ is $\widetilde{x}$-consistent and satisfies condition $1'$. We need to show that $\sim_s$ is unified and unique.

Let $z^a, z^b \in Z$ and $x_R^a \in X_R$ such that $(x_R^a, z^a) \in \widehat{Y}_R$. To show that $\sim_s$ is unified we need to demonstrate that there is also some $x_R^b \in X_R$ such that $(x_R^b, z^b) \in \widehat{Y}_R$ and $(x_R^a, z^a) \sim_s (x_R^b, z^b)$. Since $\succsim_S$ is unified, we do know that there is some $x^b \in \widehat{X}$ such that $(x_R^a, z^a) \sim_S (x^b, z^b)$. Since X_R is sufficient, there is some $x_R^b \in X_R$ such that $x_R^b \sim_{z^b} x^b$, and hence by the $\widehat{x}$-consistency of $\succsim_S$, $(x_R^b, z^b) \sim_S (x^b, z^b)$. Then by the transitivity of $\succsim_S$ we have $(x_R^a, z^a) \sim_S (x_R^b, z^b)$ and hence $(x_R^a, z^a) \sim_s (x_R^b, z^b)$. We also have $x_R^b \in \widehat{X}$ and therefore $(x_R^b, z^b) \in \widehat{Y}_R$. Thus $\sim_s$ is unified.

Let $\sim_k$ be any unified $\widehat{x}$-consistent indifference relation defined on $\widehat{Y}_R$ that satisfies condition $1'$ and is consistent with $\succsim_S$. Then from Lemma 5 part b, there is a unique $\widehat{x}$-consistent preference relation defined on $\widehat{Y}_R$ that is consistent with $\sim_k$. This preference relation is clearly $\succsim_s$. Also from Lemma 5 part b, $\sim_k$ is the unique indifference relation associated with $\succsim_s$. However $\sim_s$ is also associated with $\succsim_s$. Therefore $\sim_k = \sim_s$ and $\sim_s$ is unique.

Let $\sim_t$ be a unified $\widehat{x}$-consistent seed indifference relation on $\widehat{Y}_R$. Suppose that $\succsim_S$ can generated by $\sim_t$. Then for any $(x^a, z^a), (x^b, z^b) \in \widehat{Y}_R$, we have $(x^a, z^a) \sim_t (x^b, z^b) \Rightarrow (x^a, z^a) \sim_S (x^b, z^b) \Rightarrow (x^a, z^a) \sim_s (x^b, z^b)$. Now instead suppose that $(x^a, z^a) \sim_t (x^b, z^b) \Rightarrow (x^a, z^a) \sim_s (x^b, z^b)$ for any $(x^a, z^a), (x^b, z^b) \in \widehat{Y}_R$. Then since all the indifference information of $\sim_t$ is included in $\sim_s$, they must generate the same unique element of $\Phi(\widehat{x})$ as described by part a of this theorem, so that $\succsim_S$ can be generated by $\sim_t$. We have shown that $\succsim_S$ can generated by $\sim_t$ if and only if $(x^a, z^a) \sim_t (x^b, z^b) \Rightarrow (x^a, z^a) \sim_s (x^b, z^b)$ for any $(x^a, z^a), (x^b, z^b) \in \widehat{Y}_R$. ◻

Theorem I2

Proof. Given the demand function $\widehat{x}$ with obtainable set $\widehat{X}$:

a. Let X_R be a reference set such that $X_R \bigcap \widehat{X}$ is naturally ordered, and let $\sim_s$ be a strongly monotone seed indifference relation on $\widehat{Y}_R = (X_R \times Z) \bigcap \widehat{Y}$. By definition $\sim_s$ is consistent with a strongly monotone preference relation on $\widehat{Y}_R$, denoted $\succsim_s$. Then by part a of Theorem P2, $\succsim_s$ is $\widehat{x}$-consistent, so that by part a of Lemma 5, $\sim_s$ is also $\widehat{x}$-consistent.

Let $x^a, x^b \in X_R \bigcap \widehat{X}$ and $z \in Z$ so that $(x^a, z), (x^b, z) \in \widehat{Y}_R$. From $\widehat{x}$-consistency of $\sim_s$ we have $(x^a, z) \sim_s (x^b, z) \Rightarrow x^a \sim_z x^b$. Suppose that $x^a \sim_z x^b$. Then since X_R is non-redundant

(Theorem 2), we must have $x^a = x^b$, so that trivially $(x^a, z) \sim_s (x^b, z)$. We have shown that $(x^a, z) \sim_s (x^b, z) \Leftrightarrow x^a \sim_z x^b$ and hence $\sim_s$ satisfies condition $1'$ for indifference relation demand-consistency.

b. Let X_R be a reference set such that any strongly monotone unified seed indifference relation on $\widehat{Y}_R = (X_R \times Z) \bigcap \widehat{Y}$ is $\widehat{x}$-consistent. Then any strongly monotone preference relation on $\widehat{Y}_R$ must be consistent with at least one of these indifference relations (in particular with its unique associated indifference relation), and hence by part b of Lemma 5 is also $\widehat{x}$-consistent. Thus by part b of Theorem P2, $X_R \bigcap \widehat{X}$ is naturally ordered. □

Theorem I3

Proof. Let X_R be a reference set that is not naturally ordered and define $Y_R = X_R \times Z$, and let $\sim_s$ be a unified seed preference relation defined on on Y_R.

Since X_R is not naturally ordered, for some $x^1, x^2 \in X_R$ we have neither $x^1 \geq x^2$ nor $x^2 \geq x^1$, so that for some integers k and ℓ with $1 \leq k \leq L$ and $1 \leq \ell \leq L$, we have $x^1_k < x^2_k$ and $x^1_\ell > x^2_\ell$. Let $z^a, z^b \in Z$ with $z^a \neq z^b$. Then since $\sim_s$ is unified, there exists some $x^{1b}, x^{2b} \in X_R$ such that $(x^1, z^a) \sim_s (x^{1b}, z^b)$ and $(x^2, z^a) \sim_s (x^{2b}, z^b)$. The second condition for demand-consistent indifference relations requires $x^1 \succsim_{z^a} x^2 \Leftrightarrow x^{1b} \succsim_{z^b} x^{2b}$ (and $x^2 \succsim_{z^a} x^1 \Leftrightarrow x^{2b} \succsim_{z^b} x^{1b}$). With each of the following four cases we obtain violations of this condition.

First, suppose that $x^{1b} = x^{2b}$. Then $\sim_s$ is not demand-consistent for any demand function with an identifiable preference relation $\succsim_{z^a}$ such that either $x^1 \succ_{z^a} x^2$ or $x^2 \succ_{z^a} x^1$. Clearly there are infinitely many such demand functions for both outcomes. For each of the remaining three cases we have $x^{1b} \neq x^{2b}$. With the second case we also have $x^{1b} \geq x^{2b}$. Then $\sim_s$ is demand-inconsistent whenever $x^2 \succ_{z^a} x^1$. Similarly with $x^{1b} \leq x^{2b}$, we have demand-inconsistency whenever $x^1 \succ_{z^a} x^2$. Finally with the fourth case, we have neither $x^{1b} \geq x^{2b}$ nor $x^{1b} \leq x^{2b}$. Demand-inconsistency is then obtained with a combination of $\succsim_{z^a}$ and $\succsim_{z^b}$ identifiable preference relations. We have four such generic combinations: $(x^1 \succ_{z^a} x^2 \;\&\; x^{2b} \succsim_{z^b} x^{1b})$, $(x^1 \succsim_{z^a} x^2 \;\&\; x^{2b} \succ_{z^b} x^{1b})$, $(x^2 \succ_{z^a} x^1 \;\&\; x^{1b} \succsim_{z^b} x^{2b})$ and $(x^2 \succsim_{z^a} x^1 \;\&\; x^{1b} \succ_{z^b} x^{2b})$. Each such generic combination holds for an infinite number of actual possible $\{\succsim_{z^a}, \succsim_{z^b}\}$ combinations, and hence for an infinite number of demand functions. □

Theorem A1

Proof. Given the demand function $\widehat{x}$ define $\widehat{Y}_{WC} = (X_{WC} \times Z) \bigcap \widehat{Y}$.

a. Let $\widehat{x}$ have single-preference on X_{WC}. Also let $(x^a, z), (x^b, z) \in \widehat{Y}_{WC}$ with $z \in Z$. We need to show that $(x^a, z) \sim_{wc} (x^b, z) \Rightarrow x^a \sim_z x^b$. However by definition of this seed relation, $(x^a, z) \sim_{wc} (x^b, z) \Rightarrow x^a = x^b$ so that the first condition of indifference relation $\widehat{x}$-consistency is trivially true. Now instead let $(x^1, z^a), (x^2, z^b), (x^3, z^a), (x^4, z^b) \in (X_{WC} \times Z) \bigcap \widehat{Y}$ and $z^a, z^b \in Z$ such that $(x^1, z^a) \sim_{wc} (x^2, z^b)$ and $(x^3, z^a) \sim_{wc} (x^4, z^b)$. Then again by definition of the seed relation we have $x^1 = x^2$ and $x^3 = x^4$ so that trivially $x^1 \succsim_{z^b} x^3 \Leftrightarrow x^2 \succsim_{z^b} x^4$. Since demand is single-preferenced on X_{WC}, $x^1 \succsim_{z^a} x^3 \Leftrightarrow x^1 \succsim_{z^b} x^3$. Putting these two together we have $x^1 \succsim_{z^a} x^3 \Leftrightarrow x^2 \succsim_{z^b} x^4$, satisfying the second condition of indifference relation $\widehat{x}$-consistency. Thus $\sim_{wc}$ is $\widehat{x}$-consistent on $\widehat{Y}_{WC}$.

b. Let $\sim_{wc}$ be $\widehat{x}$-consistent on $\widehat{Y}_{WC}$, let $x^1, x^2 \in X_{WC} \bigcap \widehat{X}$ and let $z^a, z^b \in Z$. Then $(x^1, z^a) \sim_{wc} (x^1, z^b)$ and $(x^2, z^a) \sim_{wc} (x^2, z^b)$, so that by the second condition of $\widehat{x}$-consistency for indifference relations, $x^1 \succsim_{z^a} x^2 \Leftrightarrow x^1 \succsim_{z^b} x^2$. Thus $\widehat{x}$ is single-preferenced on X_{WC}. □

Theorem A2

Proof. Given the demand function $\widehat{x} : \Re^L_{++} \times Z \times \Re_{++} \to \Re^L$ with identifiable z-fixed preference relations $\{\succsim_z \mid z \in Z\}$ and obtainable set $\widehat{X}$, define $\widehat{X}_{WC} = X_{WC} \bigcap \widehat{X}$. For some element of $\Phi(\widehat{x})$, let $\succsim_i$ be the restriction of that preference relation to $\widehat{Y}_{WC} = \widehat{X}_{WC} \times Z$.

a. Let $\widehat{x}$ be single-preferenced on X_{WC}. We need to show that there exists a function $u : \widehat{X}_{WC} \to \Re$ and a preference relation $\succsim_{SX}$ defined on $\Re \times Z$ such that $\succsim_i$ can be specified with equation (3) using this function and preference relation.

Let $z^0 \in Z$, let $u : \widehat{X}_{WC} \to \Re$ be a utility function that represents the restriction of $\succsim_{z^0}$ to $\widehat{X}_{WC}$, and let the set $\mathbf{I}$ be the partition of $\widehat{X}_{WC}$ into the set of indifference sets based on u. We then have $\bigcup_{I \in \mathbf{I}} I = \widehat{X}_{WC}$; $I_a \bigcap I_b = \emptyset$ for all $I_a, I_b \in \mathbf{I}$ with $I_a \neq I_b$; and $u(x^a) = u(x^b) \Leftrightarrow I_a = I_b$ for any $x^a, x^b \in \widehat{X}_{WC}$ and $I_a, I_b \in \mathbf{I}$ such that $x^a \in I_a$ and $x^b \in I_b$. For each $I \in \mathbf{I}$ let $\bar{x}(I) \in I$ be an arbitrarily chosen fixed representative element. Let U be set of all $u(x)$ values for $x \in \widehat{X}_{WC}$. Then for each $u_0 \in U$ there is some unique $I_0 \in \mathbf{I}$ such that for any $x^0 \in I_0$ we have $u_0 = u(x^0)$. We thus have a well defined inverse utility function $\mathrm{I}_u : U \to \mathbf{I}$. We can now define our preference relation $\succsim_{SX}$ by $(u^a, z^a) \succ_{SX} (u^b, z^b) \Leftrightarrow (\bar{x}(\mathrm{I}_u(u^a)), z^a) \succsim_i (\bar{x}(\mathrm{I}_u(u^b)), z^b)$.

Let $(x^a, z^a), (x^b, z^b) \in \widehat{Y}_{WC}$. Then $(u(x^a), z^a) \succsim_{SX} (u(x^b), z^b) \Leftrightarrow (\bar{x}(\mathrm{I}_u(u(x^a))), z^a) \succsim_i (\bar{x}(\mathrm{I}_u(u(x^b))), z^b)$. By construction $(\bar{x}(\mathrm{I}_u(u(x^a))), z^0) \sim_i (x^a, z^0)$ and $(\bar{x}(\mathrm{I}_u(u(x^b))), z^0) \sim_i (x^b, z^0)$. Since $\widehat{x}$ is single-preferenced on X_{WC}, we also have $(\bar{x}(\mathrm{I}_u(u(x^a))), z^a) \sim_i (x^a, z^a)$ and $(\bar{x}(\mathrm{I}_u(u(x^b))), z^b) \sim_i (x^b, z^b)$, so that $(\bar{x}(\mathrm{I}_u(u(x^a))), z^a) \succsim_i (\bar{x}(\mathrm{I}_u(u(x^b))), z^b) \Leftrightarrow (x^a, z^a) \succsim_i (x^b, z^b)$. Thus $(u(x^a), z^a) \succsim_{SX} (u(x^b), z^b) \Leftrightarrow (x^a, z^a) \succsim_i (x^b, z^b)$.

b. Let $\widehat{x}$ be not single-preferenced on X_{WC}. We need to show that there does not exist any combination of a function $u : \widehat{X}_{WC} \to \Re$ and a preference relation $\succsim_{SX}$ defined on $\Re \times Z$ such that $\succsim_i$ can be specified with equation (3) using these two elements.

This is proved by contradiction. Suppose that these two elements exist. Let $z^a, z^b \in Z$ and $x^1, x^2 \in \widehat{X}_{WC}$. Demand-consistency gives us $(x^1, z^a) \succsim_i (x^2, z^a) \Leftrightarrow x^1 \succsim_{z^a} x^2$ and $(x^1, z^b) \succsim_i (x^2, z^b) \Leftrightarrow x^1 \succsim_{z^b} x^2$. From equation (3) we have $(x^1, z^a) \succsim_i (x^2, z^a) \Leftrightarrow (u(x^1), z^a) \succsim_{SX} (u(x^2), z^a)$ and $(x^1, z^b) \succsim_i (x^2, z^b) \Leftrightarrow (u(x^1), z^b) \succsim_{SX} (u(x^2), z^b)$. We also have $(u(x^1), z^a) \succsim_{SX} (u(x^2), z^a) \Leftrightarrow u(x^1) \geq u(x^2) \Leftrightarrow (u(x^1), z^b) \succsim_{SX} (u(x^2), z^b)$. Putting all these together we get $x^1 \succsim_{z^a} x^2 \Leftrightarrow x^1 \succsim_{z^b} x^2$, so that $\widehat{x}$ is single-preferenced on X_{WC}. Then by contradiction, the combination u and $\succsim_{SX}$ cannot exist. □

Theorem A3

Proof. Given the demand function $\widehat{x}$ with identifiable z-fixed preference relations $\{\succsim_z \mid z \in$

$Z\}$, let a function $\chi : \Re_{++} \times Z \times Z \to \Re_{++}$ exist such that for any $x_1^{\alpha a}, x_1^{\beta a} \in \Re_{++}$, $x_{-1}^{\alpha}, x_{-1}^{\beta} \in \Re_{++}^{L-1}$ and $z^a, z^b \in Z$, equation (5) is valid with $x_1^{\alpha b} = \chi(x_1^{\alpha a}, z^a, z^b)$ and $x_1^{\beta b} = \chi(x_1^{\beta a}, z^a, z^b)$. We need to prove the existence of some functions $U : \Re \times \Re_{++}^{L-1} \to \Re$ and $f : \Re_{++} \times Z \to \Re$ such that $\succsim_{rp}$ as defined by equation (4) is $\widehat{x}$-consistent.

For any fixed $z^0 \in Z$ define f by $f(x, z) = \chi(x, z, z^0)$. Let $u^0(x_1, x_{-1})$ be a utility function that represents the identifiable preference relation $\succsim_{z^0}$ and define U by $U(f, x_{-1}) = u^0(f, x_{-1})$. Finally we use these two functions in the context of equation (4) to fully define a preference relation $\succsim_{rp}$. Then for any $(x_1^{a*}, x_{-1}^{a*}), (x_1^{b*}, x_{-1}^{b*}) \in \Re_{++}^L$ and $z^* \in Z$, we have, $(x_1^{a*}, x_{-1}^{a*}, z^*) \succsim_{rp} (x_1^{b*}, x_{-1}^{b*}, z^*) \Leftrightarrow U(f(x_1^{a*}, z^*), x_{-1}^{a*}) \geq U(f(x_1^{b*}, z^*), x_{-1}^{b*}) \Leftrightarrow u^0(\chi(x_1^{a*}, z^*, z^0), x_{-1}^{a*}) \geq u^0(\chi(x_1^{b*}, z^*, z^0), x_{-1}^{b*}) \Leftrightarrow (\chi(x_1^{a*}, z^*, z^0), x_{-1}^{a*}) \succsim_{z^0} (\chi(x_1^{b*}, z^*, z^0), x_{-1}^{b*}) \Leftrightarrow (x_1^{a*}, x_{-1}^{a*}) \succsim_{z^*} (x_1^{b*}, x_{-1}^{b*})$, so that $\succsim_{rp}$ is $\widehat{x}$-consistent. The last equivalence relation comes from the properties of χ in the context of equation (5). □

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