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Existence of Nash Networks in One-Way Flow Models

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Abstract

This paper addresses the existence of Nash networks for the one-way flow model of Bala and Goyal (2000) in a number of different settings. We provide conditions for the existence of Nash networks in models where costs and values of links are heterogeneous and players obtain resources from others through the directed path between them. We find that costs of establishing links play a vital role in the existence of Nash networks. Next we examine the existence of Nash networks when there are congestion effects in the model.

JEL Classification: C72, D85

Key Words: Network Formation, Non-cooperative Games

1 Introduction

The importance of networks in economic and social activities has led to the emergence of a growing literature seeking to understand the formation of these networks. This literature in economics has focused on three main questions: (i) What are the incentives for self-interested players to form links with each others and what are the set of the stable resulting networks? (ii) What networks are efficient? and (iii) Is there a conflict between the set of stable and efficient networks?
We can discern two distinct strands in the literature differentiated by the type of stability concept used.

The first type employs the notion of pairwise stability and its variants and is inspired by Jackson and Wolinsky's (1996 [9]) work. These authors assume that a link is formed between two players if both players involved in the link agree to form that link, though link deletion occurs unilaterally. While benefits depend on the overall graph, the cost of setting up a relationship is shared equally between the two participating players. In a pairwise stable network no pair of players has an incentive to form a link and no player has an incentive to delete a link. Necessary and sufficient conditions for the existence of pairwise stable networks can be found in Jackson and Watts (2001 [8]) .

The second literature develops a non-cooperative version of network formation. This literature was initiated by Bala and Goyal (BG, 2000 [1]). These authors assume that a player can establish a link with another player without the latter's consent, as long as she incurs the cost of forming the link. They present two versions of their model: the one-way flow model and the two-way flow model. In the one-way flow model, only the (link) initiating player has access to the other player's information, whereas in the two-way flow model both players have access to each others information, regardless of who initiates the link. For both versions, the corresponding static stable networks are called Nash networks since Nash equilibrium is used to determine stability. In a Nash network, no player has an incentive to change her links, given the links formed by the other players.

The reason why Nash equilibria is an adequate concept for the one-way flow model but not for the consent model is that no coordination problem arises over setting up links in the former and each links anouncement is guaranted to change the network.

Most of the existing studies in this literature have explored the characterization of Nash networks, either in the two-way flow model (Galeotti, Goyal and Kamphorst (2005 [5]), Haller and Sarangi (2005 [7]) or in the one-way flow model (Galeotti, 2004 [4], Billand and Bravard, 2005 [2]). The existence of Nash networks however has not been studied in great detail. Indeed, although BG (2000 [1]) provide a constructive proof of the existence of Nash networks in the one-way flow model and the two-way flow model, this is done in a rather restrictive setting, since the authors assume that all costs and benefits are homogeneous across players. In a recent paper Haller, Kamphorst and Sarangi (2005 [6]) study the existence of Nash networks in two-way flow models by incorporating value, cost and link heterogeneity. However, the existence issue had remained unexplored in the one-way flow setting.

In this paper, we investigate the existence of Nash networks in BG's one-way flow model when costs and values of links are heterogeneous and players use pure strategies. More precisely, we focus on one-way flow model with linear payoffs and no decay.

The one-way flow model is worth studying since it includes some important settings. For instance, a web site can provide a link or pointer to another web site without the second web site's permission. Likewise, a researcher can generally cite another researcher without the second researcher's permission. Lastly, firms can unilaterally establish links with other firms, through intelligence economic activities, which include among others reading of industry trade press or patent literature, talking with technology vendors, sales representative or industry experts, and analyzing the competitors' product.

Moreover, the question of existence of equilibria under heterogeneity is important, since ex-ante asymmetries across players arise quite naturally in reality. For instance, in the context of information networks, it is often the case that some individuals are better informed, which makes them more valuable contacts. Similarly, as individuals differ, it seems natural that forming links is cheaper for some individuals as compared to others. For instance, players can be defined in terms of cultural, legal or geographical proximity, and it may be cheaper for a given player to set a link with a closer player.

We can discern three types of heterogeneity. The first one, value heterogeneity, concerns the value of the ressources of a given player for the other players. The second one, cost heterogeneity, concerns the cost of forming a link with a given player for the other players. The third type of heterogeneity, link heterogeneity, concerns the probability that a link formed by a player with a given player fails to transmit information from the latter player to the former player. It may also concern the loss of information that is incurred when information is transmitted from a player to another player.

The introduction of various heterogeneity conditions for costs, values and links provides a sensitivity check for the results obtained with homogeneous parameters. In other words, we can ask if the introduction of different kinds of heterogeneities in the Bala and Goyal's model alters the Nash networks existence results.

Our results concerning the existence of Nash network in the ono-way flow model under various heterogeneity assumptions complement the existing literature. Indeed, Galeotti, (2004 [4]) characterizes the (strict) Nash networks when cost and values of links are heterogeneous, but we do not know under what conditions such equilibria exist. Finally, the existence of Nash networks has never been studied, when there are congestion effects. The possibility of congestion effects was introduced by Billand and Bravard (2005 [2]) as an extension of BG's model (2000 [1]). Congestion effects exist in several situations where getting too many resources can actually prove an hindrance to agents. For instance, when researchers are seeking to get some information about a part of their field which they are unsure about, they often read a literature survey written by another scholar. This activity is costly in terms of time and effort, for instance, to identify relevant information sources. The reading effort can be expensive and tedious if they are too many sources. In extreme cases, if a survey is too exhaustive, it might have little or no value to the scholarly reader. Billand and Bravard (2005 [2]) characterize Nash networks when this assumption arises. However, they do not address the issue of existence of Nash networks.

We now provide a quick overview of our results. We show that there does not always exist a Nash network when costs and values are heterogeneous. More precisely, we find that, as in the two-way flow model, heterogeneity of cost in forming links plays a great role in the non existence of Nash network. We then provide conditions on costs of setting links to allow for the existence of Nash networks. We also show that if costs are homogeneous, then there always exist Nash networks. Finally, we show that if costs and values are homogeneous, but congestion effects can occur, then a Nash network does not always exist.

The remainder of the paper is organized as follows. In Section 2 we set the basic one-way flow model. In Section 3 we present the results about the existence of Nash networks in this model. More precisely we first study this problem under various heterogeneity conditions for costs, values and links. We then introduce the presence of congestion effects in Section 4.

2 Model Setup

Let $N = \{1, \ldots, n\}$ be the set of players. The network relations among these players are formally represented by directed graphs whose nodes are identified with the players. A network $\boldsymbol{g} = (N, E)$ is a pair of sets: the set N of players and the edges set $E(\boldsymbol{g}) \subset N \times N$ of directed links. A link initiated by player i to player j is denoted by i, j. Pictorially this is depicted as link from j to i to show the direction of information flow.[1] Each player i chooses a strategy $\boldsymbol{g}_i = (\boldsymbol{g}_{i,1}, \ldots, \boldsymbol{g}_{i,i-1}, \boldsymbol{g}_{i,i+1}, \ldots, \boldsymbol{g}_{i,n})$, $\boldsymbol{g}_{i,j} \in \{0, 1\}$ for all $j \in N \setminus \{i\}$, which describes the act of establishing links. More precisely, $\boldsymbol{g}_{i,j} = 1$ if and only if $i, j \in E(\boldsymbol{g})$. The interpretation of $\boldsymbol{g}_{i,j} = 1$ is that player i forms a link with player $j \neq i$, and the interpretation of $\boldsymbol{g}_{i,j} = 0$ is that i does not form a link with player j. We only use pure strategies. Note that $\boldsymbol{g}_{i,j} = 1$ does not necessarily imply that $\boldsymbol{g}_{j,i} = 1$. It can be that i is linked to j, but j is not linked to i. Let $\mathcal{G} = \times_{i=1}^{n} \mathcal{G}_i$ be the set of all possible networks where $\mathcal{G}_i$ is the set of all possible strategies of player $i \in N$.

We now provide some important graph theoretic definitions. For a directed graph, $\boldsymbol{g} \in \mathcal{G}$, a *path* $P(\boldsymbol{g})$ of length m in $\boldsymbol{g}$ from player j to i, $i \neq j$, is a finite sequence $i_0, i_1, \ldots, i_m$ of distinct players such that $i_0 = i$, $i_m = j$ and $\boldsymbol{g}_{i_k, i_{k+1}} = 1$ for $k = 0, \ldots, m-1$. If $i_0 = i_m$, then the path is a cycle. We denote the set of cycles in the

[1] Throughout the paper we refer to this as link from j to i. The same is true for other network components like paths.

network $\boldsymbol{g}$ by $\mathcal{C}(\boldsymbol{g})$. In the empty network, $\dot{\boldsymbol{g}}$, there are no links between any agents.

To sum up, a link from a player j to a player i ($\boldsymbol{g}_{i,j} = 1$) allows player i to get resources from player j and since we are in a one-way flow model, this link does not allow player j to obtain resources from i. Moreover, a player i may receive information from other players through a sequence of indirect links. To be precise, information flows from player j to player i, if i and j are linked by a path of length m in $\boldsymbol{g}$ from j to i. Let

$$N_i(\boldsymbol{g}) = \{j \in N | \text{ there exists a path in } \boldsymbol{g} \text{ from } j \text{ to } i\},$$

be the set of players that player i can access in the network $\boldsymbol{g}$. By definition, we assume that $i \in N_i(\boldsymbol{g})$ for all $i \in N$ and for all $\boldsymbol{g} \in \mathcal{G}$. Let $n_i(\boldsymbol{g})$ be the cardinality of the set $N_i(\boldsymbol{g})$. Information received from player j is worth $V_{i,j}$ to player i. Moreover, i incurs a cost $c_{i,j}$ when she initiates a direct link with j, i.e. when $\boldsymbol{g}_{i,j} = 1$. We can now define the payoff function of player $i \in N$:

$$\pi_i(\boldsymbol{g}) = \sum_{j \in N_i(\boldsymbol{g})} V_{i,j} - \sum_{j \in N} \boldsymbol{g}_{i,j} c_{i,j}.$$

We assume that $c_{i,j} > 0$ and $V_{i,j} > 0$ for all $i \in N$, $j \in N$, $i \neq j$. Moreover, we assume that $V_{i,i} = 0$ for all $i \in N$. The next definition introduces the different notions of heterogeneity in our model.

Definition 1 *Values (or costs) are said heterogeneous by pairs of players if there exist* $i \in N$, $j \in N$, $k \in N$ *such that* $V_{i,j} \neq V_{i,k}$ ($c_{i,j} \neq c_{i,k}$) *and there exist* $i' \in N$, $j' \in N$, $k' \in N$ *such that* $V_{j',i'} \neq V_{k',i'}$. *Values (or costs) are said heterogeneous by players if for all* $i \in N$, $j \in N$, $k \in N$: $V_{i,j} = V_{i,k} = V_i$ ($c_{i,j} = c_{i,k} = c_i$) *but there exists* $i \in N$, $i' \in N$ *such that* $V_i \neq V_{i'}$ ($c_i \neq c_{i'}$).

We now provide some useful definitions for studying the existence of Nash networks. Given a network $\boldsymbol{g} \in \mathcal{G}$, let $\boldsymbol{g}_{-i}$ denote the network obtained when all of player i's links are removed. The network $\boldsymbol{g}$ can be written as $\boldsymbol{g} = \boldsymbol{g}_{-i} \oplus \boldsymbol{g}_i$, where the operator $\oplus$ indicates that $\boldsymbol{g}$ is formed by the union of links in $\boldsymbol{g}_i$ and $\boldsymbol{g}_{-i}$. The strategy $\boldsymbol{g}_i$ is said to be a best response of player i to $\boldsymbol{g}_{-i}$ if:

$$\pi_i(\boldsymbol{g}_i \oplus \boldsymbol{g}_{-i}) \geq \pi_i(\boldsymbol{g}'_i \oplus \boldsymbol{g}_{-i}), \text{for all } \boldsymbol{g}'_i \in \mathcal{G}_i.$$

The set of player i's best responses to $\boldsymbol{g}_{-i}$ is denoted by $\mathcal{BR}_i(\boldsymbol{g}_{-i})$. Furthermore, a network $\boldsymbol{g} = (\boldsymbol{g}_1, \ldots, \boldsymbol{g}_i, \ldots, \boldsymbol{g}_n)$ is said to be a Nash network if $\boldsymbol{g}_i \in \mathcal{BR}_i(\boldsymbol{g}_{-i})$ for each $i \in N$.

Definition 2 *We say that two networks* $\boldsymbol{g}$ *and* $\boldsymbol{g}'$ *are adjacent if there is a unique player* i *such that* $g_{i,j} \neq g'_{i,j}$ *for at least one player* $j \neq i$ *and if for all player* $k \neq i$, $g_{k,j} = g'_{k,j}$, *for all* $j \in N$.

An improving path is a sequence of adjacent networks that results when players form or sever links based on payoff improvement the new network offers over the current network. More precisely, each network in the sequence differs from the previous one by the links formed by one unique player. If a player changes her links, then it must be that this player strictly benefits from such a change.

Definition 3 *Formally, an improving path from a network* $\boldsymbol{g}$ *to a network* $\boldsymbol{g}'$ *is a finite sequence of networks* $\boldsymbol{g}^1, \ldots, \boldsymbol{g}^k$, *with* $\boldsymbol{g}^1 = \boldsymbol{g}$ *and* $\boldsymbol{g}^k = \boldsymbol{g}'$, *such that the two following conditions are verified :*

1. $\boldsymbol{g}^\ell$ *and* $\boldsymbol{g}^{\ell+1}$, *are adjacent networks;*

2. *for this unique player i, we have $\boldsymbol{g}_i^{\ell+1} \in \mathcal{BR}_i(\boldsymbol{g}_{-i}^{\ell})$ and $\boldsymbol{g}_i^{\ell} \notin \mathcal{BR}_i(\boldsymbol{g}_{-i}^{\ell})$, that is $\boldsymbol{g}^{\ell+1}$ is a network where i plays a best response while $\boldsymbol{g}^{\ell}$ is a network where i does not play a best response.*

Moreover, if $\boldsymbol{g}^1 = \boldsymbol{g}^k$, then the improving path is called an improving cycle.

It is obvious that a network $\boldsymbol{g}$ is a Nash network if and only if it has no improving path emanating from it.
Finally, we define $\eta : \mathcal{G} \to \mathbb{R}$, $\eta(\boldsymbol{g}) = \sum_{i \in N} n_i(\boldsymbol{g})$ as a function.

3 Model with Heterogeneous Agents without Congestion Effect

Bala and Goyal (2000 [1]) outlines a constructive proof of the existence of Nash networks in the case of costs and values of links homogeneity. Here we begin by showing that in one-way flow models with cost and value heterogeneity by pairs of players (see Galeotti, 2004 [4]) there always exists a Nash network if the number of players is $n = 3$. This result is no longer true if the number of players is $n \geqslant 4$. However, if values of links are heterogeneous by pairs of players and costs of links are heterogeneous by players, there always exists a Nash network.

Proposition 1 *If the values and costs of links are heterogeneous by pairs and $n = 3$, then a Nash network exists.*

Proof Let $N = \{1, 2, 3\}$. We begin with the empty network $\dot{\boldsymbol{g}}$. Either $\dot{\boldsymbol{g}}$ is a Nash network and we are done, or $\dot{\boldsymbol{g}}$ is not a Nash network and there exists an improving path from $\dot{\boldsymbol{g}}$ to an adjacent network $\boldsymbol{g}^1$. That is, there exists a player, say without loss of generality player 1, such that $\dot{\boldsymbol{g}}_1 \notin \mathcal{BR}_1(\dot{\boldsymbol{g}}_{-1})$ and $\boldsymbol{g}_1^1 \in \mathcal{BR}_1(\dot{\boldsymbol{g}}_{-1})$. Since $1 \in N$ has no link in $\dot{\boldsymbol{g}}$ and forms links in $\boldsymbol{g}^1 = \boldsymbol{g}_1^1 \oplus \dot{\boldsymbol{g}}_{-1}$, we have $\eta(\dot{\boldsymbol{g}}) < \eta(\boldsymbol{g}^1)$. Now we will repeat the same step. Assume an improving path from a network $\boldsymbol{g}^1$ to a network $\boldsymbol{g}^k$ where for each player $i \in N$, we have $N_i(\boldsymbol{g}^{k-1}) \subseteq N_i(\boldsymbol{g}^k)$. We show that if there exists an improving path from $\boldsymbol{g}^k$ to $\boldsymbol{g}^{k+1}$, then for each player $i \in N$, $N_i(\boldsymbol{g}^k) \subseteq N_i(\boldsymbol{g}^{k+1})$. Let i be a player such that $\boldsymbol{g}_i^{k+1} \in \mathcal{BR}_i(\boldsymbol{g}_{-i}^k)$ and $\boldsymbol{g}_i^k \notin \mathcal{BR}_i(\boldsymbol{g}_{-i}^k)$. We show that if $j \in N_i(\boldsymbol{g}^k)$, then $j \in N_i(\boldsymbol{g}^{k+1})$. Indeed there are two possibilities for $j \in N_i(\boldsymbol{g}^k)$.

1. Either $\boldsymbol{g}_{i,j}^k = 1$, that is i directly obtains the resources of player j. Then there are two possibilities.

   - If $V_{i,j} - c_{i,j} > 0$ then $j \in N_i(\boldsymbol{g}^{k+1})$, otherwise i does not play a best response in $\boldsymbol{g}^{k+1}$.
   - If $V_{i,j} - c_{i,j} < 0$, then there is a network $\boldsymbol{g}^{k'}$, $k' < k$, such that $\ell \in N_j(\boldsymbol{g}^{k'})$ and $V_{i,j} + V_{i,\ell} - c_{i,j} > \max\{0, V_{i,\ell} - c_{i,\ell}\}$, else $\boldsymbol{g}_{i,j}^k = 0$. Since $N_j(\boldsymbol{g}^{k'}) \subseteq N_j(\boldsymbol{g}^k)$, for all $k' < k$ and for all $j \in N$, we have $\ell \in N_j(\boldsymbol{g}^k)$ and player i deletes her link with j only if $j \in N_\ell(\boldsymbol{g}^k)$ and $V_{i,j} + V_{i,\ell} - c_{i,j} < V_{i,j} + V_{i,\ell} - c_{i,\ell}$. In that case, i forms a link with ℓ and $j \in N_i(\boldsymbol{g}^{k+1})$.

2. Or $\boldsymbol{g}_{i,j}^k = 0$, $\boldsymbol{g}_{i,\ell}^k = 1$ and $\boldsymbol{g}_{\ell,j}^k = 1$, that is i indirectly obtains the resources of player j. Then, we use the same argument as above to show that player i deletes her link with ℓ only if she has an incentive to form a link with j and $j \in N_i(\boldsymbol{g}^{k+1})$.

We now show that there does not exist any cycle in an improving path $\mathcal{Q} = \{\dot{\boldsymbol{g}}, \boldsymbol{g}^1, \dots, \boldsymbol{g}^t, \dots, \boldsymbol{g}^{t+h}, \dots, \boldsymbol{g}^{t+h'}, \dots\}$, with $h' > h > 0$. In other words, we show that if $\boldsymbol{g}_{i,j}^t = 1$, $\boldsymbol{g}_{i,j}^{t+h} = 0$, and $\boldsymbol{g}_{i,j}^{t+h'} = 1$, then we have $N_i(\boldsymbol{g}^t) \subsetneq N_i(\boldsymbol{g}^{t+h'})$. We note that as $j \in N_i(\boldsymbol{g}^t)$ and $N_i(\boldsymbol{g}^t) \subseteq N_i(\boldsymbol{g}^{t+h})$, we have $j \in N_i(\boldsymbol{g}^{t+h})$. Also, as $\boldsymbol{g}_{i,j}^{t+h} = 0$, we have $\boldsymbol{g}_{i,\ell}^{t+h} = 1$ and $\ell \in N_i(\boldsymbol{g}^{t+h})$. Moreover, as $N_i(\boldsymbol{g}^{t+h}) \subseteq N_i(\boldsymbol{g}^{t+h'})$, we have $N_i(\boldsymbol{g}^{t+h'}) = \{j, \ell\}$.

Without loss of generality, we suppose that player i deletes the link i, j for the first time, between t and $t+h$, in $\boldsymbol{g}^{t+h}$. Likewise, we assume that player i forms the link i, j for the first time, between $t+h$ and $t+h'$, in $\boldsymbol{g}^{t+h'}$.
We have two cases.

1. Suppose we have $\boldsymbol{g}^t_{i,\ell} = 0$. To obtain a contradiction, assume that $\ell \in N_i(\boldsymbol{g}^t)$. It follows that $\boldsymbol{g}^{t+h}_{j,\ell} = 1$ since player i does not form the link i, ℓ between $\boldsymbol{g}^t$ and $\boldsymbol{g}^{t+h}$ if j preserves the link j, ℓ. Also j does not delete the link j, ℓ between $\boldsymbol{g}^t$ and $\boldsymbol{g}^{t+h}$ if i does not form the link i, ℓ (recall that in our process only one player changes her strategy at each period). Since player i chooses to delete the link i, j in $\boldsymbol{g}^{t+h}$, then she must form the link i, ℓ and we must have $\boldsymbol{g}^{t+h}_{\ell,j} = 1$, since $\ell \in N_i(\boldsymbol{g}^t) \subseteq N_i(\boldsymbol{g}^{t+h})$. Moreover, we note that the substitution of the link i, j by the link i, ℓ implies that $c_{i,j} > c_{i,\ell}$. Using same argument, player ℓ has not deleted the link ℓ, j between $\boldsymbol{g}^{t+h}$ and $\boldsymbol{g}^{t+h'}$. Therefore, if player i forms the link i, j in $\boldsymbol{g}^{t+h'}$ (and so deletes the link i, ℓ), then we have $c_{i,j} < c_{i,\ell}$ and we obtain the desired contradiction.

2. Next, suppose that we have $\boldsymbol{g}^t_{i,\ell} = 1$. If player i deletes the link i, j in $\boldsymbol{g}^{t+h}$, then we obtain the situation in case 1 up to a permutation of players j and ℓ. Hence the proof follows.

$\square$

We have shown that if values and costs of links are heterogeneous by pairs and $n = 3$, then there always exists a Nash network. Note that this result is not true for the model with directed links and two-way flow of resources (see Haller, Kamphorst and Sarangi 2006 [6] p. 7). We next show with an example that the above proposition is not valid for $n > 3$.

Example 1 Let $N = \{1, 2, 3, 4\}$ be the set of players and $V_{i,j} = V$ for all $i \in N$, $j \in N$. More precisely, we suppose that $c_{1,3} = V - V/16$ and $c_{1,2} = c_{1,4} = 4V$; $c_{2,1} = 2V - V/16$ and $c_{2,3} = c_{2,4} = 4V$; $c_{3,2} = 2V - V/8$, $c_{3,4} = 2V - V/6$ and $c_{3,1} = 4V$; $c_{4,1} = 3V - V/8$ and $c_{4,2} = c_{4,3} = 4V$.

1. In a best response, player 2 never forms any link with player 3 or player 4. Moreover, player 2 has an incentive to form a link with player 1 if the latter gets resources from player 3 or player 4.

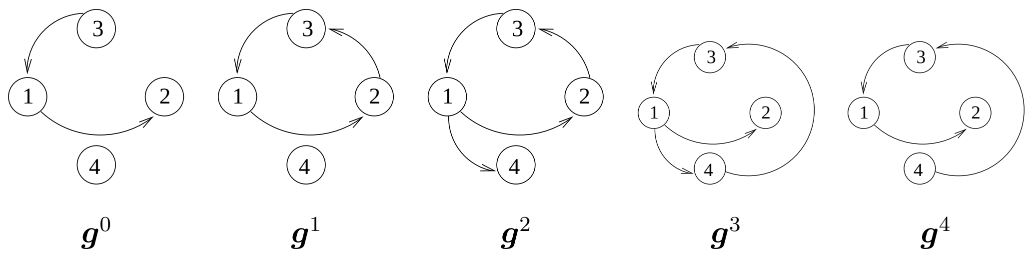
2. In a best response, player 4 never forms links with player 3 or player 2.

3. Then the unique best response of player 1 to any network in which she does not observe player 3 is to add a link with player 3 (since player 2 and player 4 never form a link with player 3). Moreover, we note that player 1 never has any incentive to form a link with player 2 or player 4.

4. In a best response, player 3 never forms any link with player 1.

Now let us take those best replies for granted and consider best responses regarding the remaining links $2, 1$; $3, 2$; $3, 4$ and $4, 1$. If player 2 initiates link $2, 1$ (see $\boldsymbol{g}^0$ in figure 1), then player 3's best response is to initiate link $3, 2$ (see $\boldsymbol{g}^1$). In that case player 4 must initiate the link $4, 1$ (see $\boldsymbol{g}^2$) and player 3 must replace the link $3, 2$ by the link $3, 4$ (see $\boldsymbol{g}^3$). Then, player 4 must delete the link $4, 1$ (see $\boldsymbol{g}^4$) and the player 3 must replace the link $3, 4$ by the link $3, 2$ (see $\boldsymbol{g}^1$). Hence there do not exist any mutual best responses. Therefore, a Nash network does not exist. Finally, by appropriately adjusting costs, it can be verified that this example holds even if we relax the assumption that $V_{i,j} = V$ for all $i \in N$, $j \in N$.

Figure 1: Best responses process of example 1



This example shows that existence results in one-way flow model with heterogeneity depends crucialy on the number of players. Indeed, the proof of existence of Nash networks with three players is based on the following fact. After a given player i has played a best response, the set of players whom she obtains resources always contains the set of players whom she obtained resources before.

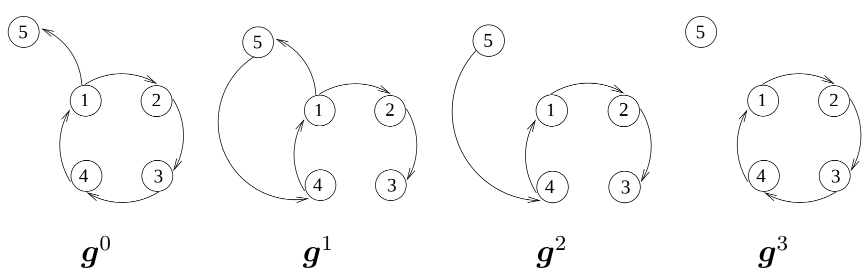
Our example stresses that this property does not hold anymore when $n > 3$. More precisely, in this example, player 3's best response leads him not to obtain resources from player 2 anymore in network $\boldsymbol{g}^4$.

3.1 Existence of Nash networks and heterogeneity of values by pairs

In this section, we present a proof of the existence of Nash network in the one-way flow model where values are heterogeneous by pairs and costs are heterogeneous by players. This proof can not be similar to the proof of Haller, Kamphorst and Sarangi (2006 [6]) who adress the Nash existence problem in the two-way flow model. Indeed, to prove that a Nash network always exists, the authors built a sequence of networks, beginning with the empty network. At each step of the sequence, a player who does not play a best response gets an opportunity to modify her links and play a best response (if no player has an incentive to modify her links, then the network is Nash). A distinctive feature of this process is that there can not exist a step in the sequence where a player has an incentive to modify her links and as a consequence to let another player get access to the resources of a smaller number of players than in the preceding step. Since the number of players is finite, there inevitably exists a step in the sequence where the corresponding network is a Nash network. In the one-way flow model, the above characteristic is no longer true. As a consequence, we cannot exclude the existence of cycles in the best response process. So we will not be able to conclude about the existence of Nash networks The following example is an illustration.

Example 2 Let $N = \{1, \ldots, 5\}$. Suppose that the best responses process lead to the network $\boldsymbol{g}^0$ in figure 2. Suppose now that player 4 is such that $V_{4,2} + V_{4,3} < V_{4,5}$. So, if player 4 revises her strategy, we obtain the network $\boldsymbol{g}^1$. We observe that player 5 does not hold resources anymore from 2 and 3 in $\boldsymbol{g}^1$. If $V_{5,1} + V_{5,4} < c$, then player 5 has an incentive to delete the link $5, 1$ and we obtain the network $\boldsymbol{g}^2$. If $V_{4,5} < c$, then player 4 has an incentive to delete the link $4, 5$ and to form the link $4, 3$. In that case, we obtain the network $\boldsymbol{g}^3$. Lastly in this network, player 5 has an incentive to form the link $5, 1$, and we obtain the network $\boldsymbol{g}^0$.

Figure 2: Best responses process of example 2



Our proof takes into account this problem of cycle, by modifying the network obtained when a player plays a best response in such a way that no player has any incentive to remove one of her links.

The profit function when values are heterogeneous by pairs and costs are heterogeneous by players is:

$$\pi_i(\boldsymbol{g}) = \sum_{j \in N_i(\boldsymbol{g})} V_{i,j} - c_i \sum_{j \in N} \boldsymbol{g}_{i,j}.$$

Let $\pi_i^j(\boldsymbol{g})$ be the marginal payoff of player i from player j in the network $\boldsymbol{g}$. If $\boldsymbol{g}_{i,j} = 1$, then $\pi_i^j(\boldsymbol{g}) = \pi_i(\boldsymbol{g}) - \pi_i(\boldsymbol{g} \ominus i,j)$. Let $\mathcal{K}(\boldsymbol{g}; i,j) = N_i(\boldsymbol{g} \ominus i,j) \bigcap N_i(\boldsymbol{g}_{-i} \oplus i,j)$, where $\boldsymbol{g} \ominus i,j$ denotes the network $\boldsymbol{g}$ without the link i,j. We can rewrite $\pi_i^j(\boldsymbol{g})$ as follows:

$$\pi_i^j(\boldsymbol{g}) = \sum_{k \in N_i(\boldsymbol{g}_{-i} \oplus i,j)} V_{i,k} - \sum_{k \in \mathcal{K}(\boldsymbol{g};i,j)} V_{i,k} - c_i. \tag{1}$$

Proposition 2 *If values of links are heterogeneous by pairs and costs of links are heterogeneous by players, then a Nash network exists.*

The proof of Proposition 2 is long, and involving a number of lemmas. So we first provide a quick overview of the proof. It consists of constructing a sequence of networks, $\mathcal{Q} = (\boldsymbol{g}^0, \ldots, \boldsymbol{g}^{t-1}, \boldsymbol{g}^t, \ldots)$ beginning with the empty network. In each subsequent network, no player should have an incentive to decrease the amount of resources she obtains. Note that this sequence of networks is not an improving path. Indeed, we go from $\boldsymbol{g}^t$ to $\boldsymbol{g}^{t+1}$ in several operations. First, in $\boldsymbol{g}^t$ we let a player $i \in N$, who is not playing a best response in $\boldsymbol{g}^t$, to play a best response (if no such player exists, $\boldsymbol{g}^t$ is a Nash network) and obtain a network called $\text{br}_i(\boldsymbol{g}^t)$. Second, we modify the network $\text{br}_i(\boldsymbol{g}^t)$ as follows: we construct a cycle using all players $j \in N$ who obtain resources from a player k who forms part of a cycle in $\text{br}_i(\boldsymbol{g}^t)$, while preserving all links in $\text{br}_i(\boldsymbol{g}^t)$ between a player $k \in N$ and a player j who is not part of a cycle in $\text{br}_i(\boldsymbol{g}^t)$. We obtain a network

called $h(\mathrm{br}_i(\boldsymbol{g}^t))$. Thirdly, we delete all links i, j which does not allow player i to obtain additional resources in $h(\mathrm{br}_i(\boldsymbol{g}^t))$. We obtain a network called $m(h(\mathrm{br}_i(\boldsymbol{g}^t))) = \overline{\boldsymbol{g}}_i^t$, and in the sequence $\mathcal{Q}$, we have $\boldsymbol{g}^{t+1} = \overline{\boldsymbol{g}}_i^t$.
When a player i receive an opportunity to revise her strategy, we go from a network $\boldsymbol{g}^{t-1}$ to a network $\boldsymbol{g}^t$, and we will show that $\eta(\boldsymbol{g}^{t-1}) < \eta(\boldsymbol{g}^t)$. Since the amount of resources that players can obtain in a network $\boldsymbol{g} \in \mathcal{Q}$ is finite, $\mathcal{Q}$ is finite and there exists a Nash network.

In the following paragraph, we define a class of networks $\mathcal{G}^3$ which contains all networks in the sequence $\mathcal{Q}$. Then, we provide a condition which implies that no player has an incentive to delete a link in a network $\boldsymbol{g} \in \mathcal{G}^3$ (Lemma 2). Finally, we show that all networks $\boldsymbol{g}^t \in \mathcal{Q}$ satisfy this condition since the empty network satisfies this condition (Lemma 6).

Let us formally define the set $\mathcal{G}^3$. Let $\mathcal{M} : \mathcal{G} \to \mathcal{P}(\mathcal{G})$, $\boldsymbol{g} \mapsto \mathcal{M}(\boldsymbol{g}) \subset \mathcal{G}$ be a correspondence. Let $m(\boldsymbol{g}) \in \mathcal{M}(\boldsymbol{g})$ be a minimal network associated to the network $\boldsymbol{g}$, $m(\boldsymbol{g})$ is a network such that, for all $i \in N$, $j \in N$, $N_i(\boldsymbol{g}) = N_i(m(\boldsymbol{g}))$ and if $m(\boldsymbol{g})_{i,j} = 1$, then $j \notin N_i(m(\boldsymbol{g}) \ominus i, j)$ and $\boldsymbol{g}_{i,j} = 1$. We note that in a network $m(\boldsymbol{g}) \in \mathcal{M}(\boldsymbol{g})$, there is at most one path from a player $i \in N$ to a player $j \in N$. In the following, we can take, without loss of generality, any element of $\mathcal{M}(\boldsymbol{g})$. Let $m(\boldsymbol{g})$ be a typical element of $\mathcal{M}(\boldsymbol{g})$. Obviously, we have $\eta(\boldsymbol{g}) = \eta(m(\boldsymbol{g}))$.

We say that $\boldsymbol{g}$ is a minimal network if $\boldsymbol{g} = m(\boldsymbol{g})$. We denote by $\mathcal{G}^m$ the set of minimal networks. Let $\mathcal{G}^1 = \{\boldsymbol{g} \in \mathcal{G}^m | i \in N_j(\boldsymbol{g}), j \notin N_i(\boldsymbol{g}), k \notin N_j(\boldsymbol{g}) \Rightarrow \boldsymbol{g}_{k,i} = 0\}$ be a subset of minimal networks. If $\boldsymbol{g} \in \mathcal{G}^2$ and $\boldsymbol{g}$ contains a cycle, then we denote by $C(\boldsymbol{g})$ the cycle in the network $\boldsymbol{g}$. We denote by $N^{C(\boldsymbol{g})}$ the set of players who belong to the cycle $C(\boldsymbol{g})$, and $E^{C(\boldsymbol{g})} \subset N^{C(\boldsymbol{g})} \times N^{C(\boldsymbol{g})}$ the set of links which belong to the cycle $C(\boldsymbol{g})$. Let $\mathcal{G}^3 = \{\boldsymbol{g} \in \mathcal{G}^2 | i \in C(\boldsymbol{g}), j \notin C(\boldsymbol{g}) \Rightarrow \boldsymbol{g}_{j,i} = 0\}$ be the set of networks which belong to $\mathcal{G}^2$ and where there does not exist any link from a player $i \in N^{C(\boldsymbol{g})}$ to a player $j \notin N^{C(\boldsymbol{g})}$.

We now present some lemmas which allow us to prove Proposition 2. The first lemma presents some properties about links that cannot arise in the set $\mathcal{G}^3$.

Lemma 1 *Suppose values of links are heterogeneous by pairs and costs of links are heterogeneous by players and $\boldsymbol{g} \in \mathcal{G}^3$.*

1. *If $\boldsymbol{g}_{j,i} = 1$, then there does not exist a player k such that $\boldsymbol{g}_{k,i} = 1$.*

2. *If $\boldsymbol{g}_{i,j} = 1$, then $\mathcal{K}(\boldsymbol{g}; i, j) = N_i(\boldsymbol{g} \ominus i, j) \bigcap N_i(\boldsymbol{g}_{-i} \oplus i, j)$ is an empty set.*

Proof We successively prove both parts of the lemma.

1. To obtain a contradiction suppose that there exist two players i and j such that $\boldsymbol{g}_{j,i} = 1$ and $\boldsymbol{g}_{k,i} = 1$ in $\boldsymbol{g} \in \mathcal{G}^3$. Then there are two possibilities:

   Suppose $i \in N^{C(\boldsymbol{g})}$. Given that $i \in N^{C(\boldsymbol{g})}$ there can be at most one link to player i. Hence it is not possible that $j \in N^{C(\boldsymbol{g})}$ and $k \in N^{C(\boldsymbol{g})}$ simultaneously. Only one of them is in $N^{C(\boldsymbol{g})}$. Without loss of generality let $j \in N^{C(\boldsymbol{g})}$. Then $\boldsymbol{g}_{k,i} = 1$ violates the fact that $\boldsymbol{g} \in \mathcal{G}^3$.

   Suppose $i \notin N^{C(\boldsymbol{g})}$. Then we know that $\boldsymbol{g}_{i,j} = 0 = \boldsymbol{g}_{i,k}$ otherwise $i \in N^{C(\boldsymbol{g})}$. From the minimality of $\boldsymbol{g}$ we know that $j \notin N_k(\boldsymbol{g})$ and $k \notin N_j(\boldsymbol{g})$. Putting all this together we have $i \in N_j(\boldsymbol{g})$, $j \notin N_k(\boldsymbol{g})$, $k \notin N_j(\boldsymbol{g}) \Rightarrow \boldsymbol{g}_{k,i} = 0$. This is a contradiction.

2. Suppose there exists a player $k \in N_i(\boldsymbol{g} \ominus i, j) \bigcap N_i(\boldsymbol{g}_{-i} \oplus i, j)$. Then, there exist two different paths from player k to player i which is impossible by the minimality of $\boldsymbol{g}$.

$\square$

It follows that if $\boldsymbol{g} \in \mathcal{G}^3$, then we can write $\pi_i^j(\boldsymbol{g})$ as follows:

$$\pi_i^j(\boldsymbol{g}) = \sum_{k \in N_i(\boldsymbol{g}_{-i} \oplus i, j)} V_{i,k} - c_i. \tag{2}$$

In the following lemma, we let $\boldsymbol{g}'_i \in \mathcal{G}_i$ be a strategy of player i, with $\boldsymbol{g}'_i \neq \boldsymbol{g}_i$. This lemma provides the best response properties of the networks $\boldsymbol{g} \in \mathcal{G}^3$.

Lemma 2 *Suppose values of links are heterogeneous by pairs, costs of links are heterogeneous by players and $\boldsymbol{g} \in \mathcal{G}^3$.*

1. *Suppose players $i \in N$, $j \in N$, $k \in N$ are such that $j \notin N_i(\boldsymbol{g})$, $i \in N_j(\boldsymbol{g})$, $k \notin N_j(\boldsymbol{g})$. If $\boldsymbol{g}'_{k,i} = 1$, then $\boldsymbol{g}'_k \notin \mathcal{BR}_k(\boldsymbol{g}_{-k})$.*

2. *Suppose $\boldsymbol{g}$ contains a cycle $C(\boldsymbol{g})$ and for all $i \in N^{C(\boldsymbol{g})}$, and for all $i, j \in E^{C(\boldsymbol{g})}$, we have $\pi_i^j(\boldsymbol{g}) > 0$. If $\boldsymbol{g}'_{i,j} = 0$, then $\boldsymbol{g}'_i \notin \mathcal{BR}_i(\boldsymbol{g}_{-i})$.*

3. *Suppose $i \in N$, $j \in N \setminus N^{C(\boldsymbol{g})}$ and $\boldsymbol{g}_{i,j} = 1 \Rightarrow \pi_i^j(\boldsymbol{g}) > 0$. If $\boldsymbol{g}'_{i,j} = 0$, then $\boldsymbol{g}'_i \notin \mathcal{BR}_i(\boldsymbol{g}_{-i})$.*

Proof We now prove each part of the lemma.

1. Let players i, j and k be such that $j \notin N_i(\boldsymbol{g})$, $i \in N_j(\boldsymbol{g})$ and $k \notin N_j(\boldsymbol{g})$. By lemma 1.1, we know that $\boldsymbol{g}_{k,i} = 0$. Either already $i \in N_k(\boldsymbol{g})$ and the formation of the link k, i is not a best response for player k, or $i \notin N_k(\boldsymbol{g})$. In the latter case, we have $j \notin N_k(\boldsymbol{g})$, $N_i(\boldsymbol{g}) \subset N_j(\boldsymbol{g})$, so $\pi_k(\boldsymbol{g} \oplus k, j) - \pi_k(\boldsymbol{g} \oplus k, i) \geq V_{k,j} > 0$. From this it

follows that player k does not play a best response if she forms a link with player i.

2. Without loss of generality, let $C(\boldsymbol{g})$ be such that $N^{C(\boldsymbol{g})} = \{1, 2, \ldots, p\}$ and $E^{C(\boldsymbol{g})} = \{p, 1; 2, 1; 3, 2; \ldots; 1, p\}$. For simplicity now consider a player $i \neq p$.

   It is straightforward from $\pi_i^{i-1}(\boldsymbol{g}) > 0$ and the minimality of $\boldsymbol{g}$ that player i does not play a best response if she deletes the link $i, i-1 \in E^{C(\boldsymbol{g})}$ and does not replace that link.

   We first show that player i cannot play a best response if she replaces the link $i, i-1$ by a link i, k, with $k \neq i-1$. Indeed, if player i replaces the link $i, i-1$ by a link i, k, $k \in N_i(\boldsymbol{g})$, then player i is not playing a best response.

   We now show that if player i replaces the link $i, i-1$ by a link i, k, $k \notin N_i(\boldsymbol{g})$, then player i does not play a best response. Indeed, since $\boldsymbol{g} \in \mathcal{G}^3$, there does not exist a player $k \notin N_i(\boldsymbol{g})$, with $k \in N \setminus N^{C(\boldsymbol{g})}$, such that $\ell \in N_k(\boldsymbol{g})$ and $\ell \in N^{C(\boldsymbol{g})}$. Otherwise, there exist a player $k' \in N \setminus N^{C(\boldsymbol{g})}$, with $k \in N_{k'}(\boldsymbol{g})$, and a player $\ell' \in N^{C(\boldsymbol{g})}$ such that $\boldsymbol{g}_{k',\ell'} = 1$. In that case, $\boldsymbol{g} \notin \mathcal{G}^3$ and we obtain a contradiction. Likewise, there does not exist a player $k \notin N_i(\boldsymbol{g})$ such that $\ell \in N_k(\boldsymbol{g})$ and $\ell \in N_i(\boldsymbol{g}) \setminus N^{C(\boldsymbol{g})}$. Indeed, if $\ell \in N_k(\boldsymbol{g})$ and $\ell \in N_i(\boldsymbol{g}) \setminus N^{C(\boldsymbol{g})}$, then there exists a player ℓ' such that $\boldsymbol{g}_{\ell',\ell} = 1$, with $\ell' \in N_i(\boldsymbol{g})$ and a player k' such that $\boldsymbol{g}_{k',\ell} = 1$, with $k' \in N_k(\boldsymbol{g})$ which is impossible by lemma 1.1. It follows that a player $i \in N^{C(\boldsymbol{g})}$ cannot obtain the resources of a player $\ell \in N_i(\boldsymbol{g}) \setminus N_i(\boldsymbol{g} \ominus i, i-1)$ from a player $k \notin N_i(\boldsymbol{g})$. Hence, if player i replaces the link $i, i-1 \in E^{C(\boldsymbol{g})}$ by a link i, k with $k \notin N_i(\boldsymbol{g})$, then player i does not play a best response.

3. It is straightforward from $\pi_i^j(\boldsymbol{g}) > 0$ and the minimality of $\boldsymbol{g}$ that player i has no incentive to delete the link i, j if she does not replace that link.

We now show that player i has no incentive to replace the link i, j. In other words, we show that there does not exist a player k who obtains a part of the resources of j and allows i to obtain more resources than j.

Let k be such that $N_k(\boldsymbol{g}) \cap N_j(\boldsymbol{g}) = \emptyset$. Then player i has no incentive to substitute the link i, k to the link i, j. Hence $N_k(\boldsymbol{g}) \cap N_j(\boldsymbol{g}) \neq \emptyset$.

First, we must show that if $N_k(\boldsymbol{g}) \cap N_j(\boldsymbol{g}) \neq \emptyset$, then either $N_k(\boldsymbol{g}) \subset N_j(\boldsymbol{g})$ or $N_j(\boldsymbol{g}) \subset N_k(\boldsymbol{g})$. If the former is true the proof is obvious and we will only focus on the latter. Note that in $\boldsymbol{g}$, $N_k(\boldsymbol{g}) \neq N_j(\boldsymbol{g})$ since $j \notin N^{C(\boldsymbol{g})}$. To obtain a contradiction, suppose that $N_k(\boldsymbol{g}) \cap N_j(\boldsymbol{g}) \neq \emptyset$, $N_k(\boldsymbol{g}) \nsubseteq N_j(\boldsymbol{g})$ and $N_j(\boldsymbol{g}) \nsubseteq N_k(\boldsymbol{g})$. Then there exist players $\ell \in N_j(\boldsymbol{g}) \cap N_k(\boldsymbol{g})$, $\ell_j \in N_j(\boldsymbol{g})$ and $\ell_k \in N_k(\boldsymbol{g})$, such that $\boldsymbol{g}_{\ell_j,\ell} = \boldsymbol{g}_{\ell_k,\ell} = 1$, which is impossible by Lemma 1.1.

Second, we must show that there does not exist a player $k \in N$, such that $N_j(\boldsymbol{g}) \subset N_k(\boldsymbol{g})$ and $N_i(\boldsymbol{g}) \nsubseteq N_k(\boldsymbol{g})$, who obtains the resources of j and allows i additional resources. If $N_i(\boldsymbol{g}) = N_k(\boldsymbol{g})$, then $i \in N^{C(\boldsymbol{g})}$, $k \in N^{C(\boldsymbol{g})}$ and in that case player i cannot obtain a part of the resources of player j due to a link with player k, since $\boldsymbol{g}$ is a minimal network. Therefore, we just need to show that the above statement is true for strict set inclusion. To obtain a contradiction, suppose there exists a player $k \in N$ such that $N_j(\boldsymbol{g}) \subset N_k(\boldsymbol{g})$ and $N_i(\boldsymbol{g}) \not\subset N_k(\boldsymbol{g})$. Then there exists a player $\ell_k \in N_k(\boldsymbol{g})$ such that $\boldsymbol{g}_{\ell_k,j} = 1$. Therefore, we have $\boldsymbol{g}_{\ell_k,j} = 1$ and $\boldsymbol{g}_{i,j} = 1$ which is impossible by Lemma 1.1. Since $N_j(\boldsymbol{g}) \subset N_k(\boldsymbol{g})$, $N_i(\boldsymbol{g}) \subset N_k(\boldsymbol{g})$, and $\boldsymbol{g}_{i,j} = 1$, by Lemma 1.2, player i cannot obtain a part of the resources of j due to her link with player k. Consequently, if player i deletes the link i, j and replaces it by the link i, k, then she does not play a best response.

$\square$

We now introduce some additional definitions that are required to complete the proof. Let $\mathcal{MBR}_i(\boldsymbol{g}_{-i})$ be a modified version of the best response function of player $i \in N$. More precisely, $\boldsymbol{g}'_i \in \mathcal{MBR}_i(\boldsymbol{g}_{-i})$ if $\boldsymbol{g}'_i$ is a best response of player i against $\boldsymbol{g}_{-i}$ and if player i does not form any links that yield zero marginal payoffs. Let $\mathrm{br}_i : \mathcal{G} \to \mathcal{G}$, $\boldsymbol{g} \mapsto \mathrm{br}_i(\boldsymbol{g})$ be a function. The network $\mathrm{br}_i(\boldsymbol{g}) = (\boldsymbol{g}'_i \oplus \boldsymbol{g}_{-i})$ is a network where $\boldsymbol{g}'_i \in \mathcal{MBR}_i(\boldsymbol{g}_{-i})$, and all other players $j \neq i$ having the same links as in the network $\boldsymbol{g}$. In other words, in $\mathrm{br}_i(\boldsymbol{g})$, we have $\mathrm{br}_i(\boldsymbol{g})_{i,j} = 1 \Rightarrow \pi_i^j(\mathrm{br}_i(\boldsymbol{g})) > 0$ and $\mathrm{br}_i(\boldsymbol{g})_{i,j} = 0 \Rightarrow \pi_i^j(\mathrm{br}_i(\boldsymbol{g})) \leq 0$.

Let $N^{\mathcal{C}(\boldsymbol{g})}$ be the set of players who belong to a cycle in $\boldsymbol{g}$. Let $\mathcal{H} : \mathcal{G} \to \mathcal{P}(\mathcal{G})$ be a correspondence. A network $h(\boldsymbol{g}) \in \mathcal{H}(\boldsymbol{g})$ is a network associated with $\boldsymbol{g}$ such that $h(\boldsymbol{g})$ contains at most one cycle, $C(h(\boldsymbol{g}))$. Moreover, if k is such that $\ell \in N_k(\boldsymbol{g})$ and $\ell \in N^{\mathcal{C}(\boldsymbol{g})}$, then $k \in N^{C(h(\boldsymbol{g}))}$. If $k \notin N^{C(h(\boldsymbol{g}))}$, then for all $\ell \in N$, we have $g_{\ell,k} = h(\boldsymbol{g})_{\ell,k}$. This is different from the networks in $\mathcal{G}^2$ since there is no minimality restriction here. This operation creates one cycle leaving unchanged the strategies of those players that do not form a part of the cycle.

Observe that for all $\boldsymbol{g} \in \mathcal{G}$ and for all $k \in N$, we have, by construction, for all $\boldsymbol{g}' \in \mathcal{M} \circ \mathcal{H}(\boldsymbol{g})$, $N_k(\boldsymbol{g}) \subseteq N_k(\boldsymbol{g}')$.

Finally, we define

$$\overline{\boldsymbol{g}}^i \in \mathcal{M} \circ \mathcal{H} \circ \mathrm{br}_i(\boldsymbol{g}), \tag{3}$$

to be a network obtained from $\boldsymbol{g}$ after performing the three operations defined above. Note that the superscript in $\overline{\boldsymbol{g}}^i$ refers to the fact that in this network player i is playing a best response.

Lemma 3 *If $\boldsymbol{g} \in \mathcal{G}^3$, then $\overline{\boldsymbol{g}}^i \in \mathcal{G}^3$.*

Proof We must show that $\overline{\boldsymbol{g}}^i$ has the following four properties: it is a minimal network, it contains at most one cycle, there does not exist a link from $j \notin N^{C(\overline{\boldsymbol{g}}^i)}$ to $k \in N^{C(\overline{\boldsymbol{g}}^i)}$

and if $\ell \in N_j(\overline{\boldsymbol{g}}^i), j \notin N_\ell(\overline{\boldsymbol{g}}^i), k \notin N_j(\overline{\boldsymbol{g}}^i)$ then $\overline{\boldsymbol{g}}^i_{k,\ell} = 0$. The first property follows from the correspondence $\mathcal{M}$ and the next two from the correspondence $\mathcal{H}$. We just need to verify that the last property is enjoyed.

First, we show that in $\mathrm{br}_i(\boldsymbol{g})$, we have $\ell \in N_j(\mathrm{br}_i(\boldsymbol{g}))$, $j \notin N_\ell(\mathrm{br}_i(\boldsymbol{g}))$, $i \notin N_j(\mathrm{br}_i(\boldsymbol{g}))$ $\Rightarrow \mathrm{br}_i\ (\boldsymbol{g})_{i,\ell} = 0$. We know that in $\boldsymbol{g}$ we have $\ell \in N_j(\boldsymbol{g}), j \notin N_\ell(\boldsymbol{g}), i \notin N_j(\boldsymbol{g}) \Rightarrow \boldsymbol{g}_{i,\ell} = 0$ since $\boldsymbol{g} \in \mathcal{G}^3$. By definition, we have $\mathrm{br}_i(\boldsymbol{g})_k = \boldsymbol{g}_k$, for all $k \in N \setminus \{i\}$. Hence, if we show that player $i \notin N_j(\mathrm{br}_i(\boldsymbol{g}))$ has not formed a link i, ℓ with a player ℓ such that $\ell \in N_j(\mathrm{br}_i(\boldsymbol{g}))$ and $j \notin N_\ell(\mathrm{br}_i(\boldsymbol{g}))$ in $\mathrm{br}_i(\boldsymbol{g})$, then we will have shown the conclusion for $\mathrm{br}_i(\boldsymbol{g})$. But, by Lemma 2.1, we know that if i has formed a link with player ℓ, then i is not playing a best response which is a contradiction.

Second, by construction, if $\boldsymbol{g}$ is such that $\ell \in N_j(\boldsymbol{g}), j \notin N_\ell(\boldsymbol{g}), k \notin N_j(\boldsymbol{g}) \Rightarrow \boldsymbol{g}_{k,\ell} = 0$, then $\boldsymbol{g}' \in \mathcal{M} \circ \mathcal{H}(\boldsymbol{g})$ is such that $\ell \in N_j(\boldsymbol{g}')$, $j \notin N_\ell(\boldsymbol{g}')$, $k \notin N_j(\boldsymbol{g}') \Rightarrow \boldsymbol{g}'_{k,\ell} = 0$. The conclusion follows. $\square$

The next lemma covers properties of networks in $\overline{\boldsymbol{g}}^i$ and $\mathrm{br}_i(\boldsymbol{g})$.

Lemma 4 *Suppose $\boldsymbol{g} \in \mathcal{G}^3$ and for all $k \in N$, $j \in N$, $\boldsymbol{g}_{k,j} = 1 \Rightarrow \pi_k^j(\boldsymbol{g}) > 0$.*

1. *If $k \in N_j(\boldsymbol{g})$, then $k \in N_j(\mathrm{br}_i(\boldsymbol{g}))$.*
2. *If $k \in N_j(\boldsymbol{g})$, then $k \in N_j(\overline{\boldsymbol{g}}^i)$.*
3. *If $\boldsymbol{g}_i \notin \mathcal{BR}_i(\boldsymbol{g}_{-i})$, then $\eta(\boldsymbol{g}) < \eta(\overline{\boldsymbol{g}}^i)$.*

Proof We successively prove each part of the Lemma.

1. Observe that for all $k \neq i$, and for all $j \in N$, we have $\boldsymbol{g}_{k,j} = \mathrm{br}_i(\boldsymbol{g})_{k,j}$. Hence, if $N_j(\boldsymbol{g}) \nsubseteq N_j(\mathrm{br}_i(\boldsymbol{g}))$, then there exists a player k such that $k \in N_i(\boldsymbol{g})$ and $k \notin N_i(\mathrm{br}_i(\boldsymbol{g}))$. Since $\boldsymbol{g} \in \mathcal{G}^3$, we know from Lemma 2.2 and 2.3, that player i will

not be playing a best response if she deletes one of her links. Hence, if $k \in N_i(\boldsymbol{g})$, then $k \in N_i(\mathrm{br}_i(\boldsymbol{g}))$, and we obtain the desired conclusion.

2. We know from the first part of the lemma that $N_j(\boldsymbol{g}) \subseteq N_j(\mathrm{br}_i(\boldsymbol{g}))$, and we know that $N_j(\mathrm{br}_i(\boldsymbol{g})) \subseteq N_j(\boldsymbol{g}')$, for all $\boldsymbol{g}' \in \mathcal{M} \circ \mathcal{H}(\mathrm{br}_i(\boldsymbol{g}))$. The result follows.

3. From the second part of the lemma, we know that $N_j(\boldsymbol{g}) \subseteq N_j(\overline{\boldsymbol{g}}^i)$ for all $j \neq i$. We now show that if $\boldsymbol{g}_i \notin \mathcal{BR}_i(\boldsymbol{g}_{-i})$, then $N_i(\boldsymbol{g}) \subset N_i(\overline{\boldsymbol{g}}^i)$. By Lemma 2.2 and 2.3, we know that player i cannot be playing a best response if she deletes links. Hence, if she is playing a best response, it must be that $N_i(\boldsymbol{g}) \subset N_i(\mathrm{br}_i(\boldsymbol{g}))$. Since, we know that, for all $\boldsymbol{g}' \in \mathcal{M} \circ \mathcal{H}(\mathrm{br}_i(\boldsymbol{g}))$, $N_i(\mathrm{br}_i(\boldsymbol{g})) \subseteq N_i(\boldsymbol{g}')$, we conclude that $N_i(\boldsymbol{g}) \subset N_i(\overline{\boldsymbol{g}}^i)$. Therefore, $\eta(\boldsymbol{g}) < \eta(\overline{\boldsymbol{g}}^i)$.

□

Let us denote by $\boldsymbol{g} \setminus \mathcal{MBR}_i(\boldsymbol{g}_{-i}) = \boldsymbol{gm}$. Then $\boldsymbol{gm} \oplus i, j$ is the network obtained from $\mathrm{br}_i(\boldsymbol{g})$ when player i forms no link except the link i, j.

Lemma 5 *Suppose* $\boldsymbol{g} \in \mathcal{G}^3$.

1. *If* $\overline{\boldsymbol{g}}^i_{i,j} = \mathrm{br}_i(\boldsymbol{g})_{i,j} = 1$, *then, for all* $j \in N \setminus \{i\}$, $N_j(\boldsymbol{gm} \oplus i, j) \subseteq N_j(\overline{\boldsymbol{g}}^i_{-i} \oplus i, j)$.

2. *Suppose for all* $i \in N$, $j \in N$, $\boldsymbol{g}_{i,j} = 1 \Rightarrow \pi_i^j(\boldsymbol{g}) > 0$. *If* $\overline{\boldsymbol{g}}^i_{k,\ell} = \boldsymbol{g}_{k,\ell} = 1$, *then* $N_\ell(\boldsymbol{g}_{-k} \oplus k, \ell) \subseteq N_\ell(\overline{\boldsymbol{g}}^i_{-k} \oplus k, \ell)$.

Proof We prove the two parts of the lemma successively.

1. If $j \notin N^{C(\overline{\boldsymbol{g}}^i)}$, then $N_j(\overline{\boldsymbol{g}}^i_{-i}) = N_j(\overline{\boldsymbol{g}}^i)$. Indeed, since $\overline{\boldsymbol{g}}^i \in \mathcal{G}^3$, $j \notin N^{C(\overline{\boldsymbol{g}}^i)}$, and $\overline{\boldsymbol{g}}^i_{i,j} = 1$, player j does not obtain any resources from player i. Moreover, we have by construction, $N_j(\mathrm{br}_i(\boldsymbol{g})) \subseteq N_j(\overline{\boldsymbol{g}}^i)$. It follows that $N_j(\boldsymbol{gm} \oplus i, j) \subseteq N_j(\mathrm{br}_i(\boldsymbol{g})) \subseteq N_j(\overline{\boldsymbol{g}}^i) = N_j(\overline{\boldsymbol{g}}^i_{-i}) \subseteq N_j(\overline{\boldsymbol{g}}^i_{-i} \oplus i, j)$.

Assume that $j \in N^{C(\overline{\boldsymbol{g}}^i)}$, $\overline{\boldsymbol{g}}^i_{i,j} = \text{br}_i(\boldsymbol{g})_{i,j} = 1$ and there exists a player ℓ such that $\ell \in N_j(\boldsymbol{gm} \oplus i, j)$ and $\ell \notin N_j(\overline{\boldsymbol{g}}^i_{-i} \oplus i, j)$. So in $\text{br}_i(\boldsymbol{g})$, player i obtains resources from player ℓ through a path containing j, and in $\overline{\boldsymbol{g}}^i$ player i obtains resources from player ℓ through a path which does not contain j, since for all $k \in N$, $N_k(\text{br}_i(\boldsymbol{g})) \subseteq N_k(\overline{\boldsymbol{g}}^i)$. Hence, there is a player j' where $j' \in N_i(\overline{\boldsymbol{g}}^i)$, $j' \notin N^{C(\overline{\boldsymbol{g}}^i)}$ and $j' \in N_j(\overline{\boldsymbol{g}}^i)$ who has formed a link with player ℓ between $\text{br}_i(\boldsymbol{g})$ and $\overline{\boldsymbol{g}}^i$. This is not possible by construction.

2. If $\ell \notin N^{C(\overline{\boldsymbol{g}}^i)}$, then $N_\ell(\overline{\boldsymbol{g}}^i_{-k} \oplus k, \ell) = N_\ell(\overline{\boldsymbol{g}}^i)$ since player ℓ does not obtain any resources from player k. Moreover, we know by Lemma 4.1 and 4.2 that $N_\ell(\boldsymbol{g}) \subseteq N_\ell(\overline{\boldsymbol{g}}^i)$. It follows that $N_\ell(\boldsymbol{g}_{-k} \oplus k, \ell) \subseteq N_\ell(\boldsymbol{g}) \subseteq N_\ell(\overline{\boldsymbol{g}}^i) = N_\ell(\overline{\boldsymbol{g}}^i_{-k} \oplus k, \ell)$.

   Suppose now that $\ell \in N^{C(\overline{\boldsymbol{g}}^i)}$. Note that $k \in N^{C(\overline{\boldsymbol{g}}^i)}$ since k has formed a link with ℓ. For a contradiction assume that $\ell \in N^{C(\overline{\boldsymbol{g}}^i)}$ and $N_\ell(\boldsymbol{g}_{-k} \oplus k, \ell) \not\subseteq N_\ell(\overline{\boldsymbol{g}}^i_{-k} \oplus k, \ell)$. Then there is a player j such that $j \in N_\ell(\boldsymbol{g}_{-k} \oplus k, \ell)$ and $j \notin N_\ell(\overline{\boldsymbol{g}}^i_{-k} \oplus k, \ell)$. Also note that $j \notin N^{C(\overline{\boldsymbol{g}}^i)}$, otherwise $j \in N_\ell(\overline{\boldsymbol{g}}^i_{-k} \oplus k, \ell)$. Moreover, if $j \in N_\ell(\boldsymbol{g}_{-k} \oplus k, \ell)$ and $j \notin N_\ell(\overline{\boldsymbol{g}}^i_{-k} \oplus k, \ell)$, then $j \notin N_k(\boldsymbol{g} \ominus k, \ell)$ and $j \in N_k(\overline{\boldsymbol{g}}^i \ominus k, \ell)$ since $\boldsymbol{g} \in \mathcal{G}^3$, and $N_\ell(\boldsymbol{g}) \subseteq N_\ell(\overline{\boldsymbol{g}}^i)$ by Lemma 4.1 and 4.2. In other words, player k obtains resources from player j in $\boldsymbol{g}$ through a path which contains ℓ, and in $\overline{\boldsymbol{g}}^i$ player k obtains resources from player j through a path which does not contain ℓ. Hence, there exists a player who has formed a link with a player ℓ' where $\ell' \in N_k(\overline{\boldsymbol{g}}^i)$, $j \in N_{\ell'}(\overline{\boldsymbol{g}}^i)$, and $k \notin N_{\ell'}(\overline{\boldsymbol{g}}^i)$ between $\boldsymbol{g}$ and $\overline{\boldsymbol{g}}^i$. This is not possible by construction of $\overline{\boldsymbol{g}}^i$.

□

Lemma 6 *Let $\overline{\boldsymbol{g}}^i$ be defined as in equation (3).*

1. *If* $\boldsymbol{g} \in \mathcal{G}^3$*, then* $\overline{\boldsymbol{g}}^i_{i,j} = 1 \Rightarrow \pi_i^j(\overline{\boldsymbol{g}}^i) > 0$.

2. *If for all* $i \in N$*,* $j \in N$*,* $\boldsymbol{g}_{i,j} = 1 \Rightarrow \pi_i^j(\boldsymbol{g}) > 0$*, then for all* $i \in N \setminus \{k\}$*,* $j \in N$*,* $\overline{\boldsymbol{g}}^k_{i,j} = 1 \Rightarrow \pi_i^j(\overline{\boldsymbol{g}}^k) > 0$.

Proof We now prove successively the two parts of the lemma.

1. (a) First, we show that this property is true if $\overline{\boldsymbol{g}}^i_{i,j} = 1$ and $j \notin N^{C(\overline{\boldsymbol{g}}^i)}$. If $j \notin N^{C(\overline{\boldsymbol{g}}^i)}$, then by construction $\mathrm{br}_i(\boldsymbol{g})_{i,j} = 1$ and so $\pi_i^j(\mathrm{br}_i(\boldsymbol{g})) > 0$. Using Lemma 5.1, Lemma 3, and the marginal profit function defined in equation (2) we have:

$$\begin{aligned} \pi_i^j(\overline{\boldsymbol{g}}^i) &= \textstyle\sum_{k \in N_j(\overline{\boldsymbol{g}}^i_{-i} \oplus i,j)} V_{i,k} - c_i \\ &\geq \textstyle\sum_{k \in N_j(\boldsymbol{gm} \oplus i,j)} V_{i,k} - \sum_{k \in \mathcal{K}(\mathrm{br}_i(\boldsymbol{g});i,j)} V_{i,k} - c_i \\ &= \pi_i^j(\mathrm{br}_i(\boldsymbol{g})) > 0 \end{aligned}$$

   (b) Second, we show that this property is true if $\overline{\boldsymbol{g}}^i_{i,j} = 1$ and $j \in N^{C(\overline{\boldsymbol{g}}^i)}$. By construction if $\overline{\boldsymbol{g}}^i_{i,j} = 1$ and $j \in N^{C(\overline{\boldsymbol{g}}^i)}$, then $i \in N^{C(\overline{\boldsymbol{g}}^i)}$. If $i \in N^{C(\overline{\boldsymbol{g}}^i)}$, then by construction of $\overline{\boldsymbol{g}}^i$, there is at least one player $\ell \in N^{C(\overline{\boldsymbol{g}}^i)}$, such that $\pi_i^\ell(\mathrm{br}_i(\boldsymbol{g})) > 0$. So for all players $\ell' \in N^{C(\overline{\boldsymbol{g}}^i)}$, there exists a network $(\overline{\boldsymbol{g}}^i)' \in \mathcal{M} \circ \mathcal{H} \circ \mathrm{br}_i(\boldsymbol{g})$ where player i forms a link with player ℓ', and by construction $\pi_i^j(\overline{\boldsymbol{g}}^i) = \pi_i^{\ell'}((\overline{\boldsymbol{g}}^i)')$. We know by Lemma 5.1, that $N_j(\boldsymbol{gm} \oplus i,j) \subseteq N_j(\overline{\boldsymbol{g}}^i_{-i} \oplus i,j)$. Finally, by Lemma 3, we know that $\overline{\boldsymbol{g}}^i \in \mathcal{G}^3$. Hence using the marginal profit function defined in equation (2) we have:

$$\begin{aligned}
\pi_i^j(\overline{\boldsymbol{g}}^i) &= \textstyle\sum_{k\in N_j(\overline{\boldsymbol{g}}^i_{-i}\oplus i,j)} V_{i,k} - c_i = \sum_{k\in N_\ell((\overline{\boldsymbol{g}}^i_{-i})'\oplus i,\ell)} V_{i,k} - c_i \\
&\geq \textstyle\sum_{k\in N_\ell(\boldsymbol{gm}\oplus i,\ell)} V_{i,k} - \sum_{k\in \mathcal{K}(\boldsymbol{gm}\oplus i,\ell;i,\ell)} V_{i,k} - c_i \\
&= \pi_i^\ell(\mathrm{br}_i(\boldsymbol{g})) > 0.
\end{aligned}$$

2. First, we show that for all $i \in N\setminus\{k\}$, and for all $j \notin N^{C(\overline{\boldsymbol{g}}^k)}$, if $\boldsymbol{g}_{i,j} = 1 \Rightarrow \pi_i^j(\boldsymbol{g}) > 0$, then $\overline{\boldsymbol{g}}^k_{i,j} = 1 \Rightarrow \pi_i^j(\overline{\boldsymbol{g}}^k) > 0$. Indeed, if player $i \in N\setminus\{k\}$ has a link with player $j \notin N^{C(\overline{\boldsymbol{g}}^k)}$ in $\overline{\boldsymbol{g}}^k$, then, by construction of $\overline{\boldsymbol{g}}^k$, player i has a link with player j in $\boldsymbol{g}$, so $\pi_i^j(\boldsymbol{g}) > 0$. We know, from Lemma 5.2, that for all $j \in N$, we have $N_j(\boldsymbol{g}_{-i}\oplus i,j) \subseteq N_j(\overline{\boldsymbol{g}}^k_{-i}\oplus i,j)$. Moreover, by Lemma 3, $\overline{\boldsymbol{g}}^k \in \mathcal{G}^3$. So using the marginal profit function defined in equation (2) we have:

$$\begin{aligned}
\pi_i^j(\overline{\boldsymbol{g}}^k) &= \textstyle\sum_{\ell\in N_j(\overline{\boldsymbol{g}}^k_{-i}\oplus i,j)} V_{i,\ell} - c_i \\
&\geq \textstyle\sum_{\ell\in N_j(\boldsymbol{g}_{-i}\oplus i,j)} V_{i,\ell} - c_i \\
&= \pi_i^j(\boldsymbol{g}) > 0.
\end{aligned}$$

Next, we show that for all $i \in N\setminus\{k\}$, and for all $j \in N^{C(\overline{\boldsymbol{g}}^k)}$, if $\boldsymbol{g}_{i,j} = 1 \Rightarrow \pi_i^j(\boldsymbol{g}) > 0$, then $\overline{\boldsymbol{g}}^k_{i,j} = 1 \Rightarrow \pi_i^j(\overline{\boldsymbol{g}}^k) > 0$. Since $\overline{\boldsymbol{g}}^k \in \mathcal{G}^3$ and there exists a link from player j to player i, we have $i \in N^{C(\overline{\boldsymbol{g}}^k)}$. If $i \in N^{C(\overline{\boldsymbol{g}}^k)}$, then there are two possibilities: either $k \in N_i(\mathrm{br}_k(\boldsymbol{g}))$ or $i \in N^{C(\boldsymbol{g})}$. We deal with these two possibilities successively.

(a) If $k \in N_i(\mathrm{br}_k(\boldsymbol{g}))$, then there exists in $\mathrm{br}_k(\boldsymbol{g})$ a link i,ℓ such that $\mathrm{br}_k(\boldsymbol{g})_{i,\ell} = \boldsymbol{g}_{i,\ell} = 1$ and $k \in N_\ell(\mathrm{br}_k(\boldsymbol{g}))$. Since, $\boldsymbol{g}_{i,\ell} = 1$, we have $\pi_i^\ell(\boldsymbol{g}) > 0$. Furthermore, by construction, player $\ell \in N^{C(\overline{\boldsymbol{g}}^k)}$, since $k \in N_\ell(\mathrm{br}_k(\boldsymbol{g}))$. We note that for all players $h' \in N^{C(\overline{\boldsymbol{g}}^k)}$, there exists a network $(\overline{\boldsymbol{g}}^k)' \in \mathcal{M} \circ \mathcal{H} \circ \mathrm{br}_k(\boldsymbol{g})$ where player i forms a link with player h', and by construction $\pi_i^j(\overline{\boldsymbol{g}}^k) = \pi_i^{h'}((\overline{\boldsymbol{g}}^k)')$. We know from Lemma 5.2 that for all $j \in N$, we have $N_j(\boldsymbol{g}_{-i} \oplus i, j) \subseteq N_j(\overline{\boldsymbol{g}}^k_{-i} \oplus i, j)$. Finally, we know by Lemma 3 that $\overline{\boldsymbol{g}}^i \in \mathcal{G}^3$. Hence, using the marginal profit function defined by equation (2), we obtain:

$$\begin{aligned}\pi_i^j(\overline{\boldsymbol{g}}^k) &= \textstyle\sum_{\ell' \in N_j(\overline{\boldsymbol{g}}^k_{-i} \oplus i, j)} V_{i,\ell'} - c_i = \sum_{\ell' \in N_\ell((\overline{\boldsymbol{g}}^k_{-i})' \oplus i, \ell)} V_{i,\ell'} - c_i \\ &\geq \textstyle\sum_{\ell' \in N_\ell(\boldsymbol{g}_{-i} \oplus i, \ell)} V_{i,\ell'} - c_i \\ &= \pi_i^\ell(\boldsymbol{g}) > 0.\end{aligned}$$

(b) If $i \in N^{C(\boldsymbol{g})}$, then we have $\pi_i^\ell(\boldsymbol{g}) > 0$ for $i, \ell \in E^{C(g)}$. We assume, without loss of generality, that player i forms in $C(\overline{\boldsymbol{g}}^i)$ a link with a player j such that $\pi_i^j(\mathrm{br}_i(\boldsymbol{g})) > 0$. By construction of $\overline{\boldsymbol{g}}^k$ we have $N^{C(\boldsymbol{g})} \subseteq N^{C(\overline{\boldsymbol{g}}^k)}$ and by Lemma 5.2, we have $N_j(\boldsymbol{g}_{-i} \oplus i, j) \subseteq N_j(\overline{\boldsymbol{g}}^k_{-i} \oplus i, j)$ for all $j \in N$. Note that for all players $h' \in N^{C(\overline{\boldsymbol{g}}^k)}$, there exists a network $(\overline{\boldsymbol{g}}^k)' \in \mathcal{M} \circ \mathcal{H} \circ \mathrm{br}_k(\boldsymbol{g})$ where player i forms a link with player h'. Also by construction $\pi_i^j(\overline{\boldsymbol{g}}^k) = \pi_i^{h'}((\overline{\boldsymbol{g}}^k)')$. We know by Lemma 3 that $\overline{\boldsymbol{g}}^i \in \mathcal{G}^3$. Again, using the marginal profit function defined by equation (2), we obtain:

$$\begin{aligned}\pi_i^j(\overline{\boldsymbol{g}}^k) &= \textstyle\sum_{\ell' \in N_j(\overline{\boldsymbol{g}}^k_{-i} \oplus i, j)} V_{i,\ell'} - c_i = \sum_{\ell' \in N_\ell((\overline{\boldsymbol{g}}^k_{-i})' \oplus i, \ell)} V_{i,\ell'} - c_i \\ &\geq \textstyle\sum_{\ell' \in N_\ell(\boldsymbol{g}_{-i} \oplus i, \ell)} V_{i,\ell'} - c_i \\ &= \pi_i^\ell(\boldsymbol{g}) > 0.\end{aligned}$$

$\square$

Proof of Proposition 2 We start with the empty network $\dot{\boldsymbol{g}} = \boldsymbol{g}^0$. It is straightforward to check that $\boldsymbol{g}^0 \in \mathcal{G}^3$. Either $\boldsymbol{g}^0$ is a Nash network, and we are done, or there exists a player, say i, who does not play a best response in $\boldsymbol{g}^0$. In that case, we construct the network $\boldsymbol{g}^1 \in \mathcal{M} \circ \mathcal{H} \circ \mathrm{br}_i(\boldsymbol{g}^0)$. We know from Lemma 4.3 that $\eta(\boldsymbol{g}^0) < \eta(\boldsymbol{g}^1)$. From Lemma 3, $\boldsymbol{g}^1 \in \mathcal{G}^3$ and from Lemma 6.1 and 6.2, we know that for all players $j \in N$ and $\ell \in N$, $\boldsymbol{g}^1_{j,\ell} = 1 \Rightarrow \pi^\ell_j(\boldsymbol{g}^1) > 0$. Either $\boldsymbol{g}^1$ is a Nash network, and we are done, or there exists a player, say j, who does not play a best response in $\boldsymbol{g}^1$. In that case, we construct the network $\boldsymbol{g}^2 \in \mathcal{M} \circ \mathcal{H} \circ \mathrm{br}_j(\boldsymbol{g}^1)$. We know from Lemma 4.3 that $\eta(\boldsymbol{g}^1) < \eta(\boldsymbol{g}^2)$. Again from Lemma 3, $\boldsymbol{g}^2 \in \mathcal{G}^3$ and from Lemma 6.1 and 6.2, we know that for all players $j \in N$ and $\ell \in N$, $\boldsymbol{g}^2_{j,\ell} = 1 \Rightarrow \pi^\ell_j(\boldsymbol{g}^2) > 0$. It follows that we can construct a sequence of networks $\{\boldsymbol{g}^0, \boldsymbol{g}^1 \ldots, \boldsymbol{g}^t, \ldots\}$ such that in $\boldsymbol{g}^{t-1}$, there exists a player, say k, who does not play a best response, and $\boldsymbol{g}^t \in \mathcal{M} \circ \mathcal{H} \circ \mathrm{br}_k(\boldsymbol{g}^{t-1})$, $\eta(\boldsymbol{g}^{t-1}) < \eta(\boldsymbol{g}^t)$, $\boldsymbol{g}^t \in \mathcal{G}^3$ and for all $j \in N$, $\boldsymbol{g}^t_{j,\ell} = 1 \Rightarrow \pi^\ell_j(\boldsymbol{g}^t) > 0$. This sequence is finite since $\eta(\boldsymbol{g}) \leq n^2$, for all $\boldsymbol{g} \in \mathcal{G}$.

$\square$

Proposition 2 establishes that if values of links are heterogeneous by pairs of players and costs of links are heterogeneous by players, then a Nash network always exists. This result is similar to the result of Haller and al. [6] in two-way flow models. We now study one-way flow models when values of links are heterogeneous by players and costs of links are heterogeneous by pairs of players.

3.2 Existence of Nash networks and heterogeneity of costs by pairs

In example 1 we have shown that a Nash network does not always exist when values of links are heterogeneous by players and costs of links are heterogeneous by pairs of players. We now state a condition which allows for the existence of Nash networks when costs of links are heterogeneous by pairs. In that case, we can write the payoff function as follows:

$$\pi_i(\boldsymbol{g}) = \sum_{j \in N_i(\boldsymbol{g})} V_i - \sum_{j \in N} g_{i,j} c_{i,j}.$$

Let $\pi_i^j(\boldsymbol{g})$ denote the marginal payoff of player i from player j in the network $\boldsymbol{g}$. If $\boldsymbol{g}_{i,j} = 1$, then $\pi_i^j(\boldsymbol{g}) = \pi_i(\boldsymbol{g}) - \pi_i(\boldsymbol{g} \ominus i, j)$. Let $\mathcal{K}(\boldsymbol{g}; i, j) = N_i(\boldsymbol{g} \ominus i, j) \bigcap N_i(\boldsymbol{g}_{-i} \oplus i, j)$. We can rewrite $\pi_i^j(\boldsymbol{g})$ as follows:

$$\pi_i^j(\boldsymbol{g}) = \sum_{k \in N_i(\boldsymbol{g}_{-i} \oplus i, j)} V_i - \sum_{k \in \mathcal{K}(\boldsymbol{g}; i, j)} V_i - c_{i,j}. \tag{4}$$

To prove the following proposition, we need an additional definition. First, we note that we cannot use our previous recomposition of the best response network. More precisely, the definition of $\mathcal{H}$ is not appropriate in the case of heterogeneous cost. Indeed, in the previous section, we could place the players in the cycle without restriction because there is no difference for player i to form a link with player j or player k since the costs are the same. However, this is not true in the case of heterogeneous costs.

So, let $\mathcal{H}_i : \mathcal{G} \to \mathcal{G}$ be a correspondence where $h_i(\boldsymbol{g}) \in \mathcal{H}_i(\boldsymbol{g})$ satisfies the following conditions.

- If $\boldsymbol{g}$ contains at most one cycle and there does not exist any link from a player $j \notin C(\boldsymbol{g})$ to a player $k \in C(\boldsymbol{g})$, then $\boldsymbol{g} = h_i(\boldsymbol{g})$.

- If player i has formed a link with no player $j \in N^{\mathcal{C}(\boldsymbol{g})}$ or with at least two players $j \in N^{\mathcal{C}(\boldsymbol{g})}$ in $\boldsymbol{g}$, then

  1. if k is such that $\ell \in N_k(\boldsymbol{g})$ and $\ell \in N^{\mathcal{C}(\boldsymbol{g})}$, then $k \in N^{C(h_i(\boldsymbol{g}))}$;
  2. if $k \notin N^{C(h_i(\boldsymbol{g}))}$, then for all $\ell \in N$, we have $g_{\ell,k} = h_i(\boldsymbol{g})_{\ell,k}$.

- If player i has formed a link with one and only one player $j \in N^{\mathcal{C}(\boldsymbol{g})}$ in $\boldsymbol{g}$, then:

  1. if k is such that $\ell \in N_k(\boldsymbol{g})$ and $\ell \in N^{\mathcal{C}(\boldsymbol{g})}$, then $k \in N^{C(h_i(\boldsymbol{g}))}$;
  2. if $k \notin N^{C(h_i(\boldsymbol{g}))}$, then for all $\ell \in N$, we have $g_{\ell,k} = h_i(\boldsymbol{g})_{\ell,k}$;
  3. player i and player j belong to $N^{C(h_i(\boldsymbol{g}))}$ and the link $i, j \in E(h_i(\boldsymbol{g}))$.

We now define $\hat{\boldsymbol{g}}^i$ as follows: $\hat{\boldsymbol{g}}^i \in \mathcal{M} \circ \mathcal{H}_i \circ br_i(\boldsymbol{g})$.

Proposition 3 *Consider a game where values of links are heterogeneous by players and costs of links are heterogeneous by pairs. There always exists a Nash network if for all* $i \in N$, $j \in N$, $j' \in N$: $|c_{i,j} - c_{i,j'}| < V_i$.

Proof The proof of this proposition is similar to the proof of the proposition 2 with $\hat{\boldsymbol{g}}^i$ playing the same role as $\overline{\boldsymbol{g}}^i$). □

Corollary 1 *Suppose a game where values and costs of links are heterogeneous by pairs. If for all* $i \in N$, $j \in N$, $j' \in N$: $|c_{i,j} - c_{i,j'}| < \min_{k \in N}\{V_{i,k}\}$, *then there is a Nash network.*

The importance of these results stems from the fact that they identify conditions under which Nash networks always exist under heterogeneity.

4 Model with Congestion Effect

In one-way flow models with homogeneous players BG [1] establish that Nash networks always exist. We show that this result is no longer true when the payoff function incorporates congestion effects – a phenomenon that frequently arises in many network settings. Billand and Bravard (2005 [2]) characterize Nash networks under congestion effects. In this section, we use their framework to show the non-existence of Nash networks.

Let us define $\phi : N \times \{0, \ldots, n-1\} \to I\!R$, $(x, y) \mapsto \phi_i(x, y)$ be such that:

$$\phi_i(x, y) > \phi_i(x, y+1).$$

Let $c_i(\boldsymbol{g}) = \sum_{j \neq i} \boldsymbol{g}_{i,j}$ be the costs incurred by i in the network $\boldsymbol{g}$. We now define the payoff function of player $i \in N$ as

$$\bar{\pi}_i(\boldsymbol{g}) = \phi_i(n_i(\boldsymbol{g}), c_i(\boldsymbol{g})).$$

As before we assume that player i obtains her own resources. We now provide an example where a Nash network does not exist.

Example 3 Let $N = \{1, 2, 3\}$, and $\phi_1(2, 1) > \phi_1(1, 0) > \phi_1(3, 1)$, $\max\{\phi_k\ (2, 1)\ , \phi_k(3, 2)\} < \phi_k(1, 0) < \phi_k(3, 1)$, for $k \in \{2, 3\}$.

First, networks in which a player forms two links are not Nash.

Second, the unique best response of player 2 (respectively 3) to any network $\boldsymbol{g}'$ in which player 1 and player 3 (respectively 2) have formed no link is to form no link. Moreover, the unique best response of player 1 to a network $\boldsymbol{g}$ in which player 2 and player 3 have formed no link is to form a link with player 2 or player 3. Therefore, the empty network is not a Nash network.

Third, a network $\boldsymbol{g}$ where $n_1(\boldsymbol{g}) \neq 2$ cannot be a Nash network. Indeed, it is obvious that $n_1(\boldsymbol{g}) = 3$ cannot be a Nash network since $\phi_1(1,0) > \phi_1(3,1) > \phi_1(3,2)$. Moreover, a network g where $n_1(\boldsymbol{g}) = 1$ cannot be a Nash network. Indeed, in a Nash network where player 1 has formed no links, players 2 and 3 cannot have established any links, since at least one of these players gets the ressources of one player only and we have $\phi_k(2,1) < \phi_k(1,0)$, for $k \in \{2,3\}$. In that case, when players 2 and 3 create no links, player 1 has an incentive to establish a link with player 2 or player 3. To sum up if there exists a Nash network $\boldsymbol{g}$, then $n_1(\boldsymbol{g}) = 2$.

Without loss of generality, we consider networks $\boldsymbol{g}$ in which player 1 has formed a link with player 2. In these networks,

1. player 2 has not formed a link with player 3 because in that case $2, 3 \in N_1(\boldsymbol{g})$ and player 1 would have an incentive to delete the link $1, 2$.
2. Player 3 has an incentive to establish a link with player 1, since $\phi_3(1,0) < \phi_3(3,1)$.
3. The networks in which a player has formed two links are not Nash networks.

Hence a Nash network does not exist.

The previous result remains true when players are homogeneous. But in that case, examples are more complicated because we need at least 7 players to show a Nash network does not always exist.

5 Discussion

Our different results lead to two questions. The first one is: Can the introduction of the decay assumption change the different results. Billand, Bravard and Sarangi (2006 [3]) show that there does not always exist a Nash network in a framework with homogeneous costs, heterogeneous values (by pairs) and decay. The second one is: How the results of the paper are sensitive to the assumption of linearity in values and costs. This question will be the subject of a future work.

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