Parameter Heterogeneity and Nonlinearities in the Aggregate Production Function: Investigating the Solow Growth Model with CES Technology

Winford H. Masanjala
Department of Economics
Louisiana State University
Baton Rouge, LA 70803
wmasanj@lsu.edu

Chris Papageorgiou*
Department of Economics
Louisiana State University
Baton Rouge, LA 70803
cpapa@lsu.edu

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Abstract

In this paper we examine whether nonlinearities in the aggregate production function can explain parameter heterogeneity in the Solow (1956) growth regressions. Nonlinearities in the production technology are introduced by replacing the linear Cobb-Douglas specification with the more general CES specification. We justify our choice of production function by showing that in the context of cross-country level regressions, we can reject the Cobb-Douglas over the CES aggregate production specification. Then, by using the endogenous threshold methodology of Hansen (2000) we show that the Solow model under CES implies robust nonlinearities in the growth process that are consistent with the presence of multiple regimes.

JEL Classification: O40, O47

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1 Introduction

Recent papers by Brock and Durlauf (2000) and Durlauf (2001) argue that the conventional Mankiw, Romer and Weil (1992) (MRW hereafter) cross-country linear regression model based on Solow (1956) imposes strong homogeneity assumptions on the growth process. Assuming *parameter homogeneity* in growth regressions is equivalent to assuming that all countries have an identical Cobb-Douglas (CD) aggregate production function. This is clearly an implausible assumption as there is nothing in the empirical or theoretical growth literature to suggest that the effect of a change in a particular variable (such as education or the savings rate) on economic growth is the same across countries. In the words of Brock and Durlauf "... the assumption of parameter homogeneity seems particularly inappropriate when one is studying complex heterogenous objects such as countries."

Not surprisingly, several empirical studies including Durlauf and Johnson (1995), Liu and Stengos (1999), Durlauf, Kourtellos and Minkin (2001), Kalaitzidakis et al. (2001) and Kourtellos (2001) find strong evidence is favor of *parameter heterogeneity* notwithstanding their different methodological approaches. Parameter heterogeneity in growth regressions has at least three possible interpretations: (a) Growth process nonlinearities: Multiple steady-state models such as Azariadis and Drazen (1990), Durlauf (1993) and Galor and Zeira (1993) suggest that parameters of a linear growth regression will not be constant across countries. Put differently, in a cross-country growth regression, countries are characterized by different coefficient estimates. (b) Omitted growth determinants: Recent models show that introduction of new variables in the standard Solow growth model may induce nonlinearities resulting in multiple equilibria and poverty traps (Durlauf and Quah (1999) enumerate a large number of such variables). (c) Nonlinearity of the production function: The identical CD aggregate production technology – a necessary condition for the linearity of the Solow growth model – assumed in the vast majority of existing studies maybe inappropriate.

This paper investigates interpretation (c) – whether nonlinearities in the aggregate production function can explain parameter heterogeneity in growth regressions. In particular, we replace the CD with the more general Constant Elasticity of Substitution (CES) aggregate production specification in the Solow growth model.[1] Our choice of the CES (nonlinear) specification is motivated,

[1]Although Solow (1957) was the first to suggest the use of the CD specification to characterize aggregate production, he also noted that there was little evidence to support the choice of such a specification. In fact, in his seminal 1956 paper, Solow presented the CES production function as one example of technologies for modeling sustainable

in part, by Duffy and Papageorgiou (2000) who find empirical support in favor of a more general CES specification of the aggregate input–output production relationship where the elasticity of substitution between capital and labor (or effective labor) is significantly greater than unity.[2] Our choice of production technology is also motivated by recent theoretical contributions, such as Ventura (1997), Klump and de La Grandville (2000), Azariadis (2001) and Azariadis and de la Croix (2001), which show that the elasticity of substitution between inputs may play an important role in the growth process.

In this paper, we first justify our choice of the production function by showing that in the context of MRW cross-country level regressions, we can reject the CD in favor of the more general CES aggregate production specification. This is an important result given that the CD is a necessary condition for the linearity of the Solow growth model. Then, by using the endogenous threshold methodology of Hansen (2000) we show that the Solow model with CES production technology implies robust non-linearities in the growth process that are consistent with parameter heterogeneity and the existence of multiple regimes. This last result suggests that using the CES aggregate production function (which is found to be empirically favorable to CD) in growth regressions does not explain away (and if anything amplifies) heterogeneity across countries, therefore shifting attention to the other two alternative interpretations mentioned above.

The rest of the paper is organized as follows. Section 2 derives the regression equations from the Solow model under CD and CES production technologies. Section 3 presents and discusses the results obtained from estimating these regressions. Section 4 employs the Hansen (2000) endogenous threshold methodology to examine the possibility of multiple regimes. Section 5 summarizes and concludes.

2 Solow Growth Model with CES Production Technology

We start by revisiting the Solow growth model with CD specification. We then replace the CD with the more general CES technology and derive the regression equations which will be estimated later on.

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economic growth.

[2]Duffy and Papageorgiou (2000) employ panel estimation techniques and aggregate data on a panel of 82 countries over 28 years to estimate a CES aggregate production function specification.

2.1 The Basic and Extended Solow-CD Models

MRW start their cross-country empirical investigation by using the basic Solow growth model where aggregate output in country i (Y_i) is determined by a CD production function, taking as arguments the stock of physical capital (K_i) and technology-augmented labor (AL_i), according to

$$Y_i = K_i^{\alpha}(AL_i)^{1-\alpha},$$

where $\alpha \in (0,1)$ is the share of capital, and A and L grow exogenously at rates g and n, respectively. Each country accumulates physical capital according to the motion equation $dK_i/dt = s_{ik}Y_i - \delta K_i$, where s_{ik} is the savings rate and δ is the depreciation rate of capital. After solving for the steady-state output per unit of augmented labor (y_i), log-linearizing and imposing the cross-coefficient restrictions on α, they obtain the *basic Solow-CD equation*

$$\ln\left(\frac{Y_i}{L_i}\right) = \ln A(0) + gt + \frac{\alpha}{1-\alpha}\ln\left(\frac{s_{ik}}{n_i+g+\delta}\right). \tag{1}$$

MRW's implied estimate of the capital share α was implausibly high relative to the capital share in national income thus motivating these authors to extend their basic model by introducing human capital (H_i) as an additional factor of production. Output in the extended model is determined by a CD production function of the form

$$Y_i = K_i^{\alpha}H_i^{\beta}(AL_i)^{1-\alpha-\beta},$$

where $\alpha \in (0,1)$ is the share of physical capital and $\beta \in (0,1)$ is the share of human capital. Physical and human capital accumulation equations take the form $dK_i/dt = s_{ik}Y_i - \delta K_i$, and $dH_i/dt = s_{ih}Y_i - \delta H_i$ respectively, where s_{ik} is the fraction of income invested in physical capital, s_{ih} is the fraction invested in human capital and δ is a common depreciation rate. Once again, solving for the steady-state output per unit of augmented labor, log-linearizing and imposing the cross-coefficient restrictions on α and β they obtain the *extended Solow-CD equation*[3]

$$\ln\left(\frac{Y_i}{L_i}\right) = \ln A(0) + gt + \frac{\alpha}{1-\alpha-\beta}\ln\left(\frac{s_{ik}}{n_i+g+\delta}\right) + \frac{\beta}{1-\alpha-\beta}\ln\left(\frac{s_{ih}}{n_i+g+\delta}\right). \tag{2}$$

[3] The cross-coefficient restrictions require that the coefficient on $\ln(n_i+g+\delta)$ is equal in magnitude and opposite in sign to the coefficient on $\ln s_{ik}$ in the basic Solow regressions (equal in magnitude and opposite in sign to the sum of the coefficients on $\ln s_{ik}$ and $\ln s_{ih}$ in the extended Solow regressions).

2.2 The Basic and Extended Solow-CES Models

Next, we replace the CD with the more general CES aggregate production specification in the Solow growth model. The production function becomes

$$Y_i = \left[\alpha K_i^{\frac{\sigma-1}{\sigma}} + (1-\alpha)(AL_i)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}},$$

where $\alpha \in (0,1)$ is now what Arrow et al. (1961) called the "distribution parameter" (rather than the share) of physical capital, and $\sigma \geq 0$ is the elasticity of substitution between capital and technology-augmented labor. It is well-known that when $\sigma = 1$ the CES production function reduces to the CD case. Assuming that the evolution of capital is governed by the same motion equation as in MRW, we derive the steady-state output per augmented labor as

$$y_i^* = \left[\frac{1}{1-\alpha} - \frac{\alpha}{1-\alpha}\left(\frac{s_{ik}}{n_i+g+\delta}\right)^{\frac{\sigma-1}{\sigma}}\right]^{-\frac{\sigma}{\sigma-1}}. \tag{3}$$

Taking logs and linearizing using a second order Taylor series expansion around $\sigma = 1$, as in Kmenta (1967), we obtain the *basic Solow-CES equation*[4]

$$\ln\left(\frac{Y_i}{L_i}\right) = \ln A(0) + gt + \frac{\alpha}{1-\alpha}\ln\left(\frac{s_{ik}}{n_i+g+\delta}\right) + \frac{1}{2}\frac{\sigma-1}{\sigma}\frac{\alpha}{(1-\alpha)^2}\left[\ln\left(\frac{s_{ik}}{n_i+g+\delta}\right)\right]^2. \tag{4}$$

There are several points worth making here. The second order linear approximation of the CES function given by equation (4) consists of two additively separable terms: The linear term $\ln A(0) + gt + \frac{\alpha}{1-\alpha}\ln\left(\frac{s_{ik}}{n_i+g+\delta}\right)$ is the first order linear approximation of the CES function that corresponds to the CD function, and the quadratic term $\frac{1}{2}\frac{\sigma-1}{\sigma}\frac{\alpha}{(1-\alpha)^2}\left[\ln\left(\frac{s_{ik}}{n_i+g+\delta}\right)\right]^2$ corresponds to a correction due to the departure of σ from unity. Our linear approximation, around $\sigma = 1$, of the CES production technology provides the CD specification with its *best opportunity* to characterize the cross-country output per worker relationship. Notice that if $\sigma = 1$ (i.e. the CD case) then the last term vanishes so that equation (4) is reduced to the *basic Solow-CD equation* (1). More importantly, notice that if σ is significantly different from unity it implies that the basic Solow-CD linear equation is mispecified. The potential specification error is associated with the choice of production function and is captured by the quadratic term of equation (4). The magnitude of the specification error depends on the extent to which σ departs from unity.

[4]See Appendix B for derivation of equations (3-4).

Next, we incorporate human capital in the CES aggregate production function as follows:

$$Y_i = \left[\alpha K_i^{\frac{\sigma-1}{\sigma}} + \beta H_i^{\frac{\sigma-1}{\sigma}} + (1-\alpha-\beta)(AL_i)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}},$$

where α and β are distribution parameters, H is the stock of human capital and σ is the elasticity of substitution between any two factors of production ($\sigma = \sigma_{j,k}$ for $j \neq k$, where $j, k = K, H, AL$).[5] Assuming the same motion equations for physical and human capital as in the *extended Solow-CD model*, we derive the steady state output per augmented labor as

$$y_i^* = \left[\frac{1}{1-\alpha-\beta} - \frac{\alpha}{1-\alpha-\beta}\left(\frac{s_{ik}}{n_i+g+\delta}\right)^{\frac{\sigma-1}{\sigma}} - \frac{\beta}{1-\alpha-\beta}\left(\frac{s_{ih}}{n_i+g+\delta}\right)^{\frac{\sigma-1}{\sigma}}\right]^{-\frac{\sigma}{\sigma-1}}. \quad (5)$$

A second order linearization of equation (5) around $\sigma = 1$ yields the *extended Solow-CES equation*[6]

$$\begin{aligned} \ln\left(\frac{Y_i}{L_i}\right) &= \ln A(0) + gt + \tfrac{\alpha}{1-\alpha-\beta}\ln\left(\tfrac{s_{ik}}{n_i+g+\delta}\right) + \tfrac{\beta}{1-\alpha-\beta}\ln\left(\tfrac{s_{ih}}{n_i+g+\delta}\right) + \\ & \tfrac{1}{2}\tfrac{\sigma-1}{\sigma}\tfrac{1}{(1-\alpha-\beta)^2}\left\{\alpha\left[\ln\left(\tfrac{s_{ik}}{n_i+g+\delta}\right)\right]^2 + \beta\left[\ln\left(\tfrac{s_{ih}}{n_i+g+\delta}\right)\right]^2 - \alpha\beta\left(\ln\tfrac{s_{ik}}{s_{ih}}\right)^2\right\}. \quad (6) \end{aligned}$$

One can easily verify that by eliminating human capital accumulation ($\beta = 0$), equation (6) reduces to the *basic Solow-CES equation* (4). It is also easy to verify that in the special case of unitary elasticity of substitution ($\sigma = 1$), equation (6) reduces to the *extended Solow-CD equation* (2).

To establish the specification of the production function that is consistent with the data we use two test. First, we investigate whether the coefficients associated with the quadratic terms are statistically significant and then we test if the elasticity of substitution σ is statistically different from unity.

3 Data, Estimation and Results

The baseline dataset employed in our estimation is identical to that of MRW (PWT version 4.0), and our discussion focuses on the non-oil sample which includes 98 countries. The variables used in our baseline estimation are: per capita output in 1985 ($\frac{Y_i}{L_i}$), the ratio of average investment to GDP

[5] In the three-factor case there is no "traditional" definition of the elasticity of substitution. Here we use the *Allen Partial Elasticity of Substitution* (APES) (see Allen 1938, pp.503-509) which asserts that if the production function is of the form $f(x_1, ..., x_n) = \left[a_1 x_1^{\frac{\sigma-1}{\sigma}}, ..., a_n x_n^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$ then $\sigma = \sigma_{j,k}$ for all $j \neq k$, where $j, k = 1, ..., n$. For an extensive discussion on the properties of APES see Uzawa (1962).

[6] See Appendix B for the derivation of equations (5-6).

over the 1960-1985 period (s_{ik}), the average percentage of working age population (population between the age of 15 and 64) in secondary education over the period 1960-1985 (s_{ih}), and the average working age population growth rate from 1960-1985 (n_i). Following MRW we assume that $g + \delta = 0.05$. As a robustness check of our baseline results we will also use the updated PWT version 6.0 which extends the coverage to 1995 for a subsample of 90 countries.[7,8]

Our estimation considers linear and nonlinear least-squares regressions to obtain parameter estimates for the *basic* and *extended Solow models*. Tables 1-2 present estimated coefficients for each of the four regression equations (1), (2), (4) and (6). The upper panels of Tables 1-2 present results from the "unrestricted" models (without cross-coefficient restrictions) while the lower panels present the implied coefficient estimates for α, β and σ from the "restricted" models (with cross-coefficient restrictions).

3.1 Basic Solow Regression Results

Table 1 presents estimates for the *basic* and *extended Solow-CD* and *-CES models* using the PWT 4.0 dataset. Columns 2 and 4 replicate the MRW results for the *basic* and *extended Solow-CD models* whereas columns 3 and 5 extend these results to the CES models.

First, we compare the regression results of the *basic Solow-CD* and *-CES models* (reported in columns 2 and 3 of Table 1). In terms of the overall fit, we find that the CD model can explain 59% whereas the CES model can explain 60% of the overall variation in per capita income. Replacing the CD with the more general CES specification does not affect the predicted signs of the coefficients, but it reduces their magnitude and significance.

In the unrestricted version of the Solow model (upper panel of Table 1, columns 2 and 3), the coefficient estimate on $\ln s_{ik}$ decreases from 1.4240 to 1.0024 remaining very significant and the coefficient estimate on $\ln(n_i + g + \delta)$ increases from -1.9898 to -1.0991 but becomes highly insignificant. In the unrestricted *basic Solow-CES model*, the quadratic term $\left[\ln\left(\frac{s_{ik}}{n_i+g+\delta}\right)\right]^2$ has a significant point estimate of 0.3345 providing evidence in favor of a two-factor CES specification over the commonly used CD specification.

Estimates from the restricted model (lower panel of Table 1, columns 2 and 3) show that

[7] For detailed explanation of the data see Bernanke and Gürkaynak (2001, pp.8-9). The data are available on-line at http://www.princeton.edu/~gurkaynk/growthdata.html.

[8] The countries with missing observations in PWT version 6.0 are Burma, Chad, Germany, Haiti, Liberia, Sierra Leone, Somalia, and Sudan.

Table 1: Cross-Country level regressions with CD and CES technologies using PWT 4.0

Specification	Basic Solow (PWT 4.0)		Extended Solow (PWT 4.0)	
	CD (Eq.1)	CES (Eq.4)	CD (Eq.2)	CES (Eq.6)
Unrestricted				
Constant	8.0353*** (1.2377)	7.1333*** (1.5056)	8.6592*** (0.8071)	6.3207*** (0.8965)
$\ln s_{ik}$	1.4240*** (0.1299)	1.0024*** (0.2088)	0.6967*** (0.1454)	1.1712** (0.5164)
$\ln(n_i + g + \delta)$	−1.9898*** (0.5368)	−1.0991 (0.8290)	−1.7452*** (0.3369)	−1.0581** (0.4887)
$\ln s_h$	—	—	0.6545*** (0.0726)	0.4814 (0.3054)
$[\ln s_{ik} - \ln(n_i + g + \delta)]^2$	—	0.3345* (0.1774)	—	0.1113 (0.1606)
$[\ln s_{ih} - \ln(n_i + g + \delta)]^2$	—	—	—	0.2586*** (0.0736)
$[\ln s_{ik} - \ln s_{ih}]^2$	—	—	—	−0.2116*** (0.0973)
s.e.e.	0.69	0.68	0.51	0.47
Adj. R^2	0.59	0.60	0.78	0.81
Obs.	98	98	98	98
Restricted				
Constant	6.8724*** (0.1027)	6.9370*** (0.0890)	7.8531*** (0.1572)	7.8749*** (0.1376)
Implied α	0.5981*** (0.0170)	0.4984*** (0.0499)	0.3082*** (0.0465)	0.2395*** (0.0406)
Implied β	—	—	0.2743*** (0.0356)	0.3582*** (0.0431)
Implied σ	1	1.5425 (0.5574)	1	1.1894††† (0.0449)
s.e.e.	0.69	0.68	0.51	—
Adj. R^2	0.59	0.60	0.78	—
Obs.	98	98	98	98

Notes: It is assumed that $g+\delta = 0.05$ as in MRW. α and β are shares of physical and human capital respectively in the CD models (distribution parameters in the CES models). All regressions are estimated using OLS with the exception of the restricted version of the *extended Solow-CES model* which was estimated using NLLS. Standard errors are given in parentheses. The standard errors for α and β were recovered using standard approximation methods for testing nonlinear functions of parameters. White's heteroskedasticity correction was used. *** (†††) Significantly different from 0 (1) at the 1% level. ** (††) Significantly different from 0 (1) at the 5% level. * (†) Significantly different from 0 (1) at the 10% level.

employing the CES specification lowers the value of α from 0.5981, to 0.4984. We also find that the implied elasticity of substitution is greater than unity ($\sigma = 1.5425$) but is statistically significant only at the 13% level.

Recall, that whereas in the CD specification α is the share of capital in output, in the CES specification it is a distribution parameter. The physical capital share of country i in the two-factor CES production function is given by $shr(K_i) = \frac{\alpha k_i^{\frac{\sigma-1}{\sigma}}}{\alpha k_i^{\frac{\sigma-1}{\sigma}} + (1-\alpha)}$, where $\frac{\partial shr(K_i)}{\partial k_i} > 0$ and $\frac{\partial shr(K_i)}{\partial \sigma} > 0$. It is possible to calculate steady-state capital shares ($shr(K_i^*)$) by using our estimated coefficients for $\alpha = 0.4984$ and $\sigma = 1.5425$, and by obtaining each country's steady-state per capita capital implied by the *basic Solow-CES model*

$$k_i^* = \left[\frac{(1-\alpha)}{\left(\frac{n_i+g+\delta}{s_{ik}}\right)^{\frac{\sigma-1}{\sigma}} - \alpha} \right]^{\frac{\sigma}{\sigma-1}}, \tag{7}$$

where n_i is population growth rate and s_{ik} is savings rate in country i.[9] As expected, we find that shares increase with per capita physical capital. More importantly, we further find that the implied physical capital shares vary considerably ranging from 0.3923 in Uganda to 0.9613 in Finland.

3.2 Extended Solow Regression Results

Columns 4 and 5 of Table 1 report results from the *extended Solow-CD* and *extended Solow-CES* regressions, respectively. All of the regressions are estimated by ordinary least squares (OLS) with the exception of the restricted version of the highly nonlinear *extended Solow-CES equation* (6) which was estimated by nonlinear least squares (NLLS).

In terms of overall fit, we find that the unrestricted and restricted Solow-CES models are slight improvements over the corresponding Solow-CD models. Coefficient estimates obtained from both the restricted and unrestricted versions of the *extended Solow-CES specification* are considerably different from those obtained under the *extended Solow-CD specification*.

In the unrestricted model (upper panel of Table 1, columns 4 and 5), the estimated coefficient for physical capital increases substantially in magnitude from 0.6967 to 1.1712 but decreases in significance level from 1% to 5%, whereas the coefficient on human capital decreases from 0.6545 to 0.4814 and becomes insignificant. Notice that two out of the three quadratic terms due to the CES

[9] Derivation of equation (7) is shown in Appendix B. Physical (and human) capital shares for all 98 countries obtained from the *basic* (and *extended*) *Solow-CES models* are reported in Table A3 in Appendix A.

specification are significant. In particular, the estimated coefficient for the quadratic human capital term $\left[\ln\left(\frac{s_{ih}}{n_i+g+\delta}\right)\right]^2$ is highly significant as is the coefficient for the quadratic term $\left[\ln\left(\frac{s_{ik}}{s_{ih}}\right)\right]^2$, whereas the quadratic physical capital term $\left[\ln\left(\frac{s_{ik}}{n_i+g+\delta}\right)\right]^2$ is insignificant. In the restricted model, the physical capital distribution parameter α equals 0.2395 whereas the human capital distribution parameter β equals 0.3582 and both are significant at the 1% level. Most importantly, the elasticity of substitution parameter, σ, equals 1.1894 and it is statistically different from unity at the 1% level.[10]

Once again, recall that under CES technology, α and β are not shares but distributions parameters. Physical capital share is now given by $shr(K_i) = \frac{\alpha k_i^{\frac{\sigma-1}{\sigma}}}{\alpha k_i^{\frac{\sigma-1}{\sigma}}+\beta h_i^{\frac{\sigma-1}{\sigma}}+(1-\alpha-\beta)}$ and human capital share by $shr(H_i) = \frac{\beta h_i^{\frac{\sigma-1}{\sigma}}}{\alpha k_i^{\frac{\sigma-1}{\sigma}}+\beta h_i^{\frac{\sigma-1}{\sigma}}+(1-\alpha-\beta)}$. We calculate steady-state physical and human capital shares ($shr(K_i^*)$, $shr(H_i^*)$) by using our estimated coefficients for $\alpha = 0.2395$, $\beta = 0.3582$ and $\sigma = 1.1894$, and by obtaining each country's steady-state per capita physical and human capital values implied by the *extended Solow-CES model*[11]

$$k_i^* = \left[\frac{1-\alpha-\beta}{\left(\frac{n_i+g+\delta}{s_{ik}}\right)^{\frac{\sigma-1}{\sigma}} - \beta\left(\frac{s_{ih}}{s_{ik}}\right)^{\frac{\sigma-1}{\sigma}} - \alpha}\right]^{\frac{\sigma}{\sigma-1}} \tag{8}$$

$$h_i^* = \left[\frac{1-\alpha-\beta}{\left(\frac{n_i+g+\delta}{s_{ih}}\right)^{\frac{\sigma-1}{\sigma}} - \alpha\left(\frac{s_{ik}}{s_{ih}}\right)^{\frac{\sigma-1}{\sigma}} - \beta}\right]^{\frac{\sigma}{\sigma-1}}. \tag{9}$$

This exercise reveals that there still exists considerable heterogeneity among the estimated physical and human capital shares across countries, but it is lower than that found in the *basic Solow-CES model*. In particular, we find that the implied physical capital shares range from 0.2283 in Ethiopia

[10] We have also estimated the restricted version of the *extended Solow-CES equation* (6) by employing a two-stage conditional estimation procedure. First, we estimated equation (6) using OLS and then recovered the implied values of the distribution parameters for physical capital (α) and human capital (β). We then re-estimated equation (6) conditional on the implied values of α and β in order to recover the implied elasticity of substitution parameter σ. The coefficient estimates from the two-stage conditional estimation are as follows:

Constant	Implied α	Implied β	Implied σ	Adj. R^2
7.5359*** (0.3252)	0.4452*** (0.1582)	0.1751 (0.1277)	1.1923††† (0.0611)	0.81

The notation in Table 1 applies to the above panel. These estimates are consistent with the NLLS estimation. In particular, the implied value of σ is slightly higher than in the NLLS estimation and significantly different from unity. Although the estimators from the two-stage conditional estimation are consistent, they are not efficient because equation (6) is over-identified.

[11] Derivation of equations (8-9) is shown in Appendix B.

to 0.3169 in Japan, whereas implied human capital shares range from 0.2232 in Rwanda to 0.4006 in Finland.[12,13]

In summary, the values of σ in both the *basic* and *extended Solow-CES models* suggest that σ is greater than unity. In the basic model, although $\sigma(= 1.5425)$ is significant at the 13% level, the coefficient associated with the quadratic term is significant. This is confirmed in the extended model where although $\sigma = 1.1894$ is lower, it is statistically different from unity at conventional levels of significance.

3.3 Robustness analysis of the results

In this section we examine the robustness of our results to the updated PWT 6.0 dataset which has recently been used in Bernanke and Gürkaynak (2001). This preliminary version of PWT extends the coverage of the data for another decade from $1960 - 1995$ for 90 out of the 98 countries in the original sample.

The results from this exercise are presented in Table 2. Columns 2 and 4 replicate the results in Bernanke and Gürkaynak for the *basic* and *extended Solow-CD models*. Qualitatively, these results are similar to those of MRW in Table 1. A noticeable difference is that using the 1960-1995 sample period increases the fit of the models (Adj. R^2 increases approximately 10% in each model). Column 3 presents results for the *basic Solow-CES model*. In general, there is stronger evidence in favor of the CES specification. For instance, in the unrestricted version of the model (upper panel of Table 2), the main difference from the baseline results is that although the quadratic term $\left[\ln\left(\frac{s_{ik}}{n_i+g+\delta}\right)\right]^2$ decreases in magnitude from 0.3345 to 0.1786, it increases in significance from the 10% to the 5% level. More importantly, in the restricted version (lower panel of Table 2) the implied elasticity of substitution parameter σ is equal to 1.3706 and is now significantly different from unity at the 5% level. This is a substantial improvement of the coefficient estimate of σ over the 13% significance level of the same coefficient in Table 1.

Column 5 presents coefficient estimates of the *extended Solow-CES model*. Results are qualita-

[12] Physical (and human) capital shares for all 98 countries obtained from the *basic* (and *extended*) *Solow-CES models* are reported in Table A3 in Appendix A.

[13] One of Kaldor's (1961) "stylized facts" of economic growth, is that the shares of income accruing to capital and labor are relatively constant over time. This view has been first challenged by the pioneer paper of Solow (1958) and remains today an open research question (i.e. see Gollin (forthcoming) who finds that labor's share of national income across 31 countries is relatively constant). As shown above, our findings suggest that relative shares vary drastically across our sample of 98 countries. Indeed, our results suggest that labor shares decline with economic development.

Table 2: Cross-Country level regressions with CD and CES technologies using PWT 6.0

Specification	Basic Solow (PWT 6.0)		Extended Solow (PWT 6.0)	
	CD (Eq.1)	CES (Eq.4)	CD (Eq.2)	CES (Eq.6)
Unrestricted				
Constant	11.4624*** (1.0444)	10.3608*** (1.2808)	11.1775*** (0.6869)	8.5420*** (0.8256)
$\ln s_{ik}$	1.0729*** (0.1112)	0.9870*** (0.0926)	0.5372*** (0.1307)	0.8826*** (0.1422)
$\ln(n_i + g + \delta)$	−2.6594*** (0.4443)	−2.0670*** (0.8290)	−2.3495*** (0.2741)	−1.3754*** (0.3352)
$\ln s_h$	—	—	0.6472*** (0.0959)	0.5138*** (0.1692)
$[\ln s_{ik} - \ln(n_i + g + \delta)]^2$	—	0.1786** (0.0880)	—	0.1414** (0.0615)
$[\ln s_{ih} - \ln(n_i + g + \delta)]^2$	—	—	—	0.2033*** (0.0725)
$[\ln s_{ik} - \ln s_{ih}]^2$	—	—	—	−0.2043*** (0.4476)
s.e.e.	0.61	0.60	0.48	0.46
Adj. R^2	0.68	0.69	0.80	0.82
Obs.	90	90	90	90
Restricted				
Constant	8.2439*** (0.0883)	8.1295*** (0.0832)	8.8431*** (0.1214)	8.5852*** (0.1071)
Implied α	0.5494*** (0.0194)	0.5035*** (0.0198)	0.2681*** (0.0526)	0.3679*** (0.0545)
Implied β	—	—	0.2963*** (0.0480)	0.2142*** (0.0633)
Implied σ	1	1.3706†† (0.1534)	1	1.1337††† (0.0404)
s.e.e.	0.63	0.61	0.50	—
Adj. R^2	0.66	0.68	0.79	—
Obs.	90	90	90	90

Notes: It is assumed that $g+\delta = 0.05$ as in MRW. α and β are shares of physical and human capital respectively in the CD models (distribution parameters in the CES models). All regressions are estimated using OLS with the exception of the restricted version of the *extended Solow-CES model* which was estimated using NLLS. Standard errors are given in parentheses. The standard errors for α and β were recovered using standard approximation methods for testing nonlinear functions of parameters. White's heteroskedasticity correction was used. *** (†††) Significantly different from 0 (1) at the 1% level. ** (††) Significantly different from 0 (1) at the 5% level. * (†) Significantly different from 0 (1) at the 10% level.

tively similar to those in Table 1. In the unrestricted version (upper panel of Table 2) notice that now all coefficient estimates are significant (even the quadratic term $\left[\ln\left(\frac{s_{ik}}{n_i+g+\delta}\right)\right]^2$ which was insignificant in Table 1). In the restricted model the implied value of σ decreases slightly from 1.1894 to 1.1337 but remains highly significant. Consistent with our baseline results regarding input shares, is our finding that physical and human capital shares in the *basic* and *extended Solow-CES models* vary considerably.[14]

Legitimate concerns can be raised on the validity of statistical inference based on test statistics with asymptotic properties when using small samples. In order to check whether specific parameter estimates or the general results are not unduly influenced by assumptions on error distribution, we also checked the sensitivity of these results by using bootstrapping. Specifically, we checked whether the linear estimation results in Tables 1 and 2 are unusual relative to $10,000$ parameter estimates obtained from randomly sampled residuals from the original model. We find that although there are slight differences in magnitudes of estimates and corresponding standard errors at two decimal places (hundredth point), our qualitative implications are robust.

Our cross-sectional analysis is subject to two additional econometric problems. First, the problem of endogeneity maybe present because variables used as regressors (i.e. physical and human capital investment) maybe influenced by the same factors that influence output. Second, the choice of variables in the regression model is not clear therefore giving rise to the "model uncertainty" problem. The most common practice to resolving the endogeneity problem has been the use of instrumental variable approaches. However, in cross-country regressions treatment of endogeneity problems is less than satisfactory because of lack of viable exogenous instruments. Brock and Durlauf (2000) and Durlauf (2001), among others, observe that studies using instrumental variables (IV) to address endogeneity are not convincing as their choice of instruments do not meet the necessary exogeneity requirements.[15] In addition, Romer (2001) shows that IV estimation potentially introduces an upward bias in the parameter estimates due to the fact that most measures of physical and human capital used in the literature vary with levels of per capita output.

Recent concerns about the appropriate choice of explanatory variables (to resolve the mispecification problem) are also valid. The vast number of potential explanatory variables that could be included in any level or growth regression creates the need for procedures that assign some level

[14] Physical and human capital shares for all 90 countries in the updated PWT 6.0 dataset obtained from the *basic* and *extended Solow-CES models* are reported in Table A3 in Appendix A.

[15] For more on this issue see Brock and Durlauf (2000, pp.9-11) and Durlauf (2001, p.66).

of confidence to each of these variables.[16] A first attempt to test the importance of explanatory variables is made by Sala-i-Martin (1997). A recent and very promising line of research for identifying effective regressors is based on Bayesian Model Averaging (see Fernàndez, Ley and Steel (2001)). Even though we are in complete agreement with these concerns, we have also tried to resolve potential mispecification error from choice of explanatory variables, by incorporating variables whose explanatory power was established to be robust by Sala-i-Martin (1997) and Fernàndez, et al. (2001). In particular, we added to our regressors a measure of longevity (life expectancy), a measure of openness (number of years the economy has been open), a measure of political stability (number of coups) and a measure for geographical externality (latitude). Quality of life, openness and latitude have positive effect on per capita output while as expected wars and coups have a negative significant impact on per capita output. The qualitative implications of our model are generally robust to inclusion of these variables, however, due to the small sample size (our sample was reduced to 70 countries) it is difficult to capture the quadratic curvature of the production function leading to smaller elasticity of substitution and negative share for human capital. These results are available from the authors upon request.

In summary, our key finding in this section is that in the context of cross-country level regressions we can reject the CD aggregate production specification over the more general CES specification. In particular, we find evidence that the elasticity of substitution parameter σ is greater than unity in both the basic and the extended models. The primary implication of our results for the empirical literature is that the vast majority of cross-country level regressions may be mispecified due to the choice of aggregate production specification. The additional quadratic term(s) appearing in the basic (extended) Solow-CES specification reflect the omitted term(s) responsible for the specification error.

4 Thresholds and Multiple Regimes in the Solow-CES Models

In our analysis so far we have shown that the identical CD aggregate production technology (a necessary condition for the linearity of the Solow growth model), assumed in the vast majority of existing studies, is rejected over the more general (and nonlinear) CES aggregate technology. In this section we investigate whether nonlinearities in the CES production function can explain the

[16] For an extensive discussion about "model uncertainty" see Brock and Durlauf (2000, pp.6-8) and Durlauf (2001, pp.67).

parameter heterogeneity evident in growth regressions. Put differently, we investigate the possibility that replacing the (identical for all countries) CD specification with an identical but nonlinear CES specification can potentially capture the differences among complex heterogenous objects such as countries.

4.1 Threshold Estimation

We follow Hansen (2000) to search for multiple regimes in the data under the Solow model with CES production technology. Hansen develops a statistical theory of threshold estimation in the regression context that allows for cross-section observations. Least squares estimation is considered and an asymptotic distribution theory for the regression estimates is developed. The main advantage of Hansen's methodology over, for instance, the Durlauf-Johnson regression-tree model is that the former is based on an asymptotic distribution theory which can formally test the statistical significance of regimes selected by the data.[17]

In much of the empirical growth literature, the cross-country growth regression equation based on the CD specification take the form

$$\ln\left(\frac{Y}{L}\right)_{i,85} - \ln\left(\frac{Y}{L}\right)_{i,60} = \ln A(0) - \theta \ln\left(\frac{Y}{L}\right)_{i,60} + \theta\frac{\alpha}{1-\alpha-\beta}\ln\left(\frac{s_{ik}}{n_i+g+\delta}\right) + \theta\frac{\beta}{1-\alpha-\beta}\ln\left(\frac{s_{ih}}{n_i+g+\delta}\right), \quad (10)$$

where $\theta = (1 - e^{-\lambda t})$, λ is the convergence rate, and $(Y/L)_{i,60}$ is the initial per capita output in country i. Under CES technology this cross-country growth regression equation now becomes

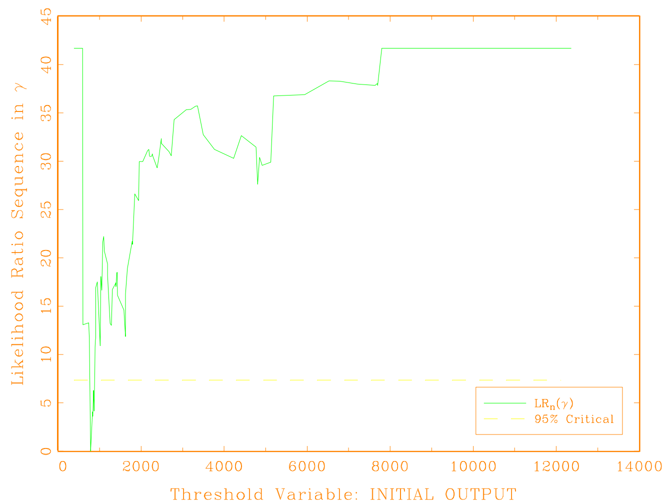
$$\begin{aligned}\ln\left(\frac{Y}{L}\right)_{i,85} - \ln\left(\frac{Y}{L}\right)_{i,60} &= \ln A(0) - \theta \ln\left(\frac{Y}{L}\right)_{i,60} + \theta\frac{\alpha}{1-\alpha-\beta}\ln\left(\frac{s_{ik}}{n_i+g+\delta}\right) + \theta\frac{\beta}{1-\alpha-\beta}\ln\left(\frac{s_{ih}}{n_i+g+\delta}\right) + \\ & \frac{1}{2}\theta\frac{\sigma-1}{\sigma}\frac{1}{(1-\alpha-\beta)^2}\left\{\alpha\left[\ln\left(\frac{s_{ik}}{n_i+g+\delta}\right)\right]^2 + \beta\left[\ln\left(\frac{s_{ih}}{n_i+g+\delta}\right)\right]^2 - \alpha\beta\left(\ln\frac{s_{ik}}{s_{ih}}\right)^2\right\}. \quad (11)\end{aligned}$$

Following Durlauf and Johnson (1995) and Hansen (2000), we search for multiple regimes in the data using initial per capita output ($(Y/L)_{60}$) and initial adult literacy rates (LIT_{60}) as potential threshold variables.[18] Since Hansen's statistical theory allows for one threshold for each threshold

[17] For a detailed discussion of the statistical theory for threshold estimation in linear regressions, see Hansen (2000).

[18] In order to compare our model predictions to those of Durlauf and Johnson (1995) and Hansen (2000) we only consider the two threshold variables considered in these papers. In future work, a variety of other potential threshold variables including openness, ethnicity, political stability etc. will be considered. In a recent contribution, Johnson and Takeyama (2001) use regression trees to examine the role of a large number of such variables in the convergence process of U.S. States since 1950. Papageorgiou (forthcoming) shows that openness, as measured by the trade share to GDP, is a threshold variable that can cluster middle-income countries into two distinct regimes that obey different statistical models.

Figure 1: First sample split

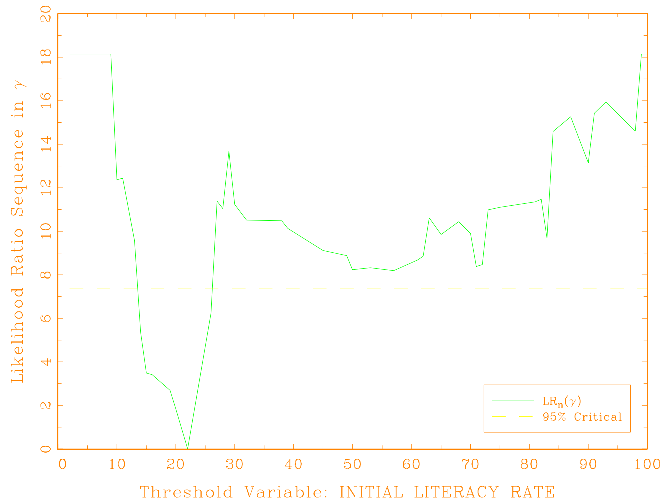


variable, we proceed by selecting between the two variables by employing the heteroskedasticity-consistent Lagrange Multiplier test for a threshold obtained in Hansen (1996). With the exception of adult literacy rates (LIT_{60}), the variables employed in this exercise are identical to those used in the regression analysis of the previous section (PWT 4.0). Adult literacy rates is defined as the fraction of population over the age of 15 that is able to read and write in 1960; data are from the World Bank's *World Report*. The sample used in this exercise includes 96 of the 98 countries in the original sample after eliminating Botswana and Mauritius for which there are no data on initial literacy rates.

In the first round of splitting, we find that the threshold model using initial output is significant with p-value at 0.025 while the threshold model using initial literacy rates is significant with p-value at 0.002. These results indicate that there maybe a sample split based on either output or literacy rate. We choose to first examine the sample split for the threshold model using output, deferring discussion on the threshold model using literacy rates for later on.

Figure 1 presents the normalized likelihood ratio sequence $LR_n^*(\gamma)$ statistic as a function of the output threshold. The least-squares estimate γ is the value that minimizes the function $LR_n^*(\gamma)$ which occurs at $\hat{\gamma} = \$777$. The asymptotic 95% critical value (7.35) is shown by the dotted line and where it crosses $LR_n^*(\gamma)$ displays the confidence set $[\$777, \$863]$. The first output threshold divides our sub-sample of 96 countries into a low-income group with 14 countries and a high-income group

Figure 2: Second sample split



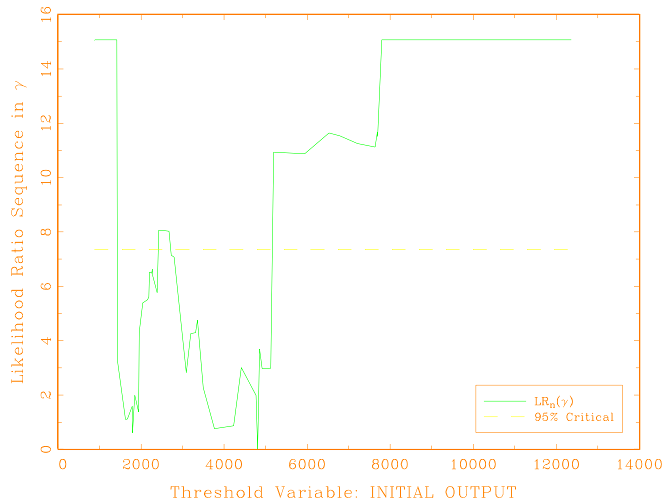
with 82 countries.

Even though further splitting of the low-income group is not possible, further splitting of the high-income group is shown to be possible. The threshold model using literacy rates is significant attaining a p-value of 0.075. Figure 2 presents the normalized likelihood ratio statistic as a function of the literacy rates threshold. The point estimate for the literacy threshold is $\hat{\gamma} = 22\%$ with the 95% confidence interval $[14\%, 26\%]$. The literacy rates threshold variable splits the high-income sub-sample of 82 countries into two additional groups; the low-literacy group with 21 countries and the high-literacy group with 61 countries.

Our third and final round of threshold model selection involves the 61 countries with initial per capita output above \$777 and initial literacy rates above 22%. We find that the threshold model using output is significant with p-value at 0.056. The output threshold value occurs at \$4802 and the asymptotic 95% confidence set is $[\$1430, \$5119]$. The normalized likelihood ratio statistic as a function of the output threshold is illustrated in Figure 3. The output threshold variable splits the high-literacy group into a high-literacy-low-income group with 40 countries and a high-literacy-high-income group with 21 countries. We have tried to further split these subsamples, but none of the bootstrap test statistics were significant and therefore no further splitting was possible using the existing threshold variables.

Figure 4 uses tree diagrams to compare our threshold estimation results obtained under the

Figure 3: Third sample split



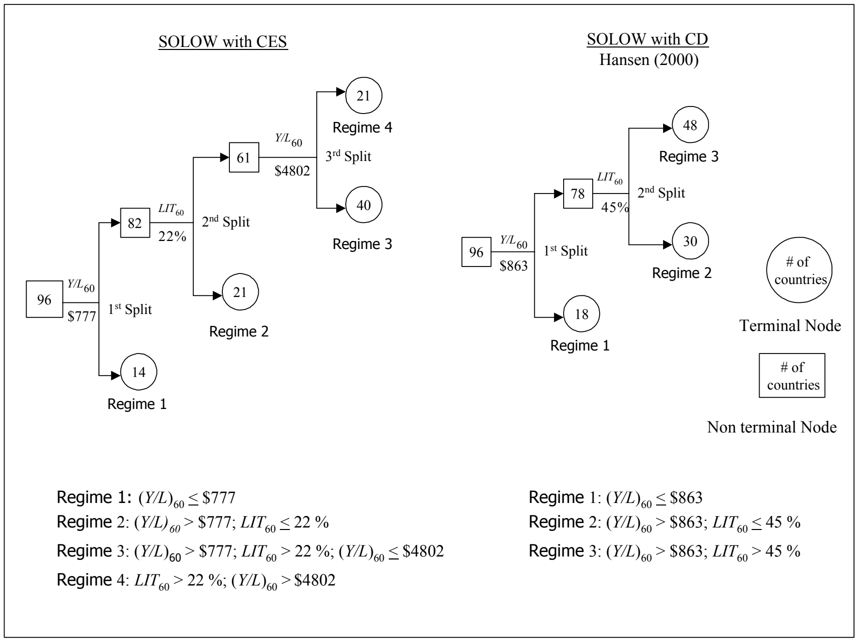
extended Solow-CES model with Hansen (2000) results obtained under the *extended Solow-CD model*. Non-terminal and terminal nodes are represented by squares and circles, respectively. The numbers inside the squares and circles show the number of countries in each node. The point estimates for each threshold variable are presented on the rays connecting the nodes. It is clear from Figure 4 that replacing the CD with the CES specification in the Solow model increases the number of endogenously determined regimes from three to four. Moreover, the composition of these regimes is different across models. Table 3 presents the countries in each regime obtained from our threshold estimation of the Solow model with CES aggregate production technology.

4.2 Regression Results

Next, we turn our attention to the estimation of equation (11) for the four regimes. Table 4 presents estimates for each regime in the unrestricted and restricted models. These estimates provide strong evidence in favor of parameter heterogeneity and the presence of multiple regimes. The heterogeneity of the coefficient estimates across regimes is evident, as coefficient estimates vary considerably in sign and magnitude.

Starting with the unrestricted model (upper panel of Table 4), in all but Regime 4 the sign of the coefficient on initial income, $\ln(Y/L)_{i,60}$, has the expected negative sign which is consistent with conditional convergence. Point estimates on $\ln(Y/L)_{i,60}$ vary from -1.2413 and significant at the

Figure 4: Threshold estimation in the Solow-CES model vs. the Solow-CD model



1% level in Regime 1, to 0.2750 and significant at the 10% level in Regime 4. There is considerable variation in the estimates associated with physical capital as well. The coefficient estimates on physical capital investment, $\ln s_{ik}$, vary from 1.3082 in Regime 1 to 2.4887 in Regime 3, and in all regimes the coefficients are significant at the 1% level. In contrast, estimated coefficients on human capital investment, $\ln s_{ih}$, provide mixed results. In three of the four regimes, the coefficients have negative sign. Estimated coefficients vary from -1.4007 in Regime 4 to 0.6860 in Regime 2. Parameter heterogeneity across regimes is equally evident in the quadratic terms $[\ln s_{ih} - \ln(n_i + g + \delta)]^2$ and $[\ln s_{ik} - \ln s_{ih}]^2$. In two of the four regimes (Regimes 1 and 2) the coefficient associated with $[\ln s_{ih} - \ln(n_i + g + \delta)]^2$ is significant and varies in magnitude from 0.1565 in Regime 1 to 0.6551 in Regime 2. In all regimes the coefficient for $[\ln s_{ik} - \ln s_{ih}]^2$ is significant and ranges from -0.6986 in Regime 4 to 0.1262 in Regime 1. Coefficient estimates for $[\ln s_{ih} - \ln(n_i + g + \delta)]^2$ are insignificant in Regime 2-4 and positive and significant in Regime 1.

Disparity in coefficient estimates across regimes in the restricted model (lower panel of Table 4) is as large as in the unrestricted model. Recall that, the coefficients of the restricted model are estimated using NLLS. The estimated distribution parameter for physical capital (α) is significant

Table 3: Country classification in the Solow-CES model

Regime 1	Regime 2	Regime 3		Regime 4
B. Faso	Algeria	Bolivia	Madagascar	Argentina
Burma	Angola	Brazil	Malaysia	Australia
Burundi	Bangladesh	Colombia	Mexico	Austria
Ethiopia	Benin	Costa Rica	Nicaragua	Belgium
Malawi	C. Afri. Rep.	Dom. Rep.	Panama	Canada
Mali	Cameroon	Ecuador	Papua N. G.	Chile
Mauritania	Chad	Egypt	Paraguay	Denmark
Niger	Congo	El Salvador	Peru	Finland
Rwanda	Haiti	Ghana	Philippines	France
Sierra Leone	I. Coast	Greece	Portugal	Italy
Tanzania	Kenya	Guatemala	S. Africa	N. Zealand
Togo	Liberia	Honduras	S. Korea	Netherlands
Uganda	Morocco	Hong Kong	Singapore	Norway
Zaire	Mozambique	India	Spain	Sweden
	Nepal	Indonesia	Sri Lanka	Switzerland
	Nigeria	Ireland	Syria	Tri. & Tobago
	Pakistan	Israel	Thailand	U.K.
	Senegal	Jamaica	Turkey	U.S.A.
	Somalia	Japan	Zambia	Uruguay
	Sudan	Jordan	Zimbabwe	Venezuela
	Tunisia			W. Germany
(14)	(21)	(40)		(21)

in three out of the four regimes (1, 3 and 4) and varies from 0.0514 in Regime 2 to 0.6770 in Regime 3. Similarly, the estimated distribution parameter for human capital (β) is substantially different across regimes ranging from 0.1768 in Regime 1 to 0.8089 in Regime 2.[19] It is worth noting that unlike the vast majority of growth regressions, under the restricted model, the distribution parameters of physical *and* human capital take economically feasible values. Finally, the coefficient estimates of the elasticity of substitution parameter (σ) vary from 0.9861 in Regime 4 to 1.9524 in Regime 1.[20,21] Of course, one should interpret these results with caution as σ (reflecting the curvature of the production function) maybe difficult to capture by our estimation given the limited number of observations in each regime.[22]

[19]This result is consistent with Kalaitzidakis et al. (2001) and Kourtellos (2001) who find strong nonlinear effects of human capital on economic growth.

[20]This result is qualitatively consistent with Duffy and Papageorgiou (2000) and Miyagiwa and Papageorgiou (forthcoming) who argue that the elasticity of substitution may vary along the development path.

[21]Physical and human capital shares for all 96 countries were calculated using regression estimates from the four regimes. As expected, these shares vary considerably more than shares estimated using an identical CES production function (presented in Table A3). These results are available by the authors upon request.

[22]Given the small number of observations in each regime, we have tried implementing the bootstrap which per-

Table 4: Cross-country growth regressions for the four regimes

Specification	Regime 1	Regime 2	Regime 3	Regime 4
Unrestricted				
Constant	7.9977^{***} (1.4756)	3.1754^{***} (0.6411)	-1.9041 (1.3274)	-0.9464 (1.1087)
$\ln(Y/L)_{i,60}$	-1.2413^{***} (0.1695)	-0.6636^{***} (0.1138)	-0.0899 (0.1041)	0.2749^{*} (0.1327)
$\ln s_{ik}$	1.3082^{***} (0.2074)	1.8882^{***} (0.4339)	2.4887^{***} (0.5310)	1.9214^{***} (0.6145)
$\ln s_{ih}$	-0.5339^{*} (0.2362)	0.6860^{*} (0.3496)	-1.1949^{***} (0.3171)	-1.4007^{*} (0.7358)
$\ln(n_i + g + \delta)$	-1.0533^{*} (0.4567)	-1.3673^{***} (0.2834)	-0.4437 (0.7727)	-1.7911^{***} (0.1832)
$[\ln s_{ik} - \ln(n_i + g + \delta)]^2$	0.3469^{**} (0.1350)	-0.0573 (0.1298)	-0.1993 (0.2175)	-0.0089 (0.1036)
$[\ln s_{ih} - \ln(n_i + g + \delta)]^2$	0.1565^{*} (0.0719)	0.6551^{***} (0.0900)	0.1889 (0.2018)	-0.0189 (0.3246)
$[\ln s_{ik} - \ln s_{ih}]^2$	-0.2595^{***} (0.0518)	0.1262^{***} (0.0299)	-0.5770^{***} (0.1268)	-0.6986^{**} (0.2546)
s.e.e.	0.14	0.10	0.32	0.13
Adj. R^2	0.78	0.81	0.51	0.85
Obs.	14	21	40	21
Restricted				
Constant	5.5065^{***} (1.3538)	3.4453^{***} (1.0862)	-0.4663 (1.0103)	-0.0784 (1.2500)
Implied α	0.2144^{***} (0.0419)	0.0289 (0.0450)	0.3779^{**} (0.1153)	0.3302^{***} (0.0808)
Implied β	0.1289 (0.1551)	0.5889^{***} (0.0623)	0.1154^{*} (0.0632)	0.2437^{***} (0.0590)
Implied σ	2.1405 (1.1196)	$1.1604^{\dagger\dagger\dagger}$ (0.0155)	0.8487 (0.1316)	0.9054 (0.2424)
Obs.	14	21	40	21

Notes: α and β are distribution parameters of physical and human capital respectively. Standard errors are given in parentheses. The standard errors for α and β were recovered using standard approximation methods for testing nonlinear functions of parameters. White's heteroskedasticity correction was used. *** (†††) Significantly different from 0 (1) at the 1% level. ** (††) Significantly different from 0 (1) at the 5% level. * (†) Significantly different from 0 (1) at the 10% level.

Next, we examine the alternative model in which the first-round threshold variable is initial adult literacy rates (recall that the bootstrap procedure obtained a p-value of 0.002). The literacy rates threshold value occurs at 25% and the asymptotic 95% confidence set is $[15\%, 26\%]$. This threshold value divides our original sample of 96 countries into a low-literacy group with 32 countries and a high-literacy group with 64 countries. We show that further splitting is possible in both of these subsamples. The low-literacy group is split using initial output obtaining a p-value equal to 0.052. The threshold value is $863 and the confidence set is $[\$846, \$863]$. The low-literacy sub-sample (32 countries) is split into a low-literacy-low-income group with 15 countries and a low-literacy-high-income group with 17 countries. The high-literacy group (64 countries) can also be split by using initial output as the threshold variable, with p-value equal to 0.003. The point estimate for the initial output threshold is $4802 and the confidence interval is $[\$1285, \$5119]$. The high-literacy sub-sample is divided into a high-literacy-low-income group with 43 countries and a high-literacy-high-income group with 21 countries. Figure A1 in Appendix A illustrates the likelihood ratio statistic as a function of the relevant threshold variables. Figure A2 presents a regression tree of this alternative splitting scheme and Table A1 presents the countries under each of the four regimes.

One of the findings that is immediately noticeable is that employing literacy rates as the first-round threshold variable obtains similar regimes (terminal nodes) to those obtained when using output as the first-round threshold variable. In fact Regime 4 is identical in both cases while Regimes 1-3 are quite similar. When using literacy for the initial splitting, Regime 1 attains 15 countries (1 country more than in the case where output is used for the initial splitting), Regime 2 attains 17 countries (4 countries less than Regime 2 in the first case), and Regime 3 attains 43 countries (3 countries more than the first case). In terms of the composition of regimes across the two alternative cases, most notable is the difference in composition in Regime 1 (compare Tables 3 and A1). As shown in Table A2, regression estimates for each of the four regimes under this alternative model vary substantially which is consistent with the original model. The lower panel of Table A2 shows that the distribution parameters of physical *and* human capital take economically feasible values and all but two estimates are significant at the 1% level.

To summarize, the key finding of this exercise is twofold: First, the Solow model with CES technology provides strong evidence in favor of parameter heterogeneity and the presence of multiple

forms inference that is more reliable in finite samples than inferences based on conventional asymptotic theory. Unfortunately, in our work bootstrap replication involves nonlinear estimation that fails to converge.

regimes. Second, whereas under the CD aggregate technology the statistical theory of threshold estimation identifies three regimes, under the CES technology it identifies four regimes. In addition to the number of regimes identified, the composition of each regime has also changed under the CES model. We conclude this section with a puzzling observation. The number and composition of the regimes identified here is surprisingly similar to those in Durlauf and Johnson (1995). We do not have an explanation to offer but we suspect that this, like many other puzzles, maybe an optical illusion.

5 Conclusion

In this paper we set out to examine whether nonlinearities in the production function can explain parameter heterogeneity in growth regressions. Our investigation involves two sequential steps. First, we question the empirical relevance of the CD aggregate production specification in cross-country linear regressions. We find that both in the basic and the extended regression models the CD specification is rejected over the more general CES specification with elasticity of substitution greater than unity. We also find that the CES specification better fits cross-country variation than the CD specification. Our findings call into question a number of earlier cross-country level regression exercises that simply assume a CD specification for the aggregate input-output relationship. In particular, we argue that the vast majority of cross-country regressions may be mispecified due to the choice of aggregate production specification. A simple test of aggregate production specification is to add the quadratic term(s) appearing in the basic (extended) Solow-CES specification and examine the significance of the estimated coefficients.

Given our first result, we then search for multiple regimes in the data by replacing the CD with the CES specification. By using the endogenous threshold methodology of Hansen (2000), we show that the Solow model under CES continues to imply robust nonlinearities in the growth process that are consistent with the presence of multiple regimes. This finding re-enforces the findings of Durlauf and Johnson (1995), Durlauf, Kourtellos and Minkin (2001) and Kourtellos (2001), and is in stark contrast with the prevalent practice in growth literature in which countries are assumed to obey a common linear international production function. Furthermore, this result suggests that an *identical* CES aggregate production function can not capture the heterogeneity that exists across countries therefore shifting attention to growth nonlinearities and omitted growth determinants as two alternative interpretations of parameter heterogeneity.

Our findings can be further enriched by extending this analysis on at least two fronts. First, use the CES specification in alternative econometric techniques relevant to parameter heterogeneity as the semiparametric varying coefficient model along the lines of Hastie and Tibshirani (1992) and Kourtellos (2001). Second, it is worth examining the quantitative and qualitative implications of our findings when different threshold variables are used. Such variables may include life expectancy, ethnicity and openness, just to name a few.

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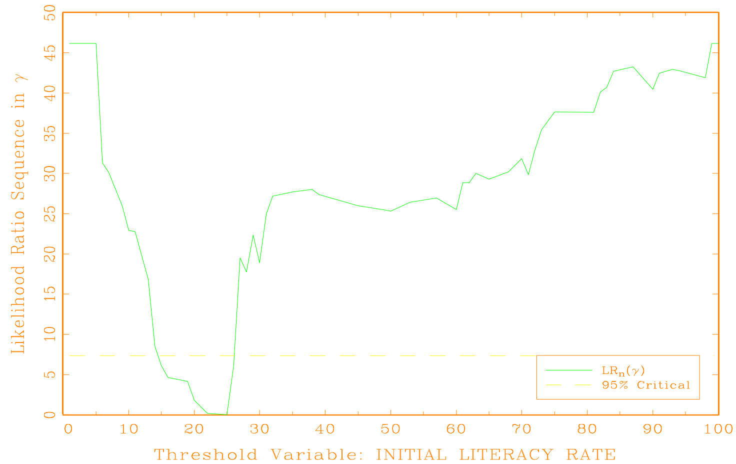
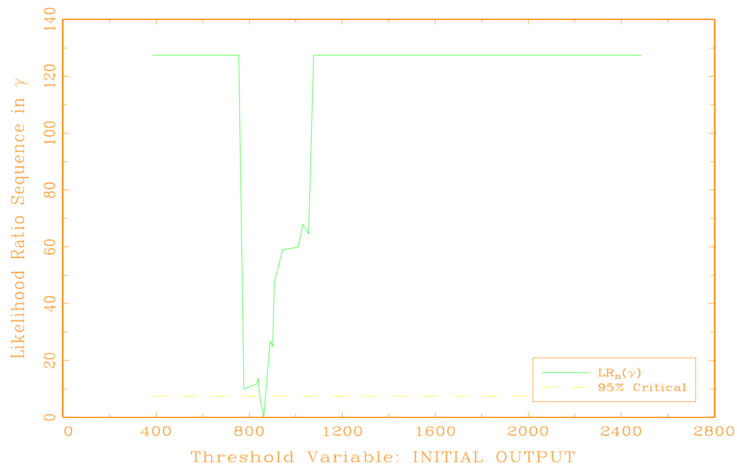
APPENDIX A

Figure A1: Likelihood ratio statistic as a function of threshold variables (alternative splitting)



First sample split



Second sample split

Third sample split

Figure A2: Threshold estimation in the Solow-CES model (alternative splitting)

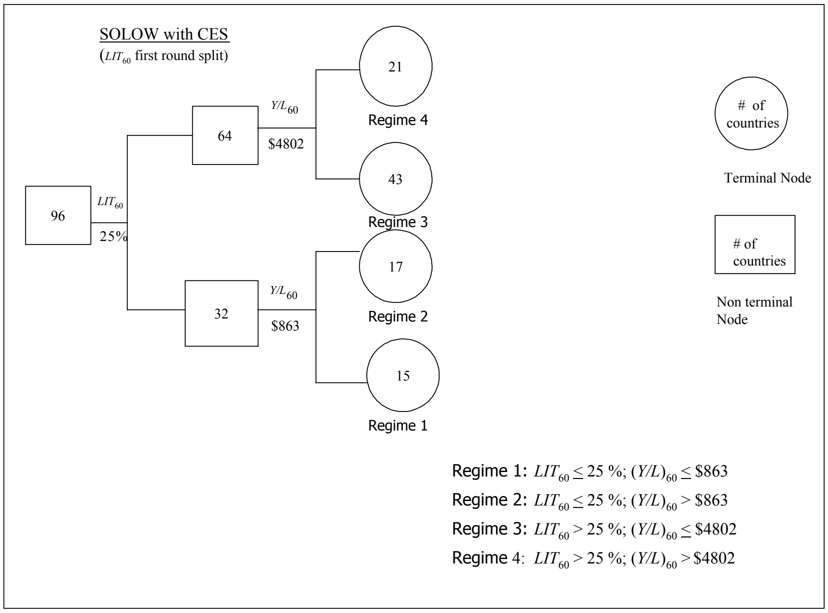


Table A1: Country classification in four regimes (alternative splitting)

Regime 1	Regime 2	Regime 3		Regime 4
B. Faso	Algeria	Bolivia	Malaysia	Argentina
Bangladesh	Angola	Brazil	Mexico	Australia
Burundi	Benin	Burma	Nicaragua	Austria
C. Afri. Rep.	Cameroon	Colombia	Panama	Belgium
Ethiopia	Chad	Costa Rica	Papua N. G.	Canada
Liberia	Congo	Dom. Rep.	Paraguay	Chile
Malawi	Haiti	Ecuador	Peru	Denmark
Mali	I. Coast	Egypt	Philippines	Finland
Mauritania	Kenya	El Salvador	Portugal	France
Nepal	Morocco	Ghana	S. Africa	Italy
Niger	Mozambique	Greece	S. Korea	N. Zealand
Rwanda	Nigeria	Guatemala	Singapore	Netherlands
Sierra Leone	Pakistan	Honduras	Spain	Norway
Tanzania	Senegal	Hong Kong	Sri Lanka	Sweden
Togo	Somalia	India	Syria	Switzerland
	Sudan	Indonesia	Thailand	Tri. & Tobago
	Tunisia	Ireland	Turkey	U.K.
		Israel	Uganda	U.S.A.
		Jamaica	Zaire	Uruguay
		Japan	Zambia	Venezuela
		Jordan	Zimbabwe	W. Germany
		Madagascar		
(15)	(17)	(43)		(21)

Table A2: Cross-country growth regressions for the four regimes (alternative splitting)

Specification	Regime 1	Regime 2	Regime 3	Regime 4
Unrestricted				
Constant	5.2380^{***}	3.9052^{***}	-1.8288^{*}	-0.9464
	(0.9456)	(0.4993)	(1.0889)	(1.1087)
$\ln(Y/L)_{i,60}$	-0.6578^{***}	1.0256^{**}	-0.1310	0.2750^{*}
	(0.1077)	(0.4293)	(0.0813)	(0.1327)
$\ln s_{ik}$	-0.3098^{**}	-0.7873^{**}	2.6145^{***}	1.9214^{***}
	(0.1264)	(0.0957)	(0.4091)	(0.6145)
$\ln s_{ih}$	0.9479^{***}	1.0905^{***}	-1.2893^{***}	-1.4007^{*}
	(0.1672)	(0.2789)	(0.3092)	(0.7358)
$\ln(n_i + g + \delta)$	-0.5614	-0.6074^{*}	-0.3967	-1.7911^{***}
	(0.3236)	(0.3302)	(0.6866)	(0.1832)
$[\ln s_{ik} - \ln(n_i + g + \delta)]^2$	-0.1165	0.1628	-0.1853	-0.0089
	(0.1132)	(0.1097)	(0.1702)	(0.1036)
$[\ln s_{ih} - \ln(n_i + g + \delta)]^2$	0.0821^{*}	0.5607^{***}	0.2876^{**}	-0.1894
	(0.0438)	(0.0864)	(0.1384)	(0.3246)
$[\ln s_{ik} - \ln s_{ih}]^2$	0.1262^{***}	-0.4325^{***}	-0.6398^{***}	-0.6986^{**}
	(0.0299)	(0.1131)	(0.1050)	(0.2546)
s.e.e.	0.10	0.13	0.31	0.13
Adj. R^2	0.81	0.93	0.57	0.85
Obs.	15	17	43	21
Restricted				
Constant	6.4971^{***}	5.0077^{***}	0.7073	0.1241
	(1.0360)	(1.2012)	(0.8613)	(1.2307)
Implied α	0.1041^{***}	0.0060	0.6551^{***}	0.5129^{***}
	(0.0433)	(0.0775)	(0.1902)	(0.1117)
Implied β	0.3368^{***}	0.7727^{***}	0.2442^{*}	0.2437^{***}
	(0.0966)	(0.0983)	(0.1441)	(0.0590)
Implied σ	$1.3236^{\dagger\dagger\dagger}$	$1.0810^{\dagger\dagger\dagger}$	1.0511	0.9861
	(0.0541)	(0.0229)	(0.0982)	(0.0256)
Obs.	15	17	43	21

Notes: α and β are distribution parameters of physical and human capital respectively. Standard errors are given in parentheses. The standard errors for α and β were recovered using standard approximation methods for testing nonlinear functions of parameters. White's heteroskedasticity correction was used. *** (†††) Significantly different from 0 (1) at the 1% level. ** (††) Significantly different from 0 (1) at the 5% level. * (†) Significantly different from 0 (1) at the 10% level.

Table A3: Shares from the *basic* and *extended Solow-CES models*

Country	Code	Basic CES (PWT 4.0)	Extended CES (PWT 4.0)		Basic CES (PWT 6.0)	Extended CES (PWT 6.0)	
		$shr(K^*)$	$shr(K^*)$	$shr(H^*)$	$shr(K^*)$	$shr(K^*)$	$shr(H^*)$
ALGERIA	1	0.7479	0.2878	0.3295	0.6182	0.4024	0.2091
ANGOLA	2	0.4642	0.2319	0.2879	0.5131	0.3709	0.1835
BENIN	3	0.5693	0.2544	0.2860	0.4615	0.3542	0.1830
BOTSWANA	4	0.7705	0.2917	0.3036	0.5987	0.3968	0.2007
BURKINA FASO	5	0.6526	0.2706	0.2334	0.5036	0.3679	0.1595
BURUNDI	6	0.4528	0.2293	0.2287	0.4463	0.3490	0.1577
CAMEROON	7	0.6132	0.2631	0.3186	0.4957	0.3654	0.1896
C. AFR. REP.	8	0.5837	0.2573	0.2792	0.4420	0.3476	0.1777
CHAD	9	0.4984	0.2395	0.2276	—	—	—
CONGO	10	0.8038	0.2974	0.3221	0.6645	0.4152	0.2166
EGYPT	11	0.6548	0.2710	0.3543	0.4760	0.3590	0.2191
ETHIOPIA	12	0.4483	0.2283	0.2650	0.4221	0.3407	0.1756
GHANA	15	0.5386	0.2481	0.3339	0.5071	0.3690	0.2053
IVORY COST	17	0.5515	0.2507	0.2867	0.4904	0.3639	0.1871
KENYA	18	0.6439	0.2689	0.2934	0.5419	0.3799	0.1914
LIBERIA	20	0.7056	0.2803	0.2976	—	—	—
MADAGASCAR	21	0.4960	0.2390	0.3046	0.3742	0.3232	0.1927
MALAWI	22	0.6109	0.2626	0.2401	0.5636	0.3864	0.1712
MALI	23	0.5008	0.2400	0.2616	0.4968	0.3658	0.1759
MAURITANIA	24	0.7786	0.2931	0.2616	0.4522	0.3510	0.1782
MAURITIUS	25	0.6629	0.2725	0.3559	0.5746	0.3897	0.2143
MOROCCO	26	0.5165	0.2434	0.3187	0.5723	0.3890	0.2007
MOZAMBIQUE	27	0.4592	0.2308	0.2445	0.3432	0.3113	0.1699
NIGER	28	0.5546	0.2514	0.2322	0.4878	0.3629	0.1610
NIGERIA	29	0.5908	0.2587	0.2974	0.4847	0.3618	0.1882
RWANDA	30	0.5006	0.2400	0.2232	0.4008	0.3331	0.1658
SENEGAL	31	0.5488	0.2507	0.2840	0.4807	0.3605	0.1844
SIERRA LEONE	32	0.5946	0.2594	0.2886	—	—	—
SOMALIA	33	0.6011	0.2607	0.2606	—	—	—
S.AFRICA	34	0.7299	0.2847	0.3109	0.5849	0.3928	0.2112
SUDAN	35	0.6052	0.2615	0.2896	—	—	—
TANZANIA	37	0.6658	0.2731	0.2308	0.6716	0.4172	0.1603
TOGO	38	0.6434	0.2688	0.3079	0.4672	0.3561	0.1941
TUNISIA	39	0.6205	0.2645	0.3285	0.6315	0.4061	0.2073
UGANDA	40	0.3923	0.2149	0.2606	0.3347	0.3079	0.1754
ZAIRE	41	0.4762	0.2346	0.3194	0.4395	0.3467	0.1883
ZAMBIA	42	0.8198	0.3000	0.2975	0.5908	0.3945	0.1900
ZIMBABWE	42	0.7073	0.2806	0.3270	0.6643	0.4152	0.2000
BANGLADESH	46	0.4793	0.2353	0.3121	0.5387	0.3789	0.1945
BURMA	47	0.6008	0.2607	0.3230	—	—	—
HONG KONG	48	0.6867	0.2769	0.3522	0.6979	0.4242	0.2131
INDIA	49	0.6650	0.2729	0.3376	0.5600	0.3854	0.2073
ISRAEL	52	0.7861	0.2944	0.3696	0.7085	0.4270	0.2216
JAPAN	53	0.9252	0.3169	0.3919	0.7779	0.4448	0.2295
JORDAN	54	0.6666	0.2732	0.3780	0.5223	0.3738	0.2243
KOREA	55	0.7244	0.2837	0.3746	0.7126	0.4281	0.2237
MALAYSIA	57	0.7185	0.2826	0.3516	0.6398	0.4084	0.2134
NEPAL	58	0.4693	0.2331	0.3000	0.5448	0.3808	0.1971
PAKISTAN	60	0.5781	0.2561	0.3064	0.5522	0.3830	0.1924

Table A3: Shares from the *basic* and *extended Solow-CES models*, continued

Country	Code	Basic CES (PWT 4.0)	Extended CES (PWT 4.0)		Basic CES (PWT 6.0)	Extended CES (PWT 6.0)	
		$shr(K^*)$	$shr(K^*)$	$shr(H^*)$	$shr(K^*)$	$shr(K^*)$	$shr(H^*)$
PHILLIPPINES	61	0.6203	0.2644	0.3746	0.5891	0.3940	0.2222
SINGAPORE	63	0.8284	0.3014	0.3680	0.7874	0.4471	0.2182
SRILANKA	64	0.6360	0.2674	0.3648	0.5354	0.3779	0.2202
SYRIA	65	0.6346	0.2672	0.3638	0.5783	0.3908	0.2163
THAILAND	67	0.6600	0.2720	0.3250	0.7098	0.4273	0.2093
AUSTRIA	70	0.8347	0.3025	0.3813	0.7632	0.4411	0.2288
BELGIUM	71	0.8294	0.3016	0.3895	0.7334	0.4335	0.2307
DENMARK	73	0.8621	0.3069	0.3971	0.7606	0.4404	0.2326
FINLAND	74	0.9613	0.3225	0.4006	0.8018	0.4507	0.2331
FRANCE	75	0.8370	0.3069	0.3831	0.7457	0.4366	0.2268
GERMANY	76	0.8889	0.3112	0.3832	—	—	—
GREECE	77	0.8864	0.3108	0.3773	0.7506	0.4379	0.2246
IRELAND	79	0.8288	0.3015	0.3957	0.7313	0.4329	0.2336
ITALY	80	0.8423	0.3037	0.3720	0.7539	0.4387	0.2221
NETHERLANDS	83	0.8138	0.2990	0.3887	0.7262	0.4316	0.2300
NORWAY	84	0.8843	0.3105	0.3917	0.8306	0.4577	0.2302
PORTUGAL	85	0.8128	0.2989	0.3602	0.7213	0.4303	0.2219
SPAIN	86	0.7291	0.2845	0.3750	0.7227	0.4307	0.2270
SWEDEN	87	0.8483	0.3047	0.3806	0.7341	0.4336	0.2275
SWITZERLAND	88	0.8852	0.3106	0.3476	0.7391	0.4349	0.2224
TURKEY	89	0.7061	0.2804	0.3409	0.6163	0.4018	0.2087
UK	90	0.7721	0.2920	0.3890	0.6999	0.4247	0.2290
CANADA	92	0.7608	0.2900	0.3827	0.6833	0.4302	0.2253
COSTA RICA	93	0.6043	0.2613	0.3473	0.5680	0.3878	0.2096
DOMINICAN REP.	94	0.6539	0.2708	0.3410	0.5495	0.3822	0.2056
EL SALVADOR	95	0.4920	0.2381	0.3176	0.4842	0.3617	0.2022
GUATAMALA	96	0.5131	0.2427	0.2951	0.5032	0.3678	0.1910
HAITI	97	0.5198	0.2441	0.2960	—	—	—
HONDURAS	98	0.6011	0.2607	0.3162	0.5454	0.3810	0.1962
JAMAICA	99	0.7438	0.2871	0.3897	0.6759	0.4183	0.2280
MEXICO	100	0.6730	0.2744	0.3454	0.6471	0.4104	0.2133
NICARAGUA	101	0.6064	0.2618	0.3383	0.5127	0.3708	0.2079
PANAMA	102	0.7554	0.2891	0.3800	0.6379	0.4079	0.2192
TRI&TOB	103	0.7297	0.2846	0.3723	0.5586	0.3849	0.2252
USA	104	0.7541	0.2889	0.3944	0.6143	0.4012	0.2236
ARGENTINA	105	0.8038	0.2974	0.3435	0.6630	0.4148	0.2175
BOLIVIA	106	0.6125	0.2629	0.3354	0.5378	0.3786	0.2065
BRAZIL	107	0.7280	0.2843	0.3298	0.6579	0.4134	0.2029
CHILE	108	0.8164	0.2995	0.3613	0.6118	0.4005	0.2179
COLOMBIA	109	0.6629	0.2725	0.3431	0.5558	0.3841	0.2110
ECUADOR	110	0.7443	0.2872	0.3537	0.6407	0.4087	0.2115
PARAGUAY	112	0.5574	0.2560	0.3277	0.5443	0.3806	0.2012
PERU	113	0.5773	0.2560	0.3589	0.6453	0.4100	0.2176
URUGUAY	115	0.6478	0.2697	0.3711	0.6476	0.4106	0.2300
VENEZUELA	116	0.5459	0.2496	0.3454	0.6286	0.4053	0.2068
AUSTRALIA	117	0.8459	0.3043	0.3779	0.7078	0.4268	0.2245
INDONESIA	119	0.6376	0.2678	0.3297	0.5794	0.3911	0.2051
NEW ZEALAND	120	0.7631	0.2905	0.3925	0.6807	0.4196	0.2296
PAPUA N.G.	121	0.6662	0.2731	0.2796	0.5468	0.3814	0.1807

APPENDIX B

Step-by-step derivation of the basic Solow-CES equation

To derive the *basic* and *extended Solow-CES equations* we use the definition of $\sigma = \frac{1}{1-\rho}$, as algebra is easier with ρ rather than σ. The aggregate production function is given by the CES specification

$$Y = [\alpha K^{\rho} + (1-\alpha)(AL)^{\rho}]^{\frac{1}{\rho}}. \tag{B1}$$

Divide through by AL to obtain the production function in its intensive form

$$y = [\alpha k^{\rho} + (1-\alpha)]^{\frac{1}{\rho}}. \tag{B2}$$

In the basic Solow model the law of motion of capital is given by

$$\dot{k} = sy - (n+g+\delta)k \overset{ss}{=} 0. \tag{B3}$$

Substitute for y and solve for k^{*}, where $(*)$ denotes steady-state values

$$s[\alpha k^{\rho} + (1-\alpha)]^{\frac{1}{\rho}} = (n+g+\delta)k \tag{B4}$$

$$[\alpha k^{\rho} + (1-\alpha)] = \left(\frac{n+g+\delta}{s}\right)^{\rho} k^{\rho}$$

$$(1-\alpha) = \left[\left(\frac{n+g+\delta}{s}\right)^{\rho} - \alpha\right] k^{\rho}$$

$$k^{\rho} = \frac{(1-\alpha)}{\left[\left(\frac{n+g+\delta}{s}\right)^{\rho} - \alpha\right]}$$

$$k^{*} = \left\{\frac{(1-\alpha)}{\left[\left(\frac{n+g+\delta}{s}\right)^{\rho} - \alpha\right]}\right\}^{\frac{1}{\rho}}. \tag{B5}$$

Substituting for k^{*} into $y = [\alpha k^{\rho} + (1-\alpha)]^{\frac{1}{\rho}}$ gives

$$\begin{aligned} y^{*} &= \left\{\alpha \frac{(1-\alpha)}{\left[\left(\frac{n+g+\delta}{s}\right)^{\rho} - \alpha\right]} + (1-\alpha)\right\}^{\frac{1}{\rho}} \\ &= \left\{(1-\alpha)\left[\frac{\alpha}{\left[\left(\frac{n+g+\delta}{s}\right)^{\rho} - \alpha\right]} + 1\right]\right\}^{\frac{1}{\rho}} \end{aligned}$$

$$
\begin{aligned}
&= \left\{ \frac{(1-\alpha)\left(\frac{n+g+\delta}{s}\right)^{\rho}}{\left[\left(\frac{n+g+\delta}{s}\right)^{\rho} - \alpha\right]} \right\}^{\frac{1}{\rho}} \\
&= \frac{(1-\alpha)^{\frac{1}{\rho}}\left(\frac{n+g+\delta}{s}\right)}{\left[\left(\frac{n+g+\delta}{s}\right)^{\rho} - \alpha\right]^{\frac{1}{\rho}}} \\
&= \frac{\left(\frac{n+g+\delta}{s}\right)}{\left[\frac{1}{1-\alpha}\left(\frac{n+g+\delta}{s}\right)^{\rho} - \frac{\alpha}{1-\alpha}\right]^{\frac{1}{\rho}}} \\
&= \left[\frac{1}{1-\alpha} - \frac{\alpha}{1-\alpha}\left(\frac{s}{n+g+\delta}\right)^{\rho}\right]^{-\frac{1}{\rho}} \\
&= \left[\frac{1}{1-\alpha} - \frac{\alpha}{1-\alpha}\left(\frac{s}{n+g+\delta}\right)^{\frac{\sigma-1}{\sigma}}\right]^{-\frac{\sigma}{\sigma-1}}.
\end{aligned}
$$

The last expression of y^* is equation (3) in the text. Define $z = -\frac{\alpha}{1-\alpha}$ and $(1-z) = \frac{1}{1-\alpha}$ and rewrite y^* as

$$
y^* = \left[z\left(\frac{n+g+\delta}{s}\right)^{-\rho} + (1-z)\right]^{-\frac{1}{\rho}}. \tag{B6}
$$

A second order Taylor series expansion around $\rho = 0$ ($\sigma = 1$) as in Kmenta (1967) yields

$$
\begin{aligned}
\ln y &= \ln A + z\ln\left(\frac{n+g+\delta}{s}\right) - \frac{1}{2}\rho z(1-z)\left[\ln\left(\frac{n+g+\delta}{s}\right)\right]^2 \\
&= \ln A(0) + gt - \frac{\alpha}{1-\alpha}\ln\left(\frac{n+g+\delta}{s}\right) + \frac{1}{2}\frac{\sigma-1}{\sigma}\frac{\alpha}{(1-\alpha)^2}\left[\ln\left(\frac{n+g+\delta}{s}\right)\right]^2,
\end{aligned}
$$

$$
\ln y = \ln A(0) + gt + \frac{\alpha}{1-\alpha}\ln\left(\frac{s}{n+g+\delta}\right) + \frac{1}{2}\frac{\sigma-1}{\sigma}\frac{\alpha}{(1-\alpha)^2}\left[\ln\left(\frac{s}{n+g+\delta}\right)\right]^2.
$$

which is equation (4) in the text.

Step-by-step derivation of the extended Solow-CES equation

The aggregate production function is now given by the CES specification

$$Y = [\alpha K^\rho + \beta H^\rho + (1-\alpha-\beta)(AL)^\rho]^{\frac{1}{\rho}}. \tag{B7}$$

Dividing through by AL gives the intensive form

$$y = [\alpha k^\rho + \beta h^\rho + (1-\alpha-\beta)]^{\frac{1}{\rho}}. \tag{B8}$$

The laws of motion for physical and human capital are give respectively by

$$\dot{k} = s_k y - (n+g+\delta)k \tag{B9}$$

$$\dot{h} = s_h y - (n+g+\delta)h. \tag{B10}$$

Substituting (B8) into (B9) gives

$$\dot{k} = s_k\left[\alpha k^\rho + \beta h^\rho + (1-\alpha-\beta)\right]^{\frac{1}{\rho}} - (n+g+\delta)k \overset{ss}{=} 0$$

$$\alpha k^\rho + \beta h^\rho + (1-\alpha-\beta) = \left[\frac{(n+g+\delta)k}{s_k}\right]^\rho$$

$$\beta h^\rho + (1-\alpha-\beta) = \left[\left(\frac{n+g+\delta}{s_k}\right)^\rho - \alpha\right]k^\rho$$

$$k^* = \left[\frac{\beta h^\rho + (1-\alpha-\beta)}{\left(\frac{n+g+\delta}{s_k}\right)^\rho - \alpha}\right]^{\frac{1}{\rho}}. \tag{B11}$$

Similarly,

$$\dot{h} = s_h\left[\alpha k^\rho + \beta h^\rho + (1-\alpha-\beta)\right]^{\frac{1}{\rho}} - (n+g+\delta)h \overset{ss}{=} 0$$

$$\alpha k^\rho + \beta h^\rho + (1-\alpha-\beta) = \left[\frac{(n+g+\delta)h}{s_h}\right]^\rho$$

$$\alpha k^\rho + (1-\alpha-\beta) = \left[\left(\frac{n+g+\delta}{s_h}\right)^\rho - \beta\right]h^\rho$$

$$h^* = \left[\frac{\alpha k^\rho + (1-\alpha-\beta)}{\left(\frac{n+g+\delta}{s_h}\right)^\rho - \beta}\right]^{\frac{1}{\rho}}. \tag{B12}$$

Substituting (B12) into (B11) obtains

$$k^* = \left\{ \frac{\alpha\beta k^\rho + \beta(1-\alpha-\beta) + (1-\alpha-\beta)\left[\left(\frac{n+g+\delta}{s_h}\right)^\rho - \beta\right]}{\left[\left(\frac{n+g+\delta}{s_h}\right)^\rho - \beta\right]\left[\left(\frac{n+g+\delta}{s_k}\right)^\rho - \alpha\right]} \right\}^{\frac{1}{\rho}}$$

$$k^\rho \left[\left(\frac{n+g+\delta}{s_k}\right)^\rho - \alpha\right] = \frac{\alpha\beta k^\rho + (1-\alpha-\beta)\left(\frac{n+g+\delta}{s_h}\right)^\rho}{\left(\frac{n+g+\delta}{s_h}\right)^\rho - \beta}$$

$$k^\rho \left[\left(\frac{n+g+\delta}{s_k}\right)^\rho - \alpha\right] - \frac{\alpha\beta k^\rho}{\left(\frac{n+g+\delta}{s_h}\right)^\rho - \beta} = \frac{(1-\alpha-\beta)\left(\frac{n+g+\delta}{s_h}\right)^\rho}{\left(\frac{n+g+\delta}{s_h}\right)^\rho - \beta}$$

$$k^\rho \left[\left(\frac{n+g+\delta}{s_k}\right)^\rho - \frac{\alpha\beta}{\left(\frac{n+g+\delta}{s_h}\right)^\rho - \beta} - \alpha\right] = \frac{(1-\alpha-\beta)\left(\frac{n+g+\delta}{s_h}\right)^\rho}{\left(\frac{n+g+\delta}{s_h}\right)^\rho - \beta}$$

$$k^\rho \left\{ \frac{\left(\frac{n+g+\delta}{s_k}\right)^\rho \left[\left(\frac{n+g+\delta}{s_h}\right)^\rho - \beta\right] - \alpha\left(\frac{n+g+\delta}{s_h}\right)^\rho}{\left[\left(\frac{n+g+\delta}{s_h}\right)^\rho - \beta\right]} \right\} = \frac{(1-\alpha-\beta)\left(\frac{n+g+\delta}{s_h}\right)^\rho}{\left(\frac{n+g+\delta}{s_h}\right)^\rho - \beta}$$

$$k^\rho \left\{ \left(\frac{n+g+\delta}{s_k}\right)^\rho \left[1 - \beta\left(\frac{n+g+\delta}{s_h}\right)^{-\rho}\right] - \alpha \right\} = 1-\alpha-\beta$$

$$k^\rho \left[\left(\frac{n+g+\delta}{s_k}\right)^\rho - \beta\left(\frac{s_h}{s_k}\right)^\rho - \alpha\right] = 1-\alpha-\beta.$$

Therefore,

$$k^* = \left[\frac{1-\alpha-\beta}{\left(\frac{n+g+\delta}{s_k}\right)^\rho - \beta\left(\frac{s_h}{s_k}\right)^\rho - \alpha}\right]^{\frac{1}{\rho}}. \tag{B13}$$

Similarly,

$$h^* = \left[\frac{1-\alpha-\beta}{\left(\frac{n+g+\delta}{s_h}\right)^\rho - \alpha\left(\frac{s_k}{s_h}\right)^\rho - \beta}\right]^{\frac{1}{\rho}}. \tag{B14}$$

Substituting (B13) and (B14) into the intensive production function $y = [\alpha k^\rho + \beta h^\rho + (1-\alpha-\beta)]^{\frac{1}{\rho}}$ yields the steady-state output per effective labor

$$
\begin{aligned}
y^* &= \left\{\alpha\left[\frac{1-\alpha-\beta}{\left(\frac{n+g+\delta}{s_k}\right)^\rho-\beta\left(\frac{s_h}{s_k}\right)^\rho-\alpha}\right]+\beta\left[\frac{1-\alpha-\beta}{\left(\frac{n+g+\delta}{s_h}\right)^\rho-\beta\left(\frac{s_k}{s_h}\right)^\rho-\beta}\right]+(1-\alpha-\beta)\right\}^{\frac{1}{\rho}} \\
&= \left\{(1-\alpha-\beta)\left[\frac{\alpha}{\left(\frac{n+g+\delta}{s_k}\right)^\rho-\beta\left(\frac{s_h}{s_k}\right)^\rho-\alpha}+\frac{\beta}{\left(\frac{n+g+\delta}{s_h}\right)^\rho-\beta\left(\frac{s_k}{s_h}\right)^\rho-\beta}+1\right]\right\}^{\frac{1}{\rho}} \\
&= \left\{(1-\alpha-\beta)\left[\frac{\alpha\left(\frac{n+g+\delta}{s_h}\right)^\rho-\alpha^2\left(\frac{s_k}{s_h}\right)^\rho-\alpha\beta+\beta\left(\frac{n+g+\delta}{s_k}\right)^\rho-\beta^2\left(\frac{s_h}{s_k}\right)^\rho-\alpha\beta}{\left[\left(\frac{n+g+\delta}{s_k}\right)^\rho-\beta\left(\frac{s_h}{s_k}\right)^\rho-\alpha\right]\left[\left(\frac{n+g+\delta}{s_h}\right)^\rho-\alpha\left(\frac{s_k}{s_h}\right)^\rho-\beta\right]}+1\right]\right\}^{\frac{1}{\rho}}.
\end{aligned}
$$

Expanding the denominator gives

$$
\frac{(n+g+\delta)^{2\rho}}{(s_h s_k)^\rho}-2\beta\left(\frac{n+g+\delta}{s_k}\right)^\rho-2\alpha\left(\frac{n+g+\delta}{s_h}\right)^\rho+2\alpha\beta+\alpha^2\left(\frac{s_k}{s_h}\right)^\rho+\beta^2\left(\frac{s_h}{s_k}\right)^\rho.
$$

Bringing all the terms in over the denominator gives the following numerator:

$$
\frac{(n+g+\delta)^{2\rho}}{(s_h s_k)^\rho}-\beta\left(\frac{n+g+\delta}{s_k}\right)^\rho-\alpha\left(\frac{n+g+\delta}{s_h}\right)^\rho,
$$

or,

$$
(n+g+\delta)^\rho\left[\left(\frac{n+g+\delta}{s_h s_k}\right)^\rho-\frac{\beta}{s_k^\rho}-\frac{\alpha}{s_h^\rho}\right].
$$

Therefore,

$$
y^*=\left\{\frac{(1-\alpha-\beta)(n+g+\delta)^\rho\left[\left(\frac{n+g+\delta}{s_h s_k}\right)^\rho-\frac{\beta}{s_k^\rho}-\frac{\alpha}{s_h^\rho}\right]}{\left[\left(\frac{n+g+\delta}{s_k}\right)^\rho-\beta\left(\frac{s_h}{s_k}\right)^\rho-\alpha\right]\left[\left(\frac{n+g+\delta}{s_h}\right)^\rho-\alpha\left(\frac{s_k}{s_h}\right)^\rho-\beta\right]}\right\}^{\frac{1}{\rho}}.
$$

Multiply top and bottom by $(s_h s_k)^\rho$ to obtain

$$
\begin{aligned}
y^* &= \left\{\frac{(1-\alpha-\beta)(n+g+\delta)^\rho\left[(n+g+\delta)^\rho-\beta s_h^\rho-\alpha s_k^\rho\right]}{\left[(n+g+\delta)^\rho-\beta s_h^\rho-\alpha s_k^\rho\right]\left[(n+g+\delta)^\rho-\beta s_h^\rho-\alpha s_k^\rho\right]}\right\}^{\frac{1}{\rho}} \\
&= \left[\frac{(1-\alpha-\beta)(n+g+\delta)^\rho}{(n+g+\delta)^\rho-\beta s_h^\rho-\alpha s_k^\rho}\right]^{\frac{1}{\rho}} \\
&= \left[\frac{(1-\alpha-\beta)}{1-\beta\left(\frac{s_h}{n+g+\delta}\right)^\rho-\alpha\left(\frac{s_k}{n+g+\delta}\right)^\rho}\right]^{\frac{1}{\rho}} \\
&= \left[\frac{1-\beta\left(\frac{s_h}{n+g+\delta}\right)^\rho-\alpha\left(\frac{s_k}{n+g+\delta}\right)^\rho}{1-\alpha-\beta}\right]^{-\frac{1}{\rho}}
\end{aligned}
$$

$$= \left[\frac{1}{(1-\alpha-\beta)} - \frac{\beta}{(1-\alpha-\beta)}\left(\frac{s_h}{n+g+\delta}\right)^{\rho} - \frac{\alpha}{(1-\alpha-\beta)}\left(\frac{s_k}{n+g+\delta}\right)^{\rho}\right]^{-\frac{1}{\rho}}$$

$$= \left[\frac{1}{(1-\alpha-\beta)} - \frac{\alpha}{(1-\alpha-\beta)}\left(\frac{s_k}{n+g+\delta}\right)^{\frac{\sigma}{\sigma-1}} - \frac{\beta}{(1-\alpha-\beta)}\left(\frac{s_h}{n+g+\delta}\right)^{\frac{\sigma}{\sigma-1}}\right]^{-\frac{\sigma-1}{\sigma}} \quad \text{(B15)}$$

which is equation (5) in the text.

Define $a_0 = \frac{1}{(1-\alpha-\beta)}$, $a_1 = -\frac{\beta}{(1-\alpha-\beta)}$, and $a_2 = -\frac{\alpha}{(1-\alpha-\beta)}$ (note that $a_0 + a_1 + a_2 = 1$) and let $\bar{H} = \frac{s_h}{(n+g+\delta)}$, $\bar{K} = \frac{s_k}{(n+g+\delta)}$.

The production function can then be written as

$$y = \left(a_0 + a_1\bar{H}^{\rho} + a_2\bar{K}^{\rho}\right)^{-\frac{1}{\rho}}. \quad \text{(B16)}$$

Taking logs gives

$$\ln(y) = -\frac{1}{\rho}\ln\left(a_0 + a_1\bar{H}^{\rho} + a_2\bar{K}^{\rho}\right). \quad \text{(B17)}$$

Let

$$f(\rho) = \ln\left(a_0 + a_1\bar{H}^{\rho} + a_2\bar{K}^{\rho}\right). \quad \text{(B18)}$$

The second order Taylor series approximation of $f(\rho)$ around $\rho = 0$ obtains $f(\rho) \approx f(0) + \rho f'(0) + \frac{\rho^2}{2}f''(0)$:

$$f(0) = \ln\left(a_0 + a_1 + a_2\right) = \ln[1] = 0 \quad \text{(B19)}$$

$$f'(\rho) = \frac{a_1\bar{H}^{\rho}\ln\bar{H} + a_2\bar{K}^{\rho}\ln\bar{K}}{a_0 + a_1\bar{H}^{\rho} + a_2\bar{K}^{\rho}} \quad \text{(B20)}$$

$$\begin{aligned} f'(0) &= \frac{a_1\ln\bar{H} + a_2\ln\bar{K}}{a_0 + a_1 + a_2} = a_1\ln\bar{H} + a_2\ln\bar{K} \\ &= -\frac{\beta}{(1-\alpha-\beta)}\ln\frac{s_h}{(n+g+\delta)} - \frac{\alpha}{(1-\alpha-\beta)}\ln\frac{s_k}{(n+g+\delta)} \end{aligned} \quad \text{(B21)}$$

$$f''(\rho) = \frac{\left(a_0 + a_1\bar{H}^{\rho} + a_2\bar{K}^{\rho}\right)\left[a_1\bar{H}^{\rho}\left(\ln\bar{H}\right)^2 + a_2\bar{K}^{\rho}\left(\ln\bar{K}\right)^2\right] - \left(a_1\bar{H}^{\rho}\ln\bar{H} + a_2\bar{K}^{\rho}\ln\bar{K}\right)^2}{\left(a_0 + a_1\bar{H}^{\rho} + a_2\bar{K}^{\rho}\right)^2} \quad \text{(B22)}$$

$$f''(0) = \frac{\left(a_0 + a_1 + a_2\right)\left[a_1\left(\ln\bar{H}\right)^2 + a_2\left(\ln\bar{K}\right)^2\right] - \left(a_1\ln\bar{H} + a_2\ln\bar{K}\right)^2}{\left(a_0 + a_1 + a_2\right)^2}. \quad \text{(B23)}$$

Expanding the numerator of equation (B23) gives

$$a_0\left[a_1\left(\ln\bar{H}\right)^2+a_2\left(\ln\bar{K}\right)^2\right]+a_1^2(\ln\bar{H})^2+a_1a_2(\ln\bar{K})^2+a_2^2(\ln\bar{K})^2+$$
$$a_1a_2(\ln\bar{H})^2-a_1^2\left(\ln\bar{H}\right)^2-a_2^2\left(\ln\bar{K}\right)^2-2a_1a_2\left(\ln\bar{K}\ln\bar{H}\right).$$

Hence,

$$f''(0)=\frac{a_0a_1\left(\ln\bar{H}\right)^2+a_0a_2\left(\ln\bar{K}\right)^2+a_1a_2\left[\left(\ln\bar{K}\right)^2-2\ln\bar{K}\ln\bar{H}+\left(\ln\bar{H}\right)^2\right]}{\left(a_0+a_1+a_2\right)^2}. \tag{B24}$$

Using that $a_0=\frac{1}{(1-\alpha-\beta)}$, $a_1=-\frac{\beta}{(1-\alpha-\beta)}$, $a_2=-\frac{\alpha}{(1-\alpha-\beta)}\Rightarrow a_0+a_1+a_2=1$ gives

$$f''(0)=-\frac{\beta}{(1-\alpha-\beta)^2}\left(\ln\bar{H}\right)^2-\frac{\alpha}{(1-\alpha-\beta)^2}\left(\ln\bar{K}\right)^2+\frac{\alpha\beta}{(1-\alpha-\beta)^2}\left(\ln\bar{K}-\ln\bar{H}\right)^2$$

$$\begin{aligned} f''(0) &= -\frac{\beta}{(1-\alpha-\beta)^2}\left[\ln\left(\frac{s_h}{n+g+\delta}\right)\right]^2-\frac{\alpha}{(1-\alpha-\beta)^2}\left[\ln\left(\frac{s_k}{n+g+\delta}\right)\right]^2+\\ &\quad \frac{\alpha\beta}{(1-\alpha-\beta)^2}\left[\ln\left(\frac{s_k}{s_h}\right)\right]^2. \end{aligned} \tag{B25}$$

Substituting (B19), (B21) and (B25) in $f(\rho)=f(0)+\rho f'(0)+\frac{\rho^2}{2}f''(0)$, obtains

$$\begin{aligned} f(\rho) &= \rho\left[-\frac{\alpha}{1-\alpha-\beta}\ln\left(\frac{s_k}{n+g+\delta}\right)-\frac{\beta}{1-\alpha-\beta}\ln\left(\frac{s_h}{n+g+\delta}\right)\right]-\frac{\rho^2\beta}{2(1-\alpha-\beta)^2}\left[\ln\left(\frac{s_h}{n+g+\delta}\right)\right]^2\\ &\quad -\frac{\rho^2\alpha}{2(1-\alpha-\beta)^2}\left[\ln\left(\frac{s_k}{n+g+\delta}\right)\right]^2+\frac{\rho^2\alpha\beta}{2(1-\alpha-\beta)^2}\left[\ln\left(\frac{s_k}{s_h}\right)\right]^2. \end{aligned} \tag{B26}$$

Finally, given that $\ln y=-\frac{1}{\rho}f(\rho)$ then

$$\begin{aligned} \ln y &= \frac{\alpha}{1-\alpha-\beta}\ln\left(\frac{s_k}{n+g+\delta}\right)+\frac{\beta}{1-\alpha-\beta}\ln\left(\frac{s_h}{n+g+\delta}\right)+\frac{\rho\alpha}{\left(1-\alpha-\beta\right)^2}\left[\ln\left(\frac{s_k}{n+g+\delta}\right)\right]^2\\ &\quad +\frac{\rho\beta}{\left(1-\alpha-\beta\right)^2}\left[\ln\left(\frac{s_h}{n+g+\delta}\right)\right]^2-\frac{\rho\alpha\beta}{\left(1-\alpha-\beta\right)^2}\left[\ln\left(\frac{s_k}{s_h}\right)\right]^2, \end{aligned} \tag{B27}$$

or,

$$\begin{aligned} \ln\left(\frac{Y}{L}\right) &= \ln A(0)+gt+\frac{\alpha}{1-\alpha-\beta}\ln\left(\frac{s_k}{n+g+\delta}\right)+\frac{\beta}{1-\alpha-\beta}\ln\left(\frac{s_h}{n+g+\delta}\right)+\\ &\quad \frac{1}{2}\frac{\sigma-1}{\sigma}\frac{1}{\left(1-\alpha-\beta\right)^2}\left\{\alpha\left[\ln\left(\frac{s_k}{n+g+\delta}\right)\right]^2+\beta\left[\ln\left(\frac{s_h}{n+g+\delta}\right)\right]^2-\alpha\beta\left[\ln\left(\frac{s_k}{s_h}\right)\right]^2\right\}. \end{aligned}$$

which is equation (6) in the text.