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(Non-)Existence and Scope of Nash Networks

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Abstract

For the connections model of strategic network formation, with two-way flow of information and without information decay, specific parameter configurations are given for which Nash networks do not exist. Moreover, existence and the scope of Nash network architectures are briefly discussed.

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1 Introduction

Jackson and Watts (2001) give necessary as well as sufficient conditions for the existence of pairwise stable networks. Jackson and Watts (2002) provide an example for non-existence of pairwise stable networks. Jackson (2005) shows existence of such networks for several prominent allocation rules. In comparison with the work on pairwise stability, existence has been less systematically explored in the literature on Nash networks. In this note, we consider a strategic model of network formation that is based on the connections model without decay and allows for heterogeneity of costs, values and links. The model permits two-way flow of information. We draw and expand on previous work by Bala and Goyal (2000a, b), Galeotti, Goyal, and Kamphorst (2004), and Haller and Sarangi (2004). We provide examples of non-existence and present instances of existence of Nash networks. We delineate the scope of Nash network architectures under various heterogeneity assumptions.

2 The Model

Let $n \geq 3$. $N = \{1, \ldots, n\}$ denotes the set of players with generic elements i, j, k. For ordered pairs $(i, j) \in N \times N$, the shorthand notation ij is used and for non-ordered pairs $\{i, j\}$ the shorthand $[ij]$ is used. The symbol $\subset$ for set inclusion permits equality. The model is specified by three families of parameters, indexed by ij, with $i \neq j$:

- Cost parameters $c_{ij} > 0$.
- Value parameters $V_{ij} > 0$.
- Link success probabilities $p_{ij} \in (0, 1]$.

In case $c_{ij} \neq c_{kl}$ ($V_{ij} \neq V_{kl}$, $p_{ij} \neq p_{kl}$) for some $ij \neq kl$, the model exhibits **cost (value, link) heterogeneity**; otherwise, it exhibits **cost (value, link) homogeneity**. In case $p_{ij} = 1$ for all ij, the model has **perfectly reliable links**; otherwise, it has **imperfectly reliable links**.

A pure strategy for player i's is a vector $g_i = (g_{i1}, \ldots, g_{ii-1}, g_{ii+1}, \ldots, g_{in}) \in \{0, 1\}^{N \setminus \{i\}}$. We only consider pure strategies. The set of all pure strategies of agent i is denoted by $\mathcal{G}_i$. It consists of 2^{n-1} elements. The joint strategy space is given by $\mathcal{G} = \mathcal{G}_1 \times \cdots \times \mathcal{G}_n$.

There is a one-to-one correspondence between the set of joint strategies $\mathcal{G}$ and the set of all directed graphs or networks with vertex set N. Namely, to a strategy profile $g = (g_1, \ldots, g_n) \in \mathcal{G}$ corresponds the graph $(N, E(g))$ with edge or node set $E(g) = \{(i,j) \in N \times N |\, i \neq j,\, g_{ij} = 1\}$. In the sequel, we shall identify a joint strategy g and the corresponding graph and use the terminology **directed graph** or **directed network** g. Since our aim is to model network formation, $g_{ij} = 1$ is interpreted to mean that a direct link between i and j is initiated by player i (node ij is formed by i) whereas $g_{ij} = 0$ means that i does not initiate the link (ij is not formed). Regardless of what player i does, player j can set $g_{ji} = 1$, i.e. initiate a link with i, or set $g_{ji} = 0$, i.e. not initiate a link with i.

To describe information flows and compute benefits associated with the network g, it is useful to introduce the closure of g. This is a non-directed network denoted $\overline{g}$ and defined by $\overline{g}_{ij} := \max\{g_{ij}, g_{ji}\}$ for $i \neq j$. Pictorially, forming the closure of a network simply means replacing every directed edge ij of g by the non-directed edge $[ij]$.

Benefits with Perfectly Reliable Links. A link between agents i and j potentially allows for **two-way (symmetric) flow of information**. Therefore, the benefits from network g are derived from its closure $\overline{g}$. Moreover, a player receives information from others not only through direct links, but also via indirect links. To be precise, information flows from player j to player i, if i and j are linked by means of a path in $\overline{g}$ from i to j. For a non-directed graph $h \in \mathcal{G}$, a **path** of length m in h from player i to player $j \neq i$, is a finite sequence $i_0, i_1, \ldots, i_m$ of pairwise distinct players such that $i_0 = i$, $i_m = j$, and $h_{i_k i_{k+1}} = 1$ for $k = 0, \ldots, m-1$. Let us denote

$$N_i(h) = \{j \in N |\, j \neq i, \text{ there exists a path in } h \text{ from } i \text{ to } j\},$$

the set of other players whom player i can access or "observe" in the non-directed network h. Information received from player j is worth V_{ij} to player i. Therefore, player i's benefit from a network g with perfectly reliable links is

$$b_i(g) = b_i(\overline{g}) = \sum_{j \in N_i(\overline{g})} V_{ij}.$$

Benefits with Imperfectly Reliable Links. Imperfect reliability of links means that $p_{ij} \in (0,1)$ for some $i \neq j$. Recall that for any $i \neq j$, the non-ordered pair $[ij]$ represents the simultaneous occurrence of ij and ji. Again, $\overline{g}$, the closure of g, determines the possible information flows. If $\overline{g}_{ij} = 0$, then as before $[ij]$ does not permit any information flow. But now if $\overline{g}_{ij} = 1$, then $[ij]$ succeeds (allows information flow) with probability $p_{ij} \in (0,1)$ and fails (does not permit information flow) with probability

$1-p_{ij}$, where p_{ij} is not necessarily equal to p_{ik} for $j \neq k$. It is assumed, however, that $p_{ij} = p_{ji}$. Furthermore, the successes of direct links between different pairs of agents are assumed to be independent events. Thus, $\overline{g}$ may be regarded as a random network with possibly different probabilities of realization for different edges. We call a non-directed network $h \in \mathcal{G}$ a realization of $\overline{g}$ ($h \subset \overline{g}$) if it satisfies $h_{ij} \leq \overline{g}_{ij}$ for all i, j with $i \neq j$. Invoking the independence assumption, the probability of the network h being realized, given $\overline{g}$ is

$$\lambda(h \mid \overline{g}) = \textstyle\prod_{[ij]\in h} p_{ij} \prod_{[ij]\notin h}(1-p_{ij}).$$

Given a strategy profile g, i's expected benefit from the random network $\overline{g}$ is

$$B_i(\overline{g}) = \textstyle\sum_{h\subset\overline{g}} \lambda(h \mid \overline{g}) b_i(h).$$

Namely, the probability of network h being realized is given by $\lambda(h \mid \overline{g})$, in which case player i obtains benefit $b_i(h)$. Summation over all possible realizations $h \subset \overline{g}$ yields expected benefits.

Costs. Player i incurs the cost c_{ij} when she initiates the direct link ij, i.e. if $g_{ij} = 1$. Hence i incurs the total costs

$$C_i(g) = \sum_{j\neq i} g_{ij} c_{ij}$$

when the network g is formed.

Payoffs. Player i's expected payoff from the strategy profile g is the net benefit

$$\Pi_i(g) = B_i(\overline{g}) - C_i(g). \quad (1)$$

Nash Networks. Given a network $g \in \mathcal{G}$, let g_{-i} denote the network that remains when all of agent i's links have been removed. Clearly $g = g_i \oplus g_{-i}$ where the symbol $\oplus$ indicates that g is formed by the union of links in g_i and g_{-i}. A strategy g_i is a **best response** of agent i to g_{-i} if

$$\Pi_i(g_i \oplus g_{-i}) \geq \Pi_i(g_i' \oplus g_{-i}) \;\text{ for all }\; g_i' \in \mathcal{G}_i.$$

Let $BR_i(g_{-i})$ denote the set of agent i's best responses to g_{-i}. A network $g = (g_1, \ldots, g_n)$ is said to be a **Nash network** if $g_i \in BR_i(g_{-i})$ for each i, that is if g is a Nash equilibrium of the strategic game with normal form $(N, (\mathcal{G}_i)_{i\in N}, (\Pi_i)_{i\in N})$. A strict Nash network is one where agents are playing strict best responses.

Graph-theoretic Concepts. We now introduce some definitions of a more graph-theoretic nature. A network with no links is called an **empty network**. A network g is said to be **connected** if there is a path in $\overline{g}$ between any two agents i and j. A connected network g is **minimally connected**, if it is no longer connected after the deletion of *any* link.

A set $C \subset N$ is called a **component** of g if there exists a path in $\overline{g}$ between any two different agents i and j in C and there is no strict superset C' of C for which this holds true. For each network g, the components of g form a partition of the player set (node set, vertex set) N into non-empty subsets. Each isolated point $i \in N$ in g, that is a player or node i with $g_{ij} = g_{ji} = 0$ for all $j \neq i$, gives rise to a singleton component $\{i\}$. In particular, the components of the empty network are the sets $\{i\}, i \in N$. N is the only component of g if and only if g is connected. If C is a component of the network g, we denote by g^C the network induced by g on the set of agents C, that is $g^C_{ij} = g_{ij}$ for $i, j \in C$, $i \neq j$. A network g is **minimal**, if g^C is minimally connected for every component C of g. Minimally connected networks are both connected and minimal.

We finally introduce the notion of an essential network. A network $g \in \mathcal{G}$ is **essential** if $g_{ij} = 1$ implies $g_{ji} = 0$. Note that if $g \in \mathcal{G}$ is a Nash network or an efficient network, then it must be essential. This follows from the fact that for each link ij, $c_{ij} > 0$ and the information flow is two-way and independent of which agent invests in forming the link, that is $h_{ij} = \max\{g_{ij}, g_{ji}\}$. Minimal networks are also essential.

3 Non-Existence of Nash Networks

Our first example constitutes a 4-player game with cost homogeneity and both value and link heterogeneity.

Example 1: Let $n = 4$, $c_{ij} = c = 0.95$ for all ij, $V_{i1} = 1$ for $i \neq 1$, $V_{i2} = 2$ for $i \neq 2$, $V_{i3} = 64$ for $i \neq 3$, $V_{i4} = 16$ for $i \neq 4$. Set $p = p_{12} = p_{21} = 0.4$; $q = p_{23} = p_{32} = 0.01473$; $r = p_{34} = p_{43} = 1/32$; $s = p_{14} = p_{41} = 1/16$; and $t = p_{ij} = 1/200 < 1/(3 \cdot 64)$ for all remaining ij. Obviously, none of the links ij with $p_{ij} = t$ will be established. Moreover, 1 will always establish the link 14, 4 will always establish the link 43, 2 will never establish the link 21 and 3 will never establish the link 32. Now the existence of a Nash network can be decided by assessing the benefits from links 12 and 23 to players 1 and 2, respectively, given that all other links have been established or not according to our foregoing account. We obtain:

- Without 23, player 1 strictly prefers not to establish 12.
- With 23, player 1 strictly prefers to establish 12.
- Without 12, the benefit to player 2 from link 23 is 0.95011 and establishing 23 is a strict best response.
- With 12, player 2's benefit from link 23 is reduced by $81pqrs = 0.00093$ (due to redundancies) and not establishing 23 is a strict best response.

Hence there are no mutual best responses regarding establishment of 12 and 23. Consequently, a Nash network does not exist.

To understand why the particular choice of q has player 2 switch back and forth, replace q by a $\widetilde{q}$ such that without 12, player 2 is indifferent between having and not having the link 23, i.e. $\widetilde{q} \cdot (64 + r \cdot 16 + rs) = c$. This yields $\widetilde{q} = 0.014728236$. Then with 12, player 2 would not want the link because of redundancies. By continuity, q slightly larger than $\widetilde{q}$ produces the best response properties exhibited above. ■

The next example constitutes a 4-player game with perfectly reliable links, cost heterogeneity and value homogeneity.

Example 2: Let $n = 4$ and $V_{ij} = V > 0$ for all ij. Suppose $c_{1k} > 3V$ for all $k \neq 1$; $c_{23} = c_{24} > 3V$ and $c_{21} < V$; $V < c_{34} < c_{32} < 2V < 3V < c_{31}$; $2V < c_{42} < 3V < c_{41} = c_{43}$. Then the unique best reply of player 1 to any network is to add no links at all. The unique best reply of player 2 to any network g_{-2} in which he does not observe player 1 is to add a link to player 1 only. Players 3 and 4 will never have a link to player 1 as part of their best reply. Moreover, in a best reply player 4 will never initiate a link to player 3.

Now let us take those best replies for granted and consider best responses regarding the remaining links 32, 34, and 42. If player 4 initiates link 42, then player 3's best response is to initiate link 34 and not 32, and in turn player 3's best response is not to form link 42. If player 4 does not initiate link 42, then player 3's best response is to form link 32 and not 34, against which player 4's best response is to initiate link 42. Hence there do not exist any mutual best responses. Therefore, a Nash network does not exist. ■

Remark 1. Example 1 can be viewed a reduced form of Example 2 in Haller and Sarangi (2004), a 83-player game with cost as well as value homogeneity and based only on link heterogeneity. The latter example is obtained by introducing additional players labelled $0, 301, \ldots, 363, 401, \ldots, 415$. Choose $c_{ij} = 0.95$ and $V_{ij} = 1$ for all ij. As before, set $p = p_{12} = p_{21} = 0.4$; $q = p_{23} = p_{32} = 0.01473$; $r = p_{34} = p_{43} = 1/32; s = p_{14} = p_{41} = 1/16$. Further put $p_{20} = p_{02} = 1$; $p_{3j} = p_{j3} = 1$ for $j = 301, \ldots, 363$; $p_{4j} = p_{j4} = 1$ for $j = 401, \ldots, 415$. For the remaining ij, set $p_{ij} < c/82$ so that the corresponding link will not be formed. The analysis of this example reduces to exactly the same numerical matching pennies game regarding links 12 and 23 as in Example 1.

Remark 2. There exists a 3-player game with perfectly reliable links, which exhibits both cost and value heterogeneity and does not have a Nash network.

Remark 3. So far the literature on strategic network formation has not considered equilibria in mixed strategies — which would overcome the existence problem in finite games.

4 Existence of Nash Networks

Bala and Goyal (2000a) outline a constructive proof of the existence of Nash networks in the case of perfect reliability of links, cost and value homogeneity. Indeed, existence can be shown under the assumption of perfect reliability of links and cost homogeneity, allowing for value heterogeneity.

Proposition. *Let links be perfectly reliable and costs be homogeneous. Then a Nash network exists.*

PROOF. We construct a minimal network which is Nash, beginning with the empty network. The empty network is minimal and has the property that no player benefits from deleting a link.

Next let g be any minimal network with the property that no player benefits from deleting a link. Since g is minimal, a link ik in g connects i with the members of k's component in g_{-i}. By assumption, i does not gain from simply severing that link. Because of cost homogeneity, player i does not strictly prefer to replace that link with a link to another member of k's component in g_{-i}. Consequently, there remain two possibilities: either (a) g is Nash or (b) some player is better off by sponsoring an additional link. In the latter case, suppose that player i is better off sponsoring the additional link ij and denote $g' = g \oplus ij$. Since player i is better off sponsoring the extra link ij, i and j belong to different components of g. Hence g' is also minimal. Moreover, adding the link ij makes the existing links more valuable. Therefore, no player benefits from deleting a link in g'.

We have shown so far: If g is a minimal network with the property that no player benefits from deleting a link and g is not Nash, then adding a suitably chosen link to g creates a larger minimal network g' with the property that no player benefits from deleting a link.

Now let us begin with the empty network and label it g^0. In case g^0 is Nash, we are done. Otherwise, by the previous argument, there exists a minimal network g^1 with one link and the property that no player benefits from deleting a link. In case g^1 is Nash, we are done. Otherwise, there exists a minimal network g^2 with two links and the property that no player benefits from deleting a link, etc. Since a minimal network with n nodes has at most $n-1$ links, in finitely many steps, say k steps with $0 \leq k \leq n-1$, a minimal network g^k is reached which has k links and is Nash. □□

To summarize, in the case of perfect reliability of links, a Nash network always exists when costs are homogeneous, whereas Nash network do not always exist when costs are heterogeneous. We have also presented examples of non-existence which exhibit link heterogeneity and cost homogeneity, with or without value homogeneity.

In addition, the literature contains assertions that for certain parameter ranges, the model admits Nash networks with specific properties. This amounts to providing sufficient conditions for the existence of certain Nash networks. If the various regions happen to cover the entire parameter space, then as a by-product, existence has been shown for the particular model. For instance, Bala and Goyal (2000b) do this for the case of $n = 3$, with imperfect reliability of links and cost, link, and value homogeneity. Existence for $n > 3$ is an open question.

5 Scope of Nash Network Architectures

Instead of addressing the existence problem directly, most of the literature is devoted to the question of which Nash network architectures may arise under specific parameter restrictions. Here we delineate the scope of Nash network architectures in our model under various heterogeneity assumptions. In each instance, we list the only possible (strict) Nash network architectures — and in each instance except (E), any such network can be obtained as a (strict) Nash network upon suitable choice of parameters.

(A) Perfectly reliable links; cost and value homogeneity. Nash networks (Bala and Goyal (2000a)): The empty network and minimally connected networks. Strict Nash networks (Bala and Goyal (2000a)): The empty network and center-sponsored stars.

(B) Perfectly reliable links; cost homogeneity; value heterogeneity. Strict Nash networks (Galeotti, Goyal, and Kamphorst (2004)): The empty network and minimal networks in which every non-singleton component is a center-sponsored star.

(C) Perfectly reliable links; cost heterogeneity; value homogeneity. Strict Nash networks (Galeotti, Goyal, and Kamphorst (2004)): Minimal networks.

(D) Perfectly reliable links; cost and value heterogeneity. Strict Nash networks (Galeotti, Goyal, and Kamphorst (2004)): Minimal networks.

(E) Imperfectly reliable links; cost, link, and value homogeneity. Nash networks (Bala and Goyal (2000b)): The empty network and connected essential networks.

(F) Imperfectly reliable links; cost and value homogeneity; link heterogeneity. Nash networks, strict Nash networks (Haller and Sarangi (2004)): Essential networks.

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