The Risk-Neutral Measure and Option Pricing under Log-Stable Uncertainty

J. Huston McCulloch*

June 27, 2003

The fact that expected payoffs on assets and call options are infinite under most log-stable distributions led both Paul Samuelson (as quoted by Smith 1976) and Robert Merton (1976) to conjecture that assets and derivatives could not be reasonably priced under these distributions, despite their attractive feature as limiting distributions under the Generalized Central Limit Theorem. Carr and Wu (2003) are able to price options under log-stable uncertainty, but only by making the extreme assumption of maximally negative skewness.

This paper demonstrates that when the observed distribution of prices is log-stable, the Risk Neutral Measure (RNM) under which asset and derivative prices may be computed as expectations is not itself log-stable in the problematic cases. Instead, the RNM is determined by the convolution of two densities, one negatively skewed stable, and the other an exponentially tilted positively skewed stable. The resulting RNM gives finite expected payoffs for all parameter values, so that the concerns of Samuelson and Merton were in fact unfounded, while the Carr and Wu restriction is unnecessary.

Since the log-stable RNM developed here is expressed in terms of its characteristic function, it enables options on log-stable assets to be computed easily by means of the Fast Fourier Transform (FFT) methodology of Carr and Madan (1999), provided a simple extension of the FFT, introduced here, is employed.

* Economics Department, Ohio State University, 1945 N. High St., Columbus OH 43210. E-mail: mcculloch.2@osu.edu. The author is indebted to participants in the Ohio State University Macroeconomics Seminar and to Álvaro Cartea, Peter Carr, Vladimir Vinogradov, Liuren Wu, and two anonymous referees for helpful comments and discussions.

JEL Fields: D81, G13. Keywords: Stable distributions, option pricing, risk-neutral measure, pricing kernel, FFT, equity premium puzzle, Romberg integration.

Revisions will be posted to <http://econ.ohio-state.edu/jhm/papers/rnm.pdf>

According to the Generalized Central Limit Theorem, if the sum of a large number of identically and independently distributed (IID) random variables has a non-degenerate limiting distribution after normalizing location and scale, the limiting distribution must be a member of the *stable* class (cf. Samorodnitsky and Taqqu 1994, Uchaikin and Zolotarev 1999). Asset price changes are the multiplicatively cumulative outcome of a vast number of contributing factors, making it natural to assume that log returns are stable, so that returns themselves are log-stable. The normal or Gaussian distribution is the most familiar member of the stable class, and the only one with finite variance. However, log returns are commonly too leptokurtic to be normal, and often are skewed as well. This consideration makes the non-Gaussian stable distributions a natural, not to mention parsimonious, choice for modeling log returns (McCulloch 1996a).

However, the heavy upper tail of most stable distributions makes the expectation of the corresponding log-stable distribution infinite. This fact left Paul Samuelson (as quoted by Smith 1976: 19) "inclined to believe in [Robert] Merton's conjecture that a strict Lévy-Pareto [stable] distribution on $\log(S^*/S)$ would lead, with $1 < \alpha < 2$, to a 5-minute warrant or call being worth 100 percent of the common." Merton (1976: 127n) further conjectured that an infinite expected future price for a stock would require the risk-free discount rate to be infinite, in order for the current price to be finite.

In a recent paper in this *Journal*, Carr and Wu (2003) are able to price options under log-stable uncertainty, but only by making the very restrictive assumption that log returns have maximally negative skewness, in order to give the returns themselves finite moments. While stock returns often exhibit negative skewness, it is only in rare instances that they appear to be maximally negatively skewed. Furthermore, prices such as foreign exchange rates obviously cannot always be negatively skewed, since their reciprocals, which are equally exchange rates, must have the opposite skewness.

The present paper shows how the Risk-Neutral Measure (RNM), a risk-adjusted density under which asset and derivative prices may be computed as expectations in an arbitrage-free market, can be derived from the underlying distribution of marginal utilities in a simple representative agent model. It then derives the characteristic function (CF) of the RNM when the Frequency Measure (FM), which governs the empirically observable distribution of returns, is a general log-stable distribution. Rather than being stable itself, the RNM for log-stable returns is shown in general to be the convolution of two densities, one a maximally negatively skewed stable density, and the other an exponentially tilted maximally positively skewed stable density. This RNM leads to finite asset and derivative prices, even when the corresponding FM has an infinite mean. It follows that above-mentioned concerns of Samuelson and Merton were unfounded, and that the finite moment restriction of Carr and Wu is not required to obtain reasonable asset and option prices under log-stable distributions.

The paper then goes on to show how the CF of the RNM can be used to quickly evaluate options for a general log-stable FM, by means of a variant on the Fast Fourier Transform (FFT) methodology introduced by Carr and Madan (1999). A mathematically equivalent formula (McCulloch 1996a) has already been used by McCulloch (1985, 1987) and Hales (1997) to evaluate options on bonds and foreign exchange rates, without the finite moment restriction of Carr and Wu, but only by means of tedious numerical integrals interpolating off tables of the maximally skewed stable distributions. The stable RNM CF presented here therefore greatly facilitates the pricing of log-stable options.[1]

I. Asset Pricing and the RNM in General

As in the representative agent model of McCulloch (1996a: §4), the price of an asset at future time T is taken to be

$$S_T = U_2/U_1,$$

where U_1 is the agent's random future marginal utility of the numeraire in which the asset is priced, and U_2 is the random future marginal utility of the asset itself. Let $g(U_1, U_2)$ be the joint pdf of U_1 and U_2 conditional on information at present time 0, so that the FM, in terms of the cumulative distribution function (cdf) of S_T, is

$$\begin{aligned} F(x) &= \Pr(S_T \leq x) = \Pr(U_2 \leq xU_1) \\ &= \int_0^\infty \int_0^{xU_1} g(U_1, U_2)\,dU_2\,dU_1, \end{aligned}$$

whence the FM probability density function (pdf) for S_T is

$$\begin{aligned} f(x) &= \int_0^\infty U_1 g(U_1, xU_1)\,dU_1 \\ &= \frac{1}{x}\int_{-\infty}^{\infty} h(v_1, v_1 + \log(x))\,dv_1, \end{aligned} \tag{1}$$

where $h(v_1, v_2) = U_1 U_2 g(U_1, U_2)$ is the joint pdf of $v_1 = \log(U_1)$ and $v_2 = \log(U_2)$. The FM, in terms of the pdf for $\log(S_T)$, is then

$$\begin{aligned} \varphi(z) &= e^z f(e^z) \\ &= \int_{-\infty}^{\infty} h(v_1, v_1 + z)\,dv_1, \end{aligned} \tag{2}$$

where $z = \log(x)$.

[1] A GAUSS program, STABOPT, which performs this evaluation, is available to researchers on the author's homepage at <http://econ.ohio-state.edu/jhm/jhm.html>.

Let F be the explicit or implicit forward price in the market at present time 0 on a contract to deliver 1 unit of the asset at future time T, with unconditional payment of F units of numeraire to be made at time T. Then the first order condition for the representative agent's expected utility to be maximized at the equilibrium 0 net position in this contract is

$$F = EU_2/EU_1. \tag{3}$$

If the asset in question is a stock with no valuable voting rights before time T that pays dividends in stock at rate d per year, and if ρ is the default-free interest rate at time 0 on numeraire-denominated loans maturing at time T, its implicit forward price can be computed from the time 0 spot price S_0 using

$$F = S_0 e^{(\rho-d)T}.$$

Let $r(x)$ be the density of the RNM in the market at time 0 for state-contingent claims at time T. By definition, $r(x)dx$ gives the value, in terms of numeraire payable unconditionally at time T, of 1 unit of numeraire payable only on the condition that $S_T \in [x, x+dx)$. The first order condition for the representative agent's expected utility to be maximized at the equilibrium 0 net position in this contract then implies that $r(x)$ is simply the FM, adjusted for the state-contingent value of the numeraire:

$$r(x) = \frac{E(U_1|U_2/U_1 = x)}{EU_1} f(x).$$

Since

$$\begin{aligned} E(U_1 \,|\, U_2/U_1 &= x) \\ &= E(e^{v_1}|v_2 = v_1 + \log x) \\ &= \frac{\int_{-\infty}^{\infty} e^{v_1} h(v_1, v_1 + \log x) dv_1}{\int_{-\infty}^{\infty} h(v_1, v_1 + \log x) dv_1}, \end{aligned}$$

it follows that

$$r(x) = \frac{1}{xEU_1} \int_0^{\infty} e^{v_1} h(v_1, v_1 + \log x) dv_1. \tag{4}$$

As may be seen by comparing (1) and (4), the FM and RNM are two different, but not unrelated, transforms of the underlying joint density $h(v_1, v_2)$.

It is often natural to assume that the distribution of S_T/S_0, and therefore that of $\log(S_T/S_0)$, is independent of S_0. In practice, therefore, Carr and

Madan (1999) in fact evaluate options in terms of the RNM for the *log* of price, whose density, in the general case, is

$$\begin{aligned} q(z) &= e^z r(e^z) \\ &= \frac{1}{EU_1} \int_{-\infty}^{\infty} e^v h(v, v+z) dv. \end{aligned} \tag{5}$$

In order to streamline the notation, the subscripts on v_1 in (5) *et seq.* have been suppressed. The Carr and Madan formula for option prices is then based on the CF or Fourier Transform (FT) of this density:

$$\begin{aligned} \mathrm{cf}_q(t) &= \int_{-\infty}^{\infty} e^{izt} q(z) dz \\ &= \frac{1}{EU_1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{v+izt} h(v, v+z) dv dz, \end{aligned} \tag{6}$$

where $i = \sqrt{-1}$.

II. The RNM with Log-Stable Distributions

A random variable Z has a *standard stable distribution* $S(\alpha, \beta)$ with density $s_{\alpha\beta}(z)$ iff its log characteristic function is

$$\log \mathrm{cf}_{\alpha,\beta}(t) = \log E e^{iZt} = \begin{Bmatrix} -|t|^\alpha \left[1 - i\beta \,\mathrm{sgn}(t) \tan(\theta)\right], & \alpha \neq 1, \\ -|t| \left[1 + i\beta \frac{2}{\pi} \mathrm{sgn}(t) \log |t|\right], & \alpha = 1, \end{Bmatrix} \tag{7}$$

where $\alpha \in (0, 2]$ is its *characteristic exponent*, $\beta \in [-1, 1]$ is its *skewness parameter*, and $\theta = \pi\alpha/2$. The random variable $X = \delta + cZ$ then has the *general stable distribution* $S(\alpha, \beta, c, \delta)$ with density

$$s(x;\ \alpha, \beta, c, \delta) = \frac{1}{c} s_{\alpha\beta}(\frac{x - \delta}{c}) \tag{8}$$

iff its log CF is

$$\begin{aligned} \log \mathrm{cf}_{\alpha,\beta,c,\delta}(t) &= E e^{iXt} \\ &= i\delta t + \mathrm{cf}_{\alpha,\beta}(ct) \qquad (9) \\ &= \begin{Bmatrix} i\delta t - |ct|^\alpha \left[1 - i\beta \,\mathrm{sgn}(t) \tan(\pi\alpha/2)\right], & \alpha \neq 1, \\ i\delta t + |ct| [1 + i\beta \frac{2}{\pi} sgn(t) \log |ct|], & \alpha = 1, \end{Bmatrix} \qquad (10) \end{aligned}$$

where $c \in (0, \infty)$ is the *standard scale*, and $\delta \in (-\infty, \infty)$ is the *location parameter*.[2] Since the case $\alpha = 1$ requires special treatment unless $\beta = 0$, its

[2] Equation (9) and therefore (10) follow DuMouchel (1971) and McCulloch (1996b), *q.v.*, in the "afocal" case $\alpha = 1$, $\beta \neq 0$, so that the location-scale relationship (8) will be valid for all α.

consideration is deferred to Appendix 2 below. Subsequent equations in the present section therefore may not apply in that special case.

Three properties of stable distributions are particularly important for the present discussion:[3]

Property 1:

$$X \sim S(\alpha, \beta, c, \delta) \Rightarrow -X \sim S(\alpha, -\beta, c, -\delta).$$

Property 2:

$$X_1 \sim S(\alpha, \beta_1, c_1, \delta_1),\ X_2 \sim \text{ind. } S(\alpha, \beta_2, c_2, \delta_2) \Longrightarrow$$

$$X_3 = X_1 + X_2 \sim S(\alpha, \beta_3, c_3, \delta_3), \tag{11}$$

where

$$\begin{aligned} c_3^\alpha &= c_1^\alpha + c_2^\alpha \\ \beta_3 &= \frac{c_1^\alpha \beta_1 + c_2^\alpha \beta_2}{c_3^\alpha} \\ \delta_3 &= \delta_1 + \delta_2. \end{aligned} \tag{12}$$

Property 3:[4]

$$X \sim S(\alpha, \beta, c, \delta),\ \lambda \text{ complex with } \Re(\lambda) \geq 0 \Rightarrow$$

$$Ee^{-\lambda X} = \begin{Bmatrix} -\infty, & \alpha < 2,\ \beta < 1 \\ \exp(-\lambda\delta - \lambda^\alpha c^\alpha \sec\theta), & \beta = 1, \end{Bmatrix} \tag{13}$$

or equivalently,

$$Ee^{\lambda X} = \begin{Bmatrix} \infty, & \alpha < 2,\ \beta > -1 \\ \exp(\lambda\delta - \lambda^\alpha c^\alpha \sec\theta), & \beta = -1. \end{Bmatrix} \tag{14}$$

Properties 1 and 2 imply that as long as v_1 and v_2 are stable with a common α, the FM of

$$\log S_T = v_2 - v_1 = v_2 + (-v_1)$$

will also be stable, with the same α. However, in order to keep the expectations in (3) etc. finite, Property 3 requires that v_1 and v_2 both have $\beta = -1$, as assumed by McCulloch(1996a). Nevertheless, this does not prevent $\log S_T$ itself from having the general stable distribution

$$\log S_T \sim S(\alpha, \beta, c, \delta),$$

[3] See Uchaikin and Zolotarev (1999), Samorodnitsky and Taqqu (1994), and McCulloch (1996a) for further properties of stable distributions.

[4] Equation (13), given by Carr and Wu (2003: Property 1.3), is equivalent, after a change in notation, to Theorem 2.6.1 of Zolotarev (1986: 112). For complex λ, λ^α is to be interpreted as $|\lambda|^\alpha \exp(\alpha \operatorname{Arg}(\lambda))$, where the *principal argument* $\operatorname{Arg}(\lambda) \in (-\pi, \pi]$.

since its skewness β will, by Property 2, be intermediate between $+1$ and -1, the exact value being determined by the relative scales c_1 and c_2 of v_1 and v_2. In the simplest case, v_1 and v_2 are independent[5], with

$$\begin{aligned} v_1 &\sim S(\alpha, -1, c_1, 0) \\ v_2 &\sim \text{ind. } S(\alpha, -1, c_2, \delta). \end{aligned} \tag{15}$$

Properties 1 and 2 then imply that c_1 and c_2 can be backed out of c and β by

$$\begin{aligned} c_1 &= \left((1+\beta)/2\right)^{1/\alpha} c, \\ c_2 &= \left((1-\beta)/2\right)^{1/\alpha} c. \end{aligned}$$

Property 3 then implies

$$\begin{aligned} EU_1 &= \exp(-c_1^\alpha \sec\theta), \quad (16) \\ EU_2 &= \exp(\delta - c_2^\alpha \sec\theta), \end{aligned}$$

whence by (3) and Property 2,

$$F = e^{\delta + \beta c^\alpha \sec\theta}. \tag{17}$$

Model (15) now implies

$$h(v_1, v_2) = s(v_1; \alpha, -1, c_1, 0) s(v_2; \alpha, -1, c_2, \delta).$$

Furthermore, (8) implies

$$\begin{aligned} s(v+z;\ \alpha, \beta, c, \delta) &= s(v+z-\delta;\ \alpha, \beta, c, 0) \\ &= s(z;\ \alpha, \beta, c, \delta - v), \end{aligned} \tag{18}$$

so that by (5) we have

$$\begin{aligned} q(z) &= \frac{1}{EU_1} \int_{-\infty}^{\infty} e^v s(v;\ \alpha, -1, c_1, 0) s(v+z;\ \alpha, -1, c_2, \delta) dv \\ &= \frac{1}{EU_1} \int_{-\infty}^{\infty} e^v s(v;\ \alpha, -1, c_1, 0) s(z;\ \alpha, -1, c_2, \delta - v) dv \end{aligned}$$

Substituting into (6), reversing the order of integration, and using (10), (16)

[5] McCulloch (1996a) generalizes (15) somewhat by allowing v_1 and v_2 to have a common, also negatively skewed, component. However, the common component has no effect on the distribution of the log price or on option values, and hence is omitted here for simplicity. See Concluding Remarks below for further discussion.

and (18), we have the following for the CF of q(z):

$$
\begin{aligned}
\mathrm{cf}_q(t) &= \frac{1}{EU_1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{v+izt} s(v;\ \alpha,-1,c_1,0)s(z;\ \alpha,-1,c_2,\delta-v)dvdz \\
&= \frac{1}{EU_1}\int_{-\infty}^{\infty} e^{v} s(v;\ \alpha,-1,c_1,0)\int_{-\infty}^{\infty} e^{izt} s(z;\ \alpha,-1,c_2,\delta-v)dzdv \\
&= \frac{1}{EU_1}\int_{-\infty}^{\infty} e^{v} s(v;\ \alpha,-1,c_1,0)\,\mathrm{cf}_{\alpha,-1,c_2,\delta-v}(t)dv \\
&= \frac{1}{EU_1}\int_{-\infty}^{\infty} e^{v} e^{i(\delta-v)t-|c_2t|^{\alpha}(1+i\,\mathrm{sgn}(t)\tan\theta)} s(v;\ \alpha,-1,c_1,0)dv \\
&= e^{c_1^{\alpha}\sec\theta+i\delta t-|c_2t|^{\alpha}(1+\mathrm{sgn}(t)\tan\theta)}\int_{-\infty}^{\infty} e^{(1-it)v} s(v;\ \alpha,-1,c_1,0)dv.
\end{aligned}
$$

Since $1-it$ has a positive real part, Property 3 now implies

$$
\begin{aligned}
\log\mathrm{cf}_q(t) &= c_1^{\alpha}\sec\theta+i\delta t-|c_2t|^{\alpha}(1+i\,\mathrm{sgn}(t)\tan\theta)-(1-it)^{\alpha}c_1^{\alpha}\sec\theta \\
&= i\delta t-|c_2t|^{\alpha}(1+i\,\mathrm{sgn}(t)\tan\theta))+c_1^{\alpha}\sec\theta(1-(1-it)^{\alpha}) \\
&= i(\log F-\beta c^{\alpha}\sec\theta)t-\frac{1-\beta}{2}|ct|^{\alpha}[1+i\,\mathrm{sgn}(t)\tan\theta] \\
&\quad +\frac{1+\beta}{2}c^{\alpha}\sec\theta(1-(1-it)^{\alpha}) \qquad (19)
\end{aligned}
$$

Since the log CF of the convolution of two densities is the sum of their respective log CFs, it may be seen from (19) that q(z) is such a convolution of two densities. The first of these, whose log CF is

$$
i(\log F-\beta c^{\alpha}\sec\theta)t-\frac{1-\beta}{2}|ct|^{\alpha}(1+i\,\mathrm{sgn}(t)\tan\theta)=i\delta t-|c_2t|^{\alpha}(1+i\,\mathrm{sgn}(t)\tan\theta)
$$

is simply the max-negatively skewed stable density of v_2. It may be seen from (29) in Appendix 1 that the second density, whose log CF is

$$
\frac{1+\beta}{2}c^{\alpha}\sec\theta(1-(1-it)^{\alpha})=c_1^{\alpha}\sec\theta(1-(1-it)^{\alpha}), \qquad (20)
$$

is an *exponentially tilted stable distribution* with parameters α, "c"$=c_1$, $\lambda=1$, and "δ"$=0$. In other words, it is the max-positively skewed stable density of $-v_1$, *that has been exponentially tilted by a factor of* e^{-x}, *and then normalized.* The RNM q(z) is therefore a hybrid combination of two different, but related, distributions, the one a negatively skewed stable distribution, and the other an exponentially tilted positively skewed stable distribution.

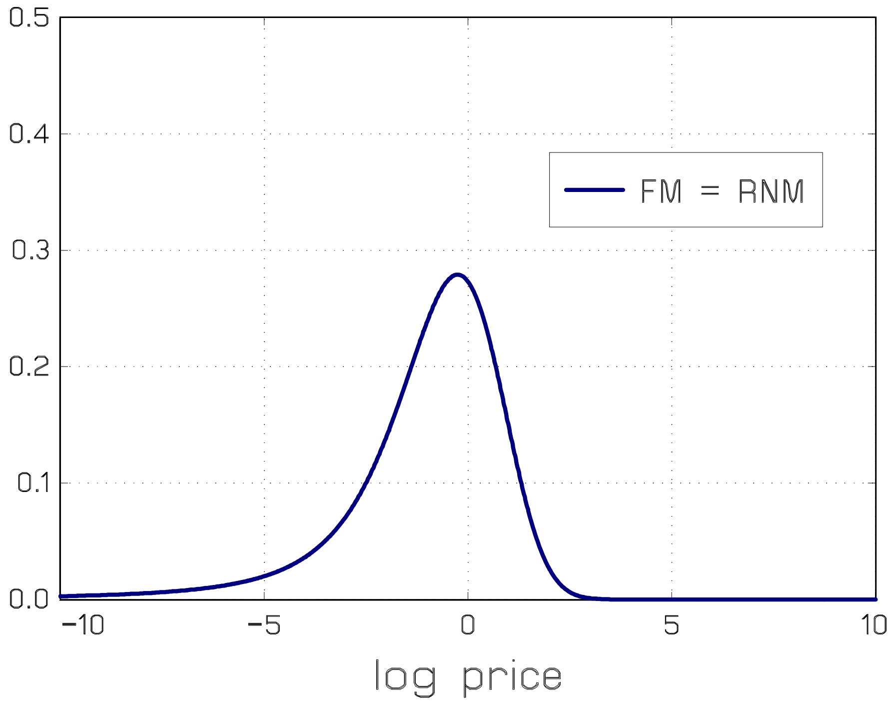

Figure 1: The coincident FM and RNM for $F = 1$, $\alpha = 1.5$, $\beta = -1$, $c = 1$.

In the case $\beta = -1$, $c_1 = 0$ and $c_2 = c$, so that the RNM and the FM coincide with

$$q(z) = \varphi(z) = s(z;\ \alpha, -1, c, \log F + c^{\alpha} \sec\theta).$$

Figure 1 shows the common RNM and FM density for the particular case $F = 1$, $\alpha = 1.5$, $\beta = -1$, $c = 1$.[6] The case $\beta = -1$ is equivalent to the "Finite Moment Log Stable Process" of Carr and Wu (2003). Their model should give exactly the same option values as McCulloch (1996a) in this case.

For $\beta > -1$, however, the RNM and FM diverge, as shown in Figures 2 and 3 for $\beta = +1$ and $\beta = 0$, resp. In both Figures, F, α, and c remain as in Figure 1.

In Figure 2, with $\beta = 1$ and therefore $c_1 = c$ and $c_2 = 0$, the maximally positively skewed FM

$$\varphi(z) = s(z;\ \alpha, +1, c, \log F - c^{\alpha} \sec\theta)$$

[6] The densities in Figures 1-3 were computed from the CFs in the paper in GAUSS, using the inverse FFT with 2^{18} points and an implied log-price step of 2^{-8}. Note, however, that GAUSS proc "FFT" generates what is generally understood by the *inverse FFT*, while "FFTI" generates *the FFT itself*.

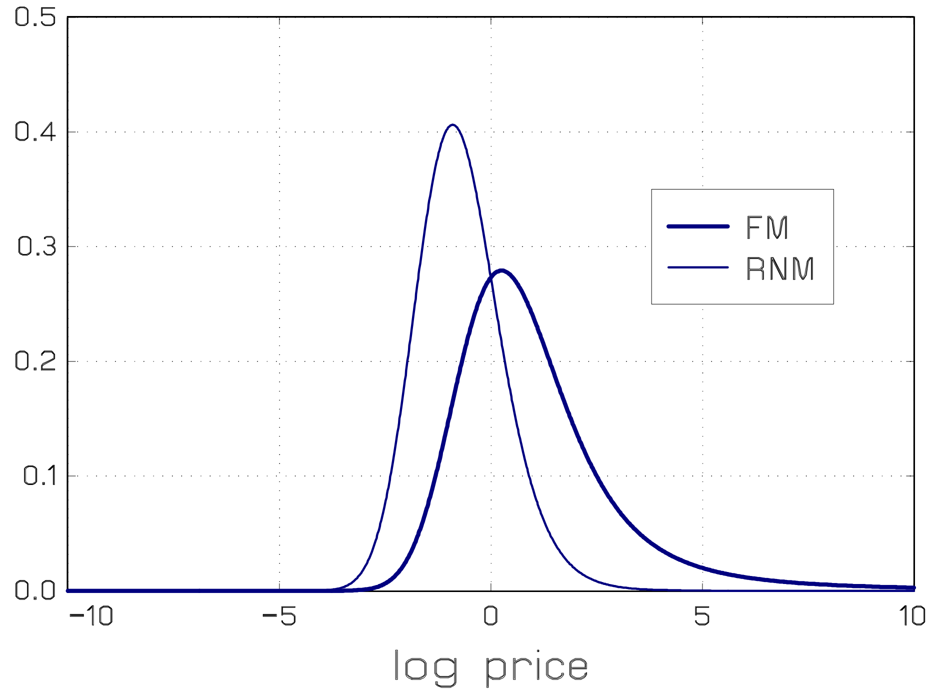

Figure 2: The divergent FM and RNM for $F = 1$, $\alpha = 1.5$, $\beta = +1$, $c = 1$.

is the mirror image of that in Figure 1, while the RNM is the exponentially tilted mutant of the FM.

In Figure 3, with $\beta = 0$ so that $c_1 = c_2 = 2^{-1/\alpha}c$, the symmetric stable FM is the convolution of two maximally skewed distributions with the same shape as the FM's in Figures 1 and 2 but somewhat smaller scale. The RNM of Figure 3 is the convolution of a reduced-scale version of the skew-stable density in Figure 1 with a similarly reduced-"c" (and therefore tighter but somewhat heavier tailed) version of the tilted stable RNM of Figure 2.

For smaller values of c, the shape difference between the RNM and FM is not so obvious to the eye, since then the primary effect of the exponential damping is far out in the upper tail, where the stable density is already small. The shape difference between the RNM and FM likewise diminishes as α increases toward 2.

When $\alpha = 2$, a stable distribution becomes normal with mean δ and variance $\sigma^2 = 2c^2$, and β loses its effect on the shape of the distribution, though not on the relation between $\log F$ and δ. The FM becomes $N(\log F + \beta\sigma^2/2, \sigma^2)$, while the RNM has log characteristic function

$$\mathrm{cf}_q(t) = i(\log F - \sigma^2/2)t - \sigma^2 t^2/2, \tag{21}$$

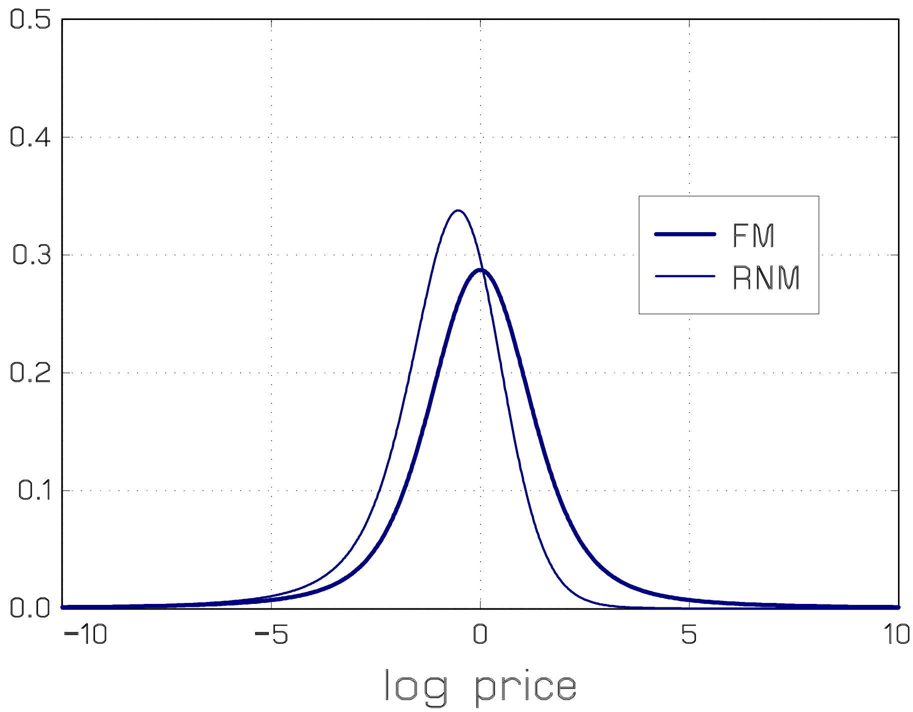

Figure 3: The divergent FM and RNM for $F = 1$, $\alpha = 1.5$, $\beta = 0$, $c = 1$.

which is that of the $N(\log F - \sigma^2/2, \sigma^2)$ distribution. In the log-normal case, the RNM and FM therefore both have the same Gaussian shape in terms of log price, with the same variance. They differ only in location, by the observable risk premium $(\beta+1)\sigma^2/2$ that is determined by β, i.e. by the relative standard deviations of $\log U_1$ and $\log U_2$. This comes about because a downwardly exponentially tilted normal distribution is just another normal back again, with the same variance but a reduced mean.

It should be noted that except in the finite moment cases $\alpha = 2$ and $\beta = -1$, the population equity premium $ES_T/F - 1$ is infinite under a log-stable FM. For any finite sample, the sample equity premium will be finite with probability 1, but a large average excess arithmetic return does not necessarily indicate an "Equity Premium Puzzle" per Mehra and Prescott (1985).

III. Option Pricing with the Stable RNM

Let $C(X)$be the value, in units of numeraire to be delivered at time 0, of a European call entitling the holder to purchase 1 unit of the asset in question at exercise (or strike) price X, at, but not before, time T. Then by definition of the RNM, its value must be the discounted expectation of its payoff under either $r(x)$or $q(z)$:

$$\begin{aligned} C(X) &= e^{-\rho T} \int_0^\infty \max(0, x - X) r(x) dx \\ &= e^{-\rho T} \int_{\log X}^\infty (e^z - X) q(z) dz. \end{aligned} \tag{22}$$

Similarly let $P(X)$ be the value of a European put option allowing the owner to sell one unit of the asset at time T at exercise price$X$, so that

$$\begin{aligned} P(X) &= e^{-\rho T} \int_0^\infty \max(0, X - x) r(x) dx \\ &= e^{-\rho T} \int_{-\infty}^{\log X} (X - e^z) q(z) dz. \end{aligned} \tag{23}$$

Equation (6) in Carr and Madan (1999) may be used to evaluate call options directly from the Fourier transform of (22), provided

$$E^Q(S_T^{a+1}) < \infty$$

for some tilting coefficient $a > 0$, where E^Q denotes the expectation under the RNM q(z). However, since the tilting factor built into the RNM is necessarily unity, the above expectation is only just finite for $a = 0$, and infinite for any larger value, so that the method of their (6) will not work in the log-stable case.

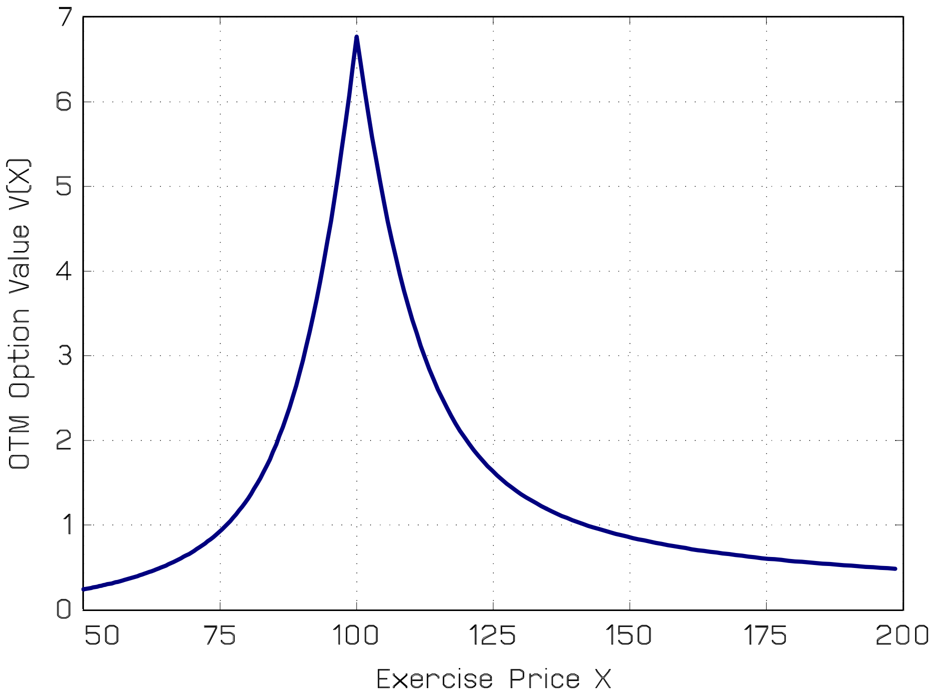

Figure 4: OTM option value function V(X) for $\alpha = 1.5$, $\beta = 0$, $c = .1$, and $F = 100$, with $\rho = d = 0$.

It is therefore instead necessary to use an approach similar to Carr and Madan's alternative formula (14). Define the out-of-the-money (OTM) option value function $V(X)$ by $V(X) = P(X)$ for $X < F$ and $V(X) = C(X)$ for $X \geq F$. By put-call parity, $P(F) = C(F)$ for any RNM, so that $V(X)$ is continuous at F. It is also known that it is monotonic and convex on either side of F. Figure 4 depicts $V(X)$ for $\alpha = 1.5$, $\beta = 0$, $c = .1$, and $F = 100$,with $\rho = d = 0$.

With no loss of generality, we may measure the asset in units such that $F = 1$. Following Carr and Madan,[7] the Fourier Transform of $v(z) = V(e^z)$ is then

$$\phi_v(t) = e^{-rT} \frac{\mathrm{cf}_q(t-i) - 1}{it(it+1)}, \ t \neq 0. \tag{24}$$

When $t = 0$, this formula takes the value 0/0, but the limit may be evaluated

[7] Carr and Madan in fact base their (14) on a function which equals $P(X)$ when X is less than the spot price S_0 and $C(X)$ otherwise. This unnecessarily creates a small discontinuity which can only aggravate the Fourier inversion. The present function $V(X)$ avoids this problem, with the consequence that (24) is in fact somewhat simpler than their (14).

by means of l'Hôpital's rule. In the stable case, using (19), this becomes

$$\phi_v(0) = -e^{-rT} c^\alpha \sec\theta \left(\beta + \frac{1-\beta}{2}\alpha \right).$$

Unfortunately, however, the function $v(z)$ has a cusp at $z = 0$ corresponding to that in $V(X)$ at $X = F$, so that numerical inversion of (24) by means of the discrete inverse FFT results in pronounced spurious oscillations in the vicinity of the cusp. The problem is that the ultra-high frequencies required to fit the cusp and its vicinity are omitted from the discrete Fourier inversion, which only integrates over a finite range of integration instead of the entire real line. Increasing the range of integration progressively reduces these oscillations, but never entirely eliminates them.

However, the fact that increasing the range of integration does give improved results allows the FFT inversion results to be "Romberged" to give satisfactory results, as follows: Start with a large number of points $N = N_1$, with a log-price step $\Delta z = c\sqrt{2\pi/N}$ (or a round number in that vicinity if desired), and a frequency-domain step $\Delta t = 2\pi/(N\,\Delta z)$. Then quadruple N to $N_2 = 4N_1$, and then again to $N_3 = 16N_1$, halving both step sizes each time, so as to double the range of integration each time, while obtaining values for the original z grid. Each of the original N_1 z values now has 3 approximate function values v_1, v_2, and v_3 that are converging on the true value at an approximately geometric rate as the grid fineness and range of integration are successively doubled. The true value may then be approximated to a high degree of precision at each of these points simply by extrapolating the geometric series implied by the three values to infinity:

$$v_\infty = v_3 + \frac{\rho}{1-\rho}(v_3 - v_2),$$

where $\rho = (v_3 - v_2)/(v_2 - v_1)$. The residual error may then be conservatively estimated by computing v_0 using $N_0 = N_1/4$, repeating the above procedure using $v_0,$ v_1, and v_2, and assuming that the absolute discrepancy between the two results is an upper bound on the error. It was found that for $\alpha \geq 1.3$, $N_1 = 2^{10}$ usually gives a maximum estimated error less than .0001 relative to $F = 1$, though occasionally $N_1 = 2^{14}$ is necessary.[8] Put-call parity may then be used to recover $C(X)$ and/or $P(X)$,as desired, from $V(X) = v(\log X)$.

The above procedure gives the value of $v(z)$ at N_1 closely spaced values of z, and therefore $V(X)$ at N_1 closely spaced values of X. Unfortunately, however, these will ordinarily not precisely include the desired exercise prices, and because of the convexity of $V(X)$ on each side of the cusp, linear interpolation may give an interpolation error in excess of the Fourier inversion computational error.

[8] For the financially less relevant values of $\alpha < 1.3$, the infinite first derivative of the imaginary part of (24) at the origin causes additional computational problems. These problems become even worse for $\alpha < 1.0$, when the imaginary part becomes discontinuous at the origin. No attempt was made in the present study to overcome these problems.

The reader may wish to experiment with recovering the Laplace density function $.5\exp(-|z|)$, which has a similar cusp at the origin, from its characteristic function $1/(1+t^2)$, by means of the Romberg-FFT inversion described in the text.

Nevertheless, cubic interpolation on $C(X)$ and/or $P(X)$ using two points on each side of each desired exercise price gives very satisfactory results.

Fourier inversion of (24) using the stable RNM (19) is mathematically equivalent to the stable option pricing formula given by McCulloch (1996a: (53)), which has already been used by McCulloch (1987) and Hales (1997) to evaluation options on foreign exchange options. However, the latter method requires tedious numerical integrals using maximally skewed stable density values interpolated off of tables. The method described in the present paper is both simpler and faster.

Maturing options create special problems for Fourier inversion of the Carr and Madan equation (14) that lead them to employ a hyperbolic sine function to transform their value function in this case. However, far-out-of-the-money stable call options (with $\log(X/F) >> c$) and put options (with $\log(X/F) << -c$) may be evaluated directly (see McCulloch 1996a: 414) using

$$\lim_{c \downarrow 0} \frac{C(X)}{c^\alpha} = e^{-rT} F(1+\beta)\Psi(\alpha, X/F)$$

and

$$\lim_{c \downarrow 0} \frac{P(X)}{c^\alpha} = e^{-rT} X(1-\beta)\Psi(\alpha, F/X),$$

where

$$\begin{aligned} \Psi(\alpha, x) &= \frac{\Gamma(\alpha)\sin\theta}{\pi} \left[(\log x)^{-\alpha} - \alpha x \int_{\log x}^{\infty} e^{-\zeta} \zeta^{-\alpha-1} d\zeta \right] \\ &= \frac{\Gamma(\alpha)\sin\theta}{\pi} \left[(\log x)^{-\alpha} - \alpha x\, \Gamma(-\alpha, \log x) \right], \end{aligned} \tag{25}$$

and

$$\Gamma(a, z) = \int_z^\infty e^{-t} t^{a-1} dt$$

is the incomplete gamma function, which is defined for $z \geq 0$ for $a > 0$ and for $z > 0$ when $a \leq 0$. Integration by parts yields the recursion[9]

$$\Gamma(a+1, z) = z^a e^{-z} + a\Gamma(a, z) \tag{26}$$

whence (25) may be further simplified to

$$\Psi(\alpha, x) = \frac{\Gamma(\alpha)\sin\theta}{\pi} x\, \Gamma(1-\alpha, \log x). \tag{27}$$

Routines which compute the gamma distribution CDF $P(a, z) = 1 - \Gamma(a, z)/\Gamma(a)$ may be used to recover $\Gamma(a, z)$, but only for $a > 0$, since $P(a, z) = 1$ for $a < 0$

[9]Note that (26) implies that the more familiar recursion $\Gamma(a+1) = a\Gamma(a)$ is valid only for $a > 0$. For $a \leq 0$, $\Gamma(a) \equiv \Gamma(a, 0) = \infty$, while (26) becomes $\Gamma(a) = \Gamma(a+1)/a + \infty$.

and $z > 0$. Nevertheless, (27) may still be evaluated using such routines, at least for $\alpha \neq 1$, by one further application of the recursion (26).[10]

In an α-stable Lévy motion, the scale c_T that accumulates in T time units is $c_1 T^{1/\alpha}$. Therefore maturing OTM stable options may be evaluated directly, without Fourier inversion or the hyperbolic sine transform, using

$$\begin{aligned} \lim_{T \downarrow 0} (C(X)/T) &= S_0(1+\beta)c_1^\alpha \Psi(\alpha, X/S_0), \\ \lim_{T \downarrow 0} (P(X)/T) &= X(1-\beta)c_1^\alpha \Psi(\alpha, S_0/X). \end{aligned} \tag{28}$$

McCulloch (1985) used (28) to evaluate the put option implicit in deposit insurance for banks and thrifts that are exposed to interest rate risk, but unnecessarily evaluated the incomplete gamma integral numerically.

It is well known that the RNM whose CF is given by (21) yields the Black-Scholes (1973) option pricing formula. The log-stable option pricing formula implied by (19) and (24) therefore nests the Black-Scholes formula in the case $\alpha = 2$.

IV. Directions for Further Research

The present paper assumes for simplicity that the log marginal utilities v_1 and v_2 are independently distributed. However, they could still be negatively skewed stable, and still lead to a general stable $\log S_T$, with a much more general bivariate stable distribution with spectral mass anywhere in the closure of the third quadrant (see, e.g., McCulloch 1996a: §2.3). Except in the rather special case of a common component that affects both v_1 and v_2 equally, considered already by McCulloch (1996a) and shown to have no effect on asset or option pricing, it is not clear whether these more general bivariate stable distributions would lead to the same relationship between the FM and the RNM presented here. If not, there may be more than one RNM for any given stable FM. This issue deserves further research.

Hurst, Platen and Rachev (1999) price options on log-symmetric stable assets using the well-known theorem of Bochner that if a normal distribution is subordinated in variance to a positive stable distribution with $\alpha' < 1$ and $\beta = 1$, the resulting distribution is symmetric stable with $\alpha = 2\alpha'$ (see, e.g., Samorodnitsky and Taqqu 1994: Proposition 1.3.1). They then evaluate options as the expectation of the Black-Scholes (1973) formula under the subordinating positive stable distribution. The formula of McCulloch (1996a: (53)), which is mathematically equivalent to Fourier inversion of (24), is not restricted to the symmetric case, but even when it is evaluated at $\beta = 0$, the two formulas do not look at all alike. It is not clear at present whether the two formulas give equivalent option values in this case, or why there would be a difference if they do not.

[10] $\Psi(1.0001, x)$ and $\Psi(0.9999, x)$ differ by at most 6 parts in 10,000 for x between 1.001 and 10, so $\Psi(1, x)$ may simply be computed as the average of these two values, to an accuracy of at least 3 parts in 10,000 in this range.

V. Conclusion

The Generalized Central Limit Theorem makes the stable distributions a particularly natural assumption for the empirically observed distribution of log asset returns. Early on, however, both Paul Samuelson and Robert Merton were discouraged from believing that assets and derivatives such as options could be reasonably priced under these distributions because of their the infinite first moment. In a recent article in this *Journal*, Carr and Wu (2003) do price options under log-stable uncertainty, but only by means of a very restrictive assumption on the stable distribution parameters.

The present paper employs a simple representative agent expected utility argument to derive the Risk-Neutral Measure (RNM), a risk-adjusted probability measure under which assets may be priced as expectations, for a general log-stable empirical distribution. It is shown that the RNM is not, as in the log-normal case Samuelson and Merton were familiar with, a simple location shift (in logs) of the empirical distribution. Instead, the RNM corresponding to a log-stable empirical distribution in general has a different shape, with an exponentially damped upper tail. This RNM has finite moments, and leads to reasonable asset and option prices.

Because the RNM for log-stable uncertainty is developed here in terms of its Fourier Transform, it is now possible to use the inverse Fourier Transform approach of Carr and Madan (1999) to numerically evaluate log-stable options by means of the Fast Fourier Transform (FFT) algorithm, without the restrictive assumption of Carr and Wu (2003). The paper introduces a simple extension of the FFT procedure in order to overcome a technicality that arises.

A GAUSS program implementing the required computations is available to researchers on the author's website.

Appendix 1:

Exponentially Tilted Stable Distributions

An *exponentially tilted positively skewed stable density* with parameters α, c, δ, and $\lambda > 0$ has density

$$f_{ts}(x; \alpha, c, \delta, \lambda) = ke^{-\lambda x} s(x; \alpha, +1, c, \delta),$$

where k is a normalizing constant to be determined. Its CF, using stable distribution Property 3 with $\alpha \neq 1$, is

$$\begin{aligned} \mathrm{cf}_{ts}(t) &= k \int_{-\infty}^{\infty} e^{ixt} e^{-\lambda x} s(x; \alpha, +1, c, \delta) dx \\ &= k \int_{-\infty}^{\infty} e^{-(\lambda - it)x} s(x; \alpha, +1, c, \delta) d \\ &= k \exp(-(\lambda - it)\delta - (\lambda - it)^{\alpha} c^{\alpha} \sec\theta), \end{aligned}$$

where, as in the text, $\theta = \pi\alpha/2$. Since for any CF, $\mathrm{cf}(0) \equiv 1$, we must have

$$k = \exp(\lambda\delta + \lambda^{\alpha} c^{\alpha} \sec\theta),$$

whence

$$\log \mathrm{cf}_{ts}(t) = i\delta t + c^{\alpha} \sec\theta(\lambda^{\alpha} - (\lambda - it)^{\alpha}) \tag{29}$$

The second density (20) in the CF (19) of the RNM q(z) is therefore tilted stable with $\delta = 0$, $c = c_1$, and $\lambda = 1$, i.e. e^{-z} times the density of $-v_1$ and normalized. Figure 2 illustrates a positively skewed stable distribution (the FM), along with the corresponding $\lambda = 1$ tilted stable distribution (the RNM).

The tilted stable class may be written as a location-scale family, with location δ, scale $s = 1/\lambda$, and shape parameters α and $\gamma = (c\lambda)^{\alpha} \sec\theta$, as follows:

$$\log \mathrm{cf}_{ts}(t) = i\delta t + \gamma(1 - (1 - its)^{\alpha})$$

Note that a change in c by itself is not a pure change in the scale of the tilted stable distribution itself, unless λ is at the same time changed in the inverse proportion. Changing c by itself does tighten or relax the distribution, but at the same time changes its shape.

One can, *mutatis mutandis*, equally tilt a maximally *negatively* skewed stable with $e^{+\lambda x}$. It is not, however, possible to tilt a stable distribution with $\beta \in (-1, 1)$ in either direction, since then $\int e^{\lambda x} s(x; \alpha, \beta, c, \delta)$ would be infinite for any value of $\lambda \neq 0$.

Tilted stable distributions have already been used in the context of option pricing by Vinogradov (2002), who points out that for $\alpha \in (1, 2]$, they are a special case of the *Tweedie distributions*, and generate what are known as

Hougaard processes. He notes that exponential tilting of a density is known as the *Esscher Transformation*. If I understand his option pricing model correctly, he is valuing options as their discounted expected payoff under an exponentially tilted stable distribution like the RNM of Figure 2, under the assumption that S_0 equals the discounted expectation of S_T under this distribution. According to the present model, this is the correct procedure for valuing options when the FM itself is *stable* with $\beta = 1$, provided the tilting coefficient λ is unity. It would not, however, be the correct procedure if the FM were *tilted stable*, unless $\log U_2$ were for some reason tilted stable and $\log U_1$ were nonstochastic, so that assets were priced as if investors were risk-neutral.

The "truncated Lévy distribution" used by Boyarchenko and Levendorskiĭ (2000) and Cartea and Howison (2002) to value options is in fact the convolution of two tilted stable distributions, one skewed left and tilted right, and the other skewed right and tilted left, with a common α and λ but perhaps different stable scales. As these authors note, both tails of the resulting density are exponentially damped. The density itself is not, however, exponentially tilted. It is not clear what FM, if any, would correspond to such an RNM.

Appendix 2:
The Case Alpha = 1.

Unless $\beta = 0$, the case $\alpha = 1$ requires special treatment of both the CF and the location parameter. This paper follows DuMouchel (1971) and McCulloch (1996b) in specifying the CF in such a way that the location-scale property (8) will hold. This in turn implies (9), and therefore (10), with c multiplying the t inside the log. Omitting this factor of c, as is done, e.g. by Samorodnitsky and Taqqu (1994), results in a non-location-scale family.

Under (9), (12) becomes

$$\delta_3 = \delta_1 + \delta_2 + \frac{2}{\pi}(\beta_3 c_3 \log(\beta_3 c_3) - \beta_1 c_1 \log(\beta_1 c_1) - \beta_2 c_2 \log(\beta_2 c_2))$$

For $X \sim S(1, +1, c, \delta)$ and $\Re(\lambda) > 0$, (13) becomes

$$Ee^{-\lambda X} = \exp(-\lambda\delta + \frac{2}{\pi}c\lambda \log(c\lambda)),$$

while for $X \sim S(1, -1, c, \delta)$ and $\Re(\lambda) > 0$, (14) becomes

$$Ee^{\lambda X} = \exp(\lambda\delta + \frac{2}{\pi}c\lambda \log(c\lambda)).$$

Furthermore, the suitably normalized tilted positively skewed stable density $ke^{-\lambda x}s(x; 1, 1, c, \delta)$ now has log CF

$$\log \mathrm{cf}_{ts}(t) = i\delta t + \frac{2}{\pi}\left[c(\lambda - it)\log(c(\lambda - it)) - c\lambda \log(c\lambda)\right],$$

in place of (29).

Setting

$$\begin{aligned} v_1 &\sim S(1, -1, c_1, \delta_1), \\ v_2 &\sim S(1, -1, c_2, \delta_2), \text{ ind.}, \end{aligned}$$

$\log S_T \sim S(1, \beta, c, \delta)$, where now

$$\begin{aligned} \delta &= \delta_2 - \delta_1 + \frac{2}{\pi}(\beta c \log c - c_1 \log c_1 + c_2 \log c_2) \\ &= \delta_2 - \delta_1 + \frac{c}{\pi}((1-\beta)\log\frac{1-\beta}{2} - (1+\beta)\log\frac{1+\beta}{2}). \end{aligned}$$

We then have

$$\begin{aligned} EU_1 &= \exp(\delta_1 + \frac{2}{\pi}c_1 \log c_1), \\ EU_2 &= \exp(\delta_2 + \frac{2}{\pi}c_2 \log c_2), \\ F &= \exp(\delta_2 - \delta_1 + \frac{2}{\pi}(c_2 \log c_2 - c_1 \log c_1)) \\ &= \exp(\delta - \frac{2}{\pi}\beta c \log c), \end{aligned}$$

and

$$\begin{aligned}\log \mathrm{cf}_q(t) &= i(\delta_2 - \delta_1) - |c_2 t|[1 - i\frac{2}{\pi} sgn(t) \log |c_2 t|] \\ &\quad + \frac{2}{\pi}[c_1(1 - it)\log(c_1(1 - it)) - c_1 \log c_1].\end{aligned}$$

As for $\alpha \neq 1$, the RNM is the convolution of the max-negatively skewed stable density of v_2 with an exponentially tilted mutation of the max-positively skewed stable density of $-v_1$, using a unitary tilting coefficient.

References

Black, Fischer, and Myron Scholes (1973): "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 81, 637-659.

Boyarchenko, Svetlana I., and Sergei Z. Levendorskiĭ (2000): "Option Pricing for Truncated Lévy Processes," *International Journal of Theoretical and Applied Finance*, 3, 549-552.

Carr, Peter, and Dilip Madan (1999): "Option Valuation Using the Fast Fourier Transform," *Journal of Computational Finance*, 2, 61-73.

Carr, Peter, and Liuren Wu (2003): "The Finite Moment Log Stable Process and Option Pricing," *Journal of Finance*, 58, 753-777.

Cartea, Álvaro, and Sam Howison (2002): "Distinguished Limits of Lévy-Stable Processes, and Applications to Option Pricing," Working Paper, University of Oxford Mathematical Institute.

DuMouchel, William H. (1971): *Stable Distributions in Statistical Inference.* Ph.D. Dissertation, Yale University.

Hales, Stanley J. (1997): *Valuation of Foreign Currency Options with the Paretian Stable Option Pricing Model.* Ph.D. dissertation, Ohio State University Department of Economics.

Hurst, S.R., E. Platen, and Svetlozar T. Rachev (1999): "Option Pricing for a Logstable Asset Price Model," *Mathematical and Computer Modelling*, 29, 105-119.

McCulloch, J. Huston (1985): "Interest-Risk Sensitive Deposit Insurance Premia: Stable ACH Estimates," *Journal of Banking and Finance*, 9, 137-156.

——————— (1987): "Foreign Exchange Option Pricing with Log-Stable Uncertainty," in Recent Developments in International Banking and Finance, Vol. 1, ed. by S.J. Khoury and A. Ghosh. Lexington, MA: Lexington, 231-245.

——————— (1996a): "Financial Applications of Stable Distributions," in *Statistical Methods in Finance*, ed. by G.S. Maddala and C.R. Rao, Vol. 14 in *Handbook of Statistics*. Amsterdam: Elsevier, 393-425. Incorporates and supplants 1985 working paper "The Value of European Options with Log-Stable Uncertainty."

——————— (1996b): "On the Parametrization of the Afocal Stable Distributions," *Bulletin of the London Mathematical Society*, 28, 651-655.

Mehra, Rajnish, and Edward C. Prescott (1985): "The Equity Premium: A Puzzle," *Journal of Monetary Economics*, 15, 145-61.

Merton, Robert C. (1976): "Option Pricing when Underlying Stock Returns are Discontinuous," *Journal of Financial Economics*, 3, 125-144.

Samorodnitsky, Gennady, and Murad S. Taqqu (1994): *Stable Non-Gaussian Random Processes. New York: Chapman and Hall.*

Smith, Clifford (1976): "Option Pricing: A Review," *Journal of Financial Economics*, 3, 3-51.

Uchaikin, Vladimir V., and Vladimir M. Zolotarev (1999): *Chance and Stability: Stable Distributions and their Applications.* Utrecht, The Netherlands: VSP.

Vinogradov, Vladimir (2002): "On a Class of Lévy Processes Used to Model Stock Price Movements with Possible Downward Jumps," *Mathematical Reports of the Academy of Science [Canada]*, 24, 152-159.

Zolotarev, Vladimir M. (1986): *One-Dimensional Stable Laws,* Vol. 65 in *Translations of Mathematical Monographs.* American Mathematical Society. (1983 Russian original).