

The Stable Demand Set General Characterization and Application to Majority Games *

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Abstract

The cooperative solution concept introduced here, the *Stable Demand Set*, contains the Core and is included in the Zhou Bargaining Set, eliminating the “dominated” coalition structures. The demand vectors belonging to the Stable Demand Set are *self-stable*. In the class of constant-sum homogeneous weighted majority games the Stable Demand Set is non-empty and predicts a *unique* stable demand vector, namely a *proportional* distribution within minimal winning coalitions. The noncooperative implementation of the Stable Demand Set is obtained for all the games that satisfy the one-stage property.

Keywords: Stable Demands, Undominated Coalition Structures, Weighted Majority Games, Bargaining Sets.

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1 Introduction

The interest in the stability properties of coalitions and payoff allocations in coalition formation games can be traced back to the first half of the 20th century, and the discussion about which solution concept we should use to predict coalition structures and payoff allocations at the same time remains open. Value concepts yield an *ex ante* evaluation, hence they cannot offer predictions about the prevailing payoff distribution within the prevailing coalitions. On the other hand, solution concepts like the bargaining set, the stable sets, and the kernel avoid the emptiness problems of the core for a large class of games and do face the problem of predicting the possible *ex post* configurations of payoffs, but the set of solutions is often too large. Most solution concepts determine the distribution of gains *within* given coalitions or coalition structures, and hence are very helpful to model arbitration problems, where coalitions are formed before the bargaining process over the distribution of payoffs begins. In contrast, as pointed out in Bennett (1985), the aspirations approach seems well suited for situations where the formation of coalitions is endogenous.

The cooperative solution concept introduced in this paper, the *stable demand set*, shares the central role of payoff demands with the aspiration approach, but it does not limit attention to the domain of aspirations. While on the one hand the traditional imputation approach determines payoff distribution after having fixed a coalition structure, and while on the other hand the aspirations approach determines coalitional outcomes after having fixed a payoff distribution, we allow payoff distribution and coalition formation to be *simultaneously determined*. In spite of this larger strategic space, we will show that in constant-sum homogeneous weighted majority games our stable demands coincide with the balanced aspirations of the game.¹

The stable demand set turns out to contain the core, and is a subset of the Zhou bargaining set. With respect to these and other solutions in the bargaining set tradition, there are two main innovations: (1) we make use of the fact that every allocation can be viewed as a *pair* consisting of a *demand vector* and a *coalition struc-*

¹The concept of balanced aspirations is well explained in Bennett (1983).

ture; and (2) for every proposed pair the set of counterobjections (to objections to such a proposal) is restricted to include only those pairs that use the *same* demand vector as in the original proposal. The second feature is the one determining the selection of a subset of the Zhou bargaining set. In particular, the selection obtained through the stable demand set has a very general property: in grand-coalition superadditive games no “dominated” coalition structure can ever be part of a solution (Theorem 2). Beside this property, we also show that the stable demand set is immune to most of the other criticisms moved against bargaining set type solutions.

To give a clear example of the meaningfulness of the selection provided by the stable demand set in applications, we show in Section 3 that in any constant-sum weighted majority game that admits a homogeneous representation the proportional payoff division is the *unique* stable outcome, and no previous bargaining set could make such a sharp prediction.² This result illustrates particularly well the role of the axiom requiring that counterobjections be limited to those that can use the original demand vector: the *social norm* of distributing payoffs proportionally to the relative contributions (within the minimal winning coalition) is the only one that can be used to counter all possible objections to a proposal where the proportional norm itself is used. This type of *self-stability* is the key idea proposed by our concept.

In Section 4 the stable demand set is implemented for all superadditive games satisfying the one-stage property, following the methodology of Pérez-Castrillo and Wettstein (2000). The mechanism is simple and serves to illustrate further the forces behind our concept.

2 The stable demand set

In this section we introduce the cooperative solution concept, defining it for any TU game and discussing some general properties.

²There is however an interesting connection with the main simple solution in the von Neumann and Morgenstern stable set. Druckman and Warwick (2002) is a good reference to see the empirical relevance of proportional payoffs in legislative bargaining contexts.

2.1 Notation and Basic Definitions

Let N be the set of players, $N \equiv \{1, 2, \dots, i, \dots, n\}$. Let $S \subseteq N$ represent a generic coalition of players, \mathcal{S} denote the set of possible coalitions, and $v : \mathcal{S} \rightarrow \mathcal{R}_+$ denote the characteristic function. If $v(i) = 0$ for all i in N , the pair (N, v) is called a zero-normalized TU game. Given any n -dimensional payoff vector x , $x(S)$ denotes the sum of the values corresponding to the components of x related to the members of S : $x(S) \equiv \sum_{i \in S} x_i$. We will denote by Σ the set of possible partitions of N (or set of coalition structures), and σ will represent a generic element of that set.

The stable demand set (henceforth SDS) is defined on the space of *norm realizations* \mathcal{X} , where

$$\mathcal{X} = \{(\alpha, \sigma) \in (\mathcal{R}^n \times \Sigma) : \sum_{j \in S} \alpha_j \leq v(S) \text{ for all } S \in \sigma, |S| > 1\}.$$

Thus, a candidate element for the SDS is a pair (α, σ) where $\alpha \in \mathcal{R}^n$ is the demand vector specifying what each player should receive for his cooperation with other players in a coalition and σ is a *coalition structure* compatible with α in the sense that coalitions of more than one player must be able to afford the demands of their members. No such restriction is imposed on singletons.³

For any given pair $(\alpha, \sigma) \in \mathcal{X}$, the corresponding *feasible allocation* α^σ is obtained using the following *payoff assignment rule*:

$$\alpha_i^\sigma = \begin{cases} \alpha_i & \text{if } i \in S \in \sigma : |S| > 1 \\ v(i) & \text{otherwise.} \end{cases} \quad (1)$$

In words, the demands are assigned as actual payoffs to the members of coalitions with more than one player; singletons receive $v(i)$ regardless of players' demands.

2.2 Solution Concept

Consider a proposed pair $(\alpha, \sigma) \in \mathcal{X}$.

³The reason for not imposing any requirement on singletons is that α is interpreted as the demand players make for cooperating *with other players*.

Definition 1 An objection of a coalition T against the proposal (α, σ) is an allocation vector y such that

$$y_i > \alpha_i^\sigma \text{ for all } i \in T$$

and $\sum_{i \in T} y_i \leq v(T)$ (i.e., y is feasible for T).

Notice that objections are against actual payoffs, not against demands; thus, an objecting player may receive less than his demand in the objection.

Definition 2 A coalition Z can counter (or, make a counterobjection to) the objection of T against the proposal (α, σ) iff

(1) $Z \cap T \neq \emptyset$ and

(2) the original demand vector α is such that

$$\sum_{i \in Z} \alpha_i \leq v(Z)$$

$$\alpha_i > y_i \text{ for all } i \in T \cap Z.$$

Definition 3 An objection to (α, σ) is justified iff it cannot be countered.

Notice that with respect to the standard definition of a counterobjection (as used in most versions of the bargaining set) we restrict the set of possible counterobjections to include only those that can be derived using the same demand vector of the original proposal. We also require the inequality within $Z \cap T$ to be strict.⁴ Because of the strict inequality, objections that use the original demand vector α are always justified.

Definition 4 A pair (α, σ) in \mathcal{X} belongs to the stable demand set iff there is no justified objection to it.

If players can counterobject using an allocation that can be derived from the same demand vector of the original proposal, it means that the demand vector itself

⁴Requiring this inequality to be strict is necessary for theorems 1 and 2, but not for theorem 3.

is *self-stable*. Intuitively, trying to obtain larger shares (with an objection) does not pay if there is the risk of being simply excluded from a coalition that distributes the unchanged demands of the other players.

2.3 Important Properties and Relation to Previous Concepts

2.3.1 The aspiration approach

Like the stable demand set, the aspiration solution concepts also study the stability of demand vectors. However, as we will now show, there is an important difference with respect to the domain where the solution is defined.

Definition 5 *A demand vector $x \in \mathcal{R}^n$ is an aspiration iff it satisfies*

1. $x(S) \geq v(S)$ for all $S \subseteq N$ (*maximality*);
2. For all $i \in N$ there exists a coalition $S \ni i$ such that $x(S) \leq v(S)$ (*feasibility*).

Limiting attention to the space of aspirations, it is possible to define a variety of solution concepts.⁵

The SDS is defined in a larger space. Neither maximality nor feasibility are imposed on demand vectors. However, maximality is obtained endogenously since otherwise there would be a justified objection.

⁵These solution concepts are constructed by adding a *stability* condition to the requirements of maximality and feasibility. The most important solution concepts are the *aspiration bargaining set*, obtained by adding the *partnership* condition (Albers (1974), Bennett (1983)) and the *aspiration core*, obtained by adding the *balancedness* condition (Cross (1967)). Denoting by $S_i(x)$ the set of coalitions containing i that are feasible given x , an aspiration vector x is partnered iff for any two players i and j either $S_i(x) = S_j(x)$ or $S_i(x) \setminus S_j(x)$ and $S_j(x) \setminus S_i(x)$ are both non-empty. An aspiration vector x is balanced iff it minimizes $\sum_{i \in N} x_i$ in the space of aspirations. Other solution concepts are the set of *equal gains* aspirations (Bennett 1983) and the *aspiration kernel* (Bennett 1985).

Remark 1 *If (α, σ) is in the SDS, it follows that α is maximal (but not necessarily feasible).^{6,7}*

2.3.2 Other bargaining sets and the core

Something similar can be said about the traditional bargaining sets, defined on the space of individually rational payoff configurations. We do not actually require (α^σ, σ) to be an individually rational payoff configuration, but this property arises endogenously.

Remark 2 *If (α, σ) is in the SDS, then (α^σ, σ) is an individually rational payoff configuration.⁸*

Since the stable outcomes in the sense of the SDS lie in the space of individually rational payoff configurations, it is interesting to relate the SDS to other solution concepts that lie in this standard domain, like the bargaining sets and the core.

Remark 3 *Since the core is the set of allocations to which there are no objections, the SDS contains the core.*

Maximality implies that if (α, σ) is in the SDS and the realized allocation and coalition structure are such that all players receive their demands, then α is in the core. Thus, the SDS *coincides* with the core for the grand coalition. If the core of v is empty, there is no allocation in the SDS such that all players receive their demands.⁹

⁶Consider the following game: $N = \{1, 2, 3\}$, $v(1, 2) = 6$, $v(1, 3) = v(2, 3) = 2$, $v(N) = 3$. The demand vector $(3, 3, x)$ is in the SDS for any $x \geq 0$, though only $x = 0$ is feasible.

⁷Even though a stable demand vector α may not be feasible for some players it cannot be unfeasible for *all* players except in the uninteresting case with $v(T) \leq \sum_{i \in T} v(i)$ for all $T \subseteq N$.

⁸The maximality of the demand vector together with the allocation rule guarantee individual rationality.

⁹Like the core, it is easy to show that the SDS may be empty. To see this, consider a four-player game where $v(S) = 1$ if $|S| = 3$, $v(S) = 0$ otherwise. Whatever (α, σ) one starts from, it is possible to find an acceptable objection where two players are given more than α , making it impossible to counterobject using α itself.

There are two bargaining sets that define objections and counterobjections as made not by individual players but by coalitions: the Mas-Colell (1989) bargaining set and the Zhou (1994) bargaining set. When compared to the Mas-Colell bargaining set, the SDS makes both objections and counterobjections more difficult, so that none of the sets is contained in the other.

We now turn to the Zhou bargaining set. The SDS and the Zhou bargaining set share the same definition of objection. As for the definition of counterobjection, they cannot be directly ranked. The SDS makes justified objections easier since it requires counterobjections to use the original demand vector and some of the counterobjecting players to be strictly better off, whereas the Zhou bargaining set requires only that the counterobjecting players are weakly better off. On the other hand, counterobjecting could seem easier in our framework, since the SDS does not impose the requirements $Z \setminus T \neq \emptyset$ and $T \setminus Z \neq \emptyset$. However we are able to show (by contradiction) that the balance of these conflicting forces implies that the SDS is included in the Zhou bargaining set.

Theorem 1 *If (α, σ) is in the SDS, then (α^σ, σ) is in the Zhou bargaining set.*

Proof. Suppose (α, σ) is in the SDS but (α^σ, σ) is not in the Zhou bargaining set. Then, there must be a coalition T that has an objection y such that any counterobjecting coalition Z satisfies either $Z \subset T$ or $T \subset Z$.

Consider first the case in which T has an objection and $Z \subset T$ has a counterobjection. Then it must be the case that $\alpha_i > y_i > \alpha_i^\sigma$ for all $i \in Z$, thus Z itself has a justified objection and (α, σ) cannot be in the SDS.

Consider now the case in which T has an objection that can *only* be countered by supersets of T . Let Z be one of those supersets. Since Z is a superset of T , it follows that $\alpha_i > y_i$ for all i in T . Maximality, together with the fact that the demand vector α is feasible for Z , implies $\sum_{i \in Z} \alpha_i = v(Z)$. Z itself has an objection in which one of the players in T (say, j) receives less than his demand and each other player in Z receives more than his demand. In order for (α, σ) to be in the SDS, there must be a counterobjection to this objection. Let Z' be one of the coalitions

that have a counterobjection to this objection. Z' must contain player j and cannot contain any other player in T . But then Z' itself has a counterobjection to the original objection by coalition T , contradicting the assumption that this objection can only be countered by supersets of T .¹⁰ **QED.**

Corollary 1 *The requirements $T \setminus Z \neq \emptyset$ and/or $Z \setminus T \neq \emptyset$ could have been imposed in definition 2 without changing the SDS.*

2.3.3 Elimination of dominated coalition structures

An important property of the SDS is that it eliminates the “dominated” coalition structures for grand-coalition superadditive games,¹¹ and therefore constitutes a meaningful selection of the Zhou bargaining set. In order to show why such selection is meaningful, let us explain what a dominated coalition structure is, and then give an example.

Definition 6 *A coalition structure σ is dominated given α if either*

$$\sum_{i \in T} \alpha_i^\sigma < v(T), \text{ for some } T \subset S \text{ for some } S \in \sigma \quad (2)$$

or

$$\exists T : v(T) > \sum_{i \in T} \alpha_i^\sigma \text{ and } T \text{ is union of elements of } \sigma. \quad (3)$$

Example 1 Consider the players' set $N = \{1, 2, 3, 4\}$ with

$$\begin{aligned} v(T) &= 4 && \text{if } |T| = 2 \\ v(S) &= |S| && \text{otherwise.} \end{aligned}$$

In this example (example 2.6 in Zhou (1994)) both the grand coalition and the all-singletons coalition structure are dominated. However, the grand coalition with equal payoff division is included in the Zhou bargaining set and in the Mas-Colell

¹⁰ Z' can only be a superset of T if T is a singleton. Notice that since the SDS only contains individually rational payoff configurations, any objecting set T must have at least two players.

¹¹A game is grand-coalition superadditive iff for any $\sigma \in \Sigma$, $v(N) \geq \sum_{S \in \sigma} v(S)$.

bargaining set. The same happens with the “all-singletons” coalition structure.¹² Neither of these unintuitive coalition structures is stable in the sense of the SDS. The general ability of the SDS to eliminate unreasonable coalition structures from being stable¹³ is established in the following theorem.

Theorem 2 *If (N, v) is grand-coalition superadditive and (α, σ) is in the SDS, then σ cannot be dominated given α .*

Proof. If (α, σ) in the SDS, maximality ensures that there is no T satisfying condition (2). As for condition (3), suppose $\exists T : v(T) > \sum_{i \in T} \alpha_i^g$ and T is the union of elements of σ . If (α, σ) is such that all players in T are receiving their demands, T itself has a justified objection. If (α, σ) is such that not all players in T are receiving their demands, call Z the set of players who are receiving their demands. Condition (3) can be written as $v(T) > \sum_{i \in T \cap Z} \alpha_i + \sum_{i \in T \setminus Z} v(i)$. Since the game satisfies grand coalition superadditivity and T is the union of elements of σ , $v(N) > \sum_{i \in Z} \alpha_i + \sum_{i \in N \setminus Z} v(i)$. Consider an objection by the grand coalition that gives all players in Z more than their demands. A counterobjecting coalition C cannot include any players from Z . Thus, if a counterobjecting coalition exists, this counterobjecting coalition itself has a justified objection. **QED.**

Corollary 2 *If $v(N) > \sum_{S \in \sigma} v(S)$ for all $\sigma \in \Sigma$, only the grand coalition can be stable in the sense of the SDS.*

Unlike the Zhou bargaining set, the SDS has the dummy player property. The SDS is also immune to two more criticisms often made to other solution concepts in the family of bargaining sets: the fact that they are “too large” and the fact that they are sometimes difficult to compute.¹⁴

¹²The traditional bargaining set (Davis and Maschler, 1967) is always nonempty for *all* coalition structures (see Peleg, 1967).

¹³Notice that the elimination of dominated coalition structures may be at the cost of emptiness of the solution concept, since there are games that have only dominated coalition structures.

¹⁴See Maschler (1992) for a lucid discussion of these two problems.

As far as the first problem is concerned, Theorems 1 and 2 hint that the SDS is less exposed to that problem than most other solution concepts of the same family. In particular, we will show below that the SDS predicts a unique payoff distribution for all constant-sum weighted majority games admitting a homogeneous representation, while other bargaining sets typically admit a continuum of payoff distributions.

As far as the second problem is concerned, the results of this paper indicate that it is possible, at least for some classes of games, to construct algorithms to generate the allocation(s) in the SDS.

2.3.4 Von Neumann-Morgenstern stable sets

Both the SDS and stable sets require counterobjections to be related to the original allocation, but not in the same way: in a stable set, a counterobjection may be very different from the original proposal but it has to belong to the same stable set; in the SDS a counterobjection does not have to belong to the SDS but it has to use the original demand vector. Unlike the stable sets (and like all other bargaining sets), the SDS is not defined in a "circular" way: the definition of stability in the sense of the SDS applies to each element separately and not to the whole set. Thus, whether a particular (α, σ) is stable does not depend on whether other pairs are stable.

In general, the SDS may not be a stable set. Since the SDS may coincide with the core, it is clear that the SDS may not be externally stable¹⁵. For a counterexample on internal stability, consider the following game.

Example 2 $N = \{1, 2, 3, 4\}$, $v(1, 2, 3) = 3$, $v(2, 4) = v(3, 4) = 2$. All other values are zero.

The SDS contains a continuum of stable demand vectors. We concentrate on two of them: $(1, 1, 1, 1)$ and $(0, 1.5, 1.5, 0.5)$. The first one is stable with coalition structure $\{\{1, 2, 3\}, \{4\}\}$, and the second one is stable with coalition structures

¹⁵Take $N = \{1, 2, 3\}$, $v(1, 2) = v(1, 3) = v(2, 3) = 1$. The only vector in the SDS is $(1, 0, 0)$, but allocations like $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ are not dominated.

$\{\{1\}, \{2\}, \{3, 4\}\}$, $\{\{1\}, \{3\}, \{2, 4\}\}$ and $\{\{1, 2, 3\}, \{4\}\}$. The allocation resulting from the first pair, $(1, 1, 1, 0)$ is dominated by the allocation resulting from the second pair, $(0, 0, 1.5, 0.5)$, thus the SDS is not internally stable. The example also shows that a stable demand vector need not be unique and that the same vector (in this case $(1, 1, 1, 1)$) may be stable with some coalition structures and not with others (coalition structure $\{\{1\}, \{2\}, \{3, 4\}\}$ is also feasible with this demand vector but it is not stable because coalition $\{1, 2, 3\}$ can object with $(1 - 2\epsilon, 1 + \epsilon, 1 + \epsilon)$ and no counterobjection is possible).

However, we will see in the next section that for the special case of constant-sum homogeneous weighted majority games the SDS makes the same predictions as one of the stable sets, called the main simple solution.

We now turn to the characterization and predictions that the SDS allows us to obtain for the important class of weighted majority games.

3 Characterization for Weighted Majority Games

3.1 Definitions, Assumptions, and Auxiliary Results

A coalitional game (N, v) is a *simple game* iff $v(\emptyset) = 0$, $v(N) = 1$, $v(S) = 0$ or 1 and $v(S) = 1$ whenever $v(T) = 1$ for some $T \subset S$.

Denote by $\Omega \equiv \{S : v(S) = 1\}$ the set of *winning coalitions* (WC) and by $\Omega^m \equiv \{S : v(S) = 1, v(T) = 0 \text{ for all } T \subset S\}$ the set of *minimal winning coalitions* (MWC).

A simple game is called *proper* iff for all $S \in \Omega$, $N \setminus S \notin \Omega$. It is called *strong* iff for all $S \notin \Omega$, $N \setminus S \in \Omega$. Proper and strong simple games are called *constant-sum*.

A simple game (N, v) is a *weighted majority game* iff there exists a vector of non-negative weights w and a number q ($0 < q \leq \sum_{i=1}^n w_i$) such that

$$S \in \Omega \iff \sum_{i \in S} w_i \geq q.$$

Assumption 1 (N, v) is a constant-sum weighted majority game.¹⁶

¹⁶A sufficient condition for a game with integer weights to be constant-sum is $p \equiv \sum_{i=1}^n w_i$ to be

Assumption 2 $w_i < q$ for all $i \in N$.¹⁷

Assumptions 1 and 2 together imply $n \geq 3$ and exclude the presence of veto players in the game.

A weighted majority game admits a *homogeneous representation* iff there exists a vector w (and an induced quota q) such that

$$\sum_{i \in S} w_i = q \quad \text{for all } S \in \Omega^m. \quad (4)$$

A weighted majority game admitting a homogeneous representation is called a *homogeneous weighted majority game*.

Example 3 Consider a four-player game where player 1 holds 3 votes, player 2 holds 2, and players 3, 4 hold 1; this representation is not homogeneous: there is one MWC with 5 votes, and three MWCs with 4 votes; however, an equivalent representation of the same characteristic function game is one where player 1 has 2 votes, players 2, 3, 4 have one vote each, and this is homogeneous.

Denote by W^i the set of winning coalitions containing player i for a given game (N, v) . Notice that $W^i \subseteq \Omega$, and $W^i = \Omega$ only if i is a veto player. Denote by μ^i the number of WCs in W^i . Similarly, denote by M^i the set of MWCs containing player i and m^i is the number of MWCs in M^i .

Lemma 1 Consider a weighted majority game satisfying assumption 1; for every player i , either $m^i = 0$ or $m^i \geq 2$.

Proof. Suppose that for some i there exists a MWC $S \in M^i$. Since the game is strong, the coalition $T \equiv \{N \setminus S\} \cup \{i\}$ must be a winning coalition. Either T itself is a MWC, or there must exist $Z \subset T$ such that $Z \in \Omega^m$, and such a coalition Z must contain i , otherwise S would not have been winning in the first place. In either case, $m^i \geq 2$. **QED.**

an odd number and $q = \frac{p+1}{2}$. If q is a smaller number the game may not be proper, whereas for a larger q , as well as for an even number of votes, the game may not be strong.

¹⁷Otherwise the coalitional game would be irrelevant.

Corollary 3 *For any MWC S containing player i , we can find another MWC S' such that $S \cap S' = \{i\}$.*

Definition 7 *Player i and player j are of the same type iff the characteristic function is unchanged when permuting them.*

Remark 4 *If (N, v) is a weighted majority game and $\mu^i = \mu^j$, then i and j are of the same type.*

Proof. Suppose not. Then, without loss of generality, $w_i > w_j$. Consider the winning coalitions that contain either i or j , $\{W^i \cup W^j\} \setminus \{W^i \cap W^j\}$. If we take the coalitions in $W^j \setminus W^i$ and replace j by i , the resulting coalition are all winning. If we take a coalition in $W^i \setminus W^j$ and replace i by j the resulting coalitions cannot be all winning, or i and j would be the same type. But then $\mu^i > \mu^j$, a contradiction.

In general, if two players, i, j , have different numbers of votes ($w_i \neq w_j$), it does not follow that they must be of a different type, i.e., they not necessarily have different bargaining power. For example, consider a three-player game, where one has 5 votes, one has 4, and one 3. 7 votes are needed to win, and hence every pair of players can make it. Every player has the same number of MWCs (and of WCs), and permuting them does not change the characteristic function, even though every player has a different weight. There is no reason why the player with 5 votes should have more bargaining power than the other two. In fact, an *equivalent* homogeneous representation of this game is one where each of the three players has 1 vote. For homogeneous representations of weighted majority games players with different numbers of votes are indeed of different types. While it is always true that if $w_i = w_j$ then $\mu^i = \mu^j$, the converse is true (as established below) if the game is strong and homogeneous.

Remark 5 *If the coalitional game (N, v) satisfies assumption 1 and admits a homogeneous representation through some vector w , then $\mu^i = \mu^j \longrightarrow w_i = w_j, \forall i : M^i \neq \emptyset, \forall j : M^j \neq \emptyset$.*

Proof. Suppose that $\mu^i = \mu^j$ and $w_i > w_j$. Consider all the WCs that contain either i or j : $\{W^i \cup W^j\} \setminus \{W^i \cap W^j\}$. These sets are always non-empty because of Corollary 3. If we take every single coalition in $W^j \setminus \{W^i \cap W^j\}$ and substitute j with i , we always obtain a WC with i , while when substituting i with j in every coalition in $W^i \setminus \{W^i \cap W^j\}$ this is not the case: $w_i > w_j$ plus homogeneity implies that there exists at least one MWC containing i but not j with exactly q votes, and this in turn implies that after the substitution the coalition would not be winning anymore. Hence $\mu^i > \mu^j$. Contradiction. **QED.**

Lemma 2 *Let (N, v) be a homogeneous weighted majority game satisfying Assumptions 1 and 2. Let i and j be two players such that each of them belongs to at least one MWC. Then there is a MWC containing both i and j .*

Proof. Let S be a MWC containing i . If it also contains j , we are done. If it does not, consider $N \setminus S \cup \{i\}$. Because the game is strong, this coalition is winning. It includes a MWC $T \ni i$. If $T \ni j$, we are done. If not, this implies $w_j < w_i$. Now we take a coalition $S' \ni j$ and repeat the same process. This time we cannot throw away i , because it would imply that $w_i < w_j$, a contradiction. **QED.**

3.2 The Proportionality Result

Theorem 2 has the following implication for the selection of coalition structures in weighted majority games:

Lemma 3 *The only “candidate pairs” (α, σ) for the SDS of any weighted majority game satisfying Assumption 1 are those where*

- (1) *For any $T \in \Omega$, $\sum_{i \in T} \alpha_i \geq v(T)$;*
- (2) *σ always includes a WC $S \in \Omega$ with $\sum_{i \in S} \alpha_i = v(S)$.*

Proof. (1) follows from maximality. The fact that σ must include a winning coalition follows from Theorem 2. Since σ must be compatible with α , $\sum_{i \in S} \alpha_i$ cannot exceed $v(S)$, which together with maximality implies $\sum_{i \in S} \alpha_i = v(S)$. **QED.**

Theorem 3 Consider a weighted majority game (N, v) satisfying Assumption 1 and 2 that admits a homogeneous representation with weights w_1, \dots, w_n . If there are no dummy players, the SDS of any such game is non-empty and only contains pairs (α^*, σ) where the unique stable demand vector has

$$\alpha_i^* = \frac{w_i}{q}, \text{ for all } i$$

and σ always contains a MWC S .

If there are dummy players, the SDS is non-empty and only contains pairs (α^*, σ) where $\alpha_i^* = \frac{w_i}{q}$ for all non-dummy players and σ contains a winning coalition.

Proof. We first provide the proof for the case in which there are no dummy players.

Consider a pair where there is a MWC $S \in \sigma$ and the demand vector is α^* . We first show that (α^*, σ) is in the SDS of the game.

Since (N, v) is proper, any blocking coalition T must contain some agents in common with S (i.e., $T \cap S \neq \emptyset$). T can make an objection to (α^*, σ) only if there exists a payoff vector y feasible for T such that $y_i > \frac{w_i}{q}$ for all $i \in T \cap S$. Define $Z \equiv \{i \in T : y_i \geq \frac{w_i}{q}\}$. Because $Z \supseteq \{T \cap S\}$ and all agents in $T \cap S$ must receive strictly more than $\frac{w_i}{q}$, it follows that Z is a losing coalition (Z winning would be unfeasible). Thus, there must be a WC $C \subseteq N \setminus Z$, and a MWC $C' \subseteq C$. Since the game is homogeneous, α^* is feasible for any MWC, thus coalition C' has a counterobjection: since the game is proper, $C' \cap T \neq \emptyset$. Moreover, the fact that C' is included in $N \setminus Z$ ensures that all players in $C' \cap T$ are strictly better-off.

We have shown that no objection to (α^*, σ) can be justified, because every objection would lead to a counterobjection using α^* once again. In order to complete the proof, we now have to show that any pair (α, σ) with $\alpha \neq \alpha^*$ is vulnerable to justified objections.

Suppose (α, σ) is in the SDS but $\alpha \neq \alpha^*$. We define the sets

$U \equiv \{i \in N : \alpha_i < \alpha_i^*\}$, the set of “underdemanding” players.

$F \equiv \{i \in N : \alpha_i = \alpha_i^*\}$, the set of players demanding exactly α^* .

$O \equiv \{i \in N : \alpha_i > \alpha_i^*\}$, the set of “overdemanding” players.

There are two possibilities:

Case 1) $U = \emptyset$. Since $U = \emptyset$ and $\alpha \neq \alpha^*$, it must be the case that $\alpha_i \geq \alpha_i^*$ for all i and $\alpha_j > \alpha_j^*$ for some j . We know from Lemma 3 that σ contains a winning coalition S such that $\sum_{i \in S} \alpha_i = v(S)$. Since $U = \emptyset$, coalitions containing overdemanding players are unfeasible, thus S must be such that $\alpha_i = \alpha_i^*$ for all $i \in S$, therefore $j \notin S$. Any MWC $T \ni j$ has a justified objection. In this objection, any player $i \in T \cap F$ receives $\alpha_i^* + \epsilon$ and any player i in $T \cap O$ receives some positive payoff. Notice that $j \in T$ and T being a MWC ensures that players in $T \cap F$ have less than q votes, so that this payoff distribution is feasible. Since $S \cap O = \emptyset$, any positive payoff makes the players in $T \cap O$ better-off, hence we indeed have an objection. There is no possible counterobjection, since it would have to include some players in $T \cap O$, and any coalition containing overdemanding players is unfeasible.

Case 2): $U \neq \emptyset$. This implies $O \neq \emptyset$ (otherwise we would contradict maximality). Again, we know from Lemma 3 that σ contains a winning coalition S such that $\sum_{i \in S} \alpha_i = v(S)$. We will distinguish two cases:

Case 2a) $S \subseteq F$. Take any player $j \in U$, and any MWC $T \ni j$. Denote $T \cup U$ by Z . Coalition Z has a justified objection in which every member of $Z \setminus O$ gets $\alpha_i^* + \epsilon$ and every member of $Z \cap O$ gets a positive payoff. To see this, notice that this payoff division is feasible for Z if ϵ is small enough.¹⁸ $S \subseteq F$ implies that all players in Z are made better-off by the objection. Finally, since a counterobjection must include at least one player from $Z \cap O$, and it can include no players from U , it follows that no counterobjecting coalition is feasible given α .

Case 2b) S is not a subset of F . Lemma 3 implies $S \cap O \neq \emptyset$ and $S \cap U \neq \emptyset$. Notice that S need not be a MWC. There is a MWC $S' \subseteq S$ such that $S' \cap O \neq \emptyset$ and $S' \cap U \neq \emptyset$, and $S \cap O \subseteq S'$. Let $Z := (N \setminus S') \cup U$. As in case 2a, Z has a justified objection in which every member of $Z \setminus O$ gets $\alpha_i^* + \epsilon$ and every member of $Z \cap O$ gets a positive payoff. Note that the difference with case 2a is that, in the

¹⁸This follows because $Z \setminus O$ is a losing coalition. If it were winning, one could find a subset of $Z \setminus O$ having exactly q votes, and containing at least one player from U (because of the way Z has been constructed, $Z \setminus U$ is losing), contradicting maximality.

construction of Z we had to make sure that $Z \cap (S \cap O) = \emptyset$.

Given that α^* is the only stable payoff vector and that dominated coalition structures are excluded from the SDS, it follows that σ must contain a minimal winning coalition.

If there are dummy players, their actual payoff is always zero but their demands are not constrained. A dummy player can be part of a winning coalition in σ if his demand is zero. **QED.**

Remark 6 *Notice that the SDS depends only on the characteristic function, and hence it is invariant to the particular representation chosen.*

Remark 7 *Note that what matters is the homogeneous representation. For example, in every 3-player weighted majority game where Assumptions 1 and 2 are satisfied, the equivalent homogeneous representations give equal weights to the three players, and hence the unique Stable Demand vector is $\alpha^* = (1/2, 1/2, 1/2)$.*

Remark 8 *If (α, σ) is in the SDS of a homogeneous game (N, v) satisfying Assumptions 1 and 2, then α solves the following program:*

$$\begin{aligned} \min \sum_{i \in N} \alpha_i \\ \text{s.t. } \sum_{i \in S} \alpha_i \geq 1 \text{ for all } S \in \Omega^m. \end{aligned} \tag{5}$$

This implies that α is a balanced aspiration.

Proof. Suppose α is a solution to the minimization problem but $\alpha \neq \alpha^*$. This implies $U \neq \emptyset$. Take $i_1 \in U$ and a MWC S containing i_1 . Because of maximality, the sum of the demands in this MWC is at least 1. In order for α to have a smaller sum than α^* , there must be at least another player from U in $N \setminus S$. Consider the set $N \setminus S \cup \{i_1\}$. Because the game is strong, this is a WC. If it is also a MWC, it contradicts maximality. Otherwise, there is a MWC included in it. In order for maximality to be satisfied, there must have been some player from U that has been thrown away. Call this player i_2 . Now take a MWC S_2 including player i_2 . Because we could take i_2 away and still have a WC, $w_2 < w_1$. Now we repeat the same

reasoning with i_2 and conclude that α cannot be in the aspiration core without the existence of a player i_3 in U with $w_3 < w_2$ and so on. Since we only have a finite set of players, this cannot continue for ever, and on the way we have to find a contradiction. **QED.**

Recall the following result shown by von Neumann and Morgenstern (1944). If for a constant-sum simple game (N, v) there is a vector $x = (x_1, \dots, x_n)$ with each $x_i \geq 0$ such that $x(S) = 1$ whenever S is a minimal winning coalition, the set of imputation vectors $\{z^S : z_i^S = x_i \forall i \in S \text{ and } z_i = 0 \forall i \in N \setminus S, S \in \Omega^m\}$ is a stable set and was called the *Main Simple Solution* by von Neumann and Morgenstern. The weighted majority games that we have studied in this section are a subset of this set of games, hence the next remark:

Remark 9 *The set of imputations $\{\alpha^{*\sigma}\}$ derived from the set of pairs $\{(\alpha^*, \sigma) \in \text{SDS}\}$ coincides with the Main Simple Solution.*

This means that among the many VNM stable sets of constant-sum weighted majority games, the SDS selects the one that was singled out by von Neumann and Morgenstern as the most meaningful one¹⁹.

3.3 The Apex Game

Consider, as an example, the so called Apex Game.²⁰

There are $n = 5$ players, where $\{1, 2, 3, 4\}$ have one vote each and player 5 has three votes. Thus, $q = 4$. The MWCs are:

I: $\{5, i\}$;

II: $\{1, 2, 3, 4\}$.

The SDS makes, we believe, the most reasonable prediction, i.e., that if **I** forms then player 5 receives $3/4$, and the other player gets $1/4$ (proportional payoffs); if

¹⁹Furthermore, the SDS coincides with the main simple solution for all constant-sum simple games admitting a main simple solution, even if they are not weighted majority games.

²⁰Davis & Maschler (1965) used this example to contrast the predictions of the existing solution concepts. The name ‘‘apex game’’ came later.

II forms they share equally (1/4 each).

The proof that the only stable demand vector (for this example) is the proportional one can be summarized as follows.

Consider first the proposal where the demand vector is $\alpha^* = (1/4, 1/4, 1/4, 1/4, 3/4)$ and the coalition structure includes the WC **I**.

One possible objection to this proposal is coalition **I** but with a different small player, and where y is $(3/4 + \epsilon, 1/4 - \epsilon)$. However, these objections are not justified, because there exists a counterobjection to each of them, with the vector α^* and the WC **II**. The second (and last) kind of objection to the pair (α^*, \mathbf{I}) , would be one with the four small players together, where the “blocker” receives $1/4 + \epsilon$ and the others share the rest. But then at least one of these other three small players must be receiving less than $1/4$, and can therefore counter by offering again α^* to player 5.

Consider then the pair (α^*, \mathbf{II}) . The objection to be considered here is one with **I** and payoffs $(3/4 - \epsilon, 1/4 + \epsilon)$; to this, however, there exists a counterobjection with $(3/4, 1/4)$ and **I** (with a simple replacement of the small agent in the WC), and hence there is no justified objection. It is finally easy to see that no other pair can be in the SDS.

The core of the Apex Game is empty, as it would be for any other constant-sum essential game. The Shapley Value only looks at the *ex ante* balance of power, which here implies $3/5$ for the big player and $1/10$ for each small player. Similarly, the nucleolus with respect to the grandcoalition has the allocation $(3/7, 1/7, 1/7, 1/7, 1/7)$ (the homogeneous representation of the game itself)²¹ and can again be interpreted as an *ex ante* evaluation. While this solution assigns each player the appropriate relative bargaining power, this property disappears when the nucleolus is computed with respect to coalition structures containing a MWC: for example, if MWC **I** forms, the nucleolus gives each of the two players a payoff of $1/2$, in spite of the very different endowments. Since we have established that the coalition structures including a MWC are the only meaningful ones, the nucleolus and the Shapley Value

²¹See Peleg (1968).

are therefore not appropriate solutions for these games if one wants an *ex post* prediction, i.e., a prediction of the payoff distribution *contingent* on one of the possible coalition structures prevailing, i.e., contingent on which MWC prevails.

The competitive bargaining set (Horowitz, 1973) predicts proportional payoffs contingent on **I** or **II** being formed, but it allows for dominated coalition structures.²² The traditional bargaining set, on the other hand, not only allows for dominated coalition structures (like the grand-coalition), but it also gives an uninformative prediction about the payoff distribution even when contingent on an undominated coalition structure. In fact, it contains **I** as one of the possible coalitions in a solution, but with a payoff vector $x = (x_5, 1 - x_5)$, where x_5 can take any value between $\frac{3}{4}$ and $\frac{1}{2}$. The same occurs with the Zhou bargaining set and with the Mas-Colell bargaining set (the latter excludes the extreme payoffs $\frac{3}{4}$ and $\frac{1}{2}$). The equal division ($x = (\frac{1}{2}, \frac{1}{2})$) corresponds to the kernel and its supersets the reactive bargaining set (Granot and Maschler, 1997) and the semireactive bargaining set (Sudhölter and Potters, 1999). The main simple solution yields the same prediction as the SDS for the apex game, but there are other stable sets. The aspiration solution concepts coincide with the SDS for apex games, but only the aspiration core coincides with the SDS for constant-sum homogeneous weighted majority games in general (the remaining aspiration solution concepts may be larger).

4 Implementation of the SDS

In this section, we provide an implementation of the SDS via a simple mechanism in which an auxiliary set of individuals compete over the n players in the cooperative game, following Pérez-Castrillo and Wettstein (2000). We will assume that the cooperative game (N, v) is zero-normalized, superadditive and has the following property: $v(S) > \sum_{i \in S} v(i)$ implies $v(N \setminus S) = \sum_{i \in N \setminus S} v(i)$. This property is called the *one-stage property* (Selten, 1981) and implies that no more than one profitable coalition can be formed at a time.

²²Moreover, the competitive bargaining set may be empty for other constant-sum homogeneous weighted majority games.

The mechanism is played by four auxiliary individuals called principals. The principals have lexicographic preferences: in the first place they want to maximize profits but, other things equal, they prefer to hire as many agents as possible.²³

The *mechanism* M is played as follows:

Stage 1. Principal 1 chooses (α, S) , with $\alpha \in \mathcal{R}^n$ and $S \subseteq N$. The bid of principal 1 for agent i , x_i , is computed in the following way: $x_i = \alpha_i$ for $i \in S$ and 0 otherwise.

Stage 2. P2 and P3 simultaneously choose y^2 and $y^3 \in \mathcal{R}^n$. We will denote $\max(y_i^2, y_i^3)$ by y_i . Given the bid vectors x , y^2 and y^3 , each agent is provisionally assigned to the principal that offers the highest price (or wage). Ties are broken in favor of the principal with the lowest index. Let T^j be the set of players provisionally assigned to principal $j = 1, 2, 3$. If one principal gets all the agents, the game ends and the principal that got the agents receives $v(N)$ less the wages of the n agents. If no principal has got all the agents, the game moves to the next stage.

Stage 3. P4 may hire any set Z of agents under the following conditions:

- 1) He has to offer them $z_i = \alpha_i$;
- 2) Z must contain elements of both T^1 and $T^2 \cup T^3$;
- 3) z must make all the players in Z weakly better-off, and players in $Z \cap (T^2 \cup T^3)$ strictly better-off.

Finally, principal j ($j = 1, 2, 3$) hires the agents in $T^j \setminus Z$ and pays them the wage offered; P4 hires the agents in Z and pays them z .

Theorem 4 *The mechanism M implements the SDS in SPE.*

Proof. We first construct an equilibrium of the mechanism M for any (α, σ) in the SDS chosen by P1.

Consider the following strategies: P1 submits (α, S) , where S is the set of players receiving their demands given (α, σ) , that is, $S = \{i \in N : \alpha_i^\sigma = \alpha_i\}$. After observing

²³This type of preferences is also assumed by Pérez-Castrillo and Wettstein in one of their mechanisms.

this choice, P2 and P3 bid $y^2 = y^3 = x = \alpha^\sigma$ (and play an equilibrium in all other subgames). If the game reaches stage 3, P4 plays a best response.

Given these strategies, P1 gets all the agents and all principals make zero profits. To show that this is an equilibrium, we prove that there is no profitable deviation at stage 1 or 2 (since at stage 3 this is the case by construction).

Let us first rule out that P1 has a profitable deviation. This cannot be the case because P2 would not be playing a best response.²⁴

Any profitable deviation by P2 or P3 corresponds to an objection to (α, σ) . The third stage would be reached after the deviation (an objection that takes all the players is excluded by theorem 2). Since (α, σ) is in the SDS, there is a counterobjection to this objection. P4 hiring some agents corresponds to a counterobjection and, since P4 prefers to hire more agents rather than less, he will always counterobject. The one-stage property ensures that the deviator (P2 or P3) would be left with a negative profit.

Now we prove that the outcome of any SPE of the mechanism M is in the SDS (provided nonempty).

First, all principals make nonnegative profits in equilibrium. Second, they all make zero profits (if some principal is making positive profits, P2 or P3 could bid ϵ more for all agents the other principals would be getting, get all the agents and end the game). Third, P1 must have all the agents. Because of the one-stage property, P1 can secure all the agents by choosing (α, S) such that the corresponding (α, σ) is in the SDS, so any strategy combination in which P1 does not get all the agents cannot be an equilibrium. Fourth, (α, σ) is in the SDS. Suppose not. Then P2 is not playing a best response, since he can make an objection, secure some agents, and P4 cannot do anything since there is no counterobjection.²⁵ **QED.**

²⁴Since P2, P3 and P4 play best responses in all subgames, after the deviation by P1 all principals make nonnegative profits. Moreover, P3 and P4 pay for the agents at least what P1 offered them. It follows that P2 cannot be playing a best response: slightly overbidding P1 (and possibly P3) and thus getting all the agents and finishing the game would be profitable.

²⁵In the mechanisms of Pérez-Castrillo and Wettstein (2000) the designer does not need to know the characteristic function; in our mechanism, the game is zero-normalized, so the designer knows

5 Concluding Remarks

Payoff distribution and coalition formation should be studied simultaneously. Since value concepts give only an *ex ante* evaluation of the prospectives of different players, they cannot be used to predict the *ex post* payoff distribution in an “equilibrium” coalition structure. Solution concepts that keep the spirit of core-like competition, respecting individual rationality as well as group rationality, seem more appropriate for this task. The stable demand set is, we believe, a meaningful addition to the set of concepts of this family. In fact, it is a selection of the Zhou bargaining set that manages to eliminate from the set of solutions all the counterintuitive solutions that are dominated, and gives a precise prediction for the important class of constant-sum homogeneous weighted majority games. The selection of undominated coalition structures makes the SDS, in our view, a very useful concept, with a much sharper predictive power than previous concepts in the family of bargaining sets. It is not always nonempty, but the proportional prediction for homogeneous weighted majority games suggests that it can be used in important allocation and distribution problems where the core is empty.

However, predictive power was not the only motivation of the paper. The main conceptual contribution of the stable demand set is the idea itself of self-stability of demands. A set of “claims” by the players is considered stable if the same claims can be used by some subset of the players to counter any possible objection to the assignment of those claims. Intuitively, this is an important requirement in distributive politics or in any distributive problem in legislatures: the advocates of any given distributive “norm” (like “to everyone according to its contribution”) should be able, if they want that norm to prevail, to counter any objection by making reference to the norm itself. Countering an objection with a completely $v(i)$ for all i in N . The extension to a completely unknown characteristic functions is straightforward: let P1 complete the vector x himself and add a final stage in which the agents accept or reject the wage offer (this would have been a problem for the original mechanisms of Pérez-Castrillo and Wettstein since they implement bargaining sets that are not individually rational). In equilibrium P1 will complete the vector x with $v(i)$ for all $i \notin S$.

unrelated counterobjection would not be convincing as much as an argument that uses the same norm originally proposed. Asking that counterobjections have to use exactly the initial demand vector is an extreme form of consistency with the initial proposed norm, and in future research it could be interesting to study weaker forms of consistency, but we believe that this concept will serve at least as an useful benchmark for this type of investigation of the way social norms are agreed upon.

Morelli (1999) studies demand bargaining games that produce similar predictions to the SDS for homogeneous weighted majority games. However, the connection between those noncooperative games and the SDS is not one-to-one. Selten (1981) implements the set of partnered aspirations, which is a superset of the SDS for constant-sum homogeneous weighted majority games. Here, instead, we managed to obtain a tight implementation result.

References

- [1] ALBERS, W. (1974). Zwei Lösungskonzepte für Kooperative Mehrpersonspiele, die auf Anspruchsniveaus der Spieler Basieren. *OR-Verfahren* (Methods of Operations Research) **21**, 1-13.
- [2] BENNETT, E. (1983), "The Aspirations Approach to Predicting Coalition Formation and Payoff Distribution in Side Payment Games," *International Journal of Game Theory* **12**, 1-28.
- [3] BENNETT, E. (1985), "Exogenous vs. Endogenous Coalition Formation", *Economie Appliquée* **37**, 611-635.
- [4] CROSS, J. (1967), "Some Theoretical Characteristics of Economic and Political Coalitions", *Journal of Conflict Resolution* **11**, 184-195.
- [5] DAVIS, M. and M. MASCHLER (1965), "The Kernel of a Cooperative Game," *Naval Research Logistics Quarterly* **12**, 223-259.
- [6] DAVIS, M., and M. MASCHLER (1967), "Existence of Stable Payoff Configurations for Cooperative Games", in M. Shubik (ed.), *Essays in Mathematical Economics in Honor of Oskar Morgenstern*, Princeton University Press, 39-52.
- [7] DRUCKMAN, J, and WARWICK (2002), "Portfolio Salience and the Proportionality of Payoffs in Coalition Governments", *British Journal of Political Science*, forthcoming.
- [8] GRANOT, D. and M. MASCHLER (1997), "The Reactive Bargaining Set: Structure, Dynamics and Extension to NTU Games", *International Journal of Game Theory* **26**, 75-95.
- [9] HOROWITZ, A.D. (1973), "The Competitive Bargaining Set for Cooperative n-Person Games", *Journal of Mathematical Psychology* **10**, 265-289.

- [10] MASCHLER, M. (1992), "The Bargaining Set, Kernel, and Nucleolus," in R. Aumann and S. Hart (eds.), *Handbook of Game Theory with Applications*, 591-667.
- [11] MAS-COLELL, A. (1989), "An Equivalence Result for a Bargaining Set," *Journal of Mathematical Economics*, **18**, 129-138.
- [12] MORELLI, M. (1999), "Demand Competition and Policy Compromise in Legislative Bargaining," *American Political Science Review* **93**, 809-20.
- [13] PELEG, B. (1967), "Existence Theorem for the Bargaining Set $\mathcal{M}_1^{(i)}$ ", in M. Shubik (ed.), *Essays in Mathematical Economics in Honor of Oskar Morgenstern*, Princeton University Press, 53-56.
- [14] PELEG, B. (1968), "On Weights of Constant Sum Majority Games," *SIAM Journal of Applied Mathematics* **16**, 527-532.
- [15] PÉREZ-CASTRILLO, D. and D. WETTSTEIN (2000), "Implementation of Bargaining Sets via Simple Mechanisms", *Games and Economic Behavior* **31**, 106-120.
- [16] SELTEN, R. (1981), "A Non-Cooperative Model of Characteristic Function Bargaining," in V. Böhm and H. Nachtkamp (eds.), *Essays in Game Theory and Mathematical Economics in Honor of Oscar Morgenstern*, 131-151. Mannheim: Bibliographisches Institut.
- [17] SUDHÖLTER, P. and J.A.M. POTTERS (1999), "The Semireactive Bargaining Set of a Cooperative Game", IMW Working Paper No 313.
- [18] VON-NEUMANN, J. and O. MORGENSTERN (1944), *Theory of Games and Economic Behavior*, Princeton N.J., Princeton University Press.