4 ELEMENTARY QUANTUM MECHANICAL PRINCIPLES AND SOCIAL SCIENCE: IS THERE A CONNECTION?

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Abstract

In this paper we provide first for a brief overview of some of the work which has been performed on the interface of quantum mechanics and macroscopic systems (such as economics). We then provide for an overview of how such quantum mechanical concepts can enter financial option pricing theory. We round off the paper with some suggestions on where this area of research can be heading in the near future.

Keywords: superposition; wave function; Black-Scholes option price; information function; probability amplitude; Schrödinger equation; Newton- Bohm trajectory; mean forward (backward) derivative

JEL classification: C00, C02; G12

1. Introduction

In 1947, John von Neumann and Oskar Morgenstern wrote a path-breaking book entitled "Theory of Games and Economic Behavior". This book contained the blueprint of game theory which subsequently would become a major discipline of economic theory.

In the 50's and 60's we saw the development of another important tool of analysis: the expected utility models. The Savage expected utility model (1954), named after its originator, the famous statistician Leonard Savage, and the Anscombe-Aumann approach (1963), were both models which attempted to enrich the celebrated von Neumann-Morgenstern model. The type of probability used in formulating von Neumann-Morgenstern expected utility can be seen, as Kreps (1988) clearly indicates, as .an objective - externally imposed probability. The Savage model, on the contrary, defines probability as used by the economic agent (i.e. subjective probability). The Anscombe-Aumann model falls in between those two models: it uses a mixture of objective and subjective probability. All of those models are build up by using an axiomatic structure. They are mathematically very elegant (especially the Savage expected utility model) but

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they are prone to experimental refutation. The Ellsberg paradox showed the world that the famous sure-thing principle, a key assumption of the Savage model can be refuted on experimental grounds. Economists however, did not sit still and important work followed after this. We note the core papers by Gilbao and Schmeidler (1989) and Ghirardato et al. (2004) and also Machina (1989).

Financial economics, a closely related ‘sister’ discipline of economics became famous in the 20th century for its celebrated asset pricing models such as the capital asset pricing model and the APT models, and their many variants. But financial economics may well be best known for a model which generated a market with trillion dollar levels of capitalization: the option market. For an excellent paper on this issue, please consult MacKenzie and Millo (2003). In the early 1970’s Black and Scholes (1973) developed a partial differential equation which would become one of the best and most widely known pde’s in the history of mankind.

Economics in all its varieties, has also shown to be very inter-disciplinary in nature. Macro-economic theory draws heavily upon statistical theory. Econometrics is a discipline in its own right. Economic theory has always had a very close association with mathematics, especially measure theory. Many of the protagonists in economic theory like Gerard Debreu (1959), well know for his work on the core of an economy, were mathematicians. However, at the time when

Debreu, Arrow (1971), Hildenbrand (1974), Georgescu-Roegen (1999) and other economics luminaries were in their heydays, the collaboration between physicists and economists was significantly much less important, but then with one exception: Fisher Black, a physicist, and Myron Scholes, an economist joined their intellectual forces in 1973 (Black and Scholes (1973)) to produce the Black-Scholes option pricing pde.

The so called ‘econophysics’ movement rose to prominence a decade ago with the heavy implication of towering physicists like Boltzmann winner, Eugene Stanley (Mantegna and Stanley (1999)). Furthermore, the heavy involvement of top level economists like Thomas Lux (1999) helped very much the movement’s momentum. Econophysics has had a high interest in modeling return distributions with distributions which reflect the data in a much closer way. We note, for instance, the work of Lisa Borland (2002), who re-defined option pricing with alternative non-Gaussian distributions. Furthermore, a lot of work in the econophysics area has been performed on the issue of modeling agent behavior in financial markets and also on modeling markets as such. Marcel Ausloos and Andrzej Pekalski (2006) provide for an important study of a closed market. Ionut Purica (2004), in a very interesting paper, analyzes the economic parameters which determine the geographical dimension of a city.

In 1997, M.I.T mathematician I.E. Segal contributed an article in the Proceedings of the National Academy of Sciences of the United States of America (PNAS) on the Black-Scholes option pricing formula in a quantum environment. In 1999, Andrei Khrennikov, a world authority on the foundations of quantum probability, wrote a path-breaking article on how quantum mechanical concepts could be applied in all kinds of macro-scopic settings. Those two articles may well have set the tone for a movement which has, since then, tried to bridge the gap between quantum mechanics and macro-scopic systems.

This paper has three aims. Our first aim consists in giving an overview of what the research on the interface of quantum mechanics and macro-scopic systems has been doing to date. A second aim consists in attempting to show where precisely, in the context of option pricing theory, this approach can be of potential benefit. Our third aim consists in providing for a hint on where this movement may be moving next.

The sections of this paper echo the aims of the paper. In the next section we give an overview of the double slit experiment and the ensuing interference concept. In the section following we provide for a bibliographic overview of research on the interface of quantum mechanics and the macroscopic movement. In section 4 we set up elementary principles relating to Bohmian mechanics and we attempt to show how those concepts could begin to be used in an economics/finance context. In section 5, we provide for the basics of option pricing theory and we discuss how Bohmian mechanics and more specifically the information wave function concept could be used in option pricing theory. We round off the paper, in section 6, with a brief discussion on possible future avenues of research.

2. The double slit experiment and probability interference

Most textbooks on basic quantum mechanics start out with discussing the so called double slit experiment. This experiment provides for the experimental rationale why one needs to use the concept of probability interference, which we discuss below in more detail.

2.1 The double slit experiment

One can imagine the experiment in the following way. In the first stage, the experimenter fires a gun containing very tiny plastic pellets onto a screen which has two equally space slits. We imagine there is a detector screen, behind the screen containing the slits, upon where the pellets land. Assume there exists a detector which can count the pellets landings in the various locations on the detector screen. Denote the top and bottom slits as respectively slits 1 and 2.

1. slit 1 is open and slit 2 is closed
2. slit 1 is closed and slit 2 is open
3. slit 1 is open and slit 2 is open

We imagine that the diameter of the pellets is substantially smaller than the slit's width. When carrying out the experiment the following (expected) result occurs. In scenario 1, pellets start accumulating behind slit 1. Some pellets also land close to slit 2, and go even further, as they are reflected on the edges of slit 1. In scenario 2, pellets accumulate behind slit 2 and some pellets land close to slit 1 and even further. When both slits are open we have an accumulation behind both slits and some scattering also. The key issue now consists in remarking that if one were to convert the pellets into electrons the result of the experiment would be altogether seriously different.

In the same experiment, but now with electrons, experimenters have ob-served that initially spots form, in a random fashion behind the slits. More strikingly however is the fact that after a while the electrons start forming an interference pattern. Morrison (1990) remarks that at first, the electrons be- have like particles. However, when time moves on they start behaving like waves! Even more surprising is the fact that there is interference between an electron and itself. This result is substantially different from what we would expect would electrons have been plastic pellets. Those results laid the basis for the formal development of quantum mechanics. We can not in this paper provide for an overview of how quantum mechanics evolved from the point onwards of this experiment. Max Jammer's (1974) book is still an excellent reference source for most of the very important developments.

2.2. Probability interference

The probabilistic description of the *electrons* landing on the detector screen is different from the probabilistic description of *plastic pellets* landing on the detector screen. In fact the difference in description is so resolutely different that a new notion was born: probability interference. In a nutshell, the observed interference in the double slit experiment when electrons are fired almost requires that we must superpose probability distributions. Note the use of the word 'almost'. As is well know such functions can not be superposed.

Let us denote with $p_1(x)$ the probability that the electron arrives at position x[1] when slit 1 is open. Similarly, for p_2 (x). We can now query what the expression would be when both slits 1 and 2 are open. Would it be:

$$p_{12}(x)=p_1(x)+p_2(x)? \quad (1)$$

This probability formula would reflect the situation when both slits are open *but* when we use plastic pellets rather than electrons. Hence, this formulation is not reflecting the interference pattern experimenters found when they used electrons.

As we have indicated above, it is not possible to capture interference by superposing probability distributions. Instead one could superpose probability waves (or also known as probability amplitudes), which we could denote as $\psi_1(x)$ for when slit 1 is open and $\psi_2(x)$ for when slit 2 is open. See Morrison (1990). We remark that those amplitudes could also be time dependent.

The multiplication of the probability amplitude and its complex conjugate yields the probability density function (pdf). Once we have obtained the pdf we can then calculate the probability value. How does the procedure work?

We note that the pdf can be denoted as $|\psi(x)|^2$ and it is obtained by writing that $|\psi(x)|^2$ is the product of the wave function with its complex conjugate. This conjugate is often denoted as ψ^*. But how can we write the probability amplitude?

We can write it (for when slit 1 is open) as:

$$\psi_1(x)=|\psi_1(x)|e^{iS_1(x)} \;, \quad (2)$$

where $S_1(x)$ is the phase of the wave and $|\psi_1(x)|$ is the amplitude of the wave function. We write out $\psi_2(x)$ in the same way. We note that *i* is a complex number. Hence, the

[1] *We could make this probability time dependent also. We omit it here.*

probability amplitude is complex valued. The transition of going from the probability amplitude to the pdf, is via the procedure of the complex conjugate. The complex conjugate of $|\psi_1(x)|e^{iS_1(x)}$ is simply $|\psi_1(x)|e^{-iS_1(x)}$.Similarly, for the amplitude function when slit 2 is open.

Finally, the probability value is obtained as:

$$\|\psi_1\|^2 = \int |\psi_1(x)|^2 dx \qquad (3)$$

At this stage we have not yet indicated how we arrive at probability interference. We need one more ingredient: superposition of the probability amplitudes. This can be defined as:

$$\psi_{12}(x) = \psi_1(x) + \psi_2(x), \qquad (4)$$

where $\psi_{12}(x)$ is the superposed state. If we write that:

$$p_{12}(x) \, \alpha \, |\psi_1(x) + \psi_2(x)|^2, \qquad (5)$$

then substituting (2) into (5) one obtains:

$$p_{12}(x) = |\psi_1(x)|^2 + |\psi_2(x)|^2 + 2|\psi_1(x)||\psi_2(x)|\cos(S1 - S2). \qquad (6)$$

This is obtained via the use of the fact that a complex number z can be denoted as z = x + iy, where x is the real part and y is the imaginary part. This same number z can also be written as $z = re^{i\theta}$ where $r = \sqrt{x^2 y^2}$ and often r can be denoted as $|z|$. The angle $\theta = \tan{-1}\frac{y}{x}$ and $x = r\cos\theta$ and $y = r\sin\theta$. We also say that $|z|$ is the amplitude and θ is the phase.

This is the probability formula, which now includes the probability interference term: $2|\psi_1(x)||\psi_2(x)|\cos(S1-S2)$ .When this term is not zero it renders the probability in a quantum context, to be either sub-or super additive.

3. Bibliographic overview on research performed on the interface of quantum mechanics and macro-scopic systems

The literature on research performed on the interface of quantum mechanics and macro-scopic systems is developing. In this section we group our discussion in the following way. We first would like to discuss some of the research centers which are active in the field. Then we provide for a brief overview of some of the representative papers in the area. Finally, we discuss some of the conferences which have had sessions devoted to the topic.

In terms of research centers active in the field, we believe there exist at least three centers. A first center is the .International Center for Mathematical Modeling in Physics, Engineering and Cognitive Sciences., at the University of Växjö in Sweden. This center is led by Andrei Khrennikov who is an international authority in the field of foundations of (quantum) probability and the applications of p-adic numbers. Andrei Khrennikov organized, under funding notably of the Swedish Royal Academy of Sciences, a series of highly successful conferences, attended by some of the foremost quantum physicists in the world. Some papers in this conference series were dedicated to the issue of bridging quantum mechanics and macro-scopic systems. The School of Mathematics and Systems Engineering at the University of Växjö also managed a project on economic modelization[1]. This project had a session devoted to the quantum mechanics modelization of stock markets. Also work is on the way on the simulation of macro-scopic quantum systems with applications in economics. This project is in collaboration with Andrei Grib of the Friedmann Lab for Theoretical Physics in St. Petersburg, Russia.

Another important center is formed by the group of researchers around Diederik Aerts of the Center Leo Apostel at the Free University of Brussels (Belgium). Members of that group such as Bart D.Hooghe have done highly important work in the quantum mechanics - macroscopic field. The particular research group, 'FUND' which is part of the Leo Apostel center succeeded in setting up a quantum formalism which could be used in both classical and quantum physics environments. An array of important papers have followed out of this endeavour.

The Oxford University Computing Laboratory, one of the world's most respected computing laboratories, where Bob Coecke is a keynote member, has hosted a few talks on topics which bridge the gap of quantum mechanics and maco-scopic systems. The laboratory also will host the quantum interaction II meeting[2] which contains papers dealing on exactly that same topic.

Individual research on the topic has become more prevalent over the years. We now provide for a brief overview (in alphabetical order) of such work.

Accardi and Boukas (2007) have done important work on re-interpreting the Black-Scholes equation with different stochastics. See also section 5.2. in this paper.

The work by Aerts et al. (2007) deals with quantum games and their applications to biology.

Arfi (2005) introduces a quantum interpretation of the trust predicament.

Baaquie (2004) wrote an important book on the subject of quantum finance. He interprets the Black-Scholes option pricing equation as the Schrödinger Equation[3] for imaginary time. He also proposes a macro-scopic Heisenberg Uncertainty Principle:" The random evolution of the stock price S(t) implies that if one knows the value of the stock price, then one has no information about its velocity...."

Bagarello (2006) looks at stock markets from an operator point of view.

Bordley (1998) introduces a generalized Heisenberg Uncertainty principle.

Broekaert, Aerts and D.Hooghe (2006) use quantum mechanics to generalize the liar paradox.

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[1] This project was in collaboration with the School of Management and Economics (Växjö University), the Institute of Applied Mathematics (Bonn University), Boston College of Management (USA), and the Academy of Economy, Summi (Ukraina).

[2] *The quantum interaction I meeting was held in March 2007 at Stanford University.*

[3] *This is the pde which describes the time evolution of the wave function.*

Busemeyer et al. (2006) in their pathbreaking paper provide for an important study on the differences between what they call quantum dynamics and Markov models. The paper also queries whether a meaningful model of human information processing can be derived from quantum dynamic principles?.

Choustova (2006, 2007) models financial processes with Bohmian mechanical principles (please, see next section for more details on this approach).

Danilov and Lambert-Mogiliansky (2006 a, 2006b) have developed i) non-classical expected utility from a quantum point of view and ii) non-classical measurement theory.

Decamps et al. (2006) refers to applications of the path integral (which can be used in option pricing) formalism.

Franco (2007) models rational ignorance with superposition of states.

Haven (2005a/b, 2007) began to apply Bohmian mechanics in option pricing theory and he also used the Wentzel-Kramers-Brillouin approximation in option pricing theory.

Eisert et al. (1999) considers quantum games.

Khrennikov and Haven (2006, 2007) did some work in providing for a psychological experiment to test for probability interference. Khrennikov also published highly important work on the applications of quantum mechanical principles in psychology and other disciplines (1999, 2002, 2004, 2007).

La Mura (2006) gives important ideas about how Allais' paradox is closely related to the double slit experiment in quantum mechanics. In the words of La Mura (p. 3): "In a sense each particle in the double-slit experiment behaves like a decision-maker who violates the Independence axiom in Allais' experiment."

Piotrowski et al. (2003) provides for an initiation to quantum games.

Schaden (2002) studies asset pricing from a quantum physical perspective.

The Segal et al. paper (1998) we alluded to in the introduction to this paper, rationalized the use of quantum principles in the option pricing context by claiming that "A natural explanation for extreme irregularities in the evolution of prices in financial markets is provided by quantum effects."

Martin Shubik, a very famous Yale University economist, wrote a very short paper on so called "quantum economics". In that paper he says that: "modern finance...has not yet provided us with either the appropriate concepts or measures for the bounds on the minimal overall uncertainty that have to be present in an economy."

In terms of conferences which have hosted the topic, we can mention the following. There is a forthcoming workshop in Sweden[1] (2007) on the topic of quantum mechanics and psychology. The AAAI (Association for the Advancement of Artificial Intelligence) will hold a meeting in March 2008 on this very topic at Oxford University. The Foundations of Probability and Physics .1, 2, 3 and 4 conferences all held in Växjö University - Sweden, hosted some papers on the quantum mechanics - macroscopic topic. In 2006, Khrennikov and Haven organized special sessions on the topic at the 3rd Feynman Festival which was held at the University of Maryland - College Park. Finally, the International Quantum Structures Association (IQSA) meetings in Denver (USA) (2004),

[1] *At the University of Växjö.*

Malta (2006) and Brussels (Belgium) (2008) all hosted (and probably will continue to host) papers on the topic.

4. Bohmian mechanics and economics

Bohmian mechanics is a particular interpretation of quantum mechanics which can be usefully applied in an economics/finance context. Historically, Bohmian mechanics can be traced back to the work of Louis de Broglie. Holland (1993), in his excellent book, indicates that de Broglie attributed two roles to the wave function[1], $\psi(q,t)$ (p. 16): "not only does it determine the likely location of a particle it also influences the location by exerting a force on the orbit." Bohmian mechanics allows the wave function to steer the particle.

In our economics/finance context the particle can be seen as the price of the asset, while the wave function can be seen as a carrier of information.

How does information affect the pricing of assets? How can the information flow be characterized? Is it random? If the price process depends on such flow, will such process then be random too? The so called efficient market hypothesis is equivalent to having a random walk. In such random walk past stock price movements can not be used to predict future price movements. Financial economics has thoroughly studied the phenomenon of asset price predictability and some studies clearly indicate that certain asset price processes do embed a memory property. In Haven (2007) we assume that information in the market can be described by a so called "information wave function". It is precisely this assumption which has led us to use this particular interpretation of quantum mechanics. It needs to be stressed that our approach is not novel as such. Andrei Khrennikov (2004) was the first to argue that such approach could be used in economics and finance. The important work by Choustova (2006, 2007) also argues for using Bohmian mechanics in a finance/economics context.

4.1 Salient features of Bohmian mechanics

Bohmian mechanics is named after its originator David Bohm (1952). Subsequent work by Bohm and Hiley (1987, 1993) has looked at many possible applications using this theory. A source which gives an excellent assessment of Bohmian mechanics is by Holland (1993). Bohmian mechanics provides for what Holland (1993) calls a 'quantum theory of motion' (see Holland (1993) (p. 18)). It needs to be stressed that Bohmian mechanics, as an interpretation of quantum mechanics, shares characteristics with classical mechanics. Bowman (2005) says it very well: "Bohmian mechanics and classical mechanics share the fundamental concepts of real particles and trajectories".

What makes Bohmian mechanics so different from classical quantum mechanics? Maybe it is the natural occurrence of the so called quantum potential which depends on the wave function. Using this potential we can understand the idea that the wave function (via the quantum potential) steers the particle.

Let us write down in a few steps how this quantum potential occurs. For more details, please consult Holland (1993). We first consider the polar form of the wave function:

[1] *We note that q indicates position and t is time.*

$\psi(q,t)=R(q,t)e^{i\frac{S(q,t)}{h}}$; where $R(q,t)=|\psi(q,t)|$ and $S(q,t)/h$ is the phase. We note that q is position, t is time and h is the Planck constant. The Schrödinger equation is the key pde in quantum mechanics which describes the time evolution of the wave function:

$$ih\frac{\partial\psi}{\partial t}=-\frac{h^2}{2m}\frac{\partial^2\psi}{\partial q^2}+V(q,t)\psi(q,t) \ , \tag{7}$$

where *h* is the Planck constant, *m* is mass, *i* is a complex number, *V* (*q; t*) is the real potential (which exists next to the quantum potential). The polar form of the wave function is now substituted into the Schrödinger equation. We skip some intermediate steps, so as to get:

$$ih\frac{\partial R}{\partial t}e^{i\frac{S}{h}}-R\frac{\partial S}{\partial t}e^{i\frac{S}{h}}=\frac{-h^2}{2m}\left[\begin{array}{l}\frac{\partial^2 R}{\partial q^2}e^{i\frac{S}{h}}+\frac{2i}{h}\frac{\partial R}{\partial q}\frac{\partial S}{\partial q}e^{i\frac{S}{h}}+\\ R\frac{i}{h}\frac{\partial^2 S}{\partial q^2}e^{i\frac{S}{h}}-\frac{R}{h^2}\left(\frac{\partial S}{\partial q}\right)^2e^{i\frac{S}{h}}\end{array}\right]+V\Psi \tag{8}$$

If the above equation is multiplied with $e-i\frac{S}{h}$, and separating the real and imaginary parts, one obtains (for the imaginary part):

$$\frac{\partial R}{\partial t}=-\frac{1}{2m}\left[2\frac{\partial R}{\partial q}\frac{\partial S}{\partial q}+R\frac{\partial^2 S}{\partial q^2}\right] \tag{9}$$

For the real part:

$$-R\frac{\partial S}{\partial t}=\frac{-h^2}{2m}\left[\frac{\partial^2 R}{\partial q^2}-\frac{R}{h^2}\left(\frac{\partial S}{\partial q}\right)^2\right]+VR \tag{10}$$

Multiply now (9) (imaginary part) (LHS and RHS) by 2R - so as to get:

$$2R\frac{\partial R}{\partial t}=\frac{-1}{2m}\left[2R2\frac{\partial R}{\partial q}\frac{\partial S}{\partial q}+2RR\frac{\partial^2 S}{\partial q^2}\right] \tag{11}$$

This can be simplified as:

$$\frac{\partial^2 R}{\partial t}+\frac{1}{m}\frac{\partial}{\partial q}\left(R^2\frac{\partial S}{\partial q}\right)=0 \ , \tag{12}$$

and this equation is also known under the name of “continuity equation”. This equation expresses the evolution of a probability distribution, since $R^2=|\psi|^2$.

Finally, we can simplify (10) (real part) a little (divided by -R):

$$\frac{\partial S}{\partial t}+\frac{1}{2m}\left(\frac{\partial\ S}{\partial q}\right)^2+\left(V-\frac{h^2}{2mR}\frac{\partial^2 R}{\partial q^2}\right)=0 \tag{13}$$

Equation (13) is central in Bohmian mechanics. Bohm interprets the above equation by indicating that $\frac{h^2}{2mR}\frac{\partial^2 R}{\partial q^2}$ is the so called quantum potential, $Q(q,t)$ .We can see that this potential contains the amplitude of the wave function (the wave function evolves according to the Schrödinger equation). Hence, it is precisely from this definition that Bohmian mechanics can claim the wave steers the particle.

How do we interpret this so called 'quantum potential' in relation to the real potential? Holland (1993 - p. 74) remarks that it is "consistent to regard (the quantum) potential on the same footing as (the real potential) in respect of the particle motion..."

What is extremely important to remark (see for instance Choustova (2006, 2007) is that since we have a quantum potential we can define a Newtonian-like (classical law of motion) formulation which includes the real $\left(V(q,t)\right)$ and quantum potential $\left(Q(q,t)\right)$ (a Newton-Bohm equation?):

$$m\frac{d^2q(t)}{dt^2}=-\frac{\partial V(q,t)}{\partial t}-\frac{\partial Q(q,t)}{\partial t} \tag{14}$$

The initial conditions are $q(t_0)=q_0$ and $q'(t_0)=g'_0$ (momentum). Thus, the price trajectory $q(t)$ can be found by solving the above equation s.t. initial conditions.

The negative partial derivative towards position of the quantum potential has an easy economics interpretation: it represents a pricing trend. This has been discussed in Choustova (2006, 2007) and Haven (2007).

The attentive reader will have already questioned how the Planck constant, mass and the real potential $V(q,t)$ can be interpreted in an economic context. The real potential describes interactions between traders as well as interactions from other factors such as macro-economic factors. In Choustova (2006, 2007) a functional form for the real potential is proposed. As an analogue of mass could be considered for instance the number of shares of an asset. The issue of interpreting the Planck constant in economics terms is an open issue. Haven (2006) and Choustova (2006, 2007) suggest it could be interpreted as a price scaling parameter.

4.2 Characteristics of the Newton-Bohm trajectory

Without going into details, it is important to make a note as to the characteristics of the Newton-Bohm trajectory. This trajectory in its bare form, as expressed by (14) does not contain the non-zero quadratic variation characteristic that price trajectories normally should contain. We add immediately that conditions can be obtained under which non-zero quadratic variation can be obtained. This has been examined in Choustova (2007). The paths traced out by (14) are smooth. We can invoke the traditional time scale argument: at very fine time scales we may have non-smoothness but at other larger time scales we may have smoothness. However, we can come up with particular types of prices which do not need the non-zero quadratic variation characteristic. See Khrennikov and Haven (2007).

5. Bohmian mechanics and option pricing

In this section we first set up the option pricing pde as it was conceived by Black and Scholes (1973). We then try to rationalize how Bohmian mechanics could be of use in the option pricing context.

5.1 The Black-Scholes option pricing pde

The option pricing pde was developed by Black and Scholes in 1973. In this subsection, we present the standard approach of deriving the option pricing Black-Scholes pde.

An option is a contract which gives the right to buy (call option) or sell (put option) an underlying asset at a certain price at a certain date in the future[1] . There exists a portfolio, Π which is short in an option, and long $((\partial o)/(\partial q))$ in shares; where *o* is the option price function $o(qt)$ and *q* is the price of the underlying asset (i.e. a share in this case), while *t* is time. The portfolio can be written as: $\Pi = -O + \frac{\partial O}{\partial q} q$. The so called Itô stochastic differential equation (Itô (1951)), in general form, is defined as: $dy = a(y,t)dt + b(y,t)dW$; where dW is a Wiener process (with mean zero and variance of unity), $a(y,t)$ is some function of a position variable, *y* and time *t*. Similarly for $b(y,t)$. The Itô Lemma on a function $G(y,t)$, with $dy = a(y,t)dt + b(y,t)dW$ is defined as:

$dG = \left(\frac{\partial G}{\partial y} a(y,t) + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial y^2} b^2(y,t) dt + \frac{\partial G}{\partial y} b(y,t) dW \right)$. Assume the underlying stock process is the geometric Brownian motion. Hence, we use *z=q* and $a(y,t) = \mu q$ and $b(y,t) = \sigma q$, where μ is the expected return and σ is the constant stock price volatility. Thus, using the Itô Lemma on $O(q,t)$ yields then: $dO = \left(\frac{\partial O}{\partial q} \mu q + \frac{\partial O}{\partial t} + \frac{1}{2} \frac{\partial^2 O}{\partial q^2} \sigma^2 q^2 \right) dt + \frac{\partial O}{\partial q} \sigma q dW$. Substituting in $d\Pi = -dO + \frac{\partial O}{\partial q} dq$ and imposing the non-arbitrage condition, one obtains the famous Black-Scholes pde:

$$\frac{\partial O}{\partial t} + \frac{\sigma^2 q^2}{2} \frac{\partial^2 O}{\partial q^2} + rq \frac{\partial O}{\partial q} = rO . \qquad (15)$$

Under appropriate boundary conditions this pde can be solved analytically (assuming that *r* and σ are constant). Analytical solutions for non-constant volatility functions can be found but the array of such functions for such case is limited. We can show that via the WKB approximation[2] (where the wave function's phase is approximated by a series of

[1] Those are the so called 'European options'.

[2] *WKB approximation is the abbreviation of Wentzel-Kramers-Brillouin approximation. See Bender and Orszag (1978) for an excellent discussion.*

powers of h) on the Schrödinger equation we can find analytical solutions (in some cases) to Black-Scholes'pde's with certain non-constant volatility functions. See Haven (2005) and Li and Zhang (2004).

5.2 Bohmian mechanics and its use in option pricing

How can we connect option pricing theory with Bohmian mechanics? One problem with using Bohmian mechanics in an option pricing context is that the price paths will be smooth. We have briefly discussed this issue above. There are possible alternatives:

- The Bohm-Vigier approach
- The Nelson stochastic differential equation approach
- The Accardi and Boukas approach
- Random mass and singular quantum potentials

The Bohm-Vigier approach (See Bohm and Hiley (1993)) seems to be one of the first equations used in the so called stochastic interpretation of quantum mechanics. The Bohm-Vigier stochastic equation is formulated as:

$$v=\frac{\nabla S(q)}{m}+\eta(t)\ , \qquad (16)$$

where v indicates the velocity of the particle; *m* is mass; *S*(*q*) is the phase of the wave function and $\nabla S(q)$ is the gradient of the phase function towards position. Finally, $\eta(t)$ is some random factor with a mean of zero. If we want to have arbitrage (a consequence of information) enter the arbitrage free portfolio we can write:

$$\frac{d\Pi}{dt}=\nabla S(\Pi)+\eta\ , \qquad (17)$$

where we make η to be time independent and where we have as information function (note that Π is the portfolio value): $\psi(\Pi)=R(\Pi)e^{iS(\Pi)}$ and $S(\Pi)=\frac{r\Pi^2}{2}$.We also define: $\eta=x\frac{\nabla S(\Pi)}{r}$ where x is some time independent arbitrage return. This extension could be made much more sophisticated. See the pathbreaking work by Fedotov and Panayides (2005) and Panayides (2005). The work by Otto (1999) is also to be mentioned. An excellent treatment on the possibility of writing the put and call as average prices is by Ilinski (2001).

The Nelson stochastic differential equation approach has been proposed in Haven (2005b). The Nelson sde was proposed by Nelson (1967). Bacciagaluppi (1999) introduces a stochastic guidance equation (see Bohm and Hiley (1989)) which is defined as follows:

$$da=\left(\frac{h}{m}\nabla S+\alpha\frac{h}{2m}\frac{\nabla|\psi|^2}{|\psi|^2}\right)dt+\sqrt{\alpha}dW\ ; \qquad (18)$$

where α is position of a particle; *h* is the Planck constant and *m* is mass. α is a parameter and dW is a Wiener process with the constraints that *E*(*dW*) and the variance is $\frac{h}{m}$. If $\alpha=1$, then we obtain the so called Nelson sde. The approach by Nelson (1967) uses the so called forward and backward mean derivatives in the derivation of the sde. Paul and Baschnagel (1999) provide for an excellent overview of this issue. They use the Itô Lemma on a function, $f(x,t)$:$df(x,t)=\frac{\partial}{\partial t}f(x,t)dt+\nabla f(x,t)dx+\frac{1}{2}\Delta f(x,t)(dx)^2$ and then substitute a Brownian motion: $dx(t)=b(x,t)dt+\sigma dW(t)$ in that Lemma - yielding:

$$df(x,t)=\begin{pmatrix}\frac{\partial f(x,t)}{\partial t}+b(x,t)\nabla f(x,t)\\ +\frac{1}{2}\sigma^2\Delta f(x,t)\end{pmatrix}dt+\sigma\nabla f(x,t)dW(t) \tag{19}$$

In the words of Paul and Baschnagel (1999): "...the Brownian path is not differentiable with respect to time, so there is no simple equivalent to the particle velocity. We therefore define the mean forward derivative…" They define this mean forward derivative (similarly for the mean backward derivative):

$$D+f(x,t)=\lim_{\Delta t\to 0}E\left[\frac{\begin{pmatrix}f(x(t+\Delta t),t+\Delta t)-\\ f(x(t),t)\end{pmatrix}}{\Delta t}\right] \tag{20}$$

which then yields using (19):

$$\frac{\partial f(x,t)}{\partial t}+b(x,t)\nabla f(x,t)+\frac{1}{2}\sigma^2\Delta f(x,t) \tag{21}$$

What is important is to remark that the term $\frac{h}{2m}\frac{\nabla|\psi|^2}{|\psi|^2}$, also called osmotic velocity, in (18) can only be obtained if both the mean forward derivative and mean backward derivatives are di¤erent from each other. In the paper by Bohm and Hiley (1989) it is said that: "...the osmotic velocity field constitutes active information which determines the average movement of the particle." In an economics and finance context, the Nelson sde could be used as a secondary sde. See Haven (2007).

The Accardi and Boukas (2007) approach use a specific type of quantum calculus, the Hudson-Parthasarathy (1984, 1992) calculus to describe quantum processes in the context of the Black-Scholes equation. This new direction holds a lot of new hope to reinterpret option pricing from a different stochastic point of view.

Last, but certainly not least, is the approach by Choustova (2006, 2007) who considers ways to obtain non-zero quadratic variation on the Bohmian paths. It is shown that by either randomizing mass or by considering quantum potentials which are singular, non-zero quadratic variation is obtained.

6. Other avenues of research

We have presented in this paper a rather general overview of how quantum physics can be of help in financial asset pricing. Where will this research possibly go next? In the area of psychology, we think that the use of probability interference in explaining violations of the sure-thing principle, can be of great benefit. The work by Busemeyer and Wang (2007) is of high importance in this respect.

Furthermore, we may also wonder whether the wave function concept we have covered, can not also be used in other areas of economics. For instance, what is the relationship between the wave function and the utility function as used in economics? Is there a relationship?

Finally, we believe there is a lot of work which can be done in the area of asset pricing and the non-arbitrage theorem, a central theorem in financial asset pricing. We hint to some of the possible work in Haven (2007).

7. Acknowledgments

benefited greatly from a two week stay at the Romanian Academy, Institute for Economic Forecasting in the Summer of 2006 during which some of the ideas presented in this paper were discussed and debated. This stay was financed under the special joint project programme between the British Academy and the Romanian Academy. I want to thank Professors Albu, Purica, Pelinescu, Chilian, Iordan, Nicolae, Lupu, Altar, Zaman and Pop and many others for their wonderful support. I benefited from a lot of important feedback during the presentation of some of the work related to this paper. I would like to thank all the participants for their important feedback.

July 19, 2007

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