Preemptive Entry in Differentiated Product Markets*

Simon P. Anderson[1] and Maxim Engers[2]

[1] Department of Economics, UVA, Charlottesville, VA 22903, USA
(e-mail: sa9w@virginia.edu)
[2]Department of Economics, UVA, Charlottesville, VA 22903, USA
(e-mail: maxim@virginia.edu)

Received: revised:

Summary: Models of spatial competition are typically static, and exhibit multiple free-entry equilibria. Incumbent firms can earn rents in equilibrium because any potential entrant expects a significantly lower market share (since it must fit into a niche between incumbent firms) along with fiercer price competition. Previous research has usually concentrated on the zero-profit equilibrium, at which there is normally excessive entry, and so an entry tax would improve the allocation of resources. At the other extreme, the equilibrium with the greatest rent per firm normally entails insufficient entry, so an entry subsidy should be prescribed. A model of sequential firm entry (with an exogenous order of moves) resolves the multiplicity problem but raises a new difficulty: firms that enter earlier can expect higher spatial rents, and so firms prefer to be earlier in the entry order. This tension disappears when firms can compete for entry positions. We therefore suppose that firms can commit capital early to the market in order to lay claim to a particular location. This temporal competition dissipates spatial rents in equilibrium and justifies the sequential move structure. However, the policy implications are quite different once time is introduced. An atemporal analysis of the sequential entry process would prescribe an entry subsidy, but once proper account is taken of the entry dynamics, a tax may be preferable.

Keywords and Phrases: Product differentiation, rent dissipation, entry deterrence, entry timing, sequential entry.

JEL Classification Numbers: L13, D43, R12.

---

*We thank Richard Arnott for alerting us to Vickrey’s contributions to this theme. We are grateful for financial support from the Bankard Fund at the University of Virginia, and to Curtis Eaton and anonymous referees for their comments and suggestions. *Correspondence to:* M. Engers

1. Introduction

Product differentiation is an important dimension of firm competition, from computer software and cable television to grocery stores and restaurants. Existing models of product differentiation are typically static, by which we mean that firms' location decisions are made simultaneously. These static models are of limited usefulness. In particular, models with endogenous location choice (in characteristics or geographical space) and potential entry typically allow multiple equilibria. This means that there is a wide range of possible firm numbers consistent with equilibrium. Roughly speaking, these range from a densest packing of firms at which all earn zero profit, to a loosest packing at which any new entrant in a niche between existing firms would just be unprofitable. Market performance is hard to evaluate because the socially optimal level of product diversity typically lies within the range of possible equilibrium levels. This leaves the analyst with no way of knowing whether entry or exit ought to be encouraged, so policy questions can scarcely be addressed without a way of choosing among the equilibria. The most usual assumption is that the equilibrium involves zero profit for all existing firms (see e.g. Salep, 1979). This is likely to be the wrong benchmark if firms can somehow influence the equilibrium selection.

One reaction to the multiple equilibrium problem is to conclude that "history matters" (Eaton and Lipsey, 1978). History can be modeled via a sequential location process of far-sighted firms (see e.g. Prescott and Visscher, 1979, and Neven, 1987). This lender modeling strategy, firms rationally anticipate how their actions affect the behavior of subsequent entrants. Early entrants recognize how their own locations affect the location (or product choice) decisions of later entrants and whether or not they will enter the market.[1] A major drawback to this approach is that a firm's profit depends on its position in the (exogenously specified) order of moves: earlier entrants earn more. Thus firms would like to move higher in the order although the models do not allow them to act on this incentive.[2]

We explicitly introduce time to model the ability and the costs of moving before others. Specifically, we suppose that it is known far in advance that a market will open.[3] Firms compete in the timing of entry into the market, with earlier entry garnering a position that has higher expected flow profit. Timing competition dissipates the rents accruing to desirable locations: an entrant must locate sufficiently early that all such rents are exhausted via early commitment of resources to the market, otherwise some other firm would pre-empt it.[4] Thus in equilibrium firms no longer have an incentive to move earlier in the order. Profits are driven to zero: even though space creates rents, time destroys them. Although profits are driven to zero, the equilibrium locations are not those of the zero-profit equilibrium of the static model. Far from it: the locations are close to the

---

[1] By contrast, the "long-run" equilibrium of the static model is usually specified as a set of locations such that no firm wishes to enter, exit, or choose a different location. These decisions could be coordinated by an imaginary "auctioneer", but the process raises the obvious problem of how firms in practice figure out which is to locate where. This is also troublesome because (in all but the zero-profit equilibrium) firms that actually enter the market earn positive profits while those that do not enter earn zero.

[2] The same criticism can be levelled against standard Stackelberg sequential output choice models: see Eaton and Ware (1987) and Anderson and Engers (1994).

[3] As we point out below, it suffices that the probability that the market opens is a continuous function of time. For example, legal restrictions are removed or a market is liberalized. Alternatively, this assumption is a stylized version of a growing market without dealing with the intricacies of the growth path (see also Eaton and Lipsey, 1979).

[4] An early exposition of this mechanism is expounded in Eaton and Lipsey (1979, 1980), where the timing of a pre-emptive decision is resolved by determining the time at which the present value of entry is zero. The source of the rent is space in Eaton and Lipsey (1979) while it is capital durability in Eaton and Lipsey (1980).

opposite extreme in the static model, that at which static rents are maximized. So, firms choose the locations at which the static rents are maximal, but to claim these rents they must commit to their decisions so early that the rents are totally dissipated.

The equilibrium locations are also essentially unique, so one can make robust policy recommendations. But these should be prescribed with care. Although the equilibrium locations are those of the (atemporal) sequential-entry model, it should not be concluded (as that model would prescribe) that incentives should be given to set up more firms, via entry subsidies for example. An entry subsidy will indeed cause more firms to enter (suggesting a welfare improvement), but competition for rents will become keener, and subsidies will be dissipated by timing competition. The net effect may be that welfare deteriorates: the appropriate policy response in a situation with insufficient product variety may actually be a tax on entry!

Our results imply that atemporal models may seriously understate the distortions inherent in differentiated product markets. Not only can the locations be worse than the worst possible case suggested by the standard static model, all rents can be lost as well.

After completing our work we became aware of a remarkable discussion by Vickrey of spatial competition. Vickrey (1964, Ch.6) anticipated several of the most important themes developed much later in the literature, including the circle model of Salop (1979), the sequential entry model of Prescott and Visscher (1977), and the Eaton and Lipsey (1978) result that profits can persist despite free entry. As later did Eaton and Lipsey, 1979 and 1980, Vickrey also realized that profit might be dissipated by early entry:

> in a dynamic world...there may be an advantage to establishing a firm in an opening gap somewhat ahead of the time at which the situation is currently profitable, in order to pre-empt the position and enjoy the later profits. The earlier losses due to the attempts at pre-emption would then have to be offset against later gains. Indeed, in a situation of perfect foresight and vigorous competition, some entrepreneur would presumably jump in as soon as the expected pre-emptional loss has diminished to the point where it will just be outweighed by the prospective profits to be enjoyed later (Vickrey, 1964, p. 334).

Section 2 describes the properties of the model of product differentiation that we use. These properties are needed in the later sections. Section 3 then determines the equilibrium of the atemporal model of sequential location choice, and Section 4 uses that analysis to determine the equilibrium entry times in the full model. Section 5 shows that the equilibrium locations in the timing game are inferior even in a static sense to those of almost any simultaneous-entry equilibrium, suggesting that performance in differentiated product markets may be much worse than implied by the usual analysis (since one must in addition allow for the rent dissipation). The equilibrium outcomes are closest to those of the simultaneous equilibrium with the greatest rent. The performance issue is addressed in Section 6, which shows that an entry subsidy is never optimal despite the small number of firms in the market (that is, the rent dissipation effect dominates). Section 7 concludes with a general discussion of rent dissipation and performance in the context of product differentiation.

2. The Static Model.

To show how our approach contrasts with the standard static equilibrium notion we use a particular specification of product differentiation.[5] This is the model of spatial price discrimination most

---

[5] Other possible formulations are discussed in the conclusion.

recently described in Eaton and Schmitt (1994), which draws on earlier work by Lederer and Hurter (1987), and MacLeod, Norman, and Thisse (1988); it was originally introduced by Hoover (1937). Several of the properties described in this section can be found in these earlier works.

Consumers are uniformly distributed around a circle of unit circumference, S^1, and each buys one unit of the good from the firm charging the lowest delivered price (consumers purchase from the closest firm in case of a price tie).[6] Each firm occupies a single location on the circle. Transport costs, $c(\cdot)$, are assumed to be a convex, increasing, and twice continuously differentiable function of distance (measured around the circle) and the same for all firms.[7] These costs are borne by firms. (Later on we concentrate on the special case of linear transport costs.) Marginal production costs are constant, and henceforth zero without further loss of generality. Each firm chooses a delivered price for each point. The Bertrand (price) equilibrium (for any given set of firm locations) involves each firm serving the points to which it delivers at lowest cost, and it serves such points at a price given by the delivery cost of the next-lowest-cost supplier.[8] This is because any higher price could be profitably undercut by another firm, and any lower price can be profitably increased without fear of another firm serving the market. The natural interpretation of the model is in geographic space; for example, cement plants (transport costs are important and firms deliver the product). The model has been interpreted in characteristics space as customization of a base product, or "flexible manufacturing" (Eaton and Schmitt, 1994). Examples include car production and clothing. We use the spatial price discrimination model to illustrate how rent dissipation ties down equilibrium locations when static models yield multiple equilibria, and how policy conclusions can be drastically altered. In the conclusions we set our results in a broader perspective.

Let there be n firms, with locations $x_i \in S^1$, $i = 1, \ldots, n$, and define Firm i's natural market, M_i, to be the set of points that i can serve at least as cheaply as any of its rivals. We apply the standard Bertrand argument to each point in S^1, assuming that if two or more firms set identical lowest prices at a point consumers buy from the closer firm (purchases are split equally in the case of two or more equally close lowest-price firms). Equilibrium prices are described by the following property:

P1: In the Bertrand price schedule equilibrium, each firm i sets a price equal to its delivered cost at each point outside its natural market, M_i and sets a price equal to the delivered cost of the firm with the second-lowest cost at each point in M_i.

Thus each Firm i serves all customers in M_i (where distinct market areas intersect - for instance because firms' actions coincide - the customers can be assigned arbitrarily to any of the lowest-cost firms since zero profits are earned on these customers). Each consumer buys from the closest firm, and so transport costs are minimized. Since demand is perfectly inelastic, transport costs are the only determinant of social welfare, so that the equilibrium decentralizes the socially optimal allocation of consumers to firms.

P2: For given firm locations, the equilibrium allocation is optimal.

The link between equilibrium and optimum is even stronger than P2 suggests, because it holds when locations are chosen endogenously. To see this, note that at the optimum, each firm is

---

[6] The analysis is readily adapted to a linear segment instead of a circle: see the discussion at the end of Section 3.

[7] As discussed below, the convexity assumption serves to simplify the analysis, and is not necessary for the key results of this Section and the next two.

[8] There are also other Nash equilibria at which some firms price below their delivered costs. These can be ruled out by introducing a small amount of product heterogeneity, and then letting the degree of heterogeneity tend to zero: see Anderson and de Palma (1987) for more details. Alternatively, trembling-hand perfection could be applied to rule these out. The equilibrium on which we focus is the standard Bertrand equilibrium (see Tirole, 1988, p.211).

equidistant from its neighbors,[9] and hence each firm serves the same size market:

P3: Given the locations of all other firms, the socially optimal location of a firm is at the midpoint of the largest interval between adjacent firms: hence the optimal locations for n firms has them equidistant ($1/n$ apart) around the circle.

The following two properties are useful for the later analysis, and are used to show that the equilibrium locations are the same as the optimal ones. From P1, the equilibrium profit of a firm depends only on the distance to each of its two neighbors. Let $\pi(\ell; r)$ denote this profit, where ℓ is the distance to the counterclockwise neighbor, and r is the distance to the clockwise neighbor. Since consumers are uniformly distributed, $\pi(\ell; r) = \pi(r; \ell)$. Note too that $\pi(\ell; r) = 0$ if either ℓ or r is zero (in this case there is Bertrand competition between two firms with identical costs at each point in space).

P4: The profit function π is:

a) increasing in each argument, and

b) convex in each argument.

The demonstration is as follows. For $r \geq \ell$, the profit $\pi(\ell; r)$ is given by (see Figure 1):

$$\pi(\ell; r) = \int_{-\ell/2}^{(r-\ell)/2} [c(x+\ell) - c(|x|)]dx + \int_{(r-\ell)/2}^{r/2} [c(r-x) - c(x)]dx \qquad (1)$$

INSERT FIGURE 1.

Using Leibniz's rule, and, after cancellation,

$$\frac{\partial \pi}{\partial \ell} = \int_{-\ell/2}^{(r-\ell)/2} c'(x+\ell)dx.$$

Hence, by the Fundamental Theorem of Calculus,

$$\frac{\partial \pi}{\partial \ell} = c\left(\frac{r+\ell}{2}\right) - c\left(\frac{\ell}{2}\right). \qquad (2)$$

Similarly,

$$\frac{\partial \pi}{\partial r} = c\left(\frac{r+\ell}{2}\right) - c\left(\frac{r}{2}\right). \qquad (3)$$

Expressions (2) and (3) also hold for $r < \ell$. Part (a) then follows immediately from (2) and (3); part (b) follows from (2) and (3) since c is convex.[10]

We shall sometimes use linear transport costs for illustration. Normalizing the transport rate per unit distance gives $\pi(\ell; r) = \ell r/2$, and profit is then the area of a rectangle with sides $r/\sqrt{2}$ and $\ell/\sqrt{2}$.

---

[9] Consider a firm located closer to its left (or counterclockwise) neighbor than to its right (or clockwise) one (see Figure 1). If the firm moves a distance dx to the right, total transport costs for serving the market between the firm and its left neighbor rise by the cost of serving the indifferent consumer (the consumer point midway between the firms) on the left (times dx). Likewise, the total cost of serving the market on its right falls by the cost of serving the indifferent consumer on the right (times dx). Thus total transport costs of serving the segment are lowest with the firm at the midpoint; moreover, total transport costs are a convex (and symmetric) function of the firm's location in the interval. The convexity property follows from the fact that each incremental move towards the center lowers total transport costs by less than the preceding move. In conjunction with P7 below, the profit of a firm is a concave function of its location in an interval between two others, zero at the endpoints and highest in the middle.

[10] The profit function is concave in each argument if c is concave.

P5: A firm's profit is a strictly concave function of its location between two neighbors, which is maximized at the midpoint: for fixed $b > 0$, $\Pi(a; b - a)$ is a strictly concave function of a with maximum at $b/2$.

From (2) and (3) above, the total derivative $d\Pi/da$ is $c([b - a]/2) - c(a/2)$, which is strictly decreasing for c strictly increasing. We can now characterize the pure-strategy equilibrium at which n firms simultaneously choose locations anticipating the price schedule equilibrium. This is a standard two-stage game with locations chosen before prices. P5 immediately implies:

P6: A best-reply location of a firm is at the midpoint of a largest interval between adjacent firms: hence the location equilibrium for n firms has them equidistant ($1/n$ units apart), which is the social optimum (P3).

A different method of showing the equivalence between optimum and equilibrium locations uses the following property. Since consumer demand is completely inelastic, each firm's profit is exactly equal to the reduction in transport costs it creates (see figure 1 for illustration):

P7: Given the location of the other firms, the profit of any firm equals its incremental contribution to social surplus.

This means that, given the locations of the other firms, each firm chooses the location that minimizes social costs (i.e., transport costs), and hence maximizes its contribution to social welfare. By P3, the socially optimal locations are also equilibrium ones, and this is the unique equilibrium (up to a rotation and relabelling of firms).[11]

We next allow for free entry and exit of firms by assuming that entry entails a sunk cost, F. The standard (static) definition of equilibrium in spatial models is that no additional firm should wish to enter, and no incumbent firm should prefer to be at a different location or out of the market. These conditions yield a fundamental multiplicity of equilibria. At each of these equilibria, firms are equidistantly spaced, but the spacing can be anywhere between a minimum spacing at which all firms earn zero profit, and a maximum at which a potential entrant's revenue would fall just short of F. It is localization of competition that is behind this result: any potential entrant would expect to earn revenues significantly lower than those earned by incumbents (indeed, one quarter of an incumbent's revenue if transport costs are linear) because an entrant must fit into a niche between a pair of incumbents.

If n is the number of firms, let $W(n)$ denote the social welfare, with firms equally spaced as per P3. For the moment, treat n as a continuous variable, and so (up to a positive constant reflecting the consumers' reservation price), $W(n) = -2n\int_0^{1/2n} c(x)dx - nF$. Differentiating with respect to n gives:

P8: The social welfare, $W(n)$, is a strictly concave function of the number of firms. The social optimum can be sustained as an equilibrium with entry.

Since welfare is simply the negative of total costs of serving the market, the concavity property states that successively adding firms (and optimally rearranging them) lowers total transport costs by smaller and smaller amounts. The social optimum is an equilibrium because existing firms must cover their fixed costs (otherwise social surplus would be increased by removing them), and no entrant could cover its fixed cost (otherwise it would be socially beneficial to add firms).

There is typically a range of equilibrium firm numbers. Ignoring integer constraints for the time being, the upper limit on the number of firms, n_U (the densest equilibrium), is twice the lower

---

[11] This in turn implies that the optimum can be sustained as an equilibrium (although as Lederer and Hurter, 1987, point out, there may be other equilibria when the location space is multi-dimensional: this observation corresponds to a result of Spence, 1976).

limit, n_L (the sparsest equilibrium), so $n_U = 2n_L$.[12] At the sparsest equilibrium, firms are just indifferent about entering midway between each adjacent pair of the n_L incumbents, and if they all do enter, each of the n_U incumbent is indifferent about leaving.

Formally, let n_L be defined by:

$$\Pi\left(\frac{1}{2n_L}, \frac{1}{2n_L}\right) = 2\int_0^{1/4n_L} c\left(\frac{1}{2n_L} - x\right) - c(x)dx = F; \tag{4}$$

i.e., the critical number of firms such that, if they are equispaced, an entrant is just unable to make a positive profit.[13] Since we are interested in competition rather than monopoly, we assume that $n_L \geq 1$: this implies that there will always be at least two equilibria in the simultaneous-entry game.[14] The social welfare is the same at the two extremes:

P9: $W(n_L) = W(n_U)$, where $n_U = 2n_L$, with $W'(n_L) > 0$ and $W'(n_U) < 0$.

The first part is shown as follows. Define $\Delta W = W(n_L) - W(n_U) = 2n_U \int_0^{1/2n_U} c(x)dx + n_U F - 2n_L \int_0^{1/2n_L} c(x)dx - n_L F$. Since $2\int_0^{1/2\,n_U} c(1/n_U - x) - c(x)dx = F$, by definition of n_L, and n_U; we can write $\Delta W = n_U \left\{ -\int_0^{1/n_U} c(x)dx + \int_0^{1/2n_U} c(x)dx + \int_0^{1/2n_U} c(1/n_U - x)dx \right\}$. The desired result then follows from rewriting the last integral using the change of variable $v \equiv 1/n_U - x$. The derivative property then follows from the first part and P8.

The intuition behind this latter property is as follows. At n_L, if we add a firm midway between a pair of existing firms, welfare is unchanged: the firm's social contribution is zero since its profits are zero (P7). Rearranging the firms to a symmetric position then raises welfare. At n_U, taking out a firm leaves welfare unchanged. Rearranging then raises welfare.

We now allow for the integer constraint. Clearly there is an equilibrium for each integer number of firms between n_L and $n_U (= 2n_L)$. That is, the smallest number is given by $m_L \equiv \lceil n_L \rceil$, where $\lceil \cdot \rceil$ denotes the ceiling function (the next integer up if n_L is not an integer); and the largest number is given by $m_U \equiv \lfloor n_U \rfloor$, where $\lfloor \cdot \rfloor$ denotes the floor function (the integer part of n_U).

INSERT FIGURE 2

P10: If n_U is an even integer, then $m_L = m_U/2$. If n_U is not an even integer, then $m_L = (m_U + 1)/2$ if m_U is odd; and $m_L = m_U/2 + 1$ if m_U is even.[15]

The explicit restriction to integers is crucial to the analysis of sequential location choice.

As long as transport costs increase in distance, all of the properties above hold true, with the exception of Property 4(b) which requires convexity. The convenience of the convexity assumption is that it simplifies the characterization of the equilibrium locations under sequential entry, although convexity is not necessary for this characterization.[16]

---

[12] When transportation costs are linear and the cost per unit distance is normalized, calculations show that the optimal number of firms is $(4F)^{-1/2}$, whereas the equilibrium number ranges from $n_L = (8F)^{-1/2}$ to $n_U = (2F)^{-1/2}$.

[13] Clearly a unique (positive) solution exists since the LHS $\to \infty$ as $n_L \to 0$; and the LHS $\to 0$ as $n_L \to \infty$, and the LHS is strictly decreasing in n_L.

[14] If (4) yields a solution $1/2 \leq n_L < 1$, then $1 \leq n_U < 2$, so that the only simultaneous-entry equilibrium involves one firm. One might think that when $n_L < 1/2$, so that $n_U < 1$, then not even a monopolist can make a profit: in fact, the assumption of inelastic demand allows a monopolist to remain profitable and the price competition implicit in the definition of n_L does not apply.

[15] The proof is straightforward. See Graham, Knuth, and Patashnik (1994) for properties of the floor and ceiling functions.

[16] Although this characterization simplifies the analysis, we do not think that the main economic insights would be fundamentally changed if the characterization were altered by sufficiently concave transport costs.

3. Equilibrium with Sequential Location Choice.

As a prelude to the timing game, we study the following atemporal game. Let there be a sufficiently large number of potential entrant firms.[17] According to a given order of moves, firms choose whether to enter, and each entrant selects a location. All entrants then simultaneously choose prices, as described in P1. We define an equilibrium to be a sub-game perfect Nash equilibrium such that in any sub-game, each entrant puts equal probability weight on each of its optimal location choices. That is, we assume that when a firm is indifferent between two (or more) profit-maximizing locations, it will choose each of them with equal probability.[18] This implies that profit is symmetric around the midpoint of a gap between previous entrants. Hence it suffices in what follows to consider location choice in the half interval up to the midpoint of a gap, and actual choice will put equal weight on optimal choices in this subinterval and their mirror images in the half interval beyond the midpoint.

Define z as the critical distance between adjacent firms such that an entrant midway between them would earn zero profit, i.e., $\pi(z/2; z/2) = F$, or, equivalently, z satisfies $2\int_0^{z/4}[c(z/2 - x) - c(x)]dx = F$ (clearly $z = 1/n_L$ and hence is uniquely determined, see (4)). The proof of the following Proposition is in the Appendix.

PROPOSITION 1. There exists an equilibrium to the sequential-entry game. In any equilibrium, there are $m_L = \lceil n_L \rceil = \lceil 1/z \rceil$ firms, which is the same number as at the simultaneous entry equilibrium with the least number of firms. The last firm locates midway in the gap of size $s \in (z; 2z]$ between its two neighbors, and there is equal spacing z between each adjacent pair of the other firms.

Thus there are three possible ex-post profit levels: that of the last firm (the niche firm), of its two neighbors, and of the remaining (shielded) firms.

The proof is by induction on an index of the size of the interval. This index will be shown to represent the equilibrium number of entrants in the interval. Accordingly we define the function $e(s) = \lceil s/z \rceil - 1$, where $\lceil . \rceil$ denotes the ceiling function as described above.[19] Note that an interval of length $s \le z$ will support no profitable entrants, and so $e(s) = 0$ in this case: also, increasing an interval length by exactly z will increase the number of entrants by one.[20]

The location of the firms in the equilibrium is unique once we normalize the position of the last firm. In the equilibrium, the entering firms fan out around the circle, and each but the last one locates the critical distance z (the market length that just deters entry) from its inside neighbor. To understand why these locations are chosen, consider the choice facing the penultimate entrant. As shown in part (iii) of the Appendix, this firm will "push" the last entrant (the niche firm) as far as possible subject to preventing entry on its own inside. (Convex transport costs are sufficient for this result. On the one hand, the penultimate firm, as "location leader", gains from the largest possible market length served when locating a distance z from its neighbor. On the other hand, it loses from not being at the center of the market it serves. With convex transport costs, the

---

[17] Sufficiently large means larger than the number of entrants given in Proposition 1.

[18] This randomization assumption rules out strategic use of indifference by firms to threaten previous players (see Dewatripont, 1987).

[19] Hence $e(1) = \lceil 1/z \rceil - 1$, so the equilibrium number of entrants on the unit circle is $e(1) + 1$ since the first firm "converts" the circle to an interval of unit length in the sense that payoffs of entrants in the circle are exactly the same as they would be at corresponding locations in a unit interval with a firm at each end.

[20] We assume that a firm enters only if it can earn strictly positive profits. Note that $e(s) = \lfloor s/z \rfloor$ except where s/z is an integer.

former effect dominates. With linear transport costs, the penultimate entrant still strictly prefers to locate a distance z away, suggesting that our main characterization result holds even for some strictly concave transport cost functions.)

A similar argument applies for earlier firms. First note that a firm will never prefer entering $2z$, $3z$, etc. away, rather than z away: entering $3z$ away (for example) will simply increase the probability of neighboring the niche firm since the next two firms will fill the intervening gap of $3z$ and be fully protected themselves, and the entrant $3z$ away has less chance of being protected because it has reduced the number of successors that could shield it.[21] Likewise, coming in at $x < z$ away is not worthwhile since by increasing x, the firm can gain more on the inside when it is protected by a future entrant, and when it is not, it gains by squeezing the niche firm as much as it can, as in the problem of the penultimate firm. Hence, since the penultimate firm prefers to squeeze the niche firm, so a fortiori do the previous firms.

Proposition 1 shows that the number of entrants in the sequential-entry equilibrium is identical to the least number of firms in a simultaneous-entry equilibrium. This is essentially because each entrant (except the last) maximizes the distance from its neighboring predecessor.

The analysis is not changed much for a linear market space. It is readily shown that the first two entrants locate inside the market boundaries, at a distance that just deters entry outside these entrants. Thereafter, the model is just like the circle, in the sense that the first two entrants on the circle also convert the remaining market space into a linear segment bounded by firms. As Vickrey noted (for his fixed-price model), the two first movers "insulate the remainder of the maneuvering from the effects of the ends" (1964, p. 332).

4. Equilibrium Entry Times: Competition for Locations via Early Commitment of Capital to the Market.

It is shown in Proposition 1 that the number of entrants on the unit circle is $\lceil 1/z \rceil$, which we shall call m and we assume $m > 1$.[22] Let R denote $1 \bmod z$; the remainder when 1 is divided by z.[23] If $R = 0$, the expected gross profit of entrant i is $\pi(z, z)$ for all $i = 1, \dots, m$. Otherwise, we can calculate expected profits as weighted averages of three expressions. These expressions are: $\Pi_S \equiv \pi(z, z)$, the profit of a firm that is shielded on both sides; $\Pi_H \equiv \pi(z, [R + z]/2)$; the profit of a firm adjacent to the niche firm; and $\Pi_B \equiv \pi([R + z]/2, [R + z]/2)$, the profit of the niche firm. The expected profit of Firm i is then

$$\begin{cases} \left[1 - 2^{3-m}\right]\Pi_S + 2^{3-m}\Pi_H & \text{for } i = 1 \\ \left[1 - 2^{1-m+i}\right]\Pi_S + 2^{1-m+i}\Pi_H & \text{for } i = 2, \dots, m-1 \\ \Pi_B & \text{for } i = m. \end{cases}$$

Note that the first and second entrants get the same expected profit since they are indistinguishable once the second has entered z away from the first. Expected profit is thereafter strictly decreasing in order of entry in this case. This is because entrant $i = 2, \dots, m-1$ is followed by $m - i - 1$ firms that could fully protect it, so the probability of protection is higher for earlier

---

[21] If $1/z$ is an integer then there is no niche firm and firms are indifferent between entering at $2z$, $3z$, etc..

[22] For notational simplicity and because of its new interpretation as the equilibrium number of firms in the sequential-entry game, we now use m instead of m_L, even though they are equal.

[23] Thus $R = 1 - \lfloor 1/z \rfloor z$. For properties of the mod function, in this general case when z need not be an integer, see Graham, Knuth, and Patashnik (1994).

entrants. Vickrey, who discussed a fixed-price model, notes a similar effect: "it is in general an advantage to be one of the earlier locators, in that an earlier seller is less likely to get crowded by the last seller, while the last and next-to-last sellers are at a definite disadvantage" (1964, p. 331). The monotone profit property allows us to solve the timing of entry game easily. Assume all firms are risk neutral and that entry entails a sunk cost F, so that entry at time t before the market opens at time zero has a time-zero cost of $Fe^{\rho t}$ (where ρ is the instantaneous discount rate).

We now argue that competition, via entry time, for the rents associated with being an early mover (that is, facing a low probability of being a neighbor to the niche firm) will drive expected rents to zero. Rent dissipation through early entry is similarly modeled in games of adoption of new technology such as Fudenberg and Tirole (1985), where again profit differentials are eliminated by early commitment of capital to the market.[24] An analogous argument is formalized in Anderson and Engers (1994), where we consider a discrete time model in which the interval between instants at which firms can move converges to zero.[25] Rent dissipation arises because if some firm were to enter at some time at which it earned positive profits in any purported equilibrium, then some other firm would do better preempting it. This requires that there be enough lead time (otherwise the first firms can earn positive profits) and that there be enough potential entrants (more than m; otherwise profits are still equalized, but not eliminated).

The rent-dissipation condition is that F equal $\exp(-\rho t_i)$ times the present value of the rent stream earned at the time the market opens. If potential entrants are uncertain about whether the market will open, and the common degree of belief at time t that it will open is denoted by $P(t)$, then we simply replace $\exp(-\rho t_i)$ by $P(t_i)\exp(-\rho t_i)$ in the rent dissipation condition. As long as P is continuous, the preceding analysis of entry times goes through under this transformation. In this way, we can describe situations in which there need not be a lengthy lead-time before the market opens, as long as its advent gradually becomes apparent. Note that if P has fallen to zero, then some firms may have committed capital to a lost cause.[26]

To illustrate the pattern of entry over time, consider the case of linear transport costs, for which $\Pi(\ell; r) = \ell r/2$. Then the critical deterrence distance z is determined by $z^2/8 = F$, and the profit of a firm protected on both sides is $\Pi_S = z^2/2 = 4F$. Recall that R denotes 1 mod z, so the profit of a neighbor to the niche firm is $\Pi_H = (R+z)z/4 = 2F(1+R/z)$ and the profit of the niche firm is $\Pi_B = (R+z)^2/8$. Then, for $R > 0$ and $i = 2, \ldots, m-1$, the entry time of firm i is given by equating full cost to gross profit:

$$\tau_i = 4[1 - 2^{1-m+i}] + 2^{1-m+i}[2(1+R/z)];$$

where we have set $\tau_i = \exp(\rho t_i)$. For Firm 1, $\tau_1 = \tau_2$, while for Firm m, $\tau_m = (1+R/z)^2$. The equilibrium entry times (for $F \le 1/8$ so that there are at least two firms) are depicted in Figure 3.

INSERT FIGURE 3.

---

[24] See Hoppe (2000) for discussion and extension.

[25] At each instant there is a set order of moves for firms. Although the order of moves is arbitrary, nothing important rides on it because all firms earn the same expected profits at the equilibrium to the timing game. Note that small differences between firms (for example, cost differences) tie down the identities of the firms that enter the market. An alternative approach would be to consider the symmetric mixed-strategy equilibria when entrants can move simultaneously at each instant. In a model without product differentiation, Sutton (1998) has shown that, as the interval between instants goes to zero, the limiting distribution of entry times is the same as in the pure-strategy equilibria in a sequential entry game.

[26] We assume here that the only uncertainty regards whether the market will open at all: a description of the equilibrium when the opening date itself is random is more complex (except for special cases).

At $F = 1/8$, two firms placed diametrically opposite each other are just profitable, and deter entry. They enter at time zero ($t = 0$, $\tau = 1$). As F falls, rent dissipation requires earlier entry, so τ rises, as shown in Figure 3. Equilibrium locations are unchanged, and both firms earn the same gross profit (and so enter at the same time) until $F = 1/32$, when two firms can no longer deter a third, which enters midway between them. Since the profits of the first two are reduced by the presence of the third, they enter substantially later once the third cannot be deterred. As F falls further, the first two entrants (which always earn the same expected profit) close up to deter entry on the shorter arc between them. The third entrant's gross profit rises both because F falls and because the market it serves increases. The higher profit implies it must enter earlier, and its entry time gets closer to that of the other two. (Nevertheless, the first two always have higher gross profits for lower F since the direct effect of falling F dominates the indirect effect of having to be closer.) For F just above $1/72$, the three firms locate one third of the circumference apart and just manage to deter a fourth. The corresponding entry time is $\tau = 4$ because the maximal gross profit that can ever be earned by deterring firms is four times that of an entrant, and this maximal profit is attained when firms are equispaced ($1 \bmod z = 0$). Hence $\tau = 4$ represents the earliest possible entry time for any number of firms, at which time they all enter simultaneously.

Now consider F just below $1/72$. Then 3 firms cannot deter a fourth. The first two can guarantee that there will be no entry between them, and so earn the greatest expected profit (hence earliest entry time), locating just less than $1/3$ apart. The next firm knows it must be adjacent to the niche firm and can fully protect only one side of its market by locating just less than $1/3$ from one of the first two firms). The niche firm then locates at the midpoint of the remaining gap. As F falls further, the first three firms must close ranks to deter entry between them. This renders gross profits and hence entry times more symmetric. Covergence to symmetry continues until F is just above $1/128$, when all firms locate $1/4$ apart, and just deter a fifth firm: all four firms enter at $\tau = 4$.

The same basic ideas apply for all smaller F. For $F \in \left(\frac{1}{8}\frac{1}{(m-1)^2}; \frac{1}{8}\frac{1}{m^2}\right]$, m firms enter in equilibrium, and entry times become earlier and closer together as F decreases within this range. The incentive for early entry stems from the higher gross profit through greater probability of being protected on both sides.

If firms differ in fixed costs, all firms with costs above some (endogenous) threshold level stay out. Those with lower costs enter earlier, and earn rents to the extent that their costs are below those of the most competitive firm kept out. Thus temporal competition leads to an order of entry that is efficient in the sense that only the lowest cost firms will produce. Lower cost firms enter earlier than higher-cost ones so that the extent of rent dissipation is lessened.

We show in the next section that the atemporal sequential entry model of Section 3 leads to too few firms in the market. Competition for rents leads to wasteful early commitment of capital to the market to stake claims on profitable slots. An entry subsidy would alleviate the first distortion, but aggravate the second. The trade-off is analyzed in Section 6.

5. Welfare Properties.

One of the problems with static location models is that they do not make tight predictions: there are multiple equilibria. The received theory is also mute on the issue of rent dissipation (it is ignored because there is no channel for competition for rents). Nevertheless, it is still instructive to compare the equilibria in a purely atemporal sense, by which we mean we can compare the welfare properties of the timeless sequential-entry game with those of the timeless simultaneous-entry one. In other words, we contrast a situation in which firms move in a given order at the date the market opens, with one in which they all move simultaneously at that date. Another way to interpret the timeless sequential-entry model is to suppose that the government auctions off slots in the order of moves, so the rents are not dissipated, but simply transferred to the government. This experiment allows us to separate out the sources of inefficiency in the dynamic model. In this section, we argue that the timeless sequential-entry location equilibrium yields lower total surplus than nearly all the simultaneous-entry equilibrium outcomes. We address the role for tax policy in reducing inefficiency in Section 6.

Recall that m_U denotes the (integer) number of firms at the simultaneous-entry equilibrium with the highest (uppermost) number of firms, and m_L is the number of firms at the simultaneous-entry equilibrium with the lowest number of firms. As we have shown, m_L is also the equilibrium number of firms in the sequential-entry equilibrium. Clearly welfare is lower with m_L firms entering sequentially than simultaneously, by P3, because symmetric locations yield higher surplus than asymmetric ones. Since the total surplus is a strictly concave function of the number of firms (P8), it suffices that it be higher at the simultaneous-entry equilibrium with m_U firms than at the sequential-entry one with m_L firms for the sequential-entry equilibrium to be worse than all the simultaneous-entry ones. As we show, this is true if m_U is even, but may not necessarily be so if m_U is odd, although in the latter case it remains true that the sequential-entry equilibrium is worse than all the other simultaneous-entry equilibria. Thus the sequential-entry equilibrium is worse than nearly all the simultaneous-entry ones.

PROPOSITION 2. Social surplus is strictly lower at the atemporal sequential-entry equilibrium than at:

a) almost any simultaneous-entry equilibrium if m_U is even;

b) any simultaneous-entry equilibrium with fewer than m_U firms if m_U is odd.

The proof is in the Appendix. The Proposition implies that the allocation attained when allowance is made for rent-seeking behavior is more inefficient than would be thought from considering the usual static (simultaneous-entry) models. Once we couple the lost rents with the pure locational inefficiency, we see that differentiated product markets may be a source of significant cause for concern regarding market performance. In the next section, we look at the case for corrective subsidies or taxes. Doing so gives further insight into the nature of the market failure.

6. Entry Tax or Subsidy?

Various governments pursue policies aimed at encouraging the start-up of new businesses. In this section we examine whether taxes or subsidies can improve the allocation of resources. We first argue that the static model is not useful for answering these questions for two reasons. First, the multiple equilibria inherent to the static model include both those equilibria that would be improved by subsidies and those that would be improved by taxes. Second, by ignoring competition for rents the static model systematically overlooks a key source of deadweight loss.

The indeterminacy due to multiplicity of equilibria is illustrated by considering the two extreme cases. These are the minimum profit equilibrium (with m_U firms), and the maximum profit equilibrium (with m_L firms). An entry tax effectively raises the fixed cost an entrant incurs and so decreases both the minimum and the maximum number of firms. From P8 through P10 (and Figure 2), welfare falls with a tax if the equilibrium with the minimum number of firms is the relevant one, and rises if the maximum number of firms is the relevant one. Hence a subsidy is optimal if one believes that the market equilibrium involves the minimum number of firms. A subsidy works in this case by rendering entry more profitable and so requiring tighter spacing between firms and hence a greater variety of products. In either case, social welfare is a step function of the tax/subsidy because welfare only changes at critical values that alter the number of firms.

Now consider the timeless version of the sequential-entry model (with m_L firms). Here the social welfare still has discontinuities where the number of firms changes, but is no longer constant between these discontinuities. This is because firms early in the sequence of moves locate so as just to deter entry on their inside; the critical distance depends on the size of the tax/subsidy. By P6, symmetric locations are preferred to asymmetric ones, so the optimal tax/subsidy will always involve symmetric locations. Second, the optimal policy will be a subsidy rather than a tax, in order to increase the number of firms to the optimal level (the first-best optimum is attainable).

The conclusion of the previous paragraph is drastically altered once proper account is taken of the competition for the rents from moving early.[27] To analyze the optimal policy, we assume for simplicity that transport costs are linear, and equal to distance. The complication that firms' successors' locations are unknown causes no difficulty because total expected gross profits equal total actual gross profits. Thus total resource costs are the same as when all but the last three firms know that they will earn $\Pi(z, z)$, the next two know they will be adjacent to the niche firm, and the last is the niche firm. With linear transport costs, the shielded firms all enter at $\tau = 4$ (since gross profits of a firm that just deters are four times those of a potential entrant). The two neighbors to the niche firm enter at $\tau = \tau_{n-1}$ as given in Section 4, and the niche firm enters at $\tau = \tau_n$ (again as given in Section 4, since both the niche firm and its neighbors earn the same in this deterministic version as in the earlier one).

Total social cost C, is the sum of transport costs and capital costs. With n firms,

$$C = TTC + (n - 3)4F + 2F\tau_{n-1} + F\tau_n,$$

where TTC represents total transport costs. From Section 3 the equilibrium number of firms will be $\lceil 1/z \rceil$, and in the sequel, the reader should interpret n as being equal to $\lceil 1/z \rceil$. For notational clarity, we do not make the substitution.

There are $(n - 2)$ gaps of length z, and two of length $[1 - (n - 2)z]/2$; since a gap of length x entails a transport cost of $x^2/4$, we have $TTC = (n - 2)z^2/4 + \frac{1}{2}[(1 - (n - 2)z)/2]^2$. Recall that the gross profit of a firm that is ℓ from its left neighbor and r from its right neighbor is $\ell r/2$. Hence τ_n and τ_{n-1} are determined by the zero profit conditions $(F + T)\tau_n = \frac{1}{2}[(1 - (n - 2)z)/2]^2$, and $(F + T)\tau_{n-1} = \frac{1}{2}[(1 - (n - 2)z)/2]z$, where T is the tax on each entrant. Total cost is thus

$$\begin{aligned} C \;=\; & (n - 2)z^2/4 + \frac{1}{2}[(1 - (n - 2)z)/2]^2 + (n - 3)4F \\ & + \left\{\frac{1}{8}[1 - (n - 2)z]^2 + \frac{z}{2}[(1 - (n - 2)z]\right\} F/(F + T). \end{aligned}$$

---

[27] The first-best optimum can be achieved if the tax/subsidy policy is allowed to be time dependent. In this case, high entry taxes can be set before the market opens, to deter completely the loss from early entry. These would be followed by the optimal subsidy (at the opening date) as in the previous paragraph. In practice, however, policies rarely vary finely with time. Instead, there are blanket policies, like the small business start-up subsidy. At this level, we ask whether such a subsidy is a move in the right direction. We shall answer no, that the optimal policy is rather a tax.

We now define $z(T)$ to be the maximal distance between neighbors that does not allow profitable entry, if tax T is imposed. Under our linear transport cost assumption, $F + T = z^2/8$, or $z(T) = \sqrt{8(F + T)}$. Substituting this in the cost expression, and then differentiating with respect to z yields

$$\frac{dC}{dz} = \left\{\frac{1}{4}(n - 2)(nz - 1) - \frac{2F[1 - (n - 4)z]}{z^3}\right\}$$

where the first term is positive (transport costs rise because firms are less symmetrically placed), and the second term is negative (less rents are dissipated). The derivative expression holds for all z such that $1/z$ is not an integer, i.e. for all values of z such that n does not jump to the next integer down. The following result will be used in the proof that a subsidy is never optimal: $dC/dz < 0$ for all $T < 0$ wherever the derivative is defined. That is, the cost function is decreasing in T, (i.e. increasing in the subsidy), except where the number of firms changes.[28]

We next consider the values of C as $1/z$ approaches an integer. For $z \downarrow 1/n$, the spatial pattern approaches n equispaced firms, and C approaches

$$C^+ = 4nF + \frac{1}{4n} = \frac{4F}{z} + \frac{z}{4},$$

where $z = 1/n$ (since the average distance travelled by consumers is $\frac{1}{4n}$ and each of n firms enters at $\tau = 4$, so incurring cost $4F$ each). Similarly, for $z \uparrow \frac{1}{n}$, the spatial arrangement allows $n + 1$ firms, with n of them almost symmetrically placed around the circle, and the last firm in the niche midway between the pair that is slightly further apart than the other adjacent pairs.

Let C^- denote $\lim_{z \uparrow 1/n} C(z) = 4nF - 4F - T + \frac{1}{4n}$.[29] Since $F + T = z^2/8 = 1/8n^2$, we can write

$$C^- = 4nF - 3F + \frac{1}{4n} - \frac{1}{8n^2} = \frac{4F}{z} - 3F + \frac{z}{4} - \frac{z^2}{8}.$$

Clearly $C^- < C^+$, C^+ is strictly convex, and C^- is minimized at a higher value of z than C^+. The relation between C^-, C^+, and C is illustrated in Figure 4.

INSERT FIGURE 4

The proof of the following Proposition (which is in the Appendix) confirms the pattern suggested by Figure 4.

PROPOSITION 3. At the equilibrium to the timing game for the spatial price discrimination model of product differentiation with linear transport costs, an entry subsidy is never optimal.

The last part of the proof (for monopoly) also indicates that one firm serves the market absent intervention if and only if one firm serves the market under the optimal tax. Moreover, if two

---

[28] To show this, note first that $dz/dT > 0$, and that, for a subsidy ($T < 0$), $F > z^2/8$ and dC/dz is less than $\frac{1}{4}\left\{- \frac{[1 - (n - 4)z]}{z} + (n - 2)(nz - 1)\right\}$, which can be written as $= \frac{1}{4z}\{(nz + 1)[(n - 2)z - 1]\}$, and is negative as required since $(n - 2)z < 1$.

[29] To understand the relation between C^- and C^+, suppose we start at a symmetric position and then the tax, T, were decreased slightly, so another firm would enter. This firm would earn zero profit, so its social cost is F, but the benefits in terms of transport costs saved would be $F + T$. In addition, its entry will halve the gross profits of its two neighbors. They used to enter at $\tau = 4$, they now do so at $\tau = 2$, so resource costs fall by $4F$. Hence $C^- = C^+ - 4F - T$.

firms serve the market under laisser-faire, then two firms will serve the market under the optimal tax (which takes away all their rents and so eliminates temporal dissipation). More generally, if m firms serve the market under laisser-faire, then the number that serve under the optimal tax is m or fewer. One characteristic of the model is that (unless the optimum entails one or two firms) a symmetric configuration of firms is never optimal. This is because a slight decrease in the tax would induce an extra entrant, reducing social cost from $C^{+}(z)$ to $C^{-}(z)$ (see also Figure 4).

To understand this result better, suppose that F is small, so that the number of entrants is large and we can effectively ignore the integer problem. In the static (simultaneous-entry) model, the equilibrium with the largest number of firms entails zero profit and hence there are $\sqrt{\frac{1}{2F}}$ entrants (since gross profit is $\frac{1}{2n^2}$), while the equilibrium with the smallest number of firms has half this number, or $\sqrt{\frac{1}{8F}}$ firms. The latter is also the number of firms at the sequential-entry equilibrium. In an atemporal setting, the optimum number minimizes $\frac{1}{4n} + nF$, where $\frac{1}{4n}$ is the average (and aggregate) transport cost and nF is the total entry cost. The socially optimal number is then $\sqrt{\frac{1}{4F}}$, which lies between the two extremes of the static equilibria. Hence, under the atemporal sequential-entry equilibrium, a subsidy would be prescribed because a subsidy raises the number of firms towards the optimum as incumbents close ranks to deter entrants. However, once we introduce entry-time competition, the social entry cost per firm is $4F$ rather than F because of entry competition: the entrant who is just deterred would earn a gross profit of one fourth of that earned by equilibrium entrants. The social cost is then $\frac{1}{4n} + 4nF$, with a minimum at $\sqrt{\frac{1}{16F}}$ firms. Since this is fewer firms than the equilibrium number, a tax is desirable.[30] For larger F, the result still holds since the C^{-} locus is minimized at a lower number of firms than the C^{+} locus (see Figure 4: note too that these loci are virtually coincident for F small).

7. Conclusions.

In spatial models there are typically multiple equilibria, ranging from a zero-profit equilibrium to one at which a potential entrant would just fail to cover its costs and active firms earn substantial pure profits. These profits persist because a potential entrant must enter a niche in the market between existing firms - even if post-entry price competition were no more intense, the entrant would serve half the market that an incumbent does before entry. Allowing for fiercer price competition renders entry even less attractive, and consequently raises the threshold profit that is immune to entry. The potential for pure profit in equilibrium in spatial (or characteristics) models was a key insight of Vickrey (1964) and Eaton and Lipsey (1978). Subsequently, faced with this multiplicity of equilibria, many authors have chosen to concentrate on the zero-profit equilibrium, and typically find that there are too many firms in equilibrium. At the other extreme, there can be too few firms at the equilibrium at which active firms' profits are maximal (see Capozza and van Order, 1980, and Eaton and Wooders, 1985). Given the multiplicity of equilibria, an obvious question is whether there is a reasonable mechanism to select one of them.

We introduce a dimension (time) that allows firms to compete for possible profits. The process of competition resolves the multiplicity problem: it determines a unique number of firms and pattern of firm locations. The equilibrium obtained is similar to the static equilibrium with maximal profits

---

[30] When F is small, the preceding analysis also applies when transport costs are non-linear, since the derivative of the transport cost function at zero provides a local approximation to the function over the relevant range.

rather than the zero-profit static equilibrium favored by many authors. But there is an important difference between the equilibrium in the model with timing and any of the static equilibria - the possibility that rents may be dissipated can radically change the policy implications of the analysis. As we showed, the atemporal sequential-entry model suggests that a subsidy is an optimal corrective policy: however, if timing is considered, a subsidy is never desired, at least under linear transport costs or sufficiently many firms. What is surprising in this result is that the inefficiencies due to suboptimal locations and insufficient entry are always outweighed by the wasteful rent dissipation. This no-subsidy result emphasizes the point that the properties of the equilibrium with endogenous entry times can be very different from those in the atemporal sequential-entry model.

Because our setting is quite specific it is worth putting our results in a broader perspective, to see how the insights generalize and point out some limitations. First, the rents in the model stem from the spatial environment and there are many ways in which the details of the spatial model can be varied. As discussed above, the assumption of a circular market does not really matter in that a linear market would give similar results, and we believe that the same is true in a market with more dimensions. It would be worthwhile to consider the spatial model with mill-pricing instead of spatial price discrimination: this is intractable since an entrant affects the equilibrium prices of all firms through the chain-linking of competition that is absent in the "extreme localization" that characterizes the discriminatory model.[31] Despite the added complications, there is no reason to expect the results to be fundamentally altered. Indeed, the mill-pricing model is tractable when F is small so that the integer problem can be ignored. Deneckere and Rothschild (1992) argue that the atemporal sequential-entry model under mill-pricing entails more firms than is socially optimal. Although this result differs from the spatial price discrimination result, it reinforces the case for an entry tax since such a tax both alleviates overentry of firms and rent dissipation.

Non-spatial models of product differentiation may give very different results. In standard symmetric models (such as the logit and CES[32]), there are virtually no rents at the free-entry equilibrium (only those minor rents arising from the integer constraint on the number of firms). Hence there is no multiplicity of equilibrium problem, and, with negligible rents, the standard analysis applies. It is quite easy to introduce asymmetries in non-spatial models by allowing different products to be associated with different production costs, or else assigning them different "quality" variables (which essentially shift demand). This leads to differential rents across products and hence to rent dissipation in our framework, but does not typically lead to fundamental multiplicity of equilibria, and hence, without multiplicity, the issue is more straightforward. Insofar as the market solution tends to err on the side of producing too many products (see Anderson and de Palma, 1999), an entry tax both deters marginal goods and alleviates rent dissipation.

We have focussed on rent dissipation via early entry, in a simple framework in which all rent dissipation is socially wasteful. Other channels of competition for rents (such as lobbying for building permits by promising infrastructure improvements) might have beneficial side effects, thereby reducing the damage from dissipation and weakening the case for an entry tax. In our model, if the market were to grow continuously (rather than discretely) over time, then early commitment has some social value because some consumers are served. However, in such situations, early firms face a trade-off between pre-emption for the later market, and optimally serving the current market. The spatial entry pattern is then rather intricate - a firm may anticipate entrants on both sides

---

[31] See Anderson, de Palma, and Thisse (1992) for further discussion of these issues, and Neven (1987) for some preliminary results on sequential entry under mill-pricing.

[32] See Spence (1976), Dixit and Stiglitz (1977), and Anderson, de Palma, and Thisse (1992).

but still enter. This is a topic for future research.

Finally, the model of this paper has firms already fully informed about future demand, and so they have only to sink the entry cost to lay claim to a product. This assumption may well characterize the location of retail stores in a growing town, but the creation of new products is a more uncertain prospect. In the latter situation, a first entrant must typically engage in product development, market research, and marketing to develop a new market. Other firms can then enter at a much lower cost once they learn that there is a potential market in the offing. The significant free-rider problem associated with the initial market development is a force that may counteract the inefficiency of rent dissipation. This is another topic that should be investigated in more detail.

References

[1] Anderson, Simon P. and Andre de Palma (1988), “Spatial Price Discrimination with Heterogeneous Products," Review of Economic Studies, 55, 573-592.

[2] Anderson, Simon P. and Andre de Palma (1999), “Product Diversity in Asymmetric Oligopoly: Is the Quality of Consumer Goods too Low? " mimeo, University of Virginia.

[3] Anderson, Simon P., Andre de Palma, and Jacques Thisse (1992), Discrete Choice Theory of Product Differentiation, MIT Press.

[4] Anderson, Simon P. and Maxim Engers (1994), “Strategic Investment and Timing of Entry," International Economic Review, 35, 833-853.

[5] Capozza, D.R. and R. van Order (1980), “Unique Equilibria, Pure Profits, and Efficiency in Location Models," American Economic Review, 70, 1046-1053.

[6] Deneckere, Raymond, and Michael Rothschild (1992), “Monopolistic Competition and Preference Diversity," Review of Economic Studies, 59, 361-373.

[7] Dewatripont, Matthias (1987), “The Role of Indifference in Sequential Models of Spatial Competition," Economics Letters, 23, 323-328.

[8] Dixit, Avinash and Joseph E. Stiglitz (1977), “Monopolistic Competition and Optimal Product Diversity," American Economic Review, 67, 297-308.

[9] Eaton, B. Curtis and Richard G. Lipsey (1978), “Freedom of Entry and the Existence of Pure Profit," Economic Journal, 88, 455-469.

[10] Eaton, B. Curtis and Richard G. Lipsey (1979), “The Theory of Market Pre-emption: The Persistence of Excess Capacity and Monopoly in Growing Spatial Markets," Economica, 46, 149-158.

[11] Eaton, B. Curtis and Richard G. Lipsey (1980), “Exit Barriers are Entry Barriers: The Durability of Capital as a Barrier to Entry," Bell Journal of Economics, Autumn, 721-729.

[12] Eaton, B. Curtis and Nicolas Schmitt (1994), “Flexible Manufacturing and Market Structure," American Economic Review, 84, 875-888.

[13] Eaton, B. Curtis and Roger Ware (1987), “A Theory of Market Structure with Sequential Entry," RAND Journal of Economics, 18, 1-16.

[14] Eaton, B. Curtis and Myrna H. Wooders (1985), “Sophisticated Entry in a Model of Spatial Competition," RAND Journal of Economics, 16, 282-297.

[15] Fudenberg, Drew, and Jean Tirole (1985), “Preemption and Rent Equalization in the Adoption of New Technology," Review of Economic Studies, 52, 383-401.

[16] Graham, Ronald L., Donald E. Knuth, and Oren Patashnik (1994), Concrete Mathematics, (2nd edition), Addison-Wesley.

[17] Hoppe, Heidrun C. (2000), \Second-mover advantages in the Strategic Adoption of New Technology under Uncertainty," International Journal of Industrial Organization, forthcoming.

[18] Hoover, Edgar M. (1937), \Spatial Price Discrimination," Review of Economic Studies, 4(3), 182-191.

[19] Lederer, Phillip J. and Arthur P. Hurter (1986) \Competition of Firms: Discriminatory Pricing and Location," Econometrica, 54(3), 623-640.

[20] MacLeod, W. Bentley, George Norman, and Jacques-Francois Thisse (1988), \Price Discrimination and Equilibrium in Monopolistic Competition," International Journal of Industrial Organization, 6(4), 429-446.

[21] Neven, Damien (1987), \Endogenous Sequential Entry in a Spatial Model," International Journal of Industrial Organization, 5, 419-434.

[22] Prescott, Edward C. and Michael Visscher (1977), \Sequential Location among Firms with Foresight," Bell Journal of Economics, 8(2), 378-393.

[23] Salop, Steven C. (1979), \Monopolistic Competition with Outside Goods," Bell Journal of Economics, 10(1), 141-156.

[24] Spence, A. Michael (1976), \Product Selection, Fixed Costs, and Monopolistic Competition," Review of Economic Studies, 43, 217-235.

[25] Sutton, John (1998), Technology and Market Structure, MIT Press.

[26] Tirole, Jean (1988), The Theory of Industrial Organization, MIT Press.

[27] Vickrey, William S. (1964), Microstatics, Harcourt, Brace and World: pages 329-336 republished as \Spatial Competition, Monopolistic Competition, and Optimum Product Diversity," International Journal of Industrial Organization (forthcoming).

Appendix.

Proof of Proposition 1.

The proof proceeds in five steps. Let $e(s) = \lceil s/z \rceil - 1$, where $\lceil \cdot \rceil$ denotes the ceiling function. In (i) we consider $e(s) = 1$, i.e., a gap of size $s \in (z, 2z]$. Thenceforth we have $s > 2z$. For the induction step, given any integer $n \geq 2$ we suppose that the result holds true for all intervals of length s, with $e(s) \leq n - 1$, and show that this implies it is true for any interval such that $e(s) = n$ (recall that $e(s)$ will be the equilibrium number of firms that will enter a gap of length s). For the remainder of the proof, we consider four kinds of subintervals; in each we show that locating at z is better than locating at any point in the subinterval. In (ii), we consider x such that $e(s - x) = n$, that is, locating so close to the left end that n further firms still enter on the right (and none on the left). In (iii), we consider $e(x) = 0$ and $e(s - x) = n - 1$, and show that profit rises with x in this interval, as x approaches its maximum value, z. In (iv) we have $e(x) = k > 0$ and $e(s - x) = n - k$; in (v), $e(x) = k > 0$ and $e(s - x) = n - k - 1$.

(i) Let $e(s) = 1$, so that $s \in (z, 2z]$. Clearly an entrant in this gap will locate at the midpoint (by P6) and no further profitable entry is then possible.

(ii) We show that any firm entering a gap of size $s > 2z$ will never enter at $x \in [0, s - nz]$, where $n = e(s) = \lceil s/z \rceil - 1$, because locating at x in this interval is less profitable than locating at z. The former strategy would yield the firm a left neighbor at distance $\ell \leq s - nz$, and a right neighbor at most z away. Locating at z gives $\ell = z$ and at worst (when neighboring the niche firm) $r = [s - (n - 1)z]/2$. It therefore suffices that $[s - (n - 1)z]/2 \geq (s - nz)$, or $s \leq (n + 1)z$, which is true since $n = e(s)$.

(iii) We next show that locating at $x \in [s - nz, z)$ is less profitable than locating at z. By the induction hypothesis, a firm entering at $x \in [s - nz, z)$ knows it will be followed by $(n - 1)$ entrants on its right and none on its left. With probability $1 - 2^{2-n}$ it will have a right neighbor at distance $r = z$; with probability 2^{2-n} it will be adjacent to the niche firm and its right neighbor is at a distance: $r(x) = [s - (n - 2)z - x]/2$. Its expected profit is then $[1 - 2^{2-n}]\pi(x, z) + 2^{2-n}\pi(x, r(x))$. Now, locating at z gives an expected profit of $(1 - 2^{2-n})\pi(z, z) + 2^{2-n}\pi(z, r(z))$. Clearly $\pi(z, z) > \pi(x, z)$, so it suffices to show that $\pi(x, r(x)) < \pi(z, r(z))$. The total derivative of $\pi(x, r(x))$ is (using (2) and (3))

$$\frac{d\pi}{dx} = \frac{\partial \pi}{\partial \ell} - \frac{1}{2}\frac{\partial \pi}{\partial r} = \frac{1}{2}\left\{ c\left(\frac{x + r(x)}{2}\right) + c\left(\frac{r(x)}{2}\right) - 2c\left(\frac{x}{2}\right) \right\}.$$

To show this is positive, first note that $r(x) > x/2$ (since $s > nz \Rightarrow s - (n - 2)z - x > 2z - x > x$, i.e., $2r(x) > x$). Because c is increasing, it then suffices that $d\pi/dx \geq 0$ when $r(x) = x/2$, i.e.

$$c\left(\frac{3x}{4}\right) + c\left(\frac{x}{4}\right) - 2c\left(\frac{x}{2}\right) \geq 0;$$

which is true by convexity of c.

It remains to consider $x \in (z, s/2]$ (where $s > 2z$; $s \leq 2z$ has already been treated). There are two cases, depending on the number of subsequent entrants that the first entrant's location choice induces. Recall that n denotes $e(s)$, the total number of firms that, in equilibrium, will enter an interval of length s. If the first entrant locates at x in $[0, s]$, then there will be $e(x)$ entrants on the left, and $e(s - x)$ entrants on the right. Hence the total number of subsequent entrants is $e(x) + e(s - x)$. This total is periodic, with period z, since increasing x a distance z increases the first term by one, and decreases the second by one. Note that $e(x) + e(s - x)$ is either equal to n (case (iv)) or equal to $n-1$ (case (v)) depending on whether $x \bmod z < s \bmod z$ or not, respectively. [To see this, it suffices to consider the subinterval $[0,z)$, by periodicity: clearly $e(x)=0$, and $e(s-x)$ is n if $x < s-nz$ (i.e., $x = x \bmod z < s \bmod z = s-nz$) and $e(s-x)$ is $n-1$ if $x \ge s-nz$.]

Suppose the first entrant comes in at $x = kz+\alpha$ where k is a positive integer and $\alpha \in [0,z)$, so $\alpha = x \bmod z$. Hence k further entrants locate to its left, i.e., $e(x)=k$. Of these entrants, all but the last will locate z from an end of the remaining subinterval, leaving the last firm to locate in the middle of the remaining gap. Since the $k-1$ firms are equally likely to locate z from either end of their subinterval, the first entrant will end up adjacent to the niche firm (i.e., "holding the baby") with probability 2^{1-k}. In this case it is $[z+\alpha]/2$ from its left neighbor. It is protected with probability $1-2^{1-k}$, and in this case it is the maximal distance z from its left neighbor.

If the entrant locates at z, it precludes further entry on its left and is z from its left neighbor with probability one. With probability 2^{2-n} its right neighbor is at distance $[s-(n-1)z]/2$, otherwise that neighbor is at distance z.

The difference between cases (iv) and (v) is the number of firms on the right.

(iv) Here $e(s-x) = n-k$. With probability 2^{1-n+k} the first entrant is not shielded on its right and is then $[s-(n-1)z-\alpha]/2 = [s \bmod z + z - x \bmod z]/2$ from its right neighbor; with probability $1-2^{1-n+k}$ it is z from its right neighbor. We wish to compare the corresponding expected profit with that obtained by entering at z. From the expressions in the preceding paragraph, expected profits are higher at z if:

$$\begin{aligned}&[1-2^{2-n}]\pi(z,z) + 2^{2-n}\pi(z,[s-(n-1)z]/2) > [1-2^{1-k}][1-2^{1-n+k}]\pi(z,z)\\ &+[1-2^{1-k}]2^{1-n+k}\pi(z,[s-(n-1)z-\alpha]/2) + 2^{1-k}[1-2^{1-n+k}]\\ &\pi([z+\alpha]/2,z) + 2^{1-k}2^{1-n+k}\pi([z+\alpha]/2,[s-(n-1)z-\alpha]/2).\end{aligned}$$

First note that $[1-2^{2-n}] \ge [1-2^{1-k}][1-2^{1-n+k}]$, so that the $\pi(z,z)$ terms (which are the highest ones) on the right of the inequality are more heavily weighted than those on the left: the probability of being shielded on both flanks is greater when the firm shields one flank with certainty. To prove the inequality, it now suffices to show that the term $\pi(z, [s-(n-1)z]/2)$ from the left-hand side is greater than each of the other profit terms (that is, excepting the $\pi(z,z)$ term) on the right-hand side. This follows by symmetry of the profit functions in their arguments, and that the smaller length is always less than $[s-(n-1)z]/2$: the only case that is not trivial is that $[s-(n-1)z]/2 > [z+\alpha]/2$, which follows since $s \bmod z > \alpha$ in Case (iv).

(v) Here $e(s-x) = n-k-1$, and $\alpha \in [s \bmod z, z)$. With probability 2^{2-n+k} the first entrant is not shielded on its right and is then $[s-(n-2)z-\alpha]/2 = [s \bmod z + 2z - x \bmod z]/2$ from its right-hand neighbor; with probability $1-2^{2-n+k}$ it is z from its right-hand neighbor. Expected profits are no lower at z if and only if:

$$\begin{aligned}&[1-2^{2-n}]\pi(z,z) + 2^{2-n}\pi(z,[s-(n-1)z]/2) \ge [1-2^{1-k}][1-2^{2-n+k}]\pi(z,z)\\ &+[1-2^{1-k}]2^{2-n+k}\pi(z,[s-(n-2)z-\alpha]/2) + 2^{1-k}[1-2^{2-n+k}]\\ &\pi([z+\alpha]/2,z) + 2^{1-k}2^{2-n+k}\pi([z+\alpha]/2,[s-(n-2)z-\alpha]/2).\end{aligned}$$

Consolidating the terms in $\pi(z,z)$ and using symmetry:

$$\begin{aligned}&[-2^{3-n} - 2^{2-n} + 2^{1-k} + 2^{2-n+k}]\pi(z,z) + 2^{2-n}\pi(z,[s-(n-1)z]/2) \ge\\ &[1-2^{1-k}]2^{2-n+k}\pi(z,[s-(n-2)z-\alpha]/2) + 2^{1-k}[1-2^{2-n+k}]\pi(z,[z+\alpha]/2)\\ &+2^{1-k}2^{2-n+k}\pi([z+\alpha]/2,[s-(n-2)z-\alpha]/2).\end{aligned}$$

To show this inequality holds, we rewrite it in the form:

$$w_0\Pi_0 + w_1\Pi_1 \geq w_2\Pi_2 + w_3\Pi_3 + w_4\Pi_4$$

where $w_0 = [-2^{3-n} - 2^{2-n} + 2^{1-k} + 2^{2-n+k}]$, $\Pi_0 = \Pi(z, z)$, etc., and we reverse the order of the first two terms on the RHS if necessary so that $\Pi_2 \geq \Pi_3$. Since $\Pi_4 \leq \Pi_3$, it suffices to show that

$$w_0\Pi_0 + w_1\Pi_1 \geq w_2\Pi_2 + w_5\Pi_3, \tag{A.1}$$

where we have defined $w_5 = w_3 + w_4$.

We now show that $w_0 \geq w_1$ and $w_0 \geq w_2$. The first amounts to showing
(a) $[-2^{3-n} - 2^{2-n} + 2^{1-k} + 2^{2-n+k}] \geq 2^{2-n}$;
while the second is established by showing both
(b) $[-2^{3-n} - 2^{2-n} + 2^{1-k} + 2^{2-n+k}] \geq [1 - 2^{1-k}]2^{2-n+k}$; and
(c) $[-2^{3-n} - 2^{2-n} + 2^{1-k} + 2^{2-n+k}] \geq 2^{1-k}[1 - 2^{2-n+k}]$.
To see (a), rewrite it as $[2^{2-k} + 2^{3-n+k}]/2 \geq 2^{4-n}$, and the note that the LHS is at least $2^{(5-n)/2}$, by the inequality between the arithmetic mean and the geometric mean. Since $n \geq 3$ for the case at hand, the RHS is no greater than this bound. Condition (b) reduces to $n \geq k + 1$, which is true since $k \leq n - 1$. Finally, (c) reduces to the trivial $k \geq 0$.

We can now show that (A1) holds. Note that Π_i can be written as $\Pi(z, r_i)$, with $r_1 < r_3 < r_2 < r_0$ and $r_3 - r_1 = r_0 - r_2$. Since $\Pi(\cdot)$ is convex in its second argument (P4b), $(\Pi_3 - \Pi_1)/(r_3 - r_1) \leq (\Pi_0 - \Pi_2)/(r_0 - r_2)$, and hence $(\Pi_3 - \Pi_1)/(\Pi_0 - \Pi_2) \leq (r_3 - r_1)/(r_0 - r_2) = 1 \leq w_0/w_1$. Thus $w_0\Pi_0 + w_1\Pi_1 \geq w_0\Pi_2 + w_1\Pi_3 \geq w_2\Pi_2 + w_5\Pi_3$, since $w_0 \geq w_2$, $\Pi_2 \geq \Pi_3$, and $w_0 + w_1 = w_2 + w_5$. □

Proof of Proposition 2.

From P10, there are three cases to consider. The first (razor's edge case) has $n_U = m_U$ an even integer, in which case $1 \bmod z = 0$, and the sequential-entry equilibrium has the same outcome as the simultaneous-entry one with the fewest firms. Welfare under sequential entry is the same as at the two extreme cases of simultaneous-entry (see P9), and lower than at any other simultaneous-entry equilibrium.

Suppose next that m_U is even but n_U is not an even integer, so $m_L = m_U/2 + 1$. Since all pairs of adjacent firms except two are z apart at the sequential-entry equilibrium, adding a firm at the midpoint of each gap of size z (and there are $m_U/2 - 1$ such gaps) will leave welfare unchanged (by P7) and bring the number of firms to m_U. Hence there are now m_U firms in both cases, and the simultaneous-entry case has them symmetrically placed and so is preferred to the other one, which does not.

Now suppose that m_U is odd, in which case $m_L = (m_U + 1)/2$. The argument of the previous paragraph can be adapted to show that the sequential-entry equilibrium is worse than any simultaneous-entry one with fewer than m_U firms. There are $m_L - 2$ gaps of size z, so when these are filled, there are $2m_L - 2 = m_U - 1$ firms in the market. The sequential-entry equilibrium is therefore worse than the simultaneous-entry one with $m_U - 1$ firms, and is worse than the one with m_L firms (P6). By P9, it is therefore worse than all of the simultaneous-entry ones, expect possibly that with m_U firms.

To show that the comparison of an odd number m_U and the sequential-entry equilibrium with m_L firms is ambiguous in terms of welfare, consider two special cases. In the first, take the limit with the m_U firms just making zero profits, so $1 = m_U z/2$. By P7, taking out a firm (and not rearranging) leaves social surplus unchanged. We now have $m_U - 2$ gaps of size $z/2$, and one of size

z. We can also bring the sequential-entry equilibrium to the same number of firms without changing its welfare level by adding firms in each of the $m_L - 2$ gaps of size z, to yield $2(m_L - 2) = m_U - 3$ gaps of size $z/2$, and two of size $3z/4$. The latter scenario dominates the former since two gaps of size $3z/4$ are preferable to one of size $z/2$ plus one of size z (by P3 applied to the subinterval of size $3z/2$).

To establish that the comparison can go the other way, take the other extreme case in which $m_U + 1$ firms are just unprofitable. Then the simultaneous-entry equilibrium has a higher welfare level than when $m_U + 1$ firms are in the market, and the sequential-entry equilibrium has the same welfare level as when there are $m_U + 1$ firms: just add a firm in the middle of each of the gaps of size z, and note that all gaps are this size. $\square$

Proof of Proposition 3.

We show here that a subsidy ($T < 0$) is never optimal. We first consider the case $z(0) < 1/2$, i.e., $F < 1/32$, so at least three firms enter if $T = 0$. To show a subsidy is never optimal, suppose one were, then $C(z(0)) \geq C(z(T^*))$ for some $T^* < 0$. Since we have already shown that $C(z(T))$ is decreasing in T for all $T < 0$ except where C jumps (i.e., where $z(T) = 1/m$ with m an integer), it suffices to consider the function $C^-(\frac{1}{m})$ at integer values of $1/z$ and to show that $C^-(\frac{1}{m})$ is increasing in m for all $m \geq n - 1$, where $n = \frac{1}{z(0)} = \frac{1}{\sqrt{8F}}$ is the number of firms if taxes are zero. That is, we show that $C^-(\frac{1}{n-1}) < C(z(T^*))$, hence lower costs than at $z(T^*)$ can be achieved. To show that $C^-(\frac{1}{m})$ is increasing as required, it suffices that $C^-(\frac{1}{m}) > C^-(\frac{1}{m-1})$ for all $m \geq n$, or equivalently

$$4mF + \frac{1}{4m} - \frac{1}{8m^2} > 4(m-1)F + \frac{1}{4(m-1)} - \frac{1}{8(m-1)^2}$$

or

$$32F > \frac{2m^2 - 4m + 1}{m^2(m-1)^2}. \tag{A.2}$$

Since $m \geq n \geq \frac{1}{\sqrt{8F}}$, then $\frac{4}{m^2} \leq 32F$, and it suffices that $\frac{4}{m^2} > \frac{2m^2-4m+1}{m^2(m-1)^2}$, or, equivalently, that $2(m-1)^2 + 1 > 0$, which is clearly true.

It remains to be shown that the no-subsidy result also holds for $F \geq 1/32$ (which corresponds to $z(0) \geq 1/2$, i.e. duopoly, monopoly, or no firm at all). First consider duopoly, or $z(0) \in [1/2; 1)$, that is, $F \in [1/32; 1/8)$. The cost $C(z)$ is decreasing on $[1/2; 1)$ since locations are diametrically opposite regardless of z in this range, so the only effect of increasing T (or equivalently, increasing z) is to reduce wasteful dissipation. This means that $C^-(1) < C(z)$ for all $z \in [1/2; 1)$. Furthermore, by (A.2) for $m \geq 3$, since $F \geq 1/32$, $C^-(z)$ is decreasing on $(0; 1/2]$, so $C^-(1/2) < C(z)$ for all $z < 1/2$. Thus, it suffices to show that $C^-(1) < C^-(1/2)$, or $2F + 1/8 < 5F + 3/32$, which is implied by $F \geq 1/32$. Hence we have shown that if there would be a duopoly in the absence of any tax or subsidy, then subsidizing to increase the number of products is not worthwhile.

The case of monopoly arises for $F \geq 1/8$. Clearly costs fall with the tax, up to the tax at which $F + T$ equals the monopolist's gross profit. At this tax, the social cost of serving the market is $F + 1/4$. Hence, by the argument of the previous paragraph, to show that a subsidy is never optimal, we need only show that this number does not exceed $C^-(1)$, i.e. $F + 1/4 \leq 2F + 1/8$, which is true exactly when $F \geq 1/8$. In other words, if there is just one producer in the market in the absence of government intervention, then optimal policy proscribes altering the number of firms, and prescribes taxing (almost) all the monopolist's net profit.[33] $\square$

---

[33] Because of our choice of tie-breaking rule (namely, that a firm stays out if it would earn zero profit entering at

---

time zero), there may be no minimum total cost, although the infimum will always exist. This problem arises at the points where $C(z)$ jumps. The planner can get arbitrarily close to the infimum by appropriate choice of T. Even though an optimum may not exist in a strict sense, this poses no real economic problem, and we shall continue to refer to the “optimal tax” in the text.