

A Beautiful Blonde: a Nash coordination game^α

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Abstract

In a memorable scene from the ...Im "A Beautiful Mind," John Nash explains to his friends how to direct their attentions to women in a bar. Game theorists who have seen the ...Im point out that the proposed solution is not a Nash equilibrium. Here we determine the Nash equilibria to the attention game. The symmetric mixed strategy equilibrium has resembles a common property resource problem. It has perverse comparative static properties that are not borne out by experimental data. Finally, we discuss alternative ways of formulating the game.

Keywords: coordination, Nash equilibrium, mixed strategy equilibrium, common property resource problem, comparative statics.

^α"If you think the title stinks, try the movie." (New Yorker review, April 8, 2002, p.26.)

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"If everyone competes for the blond, we block each other and no one gets her. So then we all go for her friends. But they give us the cold shoulder, because no one likes to be second choice. Again, no winner. But what if none of us go for the blond. We don't get in each other's way, we don't insult the other girls. That's the only way we win. That's the only way we all get [a girl.]" (from *A Beautiful Mind: The Shooting Script*, Akiva Goldsman, 2002)

1 Introduction

In a scene in the Oscar-winning movie "*A Beautiful Mind*," John Nash (played by Russell Crowe) and several male colleagues are discussing a group of women at a bar.¹ There is a blonde woman and several brunettes.² The men agree that the blonde is the most desirable, but that any one of the brunettes is better than no woman at all. First, suppose that all the men devote their full attention to the blonde. Nash explains that this is not a reasonable strategy combination because it will result in the men neutralizing their efforts with the blonde so that none is successful. Further, the brunettes feel slighted, so the men have no chance with them either. Instead, Nash suggests that the men ought to ignore the blonde and each should concentrate on a (different) brunette. As the film sequence portrays, this is a success. Any game theorist who has seen the film will point out that this scenario is actually not a Nash equilibrium either for if all other men pair off with brunettes, any individual would prefer to hit on³ the blonde.

The scenario in which one male hits on the blonde and the others on the brunettes is one type of Nash equilibrium. In this note we determine the set of Nash equilibria to the game. As we show, with n male players there are $2^n - 1$ equilibria. Each one can be characterized by the set of men who devote all their attention to the brunettes. In each such equilibrium, either a particular individual makes a play for the blonde with positive probability or does not. The "equilibrium" portrayed in the film - that all players choose brunettes and hence no player plays for the blonde with positive probability - is the only combination that cannot

¹"The phrase 'Game Theory' is never uttered" (C-ville Weekly review, March 19, 2002); "Math skills not required" (New Yorker, April 8, 2002).

²Goldsman (2002) refers to the other women as brunettes in the Cast (p.143) and we stay with this terminology. There are four non-blondes in the film sequence.

³We use the American colloquial term "hit on:" British readers may substitute "chat up."

happen. We then concentrate on the symmetric mixed strategy equilibrium to the game and interpret it as a common property resource problem. In equilibrium, too much effort is devoted to the blonde. However, the more players there are, the lower the probability that the blonde is hit on. We also show that the equilibrium masks a perverse comparative static property. As one might expect, the more attractive the blonde, the greater the likelihood that each man will hit on her. However, deeper analysis of this result indicates that it conceals a surprising property of the mixed strategy equilibrium. If only one player finds the blonde more attractive (than the others find her), his equilibrium probability of hitting on her falls, while the probability played by the others rises! We conclude with a discussion of other applications and other solution mechanisms.

2 Equilibria

Suppose that the game is as follows. There are n (male) players as well as one blonde and at least n brunettes. Each male must decide which of the women to hit on. Suppose the brunettes are equally attractive to the men, with appeal b .⁴ The blonde is more attractive, with appeal $a > b$. If more than one man attempts to hit on the blonde, they succeed neither with the blonde, nor any brunette (the latter feel slighted). Any man choosing not to hit on the blonde succeeds with a brunette, for a payoff of b . If a man is the only one who hits on the blonde, he succeeds.⁵

Clearly there are n pure strategy (asymmetric) equilibria in which any one of the men pairs with the blonde, while each of the others pairs with a brunette. These are the only pure strategy equilibria. They suffer from the drawback that there are many of them, so it is not clear to the players which one they should focus on.⁶ Faced with a similar problem of how to select among the multitude of equilibria, previous authors have suggested looking

⁴We assume they are all equivalent to all males, and do not distinguish outcomes in which different males go to different brunettes.

⁵The analysis is unchanged if each man believes he will succeed. Even if there is a probability the blonde rejects a single suitor, a man rejects the probability of success times the probability if successful.

⁶Nash's advice to his friends could be interpreted as an attempt to get them to focus on the equilibrium in which Nash gets the blonde: see below.

at the symmetric mixed strategy equilibrium, in which players independently make random choices.

At a symmetric mixed strategy Nash equilibrium, each male must be indifferent between following the blonde and brunette strategies, or else he would concentrate only on the preferred alternative. Let the probability that each other male plays "blonde" be P . Then indifference requires that

$$b = a(1 - P)^{n-1};$$

so that the equilibrium probability is

$$P = 1 - \frac{b}{a}^{\frac{1}{n-1}}.$$

We show in the Appendix that this is the only equilibrium in which all players randomize or mix their strategies; that is, all players who choose randomly do so with the same probability.

However, there are also equilibria at which some mix and the rest choose "brunette:" each player either restricts himself to a brunette, or else hits on the blond with some positive probability. With n players, and these two options, this yields 2^n possibilities. Each of these cases is an equilibrium except for the case when no player ever hits on the blonde. This means that there are $2^n - 1$ different equilibria. Ironically, the only one of the 2^n possibilities that is not a Nash equilibrium (corresponding to the " -1 ") is the one the film depicts, the one in which the blonde gets no attention.

2.1 The tragedy of the commons

We now compare the symmetric mixed strategy equilibrium to the optimal ex ante symmetric arrangement whereby the common probability is to be chosen so as to maximize the sum of the benefits to the males. That is, we take as a constraint that the blonde must be "allocated" via a contest with equal independent probabilities. This perspective enables us to interpret the inefficiency of the symmetric equilibrium independent of the inefficiency of independent random choices per se.

The optimand in this (constrained) welfare maximization problem is then

$$\max_{P \in [0;1]} W = bn(1 - P) + an(1 - P)^{n-1}P:$$

The first term here is the probability-weighted payoff per player from playing "brunette," summed over the n players. The second term is the probability-weighted payoff to a player from playing "blonde," which pays off only if no players are playing "blonde," again summed over all n players.⁷

The objective function is increasing at $P = 0$ and decreasing at $P = 1$, so the optimal choice of n then satisfies the first-order condition

$$\frac{W'(P)}{n} = -b + a(1 - P)^{n-2}(1 - nP) = 0:$$

This is equivalent to

$$b = a(1 - P)^{n-2}(1 - nP);$$

which clearly has a unique root, which is therefore the unique maximum. Where it is positive, the RHS is a strictly decreasing function of P . The solution is to be compared to that for the equilibrium, i.e., $b = a(1 - P)^{n-1}$: since the function on the RHS exceeds the former one, the social optimum solution involves a lower probability P that the players choose "blonde."

This result can be interpreted as a variant of the "tragedy of the commons" whereby the blonde takes the role of the common property resource. Each individual, in choosing to hit on the blonde with positive probability, takes account only of his personal payoff. He does not internalize the fact that his decision to pursue the blonde reduces the chance that the others might succeed with her. The equilibrium probability P is therefore too large from a social perspective.

2.2 More players reduces the chances the blonde is hit on

At the symmetric equilibrium we have $b = a(1 - P)^{n-1}$. If there are more players (higher n), the RHS of this expression must remain the same, meaning that P must fall to maintain the equality.

⁷The skeptical reader can derive the welfare maximand by enumerating over all possible outcomes and simplifying using the binomial theorem.

The probability that the blonde is pursued when there are n players is $1 - (1 - P)^n$ (which is just one minus the probability she is not hit on). To see how this changes as a function of n , denote the equilibrium probability when there are n players by P_n . We have just shown that $P_n > P_m$, when $m > n$, while $(1 - P_n)^{n+1} = (1 - P_m)^{m+1}$. This means that $(1 - P_n)^n < (1 - P_m)^m$ and hence the probability that the blonde is hit upon is decreasing in the number of players.

2.3 Comparative statics that are "right for the wrong reason"

In the symmetric mixed strategy equilibrium, the probability that a player chooses "blonde" is $P = 1 - \frac{b}{a} \frac{1}{n+1}$, as derived above. If the attractiveness of the blonde (as given by a) rises, then so does the probability P . This agrees with the simple intuition that people strive more for a prize of higher value. This conclusion obscures an odd feature underlying this result. We consider separately the case of two players and of more than two, since they differ somewhat.

In the two-player version, suppose we increase just one player's enthusiasm for the blonde. Then, by the logic of the mixed strategy, this player's equilibrium mixed strategy cannot change because the other player must remain indifferent between the two pure strategies. But the probability P played by the other player must increase in order for this player to be indifferent after the valuation rises. When we raise the valuations of the other player in turn, the same changes occur. The final outcome is that all probabilities rise. The way this is achieved though is not intuitive: when the prize value increases for one individual it is the other player who strives more while that individual does not.

The three-player (symmetric mixed strategy equilibrium) case is even stranger. Start again from the benchmark where they all find the blonde equally attractive. If one of them then becomes more enthused about the blonde, then his equilibrium probability of success with her must decrease to satisfy his indifference condition for mixing. This means that the other players must increase their probabilities of pursuing the blonde.⁸ Since their enthusiasm

⁸This relies on the symmetry result proved in the Appendix that players with the same tastes who do randomize do so with equal probabilities.

for the blonde is unchanged, their equilibrium probability of success with her cannot change, but since one rival (the other man whose tastes have not changed) has a higher probability of pursuing the blonde, the other rival (the one whose enthusiasm has increased) must have a lower probability. This is unlike the 2-player case, in which the probability was unchanged.

The end result from increasing all players' enthusiasm for the blonde is that they all try harder (in the sense of choosing "blonde" with higher probability). However, as we increase the players' enthusiasm for the blonde one at a time, the comparative static result is perverse. Namely, he who likes more strives less, while those unchanged strive more. The general case yields similar results and is presented in the Appendix.

An economic application of some of the ideas we have discussed is the analysis of rival firms into new markets. Capra (1998) has examined a game of this kind and used laboratory experiments to see whether subjects change their behavior in the strange way predicted by the model when their payoffs are altered. She looks at a version of the game phrased as whether a firm wishes to enter a market that is profitable for one but not for two firms. Entering the market corresponds to playing blonde and gives lower payoff when both enter than when both do not (i.e., play the "safe" brunette strategy). Capra then considers increasing the cost of playing the enter strategy. While this differs somewhat from our thought experiment of increasing the attractiveness of getting the blonde, this also yields perverse comparative static mixed strategy equilibrium predictions.

Capra (1998) uses 120 subjects, each in a game played only once in order to eliminate any repeated game or reputation effects. For the symmetric treatment, she finds that observed behavior is almost exactly predicted by the symmetric mixed strategy.⁹ Capra then considers a treatment in which one agent's payoffs are increased whenever she enters the market. The mixed strategy Nash equilibrium predicts that this agent's entry probability should remain unchanged while the other agent's entry probability should rise (to keep the first type indifferent because now the attractiveness of entry has risen for the first type but not for the second). The data convincingly go the other way though. Goeree and Holt (2001) present some further examples of laboratory experiments in which a change in payoffs has the subjects behaving very differently from the predictions of Nash equilibrium.

⁹In the terminology of Goeree and Holt (2001) this truly is a "treasure" treatment.

3 Discussion

We have tried to formalize the game portrayed in the movie as closely as possible to the way it is described there. Each man chooses a woman to hit on, and if several choose the blond, they all strike out. We have further assumed that the men have no problem in ensuring that no more than one vies for the same brunette.

We emphasized the common property nature of the equilibrium in our claim that excess effort is devoted to the blonde in comparison to a symmetric mixed strategy optimum. Economists might argue that one way to resolve the issue would be to specify property rights to avoid the congestion problem. One could envisage a pre-pick-up lottery among the men to determine beforehand who has the right to hit on whom. If the men's valuations differed, appropriate side-payments could be made (with the right to hit on the blonde being traded).

As in the film, we have concentrated only on the perspective of the males. We have not looked at the preferences of the women, nor how they can act strategically.¹⁰ Pareto efficiency would also account for their tastes over possible matches. The properties of various matching arrangements have long been the subject of analysis: see for example Gale and Shapley (1962).

We described a simple one-shot game. More complex would be a "war of attrition," played out in real time. Suppose that the longer a man stays trying with the blonde, the lower are his chances with any brunette, but that a man can only succeed with the blonde once all other males have given up. To make it interesting, the males would have different success probabilities, and information on these would be revealed only over time.¹¹

¹⁰Indeed, Varian (2002) notes that the Nash character "didn't look at the game from the woman's perspective, a mistake no game theorist would ever make."

¹¹If all men knew their success probabilities from the outset, the equilibrium would resemble that described in the Ghemawat and Nalebuff (1985) model of exit from a declining industry. Only the male with the best chances (assuming the brunettes are still sure things) would even bother to start expending effort to hit on the blonde because all the others realize that he will out-endure them. Introducing updating of chances makes the game more interesting.

As a prediction of a social phenomenon, another reason (in addition to it being a realization of the symmetric mixed strategy equilibrium) why all might hit on the blonde is that each could overestimate his chances. The idea goes back at least as far as Adam Smith (1776).¹² It helps explain why so many aspiring authors submit manuscripts despite miserable acceptance chances, and why so many would-be actors and screenwriters work in bars and restaurants in Los Angeles.

We have pointed out that one economic application is the entry game in which several firms are contemplating entering an industry that is profitable if there is only one firm. Another application is the simultaneous submission of similar articles to the same top journal when a new topic comes to light. A twist on the model would be to consider several journals (women) in decreasing order of attractiveness. We conjecture that there is again a symmetric mixed strategy equilibrium, and that the excess effort result is reduced as we descend the quality ladder.

Finally, the film suggests the intriguing possibility that Nash was disingenuously manipulating his buddies. Having persuaded them to think they should all go for brunettes, he then gets up and walks toward the blonde. As it turns out, he walks on past her. But if they are all convinced to play "brunette," then it is indeed a Nash equilibrium if he plays blonde.

4 Appendix

4.1 Symmetry

We show that in any equilibrium at which some players use non-degenerate mixed strategies, all such players mix with identical probability. Let K denote the set of players using such strategies, and let k be the number of these players (so that the remaining $n - k$ players play "brunette" with probability one). Then indifference among those mixing entails

$$b = a \sum_{j \in K, j \neq i} (1 - P_j) \quad \text{for all } i \in K;$$

¹²Smith argues in Chapter X of Book I of the *Wealth of Nations* that people systematically overestimate their chances of success in any venture, both because of "overweening conceit" in their own abilities to control those factors that can be controlled, as well as their "absurd presumption in their own good fortune" with respect to those factors beyond their control.

where K_{-i} denotes the set of all players in the subset K except for i . Since this condition must hold for all $i \in K$, then the expression

$$\frac{\prod_{j \in K_{-i}} (1 - P_j)}{(1 - P_i)}$$

must be the same for all mixers, so that the equilibrium probability is the same for all players in K .

Call the equilibrium probability with k such players P_k . Then

$$P_k = 1 - \frac{b}{a} \left(\frac{b}{a} \right)^{k-1}.$$

The probability that the blonde is pursued is $1 - (1 - P_k)^k$ or $1 - \left(\frac{b}{a} \right)^k$. Since $b < a$, then this probability is decreasing in k . This means that the blonde is less likely to be pursued the more males who are competing for her. The equilibrium probabilities are described more fully in the text: they are analogous to those for the symmetric mixed strategy equilibrium as the number of players is increased.

4.2 Asymmetry

Let the attractiveness of the blonde be \hat{a} for m of the players, and a for the other $n - m$ players, where $\hat{a} > a$. Let the corresponding probabilities of choosing to pursue the blonde be denoted by Q_m and P_m respectively.¹³ Then we know that $P_0 < Q_n$ which is just the result that the symmetric mixed strategy probability of pursuing the blonde is increasing in the attractiveness of the blonde. However, we are interested in what happens if we increase each player's valuation one at a time. For players with the low valuation, the indifference condition is

$$b = (1 - P_m)^{n-m-1} (1 - Q_m)^m a$$

where the first term corresponds to the $n - m - 1$ rivals with the same valuation, and the second corresponds to the m rivals with the other valuation. This expression holds for $m < n$.

¹³Symmetry within each group follows from an argument similar to that in the first part of the Appendix.

For players with the high valuation, the indifference condition is

$$b = (1 - P_m)^{n_i - m} (1 - Q_m)^{m_i - 1} a$$

and here the first term corresponds to the $n_i - m$ rivals with the other valuation, and the second to the $m_i - 1$ rivals with the same valuation. This expression holds for $m > 0$.

The ratio of these two expressions is

$$1 = \frac{(1 - Q_m)a}{(1 - P_m)b} \quad (1)$$

so that the ratio of the choice probabilities of NOT pursuing the blonde is independent of m and furthermore $Q_m < P_m$.¹⁴ That is, the blonde is pursued more intensely by those who find her less attractive. Moreover, substituting this ratio back into either of the preceding formulae shows that the pursuit probability for either type is an increasing function of the number of individuals with high valuations.¹⁵ This part at least accords with casual intuition in the sense that all strive more when some become more enthused. However, this result has again come about for an unusual reason. Increasing one man's enthusiasm means his rivals now strive more; but he has to strive less to keep them indifferent. As noted in the 3-player case, he who cares more strives less, but those who have not changed all strive more.

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¹⁴The ratio given in (1) holds for m between 1 and $n_i - 1$ and so the result that both P_m and Q_m are decreasing holds over this range. By inspection of the indifference conditions, we have $P_0 = Q_{n_i - 1}$ and for the reason that an individual with a low valuation must face the same probability the others pursue the blonde: for both $m = 0$ and $m = n_i - 1$, the other individuals all behave identically. Likewise, $P_1 = Q_n$ for an analogous reason. It remains to show that $P_1 > P_0$, implying that both P and Q are monotonically increasing in m . This relation follows from noting that the first individual whose enthusiasm is raised to \bar{a} must cause the symmetric probability of the others to rise to keep him indifferent.

¹⁵Substitution gives: $P_m = 1 - \frac{b}{a} \left(\frac{a}{b}\right)^{m_i - 1} \frac{1}{(1 - P_m)^{n_i - 1}}$, which is increasing in m .

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