AN EXPLANATION OF ANOMALOUS BEHAVIOR IN BINARY-CHOICE GAMES: ENTRY, VOTING, PUBLIC GOODS, AND THE VOLUNTEERS' DILEMMA

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Abstract. This paper characterizes behavior with "noisy" decision making for a general class of *N*-person, binary-choice games. Applications include: participation games, voting, market entry, binary step-level public goods games, the volunteer's dilemma, etc. A simple graphical device is used to derive comparative statics and other theoretical properties of a "quantal response" equilibrium, and the resulting predictions are compared with Nash equilibria that arise in the limiting case of no noise. Many anomalous data patterns in laboratory experiments based on these games can be explained in this manner.

Keywords: participation games, entry, voting, step-level public goods games, volunteers' dilemma, quantal response equilibrium.

I. Introduction

Many games have a simple binary-choice structure, with payoffs for each decision depending on others' decisions. For example, entry games often have the property that entrants' profits depend on the number of entrants, e.g. the awarding of a prize randomly to one of the entrants. Similarly, the decision of whether to go to a particular restaurant or bar may depend on how likely it is that too many others also decide to go. Entry decisions in this context are known as the "El Farol" dilemma, named after a popular bar in Santa Fe (Morgan, Dylan, Bell, and Sethares, 1999). In other examples, the payoffs for each decision may be contingent on getting a minimal number of decisions of a certain type, e.g. a minimal number of contributors to a step-level public good, or a majority of legislators voting in favor of a legislative pay raise

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that they all want but prefer not to support if it will pass otherwise (Ordeshook, 1986, Chapter 3). Sometimes the minimal number of participants needed is only one, as in the "volunteer's dilemma," where all players are better off if at least one of them incurs a cost from vetoing an option, attempting a dangerous rescue, or volunteering to perform a task that benefits them all.

There have been a number of laboratory experiments involving binary choice participation games. Kahneman (1988) first reported an experiment in which the number of entrants was approximately equal to the market "capacity" parameter that determined whether or not entry was profitable. He remarked: "To a psychologist, it looks like magic." Subsequent experiments have been based on similar models, and the general finding is that players are able to coordinate entry decisions in a manner that roughly equates expected profits for entry to the opportunity cost (Ochs, 1990; Sundali, Rapoport, and Seale, 1995). This successful coordination has been explained by models of adaptation and learning (Meyer, Van Huyck, Battalio, and Saving, 1992; Erev and Rapoport, 1998). However, the "magic" of efficient entry coordination has been called into question by recent experimental results. For example, Fischbacher and Thöni (1999) conducted an experiment in which a monetary prize is awarded to a randomly selected entrant, so the expected prize amount is a decreasing function of the number of entrants. Over-entry was observed, and it was more severe for larger numbers of potential entrants. This over-entry pattern is somewhat intuitive but is not predicted in a mixed-strategy Nash equilibrium where the number of actual entrants reduces expected profits to a constant, independent of the number of potential entrants. In contrast, Camerer and Lovallo (1999) report under-entry and positive net payoffs in the baseline treatment of a market entry game. In addition, they find over-entry in a treatment where post-entry payoffs depend on a skill-based competition. Rapoport, Seale, and Ordonez (1998) review some of these studies and conclude that over-entry is more likely when the Nash probability of entry is low and under-entry when the Nash entry probability is high.

Another participation game that has received attention from experimenters is the "volunteer's dilemma," in which all players receive a benefit B if at least one of them incurs a cost, $C < B$, of "volunteering" to perform a task, like attempting a dangerous rescue, or issuing a politically risky veto. In a mixed-strategy Nash equilibrium, an increase in the number of potential volunteers will reduce the probability that any one person volunteers, which is intuitive, and will decrease the probability that *at least* one person volunteers, which is unintuitive.

Experimental data support the intuitive prediction but not the unintuitive one (Franzen, 1995). Similarly, laboratory results for binary coordination and public goods games support some theoretical Nash predictions, but also generate data patterns that are intuitive but not explained by standard game theory, as discussed below.

The objective of this paper is to explore the common structural elements of a wide class of binary-choice games, and to provide a unified theoretical perspective on seemingly contradictory results, like the positive relationship between over-entry (or the probability of getting a volunteer) with the number of potential entrants (or volunteers). The analysis is based on the incorporation of "noisy" behavior into models of equilibrium and adaptive adjustment (Palfrey and Rosenthal, 1985, 1988; McKelvey and Palfrey, 1995; Goeree and Holt, 1999). Section II presents the model of a general class of N-person binary choice games. Market entry games and the volunteer's dilemma are considered in Sections III and IV respectively, and the resulting participation probabilities are compared with those of a Nash equilibrium. This comparison allows an analysis of anomalies like excess participation relative to the Nash benchmark. Models with positive externalities, e.g. step-level public goods games, are considered in section V, where the object is to explain data patterns that are not predicted by Nash. Section VI applies this approach to the analysis of a voting participation game, where voters are of two types and must decide whether or not to incur the cost of voting. The focus is on explaining the effects of the outcome rule (majority or proportional) on the participation probabilities. The final section concludes.

II. To Participate or Not?

A symmetric N-person participation game is characterized by two decisions, which we will call participate and exit. The payoff from participation is a function of the total number, n, who decide to participate, which is denoted by $\pi(n)$, defined for $n \le N$. In a market entry game, for example, the payoff for all entrants may be a decreasing function of the number, n, who enter. The expected payoff for the exit decision will be denoted by $c(n)$, which is typically non-decreasing in n (the number of players that enter). In many applications, $c(n)$ is simply a constant that can be thought of as the opportunity cost of participation, but we keep the more general notation to include examples where a higher number of participants has external benefits to all, including those who exit (e.g. step-level public goods games).

A strategy in this game is a participation probability, $p \in [0, 1]$. In order to characterize a symmetric equilibrium, consider one player's decision when all others participate with probability p. Since a player's own payoff is a function of the number who actually participate, the *expected* payoff for participation is a function of the number of other players, $N - 1$, and the probability p that any one of them will participate. Assuming independence, the distribution of the number of other participants is binomial with parameters $N - 1$ and p. This distribution, together with the underlying $\pi(n)$ function, can be used to calculate the expected participation payoff, which will be denoted by $\pi^e(p, N\text{-}1)$. More precisely, $\pi^e(p, N\text{-}1)$ is defined to be the expected payoff if a player participates (with probability 1) when all $N - 1$ others participate with probability p. Similarly, $c^e(p, N\text{-}1)$ is the expected payoff from exit when the $N - 1$ others participate with probability p.

Equilibrium

In a Nash equilibrium, players choose the participation decision that yields the highest expected payoff, or randomize in the case of indifference. Our goal is the explanation of "anomalous" data from laboratory experiments, so it is convenient to model a type of noisy behavior that includes the rational-choice Nash predictions as a limit case. One way to relax the assumption of noise-free, perfectly rational behavior is to specify a utility function with a stochastic component. Thus the expected payoff for participation, π^e, and the expected payoff for exit, c^e, are each augmented by adding a stochastic term $\mu\, \varepsilon_i$, where $\mu > 0$ is an "error" parameter and the ε_i represent identically and independently distributed realizations of a random variable for decision i = 1 (participate) or 2 (exit). The utility of participation is greater if $\pi^e + \mu\, \varepsilon_1 > c^e + \mu\, \varepsilon_2$, so that when $\mu = 0$ the decision with the highest expected payoff is selected, but higher values of μ imply more noise relative to payoff maximization. This noise can be due to either errors (e.g. distractions, perception biases, or miscalculations that lead to non-optimal decisions), or to unobserved utility shocks that make rational behavior look noisy to an outside observer. Regardless of the source, the result is that choice is stochastic, and the distribution of the random variable determines the form of the choice probabilities.[1] The participation decision is selected if $\pi^e + \mu\varepsilon_1 > c^e + \mu\varepsilon_2$, or equivalently, if $\varepsilon_2 - \varepsilon_1 < (\pi^e - c^e)/\mu$, which occurs with probability:

$$p = F\left[\frac{\pi^e(p, N-1) - c^e(p, N-1)}{\mu}\right], \tag{1}$$

where F is the distribution function of the difference $\varepsilon_1 - \varepsilon_2$. Since the two random errors are identically distributed, the distribution of their difference will be "symmetric" around 0, so $F(0) = 1/2$.[2] The error parameter, μ, determines the responsiveness of participation probabilities to expected payoffs. Perfectly random behavior (i.e. $p = 1/2$) results as $\mu \to \infty$, since the argument of the $F(\cdot)$ function on the right side of (1) goes to zero and $F(0) = 1/2$ as noted above. Perfect rationality results in the limit as $\mu \to 0$, since the choice probability converges to 0 or 1, depending on whether the expected participation payoff is less than or greater than the expected exit payoff.

Equation (1) expresses the participation probability as a "noisy best response" to the expected payoff difference. This equation characterizes a quantal response equilibrium (McKelvey and Palfrey, 1995) if the participation probability p in the expected payoff expressions on the right is equal to the choice probability that emerges on the left.[3] Without further parametric assumptions, there is no closed-form solution for the equilibrium participation probability, but a simple graphical device can be used to derive theoretical properties and characterize factors that might cause systematic deviations from Nash predictions. To this end, apply the inverse of the F function to both sides of (1) to obtain: $\mu F^{-1}(p) = \pi^e(p, N\text{-}1) - c^e(p, N\text{-}1)$. The determination of the equilibrium participation probability is illustrated in Figure 1. As p

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[1] For instance, a normal distribution yields the probit model, while a double exponential distribution gives rise to the logit model, in which case the choice probabilities are proportional exponential functions of expected payoffs.

[2] More formally, $\Pr(\varepsilon_1 \le \varepsilon_2) = 1/2$, so $\Pr(\varepsilon_1 - \varepsilon_2 \le 0) = F(0) = 1/2$.

[3] The quantal response equilibrium, developed by political scientists (McKelvey and Palfrey, 1995), has been applied to the study of international conflict by Signorino (1999). A general introduction to the usefulness of the quantal response approach in the analysis of political data can be found in Morton (1999).

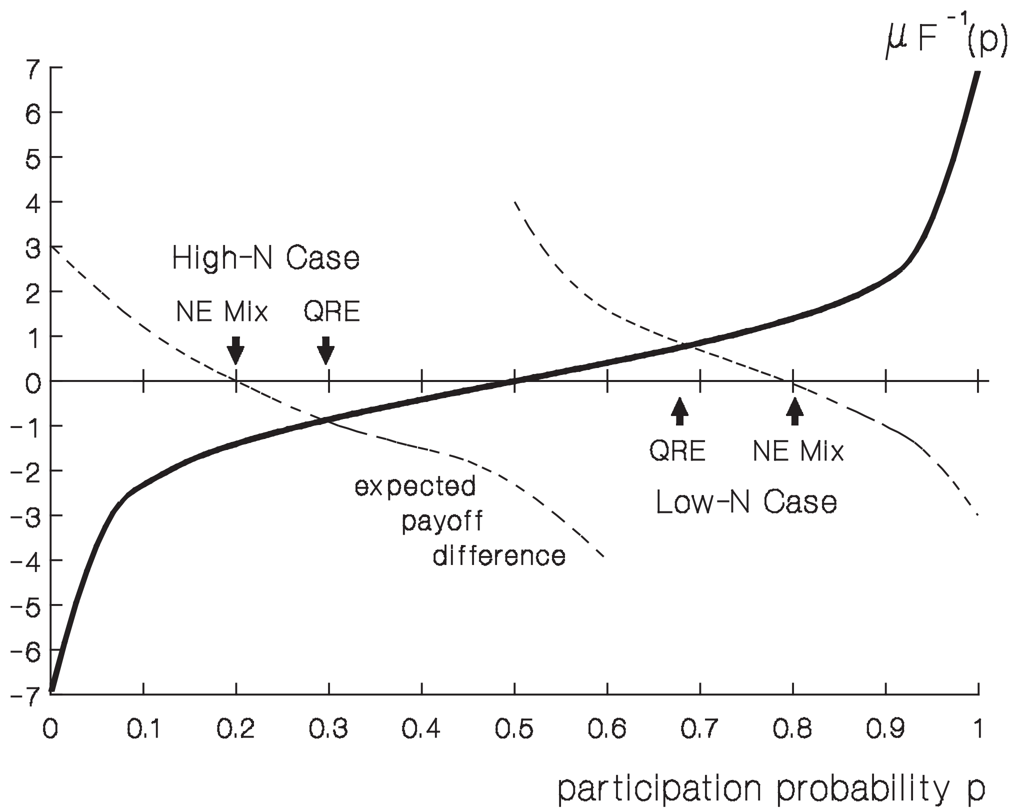

Figure 1. Quantal Response and Nash Mixed Participation Probabilities for Low-N and High-N Cases

goes from 0 to 1, $\mu F^{-1}(p)$ increases from $-\infty$ to $+\infty$, as shown by the curved dark line with a positive slope in the figure.[4] Since the expected payoff difference is continuous in p, it has to cross the $\mu F^{-1}(p)$ line at least once, which ensures existence of a symmetric equilibrium.[5] If the expected payoff difference $\pi^e(p, N\text{-}1)$ - $c^e(p, N\text{-}1)$ is decreasing in p, the intersection will be unique. This case is illustrated in Figure 1, where the negatively sloped dashed line on the left

4 To see this, note that an expected payoff difference of $-\infty$ on the vertical axis will cause the participation probability to be 0, and an expected payoff difference of $+\infty$ will cause the participation probability to be 1. This is why the dark "inverse distribution" line starts at $-\infty$ on the left side of Figure 1 and goes to $+\infty$ on the right.

5 The existence of quantal response equilibria for normal-form games with a finite number of strategies is proved in McKelvey and Palfrey (1995) and for normal-form games with a continuous strategy space in Anderson, Goeree, and Holt (1999).

side of the figure represents the expected payoff difference. This line intersects the "inverse distribution" line at the equilibrium probability labeled QRE on the left. Also, notice that the point where the dashed expected payoff difference crosses the zero-payoff line constitutes a mixed-strategy Nash equilibrium, since players are only willing to randomize if expected payoffs for the two decisions are equal. This crossing point is labeled "NE Mix" in the figure.

Next consider the intuition for why the quantal response equilibrium is not typically at the intersection of the expected-payoff-difference line and the zero-payoff horizontal line in the figure. With equal expected payoffs for participation and exit, the person is indifferent and since $F(0) = 1/2$, the stochastic best response to such indifference is to participate with probability 1/2. In the figure, this result can be seen by starting where expected payoffs are equal at the NE Mix point on the left and moving horizontally to the right, crossing the dark line at $p = 1/2$. This is not a quantal response equilibrium since the p we started with (at the NE Mix) is not the stochastic best response to itself. To find a stochastic best response to any given entry probability p on the horizontal axis, first move in the vertical direction to find the associated expected payoff difference, and then move horizontally (left or right) to the dark line, which determines the stochastic best response to that expected payoff difference. Equilibrium requires that the stochastic best response to the others' participation probability is that same probability, which occurs only at the intersection of the expected-payoff-difference and inverse distribution lines in the figure. To summarize, a symmetric quantal response probability is a *stochastic best response* to itself, whereas a symmetric Nash equilibrium probability is a *best response* to itself.[6]

As long as the expected payoff difference is decreasing in p, it is apparent from Figure 1 that any factor that increases the expected payoff difference line for all values of p will move the intersection with the dark inverse distribution line to the right, and hence raise the quantal response equilibrium probability. In an entry game, for example, the original $\pi(n)$ function would be decreasing if profits are decreasing in the number of competitors, and it is then

[6] At the "NE Mix" point in Figure 1, expected payoffs are equal and any probability is a best response, so the NE Mix probability is a best response to itself.

straightforward to show that $\pi^e(N\text{-}1, p)$ is a decreasing function of both arguments.[7] When the opportunity cost payoff from not entering is constant, it follows that the expected payoff difference $\pi^e(p, N\text{-}1)$ - $c^e(p, N\text{-}1)$ is decreasing in p and N, so a reduction in the number of potential entrants will shift the dashed line in the figure upward and raise the quantal response (QRE) probability, as represented by a comparison of the high-N case on the left with the low-N case on the right.

The effect of additional "noise" in this model is easily represented, since an increase in the error parameter μ makes the $\mu F^{-1}(p)$ line steeper, although it still passes through the zero-payoff line at the midpoint, p = 1/2, in Figure 1. This increase in noise, therefore, moves the quantal response equilibrium closer to 1/2, as would be expected. In contrast, as a reduction in μ makes the $\mu F^{-1}(p)$ line flatter, and in the limit it converges to the horizontal line at zero as the noise vanishes. In this case, the crossings for the QRE and mixed Nash equilibria match up, as would be expected.

Next, consider coordination-type games where participation can be interpreted as an individual decision of whether or not to help with a group production process that will only succeed if enough people help out. In such games, it does not pay to participate unless enough others do, so $\pi(n)$ will be less than $c(n)$ for low n and greater than $c(n)$ for high n. Thus the right side of (1) is increasing in the probability of participation. This property may result in multiple quantal response equilibria since there can be multiple intersections when both the expected-payoff-difference and the inverse distribution lines are increasing in p (see Figure 2). With multiple crossings, any factor that shifts the expected payoff difference line upward will move some intersection points to the left and others to the right. Thus the comparative statics effects are of opposite signs at adjacent equilibria, and we need to use an analysis of dynamic

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[7] Intuitively, holding N fixed, a higher probability of entering means that more people enter, which results in a lower expected payoff of entry. Similarly, holding p fixed, a higher number of potential entrants results in more entry. This can be made more precise as follows: suppose N is fixed and the entry probability is p_1. Let the number of entrants be determined by drawing a random number that is uniformly distributed on [0, 1] for each player. If the number is less than p_1 a player enters, otherwise the player stays out. When the probability of entering increases to $p_2 > p_1$, the number of entrants is at least the same as before for all possible realizations of the random variables, and greater for some realizations. (When a player's random variable is less than p_1 it is certainly less than p_2, leading to the same entry decision, and when it lies between p_1 and p_2, the player's decision changes from staying out to entering.) Likewise, when p is fixed, an increase in the potential number of entrants means that for all possible realizations of players' random draws, the number of entrants is the same or higher, which makes the expected payoff from entry be the same or lower.

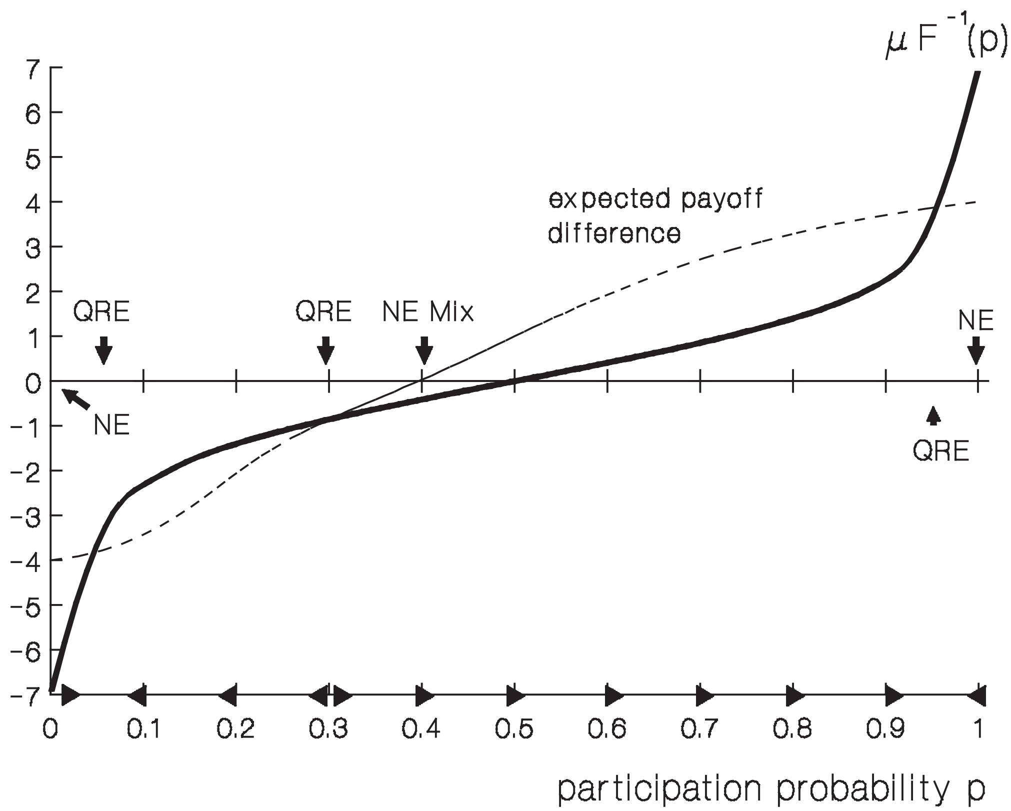

Figure 2. Quantal Response and Nash Mixed Participation Probabilities for a Game with Positive Externalities

adjustment to restrict consideration to equilibria that are stable (the Samuelsonian "correspondence principle"). A simple dynamic model can be based on the intuitive idea that the participation probability will increase over time when the "noisy best response" to a given p is higher than p. Thus $dp/dt > 0$ when $F((\pi^e(p, N\text{-}1) - c^e(p, N\text{-}1))/\mu) > p$, or equivalently, p would tend to increase when $\pi^e(p, N\text{-}1) - c^e(p, N\text{-}1) > \mu F^{-1}(p)$ and decrease otherwise. For example, start at $p = .6$ in Figure 2, which gives a positive expected payoff difference and a stochastic best response of almost .9, found by moving horizontally to the right. For this reason, a rightward arrow is present at $p = .6$ on the horizontal axis. The other directional arrows are found similarly, so there is an unstable QRE at about .3, with arrows pointing away. In this manner it can be seen that the quantal response equilibrium will be stable whenever the expected payoff difference line cuts the inverse distribution line from above.

Notice that any factor that raises the payoff from participation, and hence shifts the expected-payoff-difference line upward in Figure 2, will raise the QRE participation probability if the equilibrium is stable and not otherwise. To summarize:

Proposition 1. There is at least one symmetric quantal response equilibrium in a symmetric binary-choice participation game. The equilibrium is unique if the difference between the expected payoff of participating and exiting is decreasing in the probability of participation. In this case, any exogenous factor that increases the participation payoff or lowers the exit payoff will raise the equilibrium participation probability. The same comparative statics result holds when there are multiple equilibria and attention is restricted to stable equilibria.

It is useful to begin with a discussion of entry games since they are the simplest application. Moreover, the quantal response properties for these games also apply to the stable equilibria in more complex applications such as step-level public goods, volunteer's dilemma or voting. The reader who is primarily in one of these subsequent applications may wish to skip any of the later sections after reading as far as Proposition 2.

III. Entry Games: Under-Entry and Over-Entry Relative to Mixed-Nash Predictions

A widely studied example that fits the binary choice framework is an entry game, in which players choose between a "risky" entry decision with high potential payoffs but only if few others enter and a "secure" exit payoff. For example, entry may correspond to the purchase of a lottery ticket or the filing of an application for a limited number of public broadcast licenses. There are N potential entrants, and we assume that if all others enter with probability 1, the representative player would prefer to exit due to congestion, but if nobody else enters, then the player would prefer to enter: $\pi^e(1, N\text{-}1) < c^e(1, N\text{-}1)$ and $\pi^e(0, N\text{-}1) > c^e(0, N\text{-}1)$. Consider a simple three-person version of the "El Farol" problem mentioned in the introduction: each person's payoff from participation (going to the bar) is 1 unless both of the other people also happen to show up, in which case the congestion reduces the payoff to 0. The exit payoff for staying home is c, with $0 < c < 1$. When both others participate with probability p, the probability of congestion is p^2, so $\pi^e = 1 - p^2$, which is less than the exit payoff c when $p = 1$

and greater than the exit payoff when $p = 0$. In this example and in all other applications considered below, the expected payoff difference will be continuous and decreasing in p, so there is a unique p^* for which

$$\pi^e(p^*, N-1) = c^e(p^*, N-1). \tag{2}$$

(For instance, in the three-person "El Farol" problem $p^* = (1 - c)^{1/2}$.) Since (2) implies indifference, it characterizes the unique Nash equilibrium in mixed strategies. The line representing the net payoff for participation, $\pi^e(p, N\text{-}1) - c^e(p, N\text{-}1)$, is decreasing in p, as shown by the "expected payoff difference" line on the left side of Figure 1. As noted above, the crossing of this dashed line and the horizontal line at 0 represents the solution to (2), and is labeled "NE Mix" on the left side of the figure.

In order to compare the Nash and quantal response equilibria, note that the dashed lines representing the differences in expected payoffs are always negatively sloped in an entry game. First consider the high-N case in the left side, where the large number of potential entrants lowers the expected payoff associated with a given participation probability, and the resulting mixed equilibrium is less than 1/2. The intersection of the negatively sloped dashed line and the increasing inverse distribution line determines the quantal response participation probability, and this intersection will be to the right of the mixed Nash probability. The opposite occurs for the low-N case on the right side of the graph, where the low number of potential entrants results in a mixed equilibrium that is greater than 1/2. In this low-N case, the QRE probability is biased downward from the Nash probability. One way to understand both cases is to note that the effect of adding noise is to push the equilibrium towards 1/2.[8]

Finally, recall that the dashed lines in Figure 1 represent the expected payoff difference on the right side of (1). At the QRE probability on the left, net expected payoffs are negative

[8] In some games with strong strategic interactions, the "snowball" effects of small amounts of noise can push decisions away from the unique Nash equilibrium so strongly that they overshoot the midpoint of the strategy space, with most of the theoretical density at the opposite end of the set of feasible decisions from the Nash prediction. This is the case for some parameterizations of the "traveler's dilemma" (Capra, *et al*., 1999). This prediction, that the data will be clustered on the opposite side of the midpoint decision from the Nash equilibrium, is borne out by the experimental evidence.

and there is over-entry in this case of a high number of potential entrants. In contrast, the dashed line lies above the zero line at the QRE probability on the right side, for the low-N case. This negative relationship between the number of potential entrants and net returns from participation is consistent with the experimental results of Fischbacher and Thöni (1999) that were discussed in the introduction.[9] To summarize:

Proposition 2. In the quantal response equilibrium for the entry game, there is over-entry resulting in negative net expected payoffs when the mixed-strategy Nash equilibrium is less than 1/2. The reverse effect, under-entry, occurs when the mixed Nash equilibrium is greater than 1/2.

Meyer *et al*. (1992) report an experiment in which subjects choose to enter one of two markets. With a group size of six, profits are equalized with three in each market, so the equilibrium probability of entry is 1/2. An immediate corollary to Proposition 2 is that in this case QRE coincides with Nash and both predict an entry probability of 1/2. This prediction is borne out by their data: the average of the number of people that enter each market is never statistically different from three in the eleven baseline sessions that they report (see their Table 3), even when the game is repeated for as many as sixty periods (see their Table 5).[10]

Camerer and Lovallo (1999) provide support for the QRE under-entry prediction when the Nash probability of entry is greater than 1/2. In their experiment subjects decide whether or not to enter a market with a fixed capacity, c. The entrants were randomly ranked and the top c entrants divide \$50 according to their rank, while all other entrants lose \$10. The exit payoff is simply 0, and the equilibrium number of entrants is (close to) $c + 5$. The capacities and potential numbers of entrants were chosen such that the Nash entry probability was greater than

[9] In their game, a prize worth V is awarded randomly to one of the n players who purchase a lottery ticket at cost c, so $\pi(n) = V/n - c$. From this it can be shown that the expected payoff difference is decreasing in p and N.

[10] Meyer *et al.* (1992) also report some evidence that does not square with either the symmetric Nash or the quantal response predictions of our model. In particular, the frequency with which subjects switch markets is less than the predicted frequency (50 percent). We conjecture that this "inertia" could be explained by an asymmetric quantal response equilibria in which some people tend to enter with higher probability than others.

or equal to 1/2 in all treatments.[11] Under-entry occurred in all of the eight sessions in their baseline treatment, which resulted in positive expected payoffs for entry (see their Table 4). The net expected payoff of entry across sessions and periods was $15, which translates into under-entry of 1-2 subjects per round.[12]

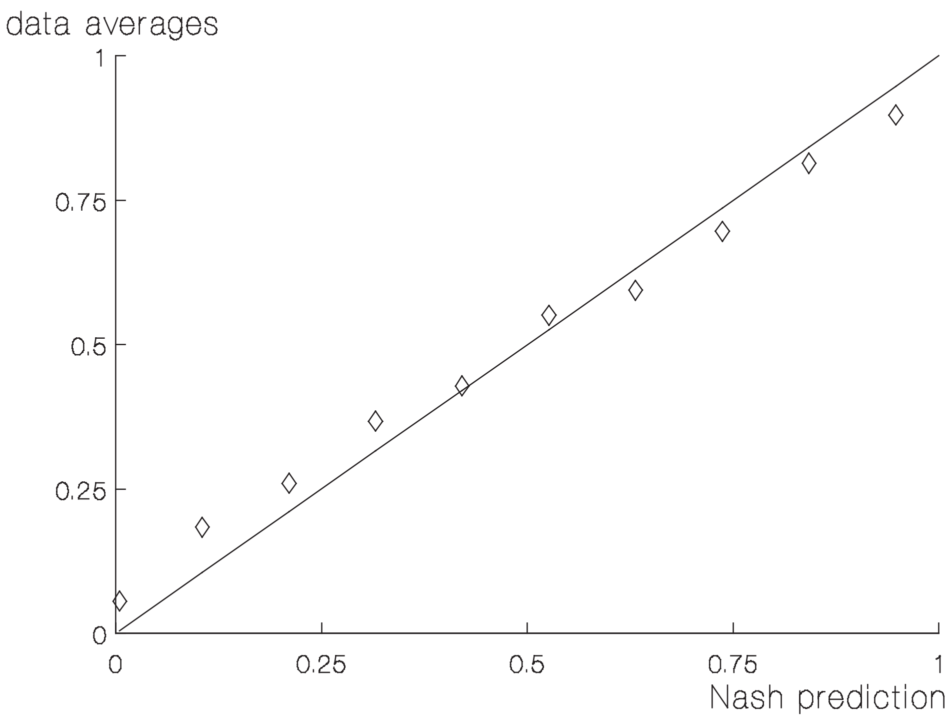

Figure 3. Nash Predictions (solid line) and Observed Entry Probabilities (diamonds)
Source: Sundali, Rapoport, and Seale (1995)

The strongest evidence for the quantal response predictions in Proposition 2 can be found

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[11] Capacities were 2, 4, 6, or 8, yielding equilibrium numbers of entrants (c + 5) of 7, 9, 11, or 13 respectively, which are always greater than or equal to half the group size (14-16).

[12] Camerer and Lovallo (1992) also report a second treatment in which subjects are told beforehand that their performance on sports or current events trivia will determine their payoff. This creates a selection bias, since people that participate in the experiment are more likely to think they will rank high when they enter (i.e. they are "overconfident"), neglecting the fact that other participants think the same ("reference group neglect"). Camerer and Lovallo propose overconfidence and reference group neglect as a possible explanation of the over-entry that occurs in this second treatment. This explanation is quite plausible, in that it is analogous to the failure to perceive a selection bias that causes winners in a common-value auction to be the ones who overestimated its value. Note that overconfidence cannot be the whole story, however, since this bias does not explain under-entry in their baseline treatment.

in Sundali, Rapoport and Seale (1995). In their experiments, subjects received a fixed payoff of 1 for exit and an entry payoff that is increasing in market capacity, c, and decreasing in the number of entrants: $\pi(n) = 1 + 2\,(c - n)$. Thus entry in excess of capacity reduces payoffs below 1, the payoff for exit. It is straightforward to derive the mixed Nash entry probability: $p^* = (c\text{-}1)/(N\text{-}1)$, which is approximately equal to the ratio of capacity to the number of potential entrants.[13] The capacities for the various treatments were: $c = 1, 3, ..., 19$, and with groups of $N = 20$ subjects, the Nash equilibrium probability ranged from $p^* = 0$ to $p^* = 18/19$. Figure 3 shows the entry decisions averaged over all subjects, with the Nash predictions shown as the 45 degree line. Since each subject participated in ten "runs" and there were three groups of twenty subjects, a data point in the figure is the average of 10*3*20 = 600 entry decisions. Note that the entry frequency is generally higher than predicted by Nash for $p^* < 1/2$ and lower than predicted for $p^* > 1/2$, in line with the quantal response predictions.

To summarize, the quantal response analysis explains the "magical" conformity to Nash entry predictions (e.g. Meyer et al., 1992), the under-entry in the Camerer and Lovallo (1999) baseline, the over-entry with many potential entrants observed by Fischbacher and Thöni (1999), and the systematic pattern of deviations from Nash predictions reported by Sundali, Rapoport and Seale (1995). This general approach can be adapted to evaluate behavior in other contexts where payoffs for one decision are diminished as a result of congestion effects, as the next section illustrates.[14]

IV. The Volunteer's Dilemma

There are many situations in which a player's decision to participate benefits others.

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[13] To derive this symmetric mixed equilibrium, note that the expected number of *other* people who enter is $(N\text{-}1)p$, so if a person enters, the expected total number of entrants is $1 + (N\text{-}1)p$. Then $\pi(n)$ can be used to calculate the expected payoff for entering: $\pi^e(p, N\text{-}1) = 1 + 2\,(c - 1) - (N\text{-}1)\,2\,p$ and the Nash equilibrium probability of entering follows by equating this expected payoff to the exit payoff of 1, which yields the result in the text.

[14] The analysis presented here does not apply directly to the experiments reported in Ochs (1990), since his experiments involved more than two market locations, each with different "capacities" that determined the number of entrants which could be accommodated profitably. Nevertheless the data patterns with random regrouping ("high turnover") are suggestive of the quantal response results derived here. The locations with the most capacity (and high probabilities) consistently have a lower frequency of entry than required for a mixed-strategy Nash equilibrium, whereas the opposite tendency was observed for locations with the capacity to accommodate only one entrant profitably.

Contributions to a public good, for instance, have positive returns for everyone involved, and these returns are increasing in the number of contributions. In some contexts, the optimal number of participants is one, e.g. when a volunteer is needed to perform a task like issuing a politically risky veto or sanctioning a group member that violated a norm. The dilemma in these situations is that volunteering is costly and players have an incentive to free ride on others' benevolence.

In the *volunteer's dilemma* game studied here (Diekmann, 1986), all players receive a benefit B if at least one of them incurs a cost, $C < B$. In this case, the expected payoff of participation, or "volunteering" is simply a constant, $B - C$. The expected payoff from "exiting" follows from the observation that when the $N - 1$ others volunteer with probability p, there is a $(1 - p)^{N-1}$ chance that no one volunteers, so $c^e(p, N\text{-}1) = B\,(1 - (1 - p)^{N-1})$. Notice that the volunteer's dilemma game satisfies the assumptions underlying Figure 1, i.e. the difference between the expected payoffs of participating and exiting is decreasing in p. The Nash probability of volunteering follows by equating these expected payoffs (as per (2)) to obtain:

$$p^* = 1 - \left(\frac{C}{B}\right)^{\frac{1}{N-1}} . \tag{3}$$

This probability of volunteering has the intuitive properties that it is increasing in the benefit, B, decreasing in the cost, C, and decreasing in the number of potential volunteers, N. However, the probability of getting no volunteers is $(1\text{-}p^*)^N$. By (3) the probability of getting no volunteers in a Nash equilibrium is $(C/B)^{N/(N-1)}$, which is *increasing* in N, with $\lim_{N\to\infty} P(\textit{No Volunteer}) = C/B > 0$. Unlike the intuitive comparative statics properties mentioned before, this prediction is not supported by experimental data. Table I reports experimental results for a one-shot volunteer's dilemma game with $B = 100$ and $C = 50$ (Franzen, 1995). Notice that the probability that any person volunteers is generally declining with N, as predicted by Nash.[15] The probability that no one volunteers, however, is *decreasing* in N and converges to 0 instead of $C/B = 1/2$.

Next, consider the quantal response equilibrium for the volunteer's dilemma. Since the

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[15] Franzen (1995) reports that the group-size effect is significant at the five percent level using a chi-square test with seven degrees of freedom.

Table I. Frequencies of Individual Volunteer Decisions (*p*) and of "No Volunteer" Outcomes
Source: Franzen (1995)

N	*p*	*P*(*No Volunteer*)
2	.65	.12
3	.58	.07
5	.43	.06
7	.25	.13
9	.35	.02
21	.30	.00
51	.20	.00
101	.35	.00

difference between the expected payoffs of volunteering and exiting is decreasing in the probability of volunteering, Proposition 1 implies that the QRE probability of volunteering is unique, decreasing in *N* and *C*, and increasing in *B*. Interestingly, the introduction of (enough) endogenous noise reverses the unintuitive Nash prediction that the probability of "no volunteer" increases with *N*.

Proposition 3. In the quantal response equilibrium for the volunteer's dilemma game, the probability that no one will volunteer is decreasing in the number of potential volunteers for a sufficiently high error rate, μ. Furthermore, $\lim_{N\to\infty} P(\text{No Volunteer}) = 0$ *for any* $\mu > 0$.

The proof of Proposition 3 is contained in the Appendix. The intuition is that, in the presence of noise, the addition of potential volunteers only results in a small reduction in the probability of volunteering, and the net effect is that the chance that *someone* volunteers will rise.[16]

V. Games with Multiple Equilibria: Step-Level Public Goods Games

In some participation games the expected payoff function for participating is not decreasing in p. For example, in a production game with shared output that is increasing in the number of participants, the expected payoff function will typically be increasing, which permits multiple crossings. This is intuitive, since there may exist both low-participation equilibria and high-participation equilibria in such coordination games.[17] Another example of a game with multiple equilibria is a step-level public goods game, where each of N players must decide whether or not to "contribute" at cost c. If the total number of contributions meets or exceeds some threshold n^*, then the public good is provided and all players receive a fixed return, V, whether nor not they contributed. Here we assume that the contribution is like an effort that is lost if the threshold is not met, so there is "no rebate."

In the standard linear public goods games without a step, observed contributions in experiments are positively related to the marginal effect of a contribution on the value of the public good, known as the "marginal per capita return" (MPCR). Anderson, Goeree, and Holt (1998) have shown that a logit quantal response analysis predicts this widely observed MPCR effect. This raises the question whether there is a similar measure or index that would predict the level of contributions in step-level public goods games. One would intuitively expect that contributions are positively related to the total (social) value of the public good (NV) and negatively related to the minimum total cost of providing it (n^*c). Croson (1999) has proposed using the ratio of social value to cost, which she calls the "step return:" $\mathrm{SR} = NV/n^*c$. Based on a meta-analysis of several step-level public goods games, she concludes "... subjects respond to

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[16] In the extreme case when $\mu \to \infty$, players volunteer with probability one-half, irrespective of the number of potential volunteers, and the chance that no one volunteers falls exponentially, since the probability of no volunteer is 2^{-N}.

[17] Stability arguments can often be used to rule out the middle equilibrium if there are three crossings as in Figure 2. For low μ, this middle equilibrium is usually close to a mixed Nash equilibrium with "perverse" comparative statics properties. The high- and low-participation equilibria then correspond to low-effort and high-effort pure-strategy Nash equilibria that often arise in coordination games.

the step return just as they correspond to the marginal per capita return (MPCR) in linear public goods games: higher step returns lead to more contributions."

First we consider whether there is a clear theoretical basis for expecting contributions to be positively related to step return measures. A contribution in this game pays off only when it is pivotal, i.e. when exactly $n^* - 1$ others contribute, which happens with probability

$$\binom{N-1}{n^*-1} p^{n^*-1} (1-p)^{N-n^*}, \tag{4}$$

where, as before, p denotes the probability that *others* participate. The difference between the expected payoff of contributing or not contributing is therefore:

$$\pi^e(p, N-1) - c^e(p, N-1) = V \binom{N-1}{n^*-1} p^{n^*-1} (1-p)^{N-n^*} - c. \tag{5}$$

The right side is a single-peaked function of p, and equating its derivative to 0 yields a unique maximum at $p = (n^* - 1)/(N - 1)$. Figure 4, drawn for $V = 6$, $c = 1$, and $N = 10$, shows these "hill shaped" expected payoff difference lines for three values of the threshold: $n^* = 3, 5, 8$. (Please ignore the "n^*={5,8}" line, which pertains to a multiple step case to be considered later.) In each case there are two Nash equilibria in mixed strategies, determined by the crossings of the dashed line with the horizontal line at zero. The inverse distribution line is plotted for the case of a logistic distribution, i.e. $F(x) = 1/(1+\exp(-x))$, and $\mu = 1$. As before, the intersection of the inverse distribution line with the dashed lines determines the quantal response equilibrium, which is unique for all three values of the threshold.

Recall that the step return is NV/n^*c, which is increasing in N and V, and decreasing in n^* and c. In order to evaluate these properties in the context of the quantal response predictions, note that the bell shaped nature of the expected payoff differences imply that there may be multiple quantal response equilibria. It follows from Proposition 1, however, that any factor that shifts the expected payoff difference line upwards will raise the equilibrium probability in a stable equilibrium. Since the difference in (5) is increasing in V and decreasing in c, we conclude that the equilibrium contribution probability will be increasing in V and decreasing in

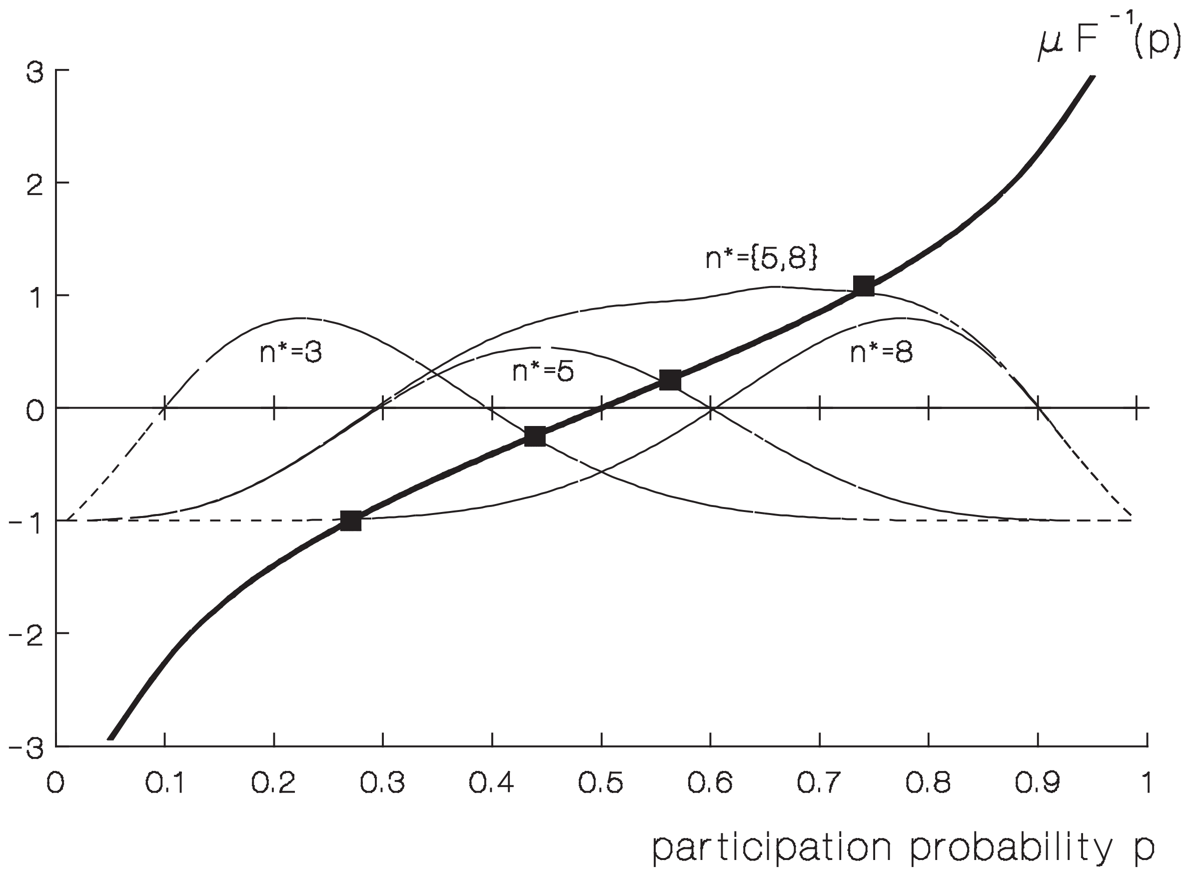

Figure 4. Expected Payoffs Differences and the Inverse Distribution Line for Different Thresholds in Step-Level Public Goods Games

c, just as indicated by the step-return effect. Next consider the effects of the numbers variables, *N* and *n**, beginning with a somewhat informal graphical analysis (precise results are presented in Proposition 4 below). Recall that the maximum of the expected payoff difference "hill" is at a probability of $(n^* - 1)/(N - 1)$, so an increase in *N* tends to shift this function to the left. Notice that a leftward shift in the dashed line labeled $n^* = 3$ in Figure 4 will lower the equilibrium probability, but a slight leftward shift in the line labeled $n^* = 8$ will move the intersection point up along the dark line, and hence will raise the quantal response equilibrium probability. Thus an increase in *N* can result in a *decrease* in the equilibrium probability when the threshold is low and an increase when the threshold is high.[18] The effect of changes in the threshold, n^*, are similar. Note that the quantal response probability of contributing does not decrease monotonically with the threshold: when n^* increases from 3 to 5, the equilibrium probability *increases* from .43 to .56, and then drops to .27 when $n^* = 8$. The intuition is that when the threshold rises and it is still likely that the public good will be provided, individual contributions will rise, but contributions drop dramatically when too many contributions are needed for provision. To summarize, in a quantal response equilibrium, a higher step return ratio leads to more contributions when it is due to a higher total value of the public good or a lower cost of provision, but not necessarily when it is due to an increase in the number of potential contributors or to a lower threshold. Thus the (admittedly theoretical) analysis here yields only qualified support for the use of the step return as a rough measure of the propensity to contribute in a binary step-level public goods game.[19]

Of course, even when individual contributions rise in response to the increased threshold, the probability that the public good is actually provided may decrease, since more people are needed to meet the threshold. For the numeric example represented in Figure 4, the probability of success drops from .83 to .62 to practically 0 when n^* is increased from 3 to 5 to 8. Van de Kragt, Orbell, and Dawes (1983) report an experiment that implemented a step-level public goods game with binary contributions and found that increasing the number of contributors needed for success reduced the incidence of successful provision. The next proposition shows that these findings are in line with QRE predictions when there is sufficient noise.

Proposition 4. For a high enough error rate, μ, the quantal response equilibrium for the step-level public goods game is unique and predicts that individual contributions first rise and then fall with the threshold, n^*, while the probability of successful provision always decreases with n^*.

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[18] See, for instance, Offerman, Schram, and Sonnemans (1997) for experimental evidence on some of these comparative static results.

[19] Nor are the numbers effects in a Nash equilibrium necessarily consistent with the qualitative properties of the step return ratio. This is because an increase in the threshold n^* shifts the maximum of the expected-payoff-difference line to the right in Figure 4, which is likely to shift the right-most (stable) mixed Nash equilibrium to the right. Thus a rise in n^*, which lowers the step return, can raise the mixed Nash contribution probability.

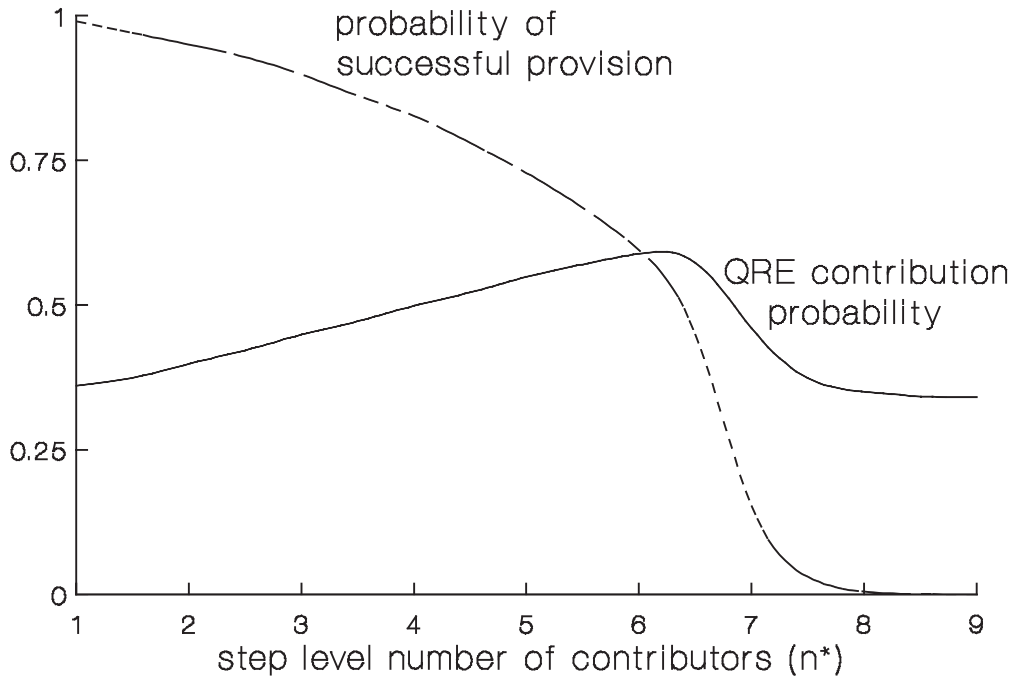

Figure 5. QRE Probabilities of Individual Contribution and Successful Group Provision of a Step Level Public Good, as a Function of the Provision Point

This proposition, which is proved in the Appendix, is illustrated in Figure 5, which was drawn for the case where $V = 6$, $c = 1$, $N = 10$, $\mu = 1.5$, and with the provision point, n^*, varying from 1 to 9. A movement to the right in the figure corresponds to an increase in the number of contributors needed for successful provision, which reduces the probability of success in a quantal response equilibrium. As the step level is increased, individual contributions first increase to meet the challenge, and then fall as the threshold becomes more unattainable. Interestingly, Palfrey and Rosenthal (1988) derive this result in an equivalent manner by introducing random, individual-specific "joy of giving" (or "warm-glow" altruism) shocks that are added to a person's payoff for a contribution decision.[20] Proposition 4 extends their analysis by showing that the

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[20] The Nash equilibrium for the resulting game of incomplete information is mathematically equivalent to a quantal response equilibrium. Palfrey and Rosenthal (1988) prove that individual contributions first rise and then fall with the threshold (see their Table 2). They also show that the number of potential contributors, N, has the reverse effect: individual contributions first fall and then rise with increases in N.

probability of successful provision is decreasing in n^*.

Finally, it is interesting to see how contribution behavior changes as multiple steps, or thresholds are introduced. Suppose, for instance, that in addition to the $n^* = 5$ threshold, there is another threshold at $n^* = 8$: with five or more contributions, everyone receives a return of 1 from the public good, while with eight contributions or more, the return is 2. This multiple-step case can be analyzed in the same manner as before. Now there are two points at which one's contribution can be pivotal, and the expected payoff is the sum of the two effects. In terms of Figure 4, the expected payoff lines for $n^* = 5$ and $n^* = 8$ get "summed," as indicated by the $n^* = \{5,8\}$ line in Figure 4 (the cost of contributing only enters once, which is why the endpoints of the dotted line are still at -1). The introduction of the extra threshold at $n^* = 8$, which by itself leads to a low contribution probability, dramatically increases contributions: the QRE contribution probability is .73 and the probability that at least five people contribute is as high as .97. An immediate extension of this analysis is that adding more steps, without reducing the payoff increment at any of the existing steps, will increase quantal response contribution probabilities in a binary public goods game.

VI. Voting Participation Games

Another binary choice of considerable interest is the decision of whether or not to vote in a small-group situation where voting is costly and a single vote has a non-negligible effect on the final outcome, e.g. the decision of whether to attend a faculty meeting on a busy day. The analysis is similar to that of a step-level public goods game, since the threshold contribution, n^*, corresponds to the number of votes needed to pass a favored bill. In a real voting contest, however, the vote total required to win is endogenously determined by the number of people voting against the bill. If there are two types of voters, those who favor a bill and those who oppose, then the equilibrium will be characterized by a participation probability for each type. Here we restrict attention to a symmetric model with equal numbers of voters of each type, equal costs of voting, c, and symmetric valuations: V if the preferred outcome receives more votes and 0 otherwise. Ties in this majority rule game are decided by the flip of a coin. Note that the public goods incentives to free ride are still present in this game, since voters benefit when their side wins, regardless of whether or not they incurred the cost of voting.

The analysis of the majority voting game is a straightforward application of the approach taken in the previous sections. The gain from a favorable outcome is V, so the expected payoff difference is V times the probability that one's vote affects the outcome minus the cost of voting. (Obviously, the net cost of voting could be small or even negative if voting is psychologically rewarding or if there are social pressures to vote, e.g. to attend a faculty meeting.) Since a tie is decided by the flip of a coin, the probability that a vote is pivotal is one-half times the probability that it creates or breaks a tie. In a symmetric equilibrium with common participation probability, p, it is straightforward to use the binomial formulas to calculate these probabilities, and the expected payoff difference for voting is then $V/2$ times this "influence probability" minus the cost of voting.[21]

Figure 6 shows the expected payoff difference as a function of the common participation probability, which is labeled "majority rule." The parameters that were used to construct this figure are taken from Schram and Sonnemans (1996b) who conducted an experiment based on this game form with $N = 6$, $V = 2.5$, and $c = 1$. The "U" shape of the expected payoff difference reflects the fact that a costly vote is wasted when the preferred outcome is already winning, or when it cannot win even with an extra vote. Indeed, the expected value of a vote is highest when either no one else or everyone else votes, since a vote is then guaranteed to be pivotal by breaking or creating a tie. In contrast, when all others vote with probability 1/2, one extra vote is likely to be superfluous or not enough and its expected value is therefore small. As in previous sections, the mixed Nash prediction is determined by where the expected payoff

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[21] Suppose there are two groups of equal size, N, and consider a player in group 1. The player's vote is pivotal only when the number of voters in group 1 is equal to $n_2 - 1$ or n_2, where n_2 denotes the number of voters in group 2, which happens with probability:

$$\sum_{n_2=1}^{N} \binom{N}{n_2}\binom{N-1}{n_2-1} p^{2n_2-1}(1-p)^{2N-2n_2} + \sum_{n_2=0}^{N-1} \binom{N}{n_2}\binom{N-1}{n_2} p^{2n_2}(1-p)^{2N-2n_2-1},$$

where, as before, p denotes that probability with which all others (in both groups) vote. The first term represents the probability that a tie is created and the second term is the probability that a tie is broken. A player's expected payoff is $V/2$ times this "influence probability" minus c, the cost of voting.

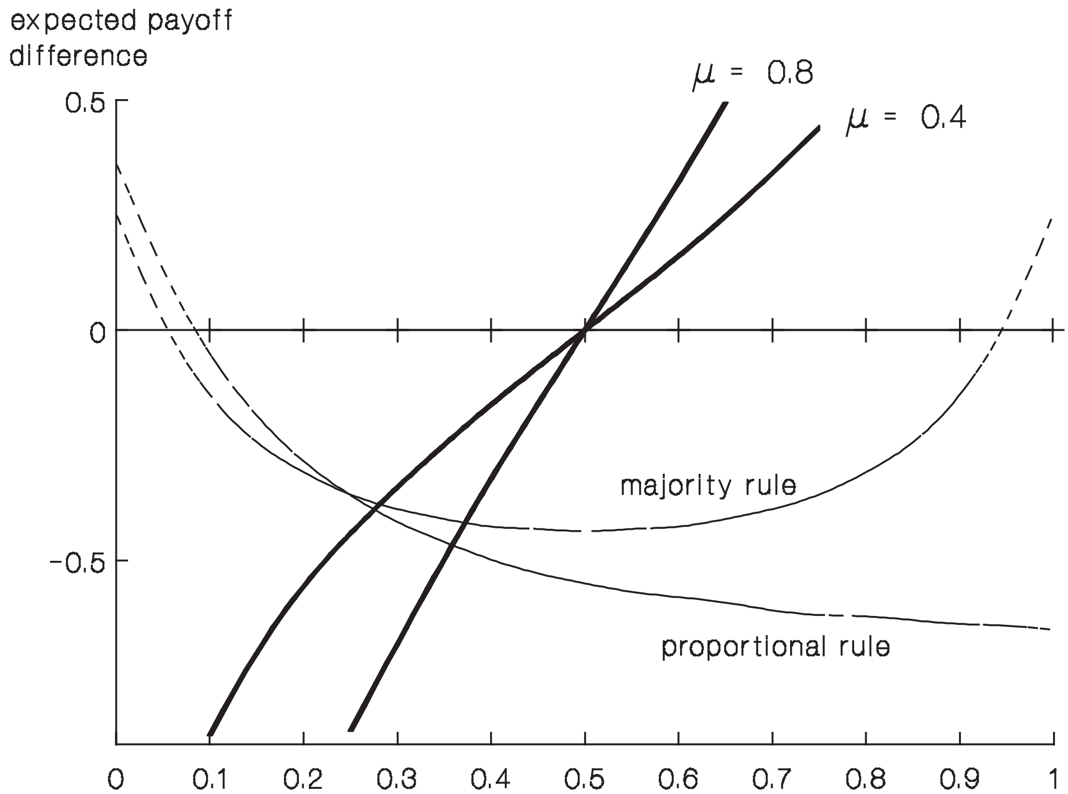

Figure 6. Nash and Quantal Response Voting Probabilities
Under Majority and Proportional Rules

difference line crosses the zero line: there are two Nash equilibria, one in which almost no one votes and another in which almost everyone votes (Palfrey and Rosenthal, 1983).

The quantal response equilibrium is determined by the intersection of the expected payoff difference line and the inverse distribution function (dark lines).[22] The μ parameter of 0.8 used to construct the steeper line was selected so that the QRE prediction would be at about the same level (30 to 50 percent) as the vote participation probabilities reported by Schram and Sonnemans (1996b) in the initial periods of their experiment. Interestingly, the voting probabilities started

[22] Palfrey and Rosenthal (1985) use essentially the same techniques to determine the Bayesian-Nash equilibrium in a voting game with incomplete information. In their paper, individual cost-of-voting shocks are added to each person's payoffs. The resulting Bayesian-Nash equilibrium is mathematically equivalent to a quantal response equilibrium.

high and then decreased to stabilize somewhere in the 20 to 30 percent range. This downward trend is crudely captured by a reduction in the noise parameter μ to 0.4 as indicated by the second inverse distribution line in Figure 6.[23]

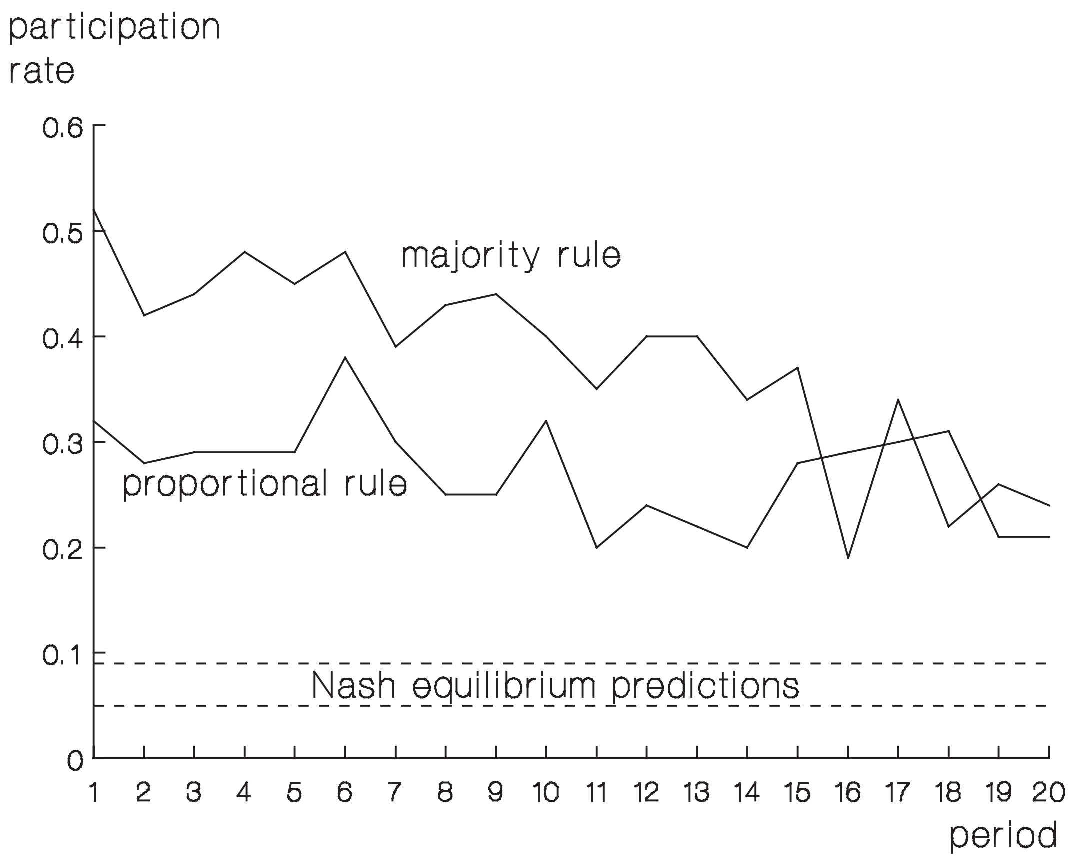

Figure 7. Voting Participation Rates with Random Matching
Source: Schram and Sonnemans (1996b)

Schram and Sonnemans (1996b) also considered a "proportional rule" game in which each person's payoff is the proportion of votes for their preferred outcome, minus the cost of voting if they voted. Again, it is straightforward to use the binomial formula to calculate the expected proportion of favorable votes, contingent on one's own decision of whether to vote, as a function

[23] Alternatively, this downward adjustment could be explained by the $\mu = .4$ line, together with the dynamic stability argument in Section II that produces directional movements of the type indicated by the arrows on the horizontal axis of Figure 2.

of the common participation probability, p.[24] The expected payoff difference for this proportional representation game is the increase in the expected proportion of favorable votes, minus the cost of voting. This difference is declining everywhere because one's vote has a smaller impact on the vote proportion as the probability of others' participation increases. The expected payoff difference line is labeled "proportional rule" in Figure 6, where the parameters are again taken from Schram and Sonnemans (1996b): $N = 6$, $V = 2.22$, and $c = 0.7$. The Schram and Sonnemans data for the proportional rule experiments, plotted as the lower line in Figure 7, started in the 30 to 40 percent range and ended up between 20 and 30 percent in the final periods. Note that participation is initially higher with the majority rule than with the proportional rule, while this difference disappears in the final periods of the experiment when the voting probabilities are close to 25 percent, well above the Nash predictions for these games. This result is not surprising from a QRE point of view, since the two expected payoff difference lines cross at $p = .25$ at which they intersect with the inverse distribution line (for $\mu = 0.4$). The result, however, is unexplainable by a Nash analysis for which the intersection of the two expected payoff difference lines plays no role and only "crossings at zero" matter. For the parameter values of the experiment, these crossings are at $p = .05$ and $p = .95$ for the majority rule treatment and at $p = .09$ for the proportional rule treatment, and seem to have little predictive power for the results of the Schram and Sonnemans (1996b) experiment.[25] To summarize, both the qualitative data patterns as well as the magnitude of the observed voting probabilities are consistent with a QRE analysis (but not with Nash), as can be seen from Figures 6 and 7.

This general approach may be extended to cover cases with asymmetries, e.g. when one type is more numerous than another. With asymmetries, the equilibrium will consist of a

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24 Using the same notation as before, the expected payoff difference for a player in group 1 is:

$$V \sum_{n_1=0}^{N-1} \sum_{n_2=0}^{N} \binom{N-1}{n_1} \binom{N}{n_2} \left[\frac{n_1+1}{n_1+n_2+1} - \frac{n_1}{n_1+n_2} \right] p^{n_1+n_2} (1-p)^{2N-n-n_2-1} - c \, .$$

where the outside sum pertains to the decisions of the N-1 others of one's own type, and the inside sum pertains to the N voters of the other type.

25 See also Schram and Sonnemans (1996a) for a similar experiment with slightly different parameter values.

participation probability for each type. These two probabilities will be determined by two equations analogous to (1), with the expected payoff for participation (voting) being a function of the number of potential voters of each type and the equilibrium participation probabilities. While a simple graphical analysis of this asymmetric model is not possible, it is straightforward to proceed with numerical calculations, for example, to show that the smaller group is more likely to vote when the costs of voting are symmetric.

VII. Conclusion

Many strategic situations are characterized by binary decisions, e.g. whether or not to vote, volunteer, attend a congested event, or perform a costly task with public benefits. In this paper we present a simple model of equilibrium behavior that applies to a wide variety of seemingly unrelated binary-choice games, including coordination, public goods, entry, participation, and volunteer's dilemma games. The model captures the feature that the decision of whether or not to participate may be affected by randomness, either in preferences (e.g. entry or voting costs) or in decision making (due to perception or calculation errors). The resulting quantal response equilibrium (McKelvey and Palfrey, 1995) incorporates this randomness in the form of an error parameter and nests the standard rational-choice Nash equilibrium as a limiting case. The quantal response equilibrium tracks many behavioral deviations from Nash predictions, e.g. the tendency for entry to match the Nash predictions when the prediction is 1/2, and for excess entry when the Nash prediction is below 1/2. In other words, a model with behavioral noise is capable of explaining the "magical" accuracy of Nash predictions in some experiments and the systematic deviations in others. The observed over-entry when Nash predictions are low is analogous to the over-participation in voting experiments, which is explained by a quantal response analysis. The participation rates in these experiments are roughly the same for the majority and proportional outcome rule treatments, which are consistent with theoretical calculations for the parameters used in the experiments. Similarly, the quantal response model tracks intuitive "numbers effects" observed in volunteers' dilemma and step-level public goods experiments, both when these effects are consistent with Nash predictions and when they are not. Taken together, these results indicate that standard "rational-choice" game theory can be enriched in a manner that increases its behavioral relevance for a wide class of situations. Moreover, the simple nature of the graphical equilibrium analysis will aid researchers in other binary choice applications.

Appendix

Proof of Proposition 3. The probability, P, that no one volunteers is given by $(1-p)^N$, where the QRE probability of volunteering, p, satisfies:

(A1)
$$\mu\, F^{-1}(p) = B(1-p)^{N-1} - C.$$

Combining these equations and using the fact that $F^{-1}(p)$ is symmetric, i.e. $F^{-1}(p) = -F^{-1}(1-p)$, allows one to express (A1) in terms of the probability that no one volunteers:

(A2)
$$\mu\, F^{-1}(P^{\frac{1}{N}}) = C - BP^{\frac{N-1}{N}}$$

from which the derivative of P with respect to N can be established as:

$$\frac{dP}{dN} = -\frac{P\,Log(P)}{N}\;\frac{Bf(F^{-1}(P^{1/N})) - \mu\, P^{-1+2/N}}{(N-1)Bf(F^{-1}(P^{1/N})) + \mu\, P^{-1+2/N}}$$

Note that dP/dN can only be non-negative when $\mu \leq P^{1-2/N} B f(F^{-1}(P^{1/N}))$. The right side of this inequality is bounded by $B\max(f)$, so dP/dN has to be negative for large enough μ. Finally, suppose, in contradiction, that $\lim_{N\to\infty} P > 0$. This implies that $P^{1/N}$ tends to 1, so $\mu F^{-1}(P^{1/N}) \to \infty$ when $\mu > 0$. This contradicts (A2) since the right side limits to $C - B\,P$, which is finite. Hence, P tends to zero when N tends to infinity. In fact, from (A2) it follows that for large N, P converges to $F(C/\mu)^N$, which tends to zero since $F(C/\mu) < 1$ for $\mu > 0$. Q.E.D.

Proof of Proposition 4. The QRE probability of contributing, p, satisfies:

(A3)
$$\mu\, F^{-1}(p) = VP_w^{\,N}(p) - c,$$

where $w \geq 1$ denotes the threshold and $P_w^N(p)$ is the probability that w-1 out of the N-1 others contributed (see equation (6)). The solution to (A3) is unique when the derivative of the left side is everywhere greater than that of the right side. The derivative of $P_w^N(p)$ with respect to p is given by $((w-1)/p - (N-w)/(1-p))\, P_w^N(p)$ and the relevant condition for uniqueness is therefore:

(A4) $$\mu \; > \; Vf(F^{-1}(p))\,((w-1)/p \; - \; (N-w)/(1-p))\,P_w{}^N,$$

Note that the right side is negative when $w = 1$ and for $w \geq 2$ it is less than $Vf(F^{-1}(p))P_w{}^N\,(w\text{-}1)/p$. The latter expression can be rewritten as $(N\text{-}1)\,V f(F^{-1}(p))P_{w-1}{}^{N-1}$, which is bounded by $(N\text{-}1)\,V\max(f)$. So for $\mu > (N\text{-}1)\,V\max(f)$, the quantal response probability of contributing is unique for all values of the threshold. The derivative of $P_w{}^N(p)$ with respect to $w < N$ (holding p fixed) is $P_{w+1}{}^N(p)$ - $P_w{}^N(p)$, which simplifies to: $P_w{}^N(p)\,(1 - w/(N\text{-}w)\,(1\text{-}p)/p)$. Together with (A3) this implies that the derivative of the QRE probability, p, with respect to the threshold, w, is given by:

(A5) $$\frac{dp}{dw} \; = \; \frac{1-p}{N-w}\,\frac{Vf(F^{-1}(p))\,((N-w)/(1-p)\; - \; w/p)\,P_w{}^N}{\mu \; + \; Vf(F^{-1}(p))\,((N-w)/(1-p)\; - \;(w-1)/p)\,P_w{}^N},$$

Note that the denominator of the second term is positive when the condition for a unique QRE (eq. (A4)) is satisfied. The sign of dp/dw is then determined by the numerator which is positive iff $p \geq w/N$. The intuition for this result is straightforward: as long as the "inverse distribution" line intersects the "expected payoff difference" line to the right of its maximum (i.e. $p > w/N$), an increase in w shifts the expected payoff difference to the right and moves the intersection point upwards. The reverse happens for higher values of w when the inverse distribution line cuts the expected payoff difference line to the left of the maximum (see also Palfrey and Rosenthal, 1982).

The probability, $Q_w{}^N$, that the public good is provided is given by

$$Q_w{}^N \; = \; \sum_{k\,=\,w}^{N} \binom{N}{k} p^k\,(1-p)^{N-k},$$

and its derivative with respect to w (for $w < N$) is:

(A6) $$\frac{dQ_w{}^N}{dw} \; = \; Q_{w+1}{}^N \; - \; Q_w{}^N \; + \; \frac{dQ_w{}^N}{dp}\,\frac{dp}{dw} \; = \; NP_w{}^N\left(\frac{dp}{dw} \; - \; \frac{1-p}{N-w}\right).$$

Combining (A5) and (A6) shows that $Q_w{}^N$ is decreasing in w. Q.E.D.

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