University of Michigan Retirement Research Center

Working Paper
WP 2004-072

Life, Death, and the Economy: Mortality Change in Overlapping-Generations Model

Qi Li and Shripad Tuljapurkar

MRRC

Project #: UM01-03

“Life, Death, and the Economy: Mortality Change in Overlapping-Generations Model”

Qi Li
Stanford University

Shripad Tuljapurkar
Stanford University

January 2004

Michigan Retirement Research Center
University of Michigan
P.O. Box 1248
Ann Arbor, MI 48104

Acknowledgements

This work was supported by a grant from the Social Security Administration through the Michigan Retirement Research Center (Grant # 10-P-98358-5). The opinions and conclusions are solely those of the authors and should not be considered as representing the opinions or policy of the Social Security Administration or any agency of the Federal Government.

Life, Death, and the Economy: Mortality Change in Overlapping-Generations Model

Qi Li
Shripad Tuljapurkar

Abstract

Demographers have shown that there are regularities in mortality change overtime, and have used these to forecast changes due to population aging. Such models leave out potential economic feedbacks that should be captured by dynamic models such as the general-equilibrium, overlapping-generations model first studied by Yaari and Blanchard. Previous analytical and simple numerical work by economists has focused on comparative statics and used simplistic representations of mortality, such as the assumption of a constant age-independent death rate, or some parametric approximation to a survival curve. We show that it is straight forward to analyze equilibria in such models if we work with the probability distribution of the age at death. US and other data show that this distribution can be plausibly described by a normal distribution {for this case we obtain analytical results. For the general case we have numerical results. We show that a proper accounting for the uncertainty of when one dies has significant qualitative and quantitative effects on the equilibria of such economic models. There are, in turn, significant lessons to be drawn for models of future fiscal policy.

Authors' Acknowledgements

The authors are grateful for helpful comments from Ryan Edwards, Ronald Leeand seminar participants at Policy and Economics Research Roundtable at Department of Management Science and Engineering, Research in Progress Seminar at Department of Health Research and Policy, both at Stanford University and Demographic and Economic Dynamics Conference and Workshop at PaloAlto in 2003. This paper is partially supported by the Morrison Institute for Population and Resource Studies at Stanford University, the Michigan Retirement Research Center, and the Center for the Economics and Demography of Aging at the University of California at Berkeley.

1 Introduction

Mortality rates in the U.S. have been declining for over a century. In figure 1,[1] we combine historical data and forecasts to show past and expected future increases in cohort life expectancy at birth. Demographic aging in the US, as in many other countries, is due to both this decline of death rate and swings in fertility rate leading to large birth cohorts as in the US baby boom.[2]

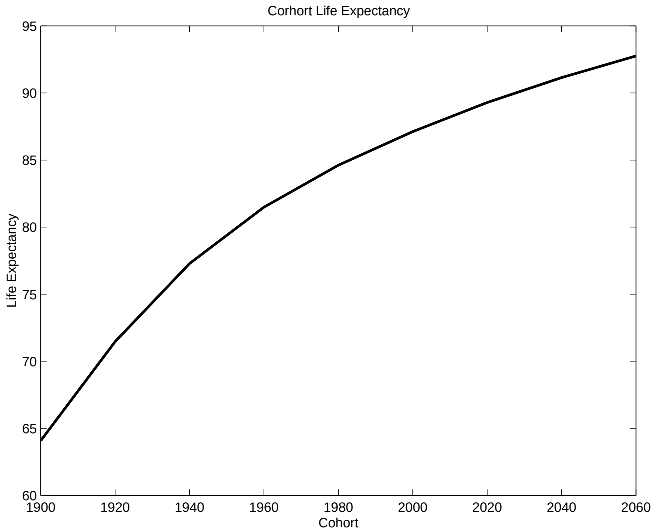

Figure 1: Cohort Life Expectancy in U.S.

Demographic aging in the industrialized countries has stimulated much work on the economic effects of demographic change. The economic setting for such analyses, beginning with Yaari (1965) and Blanchard (1985), is a general equilibrium model with overlapping generations. Published work on the effects of population aging in such models has typically sacrificed the details of age-dependent mortality in order to make analytical progress. Indeed, observed mortality patterns are bracketed by two stylized patterns that have been made in the past. The first assumes that all persons die at a fixed age and studies the effect of increases in this age of death, as in Futagami and Nakajima (2001). The second follows Yaari and Blanchard in assuming a fixed age-independent death rate, as in Kalemli-Ozcan, Ryder and Weil (2000). A more realistic two-parameter survival function was used by Boucekkine, Croix and Licandro (2002), but their function has limitations that we will discuss.

We study economic steady states in the overlapping-generations framework including age-dependent mortality decline and economic feedback. Our economic assumptions are most similar to those used by Kalemli-Ozcan, Ryder and Weil (2000). We assume constant

[1]The data before 1996 is history data of U.S.. The data after 1996 is forecasted by Lee-Carter Model (Lee and Carter, 1992).

[2]US fertility rates are analyzed and projected in Lee and Tuljapurkar (1994).

relative risk aversion utility function and use a aggregate Cobb-Douglas production function. To study schooling, we choose a wage profile that is only a function of years spending in school. Our focus is on the realistic treatment of mortality in modern populations, and we have three goals.

First, we show that Yaari's formulation neatly incorporates any realistic mortality pattern, if we work in terms of the probability distribution of the age at death (we call it the death age). This approach naturally allows us to think about individual decision-making in response to life extension, in which the average death age increases; and changes in the uncertainty of the timing of death which, as Yaari pointed out, are described by changes in the variance of the death age. to yield an analytical treatment of mortality decline in the industrialized countries. Second, we summarize evidence that mortality decline in the 20th century has resulted in a tightening of the distribution of the death age in all industrialized countries. Indeed, we show that this distribution can be usefully approximated by a normal distribution. Finally, we use a normal distribution for death age to obtain analytical results for the equilibrium in our model. We show that the use of a realistic mortality pattern yields significantly different implications, relative to existing stylized studies, for the effect of aging on consumption, interest rate, wage and wealth. We present numerical results that show the accuracy of the normal approximation.

Our work is only a step towards a fully demographic analysis because we assume, as have past studies, that birth rates are fixed, and because we assume that the population age structure is stationary.

2 Modelling age patterns of mortality

2.1 Alternative Models of Mortality

At time z define the instantaneous death rate $\mu(s, z)$ at age $z - s$ for members of a cohort (generation) born at time s. Survivorship $l(s, z)$ is the probability that an individual from a cohort born at time s will be alive at time z. Defining T to be the random age at death, the probability density of T is given by the product $\phi(s, z) = \mu(s, z)\, l(s, z)$. If we assume that death rate μ is independent of age, as in many economic analyses, we have exponential survivorship, and an exponential density for T. If we assume that everyone dies at the same age T_0, the survivorship function l is a step function, constant at $l = 1$ until age T_0, and falling to $l = 0$ thereafter; the distribution of T is a delta function at T_0.

Demographers have studied extensively the age pattern of mortality. In modern industrialized countries, deaths mainly occur at ages over 45 years, and death rates are falling at all ages including old ages (i.e., ages 65+ and 85+). Since most death is increasingly concentrated at late ages, the shape of survivivorship changes rapidly at these ages. In terms of the distribution of death age T, the mean age at death has been increasing while the variance of the age at death has been decreasing, as shown for several countries by Wilmoth and Horiuchi (1999).

Past stylized assumptions about mortality contrast sharply with realistic patterns of mortality. We can see the differences clearly by comparing three mortality patterns: age-independent constant death rate, a fixed death age, and US projections. We compare these by displaying three functions, death rate μ as a function of age, survivorship l as a function of age, and the probability and distribution ϕ of death age in figure 2. The plots are shown for a life expectancy (i.e., average age at death) of 80 years. Assumptions of a constant death rate or a fixed death age are certainly analytically convenient, but neither captures the essential age-dependance of realistic death rates.

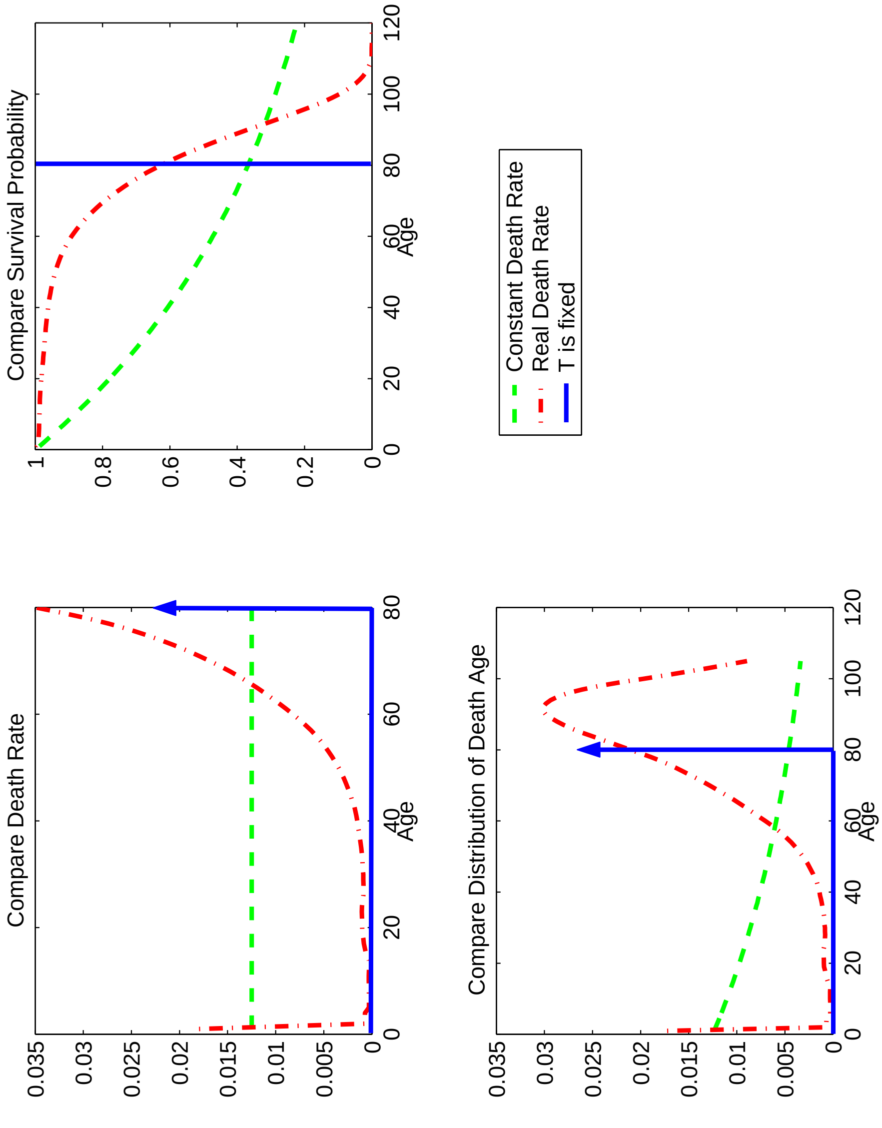

Figure 2: Alternative models of mortality

A recent paper by Boucekkine, Croix and Licandro (2002) does a better job of modeling mortality than the two stylized assumptions we have considered. They fit realistic survival probability using a two parameter, age-dependent survival probability l. Although their model yields shapes of death rate and survival curve that are approximately right, their model generates a density function $\phi(a)$ of death age which always increases with age. This behavior contradicts the fact that the empirical distribution has a hump shape with a well-defined peak in modern industrial countries such as U.S. and Sweden.[3]

2.2 Using Mortality Models in OLG Models

In the overlapping-generations framework, a major analytical task is to aggregate individual consumption and saving across cohorts. As Yaari showed, one can use either the functions μ and l together, or simply work with the distribution ϕ of the death age. The stylized assumptions of past studies have, we believe, aimed to yield analytically tractable form of μ and l. Instead, we propose to work directly with ϕ, and as we will show, makes possible the use of richer and more realistic analytical models.

Let $\phi(x)$ be the distribution function of death age T, age-dependent survival curve $l(a)$ is

$$l(a) = \int_a^\infty \phi(t)dt \tag{1}$$

In the analysis of OLG models, we need to aggregate variables such as consumption and saving, using as weights the probabilities of death. A typical aggregate of some function $j(a)$ of age a is defined with respect to survivorship,

$$J = \int_0^\infty j(x)l(x)dx = \int_0^\infty j(x)\int_x^\infty \phi(t)dtdx \tag{2}$$

But a change the order of integration turns this into

$$J = \int_0^\infty \int_0^t j(x)dx\phi(t)dt = E_T[\int_0^T j(x)dx] \tag{3}$$

This transformation, which Yaari performed in the reverse direction, expresses the aggregate as an expectation over the distribution of death age T. We see immediately that a variety of analytical forms of the distribution ϕ can lead to tractable aggregations. Furthermore, the details of mortality change in industrialized countries in the past 50 years are accurately captured by a relatively choice of ϕ, as we now show.

2.3 Distribution of death age

To describe cohort mortality for cohorts that are still alive, we must use a forecasting model for mortality, specifically, the model of Lee and Carter (1992). Distribution curves of death age based on their model shows a linear relationship between life expectancy e_0 and variance of death age v_0 as in figure 4. When life expectancy increases, the variance of death age decreases almost linearly, consistent with the study of Wilmoth and Horiuchi (1999). Wilmoth

[3]See Wilmoth and Horiuchi (1999).

and Horiuchi report that the standard deviation of population death age has decreased from about 29 in 1901-1905 to about 17 in 1990-1995. This trend is not only true in U.S. but also true in many other countries such as Sweden and Japan. In Sweden and Japan, the standard deviation has decreased to about 14 in 1991-1995. The Lee-Carter model and U.S. historical data together yield the linear relationship

$$v_0 = B_0 + B_1 e_0 \tag{4}$$

where $B_0 = 2582$ and $B_1 = -26.7$.[4]

Turning to the shape of the distribution ϕ of the death age T, we see from figure 3 that death age for US cohorts is distributed roughly as a normal distribution together with a long but modest left tail.

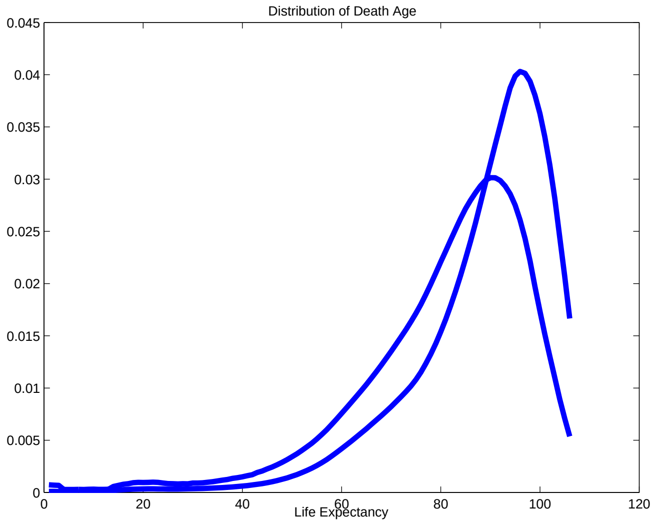

Figure 3: Distribution of U.S. cohort life expectancy for cohort 1950 and 2010.

It turns out that the left tail can be ignored in the analysis we do here, as we later demonstrate. We thus propose to approximate ϕ simply by a normal distribution,

$$\phi(a) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(a-e_0)^2}{2\sigma^2}}.$$

The corresponding survivorship is

$$l(a) = 1 - [\Phi(\frac{a-e_0}{\sigma}) - \Phi(-\frac{e_0}{\sigma})]$$

[4]This regression is based on the historical data and our forecast of U.S. population data. The R^2 of this regression is 0.985.

and the death rate may be obtained as

$$\mu(a) = \phi(a)/l(a)$$

This approximation is much more realistic than the stylized assumptions of constant death rate or a fixed death age. A major advantage of this approach is that the parameter e_0 captures the length of life whereas the standard deviation σ is a direct measure of the uncertainty in the age at death. These parameters enable us to examine the separate effects of the need to spread consumption over a longer life, and the need to adjust savings as a precaution against living for fewer or more years than one might expect. In addition, we can study economic responses to the particular trajectory of increasing e_0 and decreasing σ that is seen in the past data.

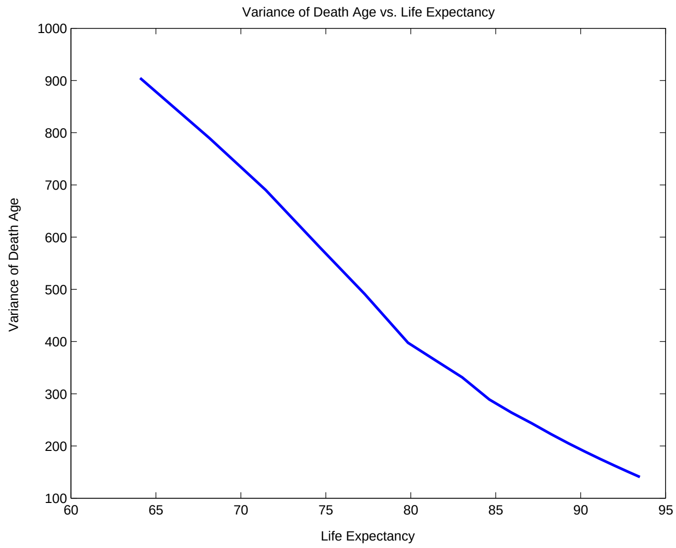

Figure 4: Variance of death age vs. life expectancy

The normal approximation does not capture two aspects of historical death rates: the long left tail mentioned above, and high death rate at very early ages between 0 and 1 year. Historically, death rates have fallen faster at the youngest ages, and this is likely to reduce the error we make with a normal assumption. However, our approach allows us to use many alternatives should we wish to capture the details we have left out of the simple normal assumption. For example, a better approximation is the two term distribution function

$$f(T) = \alpha \cdot \lambda e^{-\lambda T} + \beta \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(T-\nu)^2}{2\sigma^2}}$$

In this model, there are five parameters, α, β, ν, σ and λ. To make $f(T)$ a distribution function, we require $\alpha + \beta = 1$. We can fit this model to the data, and more important, we can also produce analytical solutions for this approximation (we do not discuss them here).

3 Life cycles and the economy

We use a continuous time overlapping generations model based on Yaari (1965) and Blanchard (1985). We assume that the population age structure is stationary, with a constant birth rate b, a constant total population size N and a probability distribution ϕ for an individual's age at death. We use a Constant Relative Risk Aversion(CRRA) utility function,

$$u(c(z)) = \frac{c(z)^{1-\gamma}}{1-\gamma} \tag{5}$$

where γ is the relative risk aversion coefficient and $c(z)$ is consumption at time z. We set the subjective discount function for people at age a to be $e^{-\theta \cdot a}$ where θ is a rate of time discount. In this model, the only source of uncertainty is the age at death, and the expected utility to be maximized is

$$\int_t^\infty l(z-t) \cdot u(c(z)) \cdot e^{-\theta \cdot (z-t)} \cdot dz \tag{6}$$

We assume the economy is in a steady state, and that aggregate output follows the Cobb-Douglas production function,

$$Y = AK^\alpha H^{1-\alpha} \tag{7}$$

where K denotes aggregate physical capital stock and H denotes total human capital. A is a positive constant representing productivity level and $0 < \alpha < 1$. The equilibrium interest rate r and the wage level w are determined by the marginal rates of change of output with respect to capital and human capital.

Human capital here depends on the number of years of schooling. We assume that individuals choose an age a_s when they finish schooling, and that they then work until a fixed age of retirement a_r. The relative wages of an individual only depend on years of schooling

$$y(a_s) = w\ e^{f(a_s)} \tag{8}$$

when $a_s \leq x \leq a_r$ and zero otherwise.[5] With this assumption aggregate human capital is just $e^{f(a_s)}\ L$ where L is the total labor force between ages a_s and a_r.

The random death age T has a known distribution, and we use equation (3) to define a series of aggregated functions. These aggregates provide explicit equilibrium conditions for the model.

$$g(z) \equiv E_T[e^{zT}] \tag{9}$$

[5] This is the same assumption as in Kalemli-Ozcan, Ryder and Weil (2000). A useful extension would incorporate a hump shaped age profile, for example at at age x wages are

$$y(x) = w\ e^{f(a_s)}\ (a_1 e^{-\beta_1 x} + a_2 e^{-\beta_2 x})$$

where $\beta_2 \geq \beta_1 \geq 0$, $a_1 \geq 0 \geq a_2$. This form of human capital is initially an increasing function of age and then a decreasing function of age. It is possible to extend our analytical results to this case.

$$P(z) \equiv E_T[e^{z(T \wedge a_s)}] \tag{10}$$

$$Q(z) \equiv E_T[e^{z(T \wedge a_r)}] \tag{11}$$

$$\lambda(a) \equiv E_T[T \wedge a] \tag{12}$$

Notice when $a_s = 0$, $P(z) = 1$. When $a_r = \infty$, $Q(z) = g(z)$. $\lambda(\infty) = E_T(T) \equiv e_0$ where e_0 is life expectancy. In the Appendix A, we calculate the closed form solutions for all g, P, Q, and λ under the assumption that death age T is normally distributed.

For a fixed level of schooling a_s the standard optimality conditions yield an optimal individual lifetime consumption path $c(a)$

$$c(a) = c_0 e^{ka}, \tag{13}$$

and the budget constraint yields consumption at birth c_0

$$c_0 = \frac{(k-r)we^{f(a_s)}}{r} \cdot \frac{P(-r) - Q(-r)}{g(k-r) - 1}. \tag{14}$$

Here

$$k \equiv (r - \theta)/\gamma. \tag{15}$$

Thus the initial consumption c_0 is a function of a_s, and the relevant condition leads to an equation for the optimal level of schooling

$$f'(a_s)[P(-r) - Q(-r)] + \frac{dP(-r)}{da_s} = 0 \tag{16}$$

Finally, the aggregate consumption is

$$C(t) = \frac{bN(k-r)we^{f(a_s)}}{rk} \cdot \frac{P(-r) - Q(-r)}{g(k-r) - 1} \cdot [g(k) - 1] \tag{17}$$

As shown in Appendix B, we find that the aggregate human capital H and aggregate capital stock K are:

$$K(t) = \frac{bN}{r} we^{f(a_s)} \varphi(r, e_0, \sigma_0)$$

$$H(t) = bNe^{f(a_s)}[\lambda(a_r) - \lambda(a_s)] = bNe^{f(a_s)}\xi$$

where

$$\varphi(r, e_0, \sigma_0) = \frac{r-k}{kr} \frac{(Q(-r) - P(-r))(g(k) - 1)}{g(k-r) - 1} - \lambda(a_r) + \lambda(a_s) \tag{18}$$

and

$$\xi = \lambda(a_r) - \lambda(a_s)$$

These lead to

$$\frac{K}{H} = \frac{w\varphi(r, e_0, \sigma_0)}{r\xi}$$

The first order conditions of production function give

$$\varphi(r, e_0, \sigma_0) = \frac{\alpha\xi(r, e_0, \sigma_0)}{1-\alpha} \tag{19}$$

Substitute φ and ξ into equation (19), it becomes

$$\frac{r-k}{kr}\frac{(Q(-r)-P(-r))(g(k)-1)}{g(k-r)-1} = \frac{\lambda(a_r)-\lambda(a_s)}{1-\alpha} \tag{20}$$

By solving equations (16) and (20), we can find age of schooling a_s and equilibrium interest rate r. All the other variables can be found accordingly.

The two equations (16) and (20) characterize the steady state of the economy. Note that these two equations do not depend on any specific forms of distribution function of death age T.

4 Comparative statics: the simplest case

In the simplest case, people work for a fixed wage w over their entire lives. The goal of our analysis is to understand how changes in the length and uncertainty of life affect consumption and wealth over the life cycle. We present analytical and numerical results based on the following assumption.

Assumption 1. *The death age is normally distributed and*

$$\gamma = 1, \theta = 0.03, \alpha = 0.03, T_{max} = 120$$

where T_{max} is the maximum possible age at death.

The value of T_{max} is reasonable for current and projected mortality.[6] The value of relative risk aversion coefficient is chosen to be 1. This is equivalent to assume a log utility function. Since we will compare our results with those under the assumptions of constant death rate, fixed death age and fitted distribution of death age, in particular, the work of Kalemli-Ozcan, Ryder and Weil (2000), we choose the same values of parameters θ and α as what they have calibrated.

One of the main questions in the OLG model is: how do consumption and wealth change as a population ages? It is also interesting to examine other macroeconomic variables such as interest rate and wage. To answer these questions, we need first to characterize demographic aging. In the stylized assumptions of constant death rate or a fixed death age, there exists only one parameter which can not be used to capture the important two aspects of demographic aging: life expectancy and variance of death age. Although a two parameter model has been proposed by Boucekkine, Croxix and Licandro (2002), as we have discussed in the former section, their density function of death age does not capture the empirical distribution which there is a hump shape around age 80. In Assumption 1, the age at death is

[6] See Wilmoth, Deegan, Lundstrom, and Horiuchi (2000) for details.

assumed normally distributed. This gives us a convenient way to focus on the two important aspects of demographic aging.

In this section, we first study the effects of changing life expectancy on consumption and wealth. Then we explore the effects of changing variance of death age. Finally, we examine the joint effects of these two important aspects of demographic aging and compare our results with those stylized assumptions in the literature.

4.1 Understanding changes in life expectancy

Suppose life expectancy increases and variance of death age is constant. What happens to aggregate wealth? To answer this question, we need first to study individuals' wealth over their life cycle. In Yaari-Blanchard framework, there exists a life insurance company. Individuals contract to make a payment contingent on their death. In exchange, they will receive a fair rate of payment from insurance company.[7] Although it is not necessary for individual net asset to be a monotone increasing function of age, individual net asset eventually increases when age is large due to the existence of insurance company to remove bequest motive. Thus, individual net asset between birth and death has a trend to increase in old age. As life expectancy increases, birth rate is adjusted such that total population size is fixed. Therefore, the aggregate human capital H is fixed. As a result, there is a higher percentage people in old age. Since the old have much larger asset than the young, the total wealth increases as life expectancy e_0 increases. This analysis is consistent with what we find in figure 5.

Figure 5 shows surface plots of interest rate, consumption, wage, wealth with respect to life expectance and variance of death age. Note that in the simplest case, as total wealth changes, wage will change in the same direction and interest rate will change in the opposite direction. This can be derived analytically using the first order conditions of the Cobb-Douglas production function to find

$$w(t) = A(1-\alpha)(\frac{K(t)}{H(t)})^{\alpha} \tag{21}$$

$$r(t) = A\alpha(\frac{K(t)}{H(t)})^{\alpha-1} \tag{22}$$

In the simplest case, the aggregated human capital $H(t)$ is constant. Since $0 < \alpha < 1$, wage will change in the same direction as that of total wealth $K(t)$ and the interest rate will change in the opposite direction.[8]

[7] This fair rate is the cohort death rate. Since the cohort size is large, the cohort size is deterministic even though the individual death age is uncertain.

[8] In the simplest case with Assumption 1, we find that if there exists a steady interest rate in the rage (θ, ∞), it may not be unique as shown in Appendix C. This is very different from other cases such as the constant death rate in Kalemli-Ozcan, Rder and Weil (2000). Although the steady interest rate is not unique, there is only one in the reasonable region. In particular, we choose minimum value of the solved interest rates which are greater than θ as the steady interest rate. The steady state values for other variables can

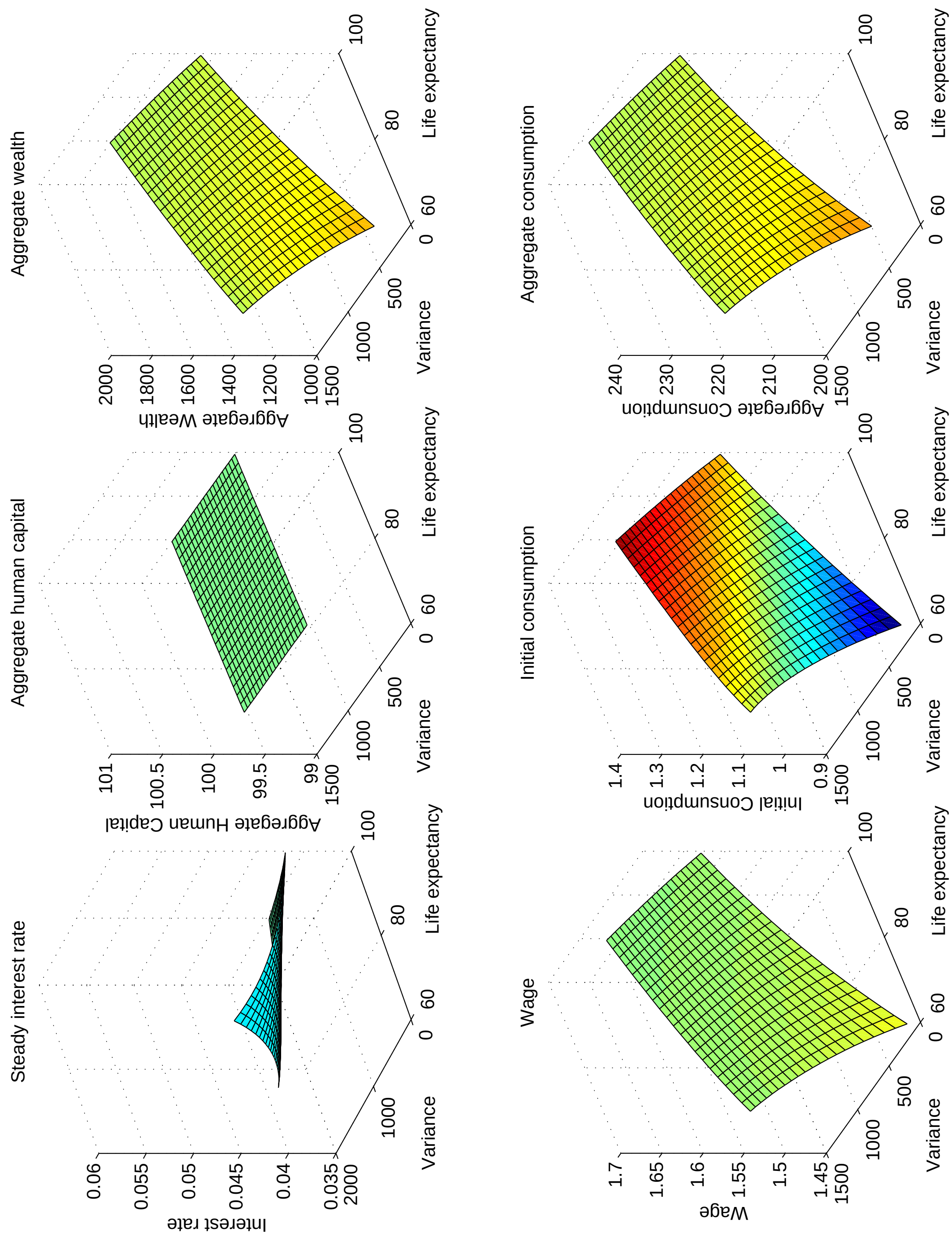

Figure 5: The simplest case: understanding changes in life expectancy and variance of death age

Initial consumption c_0 is an increasing function of life expectancy e_0 if variance of death age v_0 is fixed. As life expectancy increases, individual's lifetime labor income increases since wage w increases and the steady interest rate decreases. [As e_0 increases, survivorship $l(a)$ increases for all age a and the interest rate r decreases]. Thus, lifetime consumption must also increase because it is equal to lifetime labor income as in equation (23).

$$c_o \int_0^{\infty} e^{-\theta a} l(a)da = w \int_0^{\infty} e^{-rs} l(s)ds \tag{23}$$

Therefore, as life expectancy increases, total income increases which implies higher initial consumption c_0.

Total consumption C is an increasing function of life expectancy e_0 given a fixed variance v_0 of death age. As e_0 increases, there are more old people in the population. The consumption for individual at age a is $c(a) = c_0 e^{(r-\theta)a}$. Thus the consumption for the old is much larger than the consumption for the young. This implies that the total consumption C increases as life expectancy increases.

4.2 Understanding changes in variance of death age

We have studied the effect of life expectancy given that the variance of death age is constant, what is the effect of changes in the variance of death age for a fixed life expectancy on aggregate wealth, consumption and others? Again we start our analysis by studying the effects on aggregate wealth.

We find that aggregate wealth will increase generally. Assume that life expectancy is not small and variance of death age is not very high.[9] As variance increases, there is a higher probability for people to die at old age and at young age. Since young people have much less wealth than the old people, the loss at young age is less than gain at old age and total wealth will increase as variance of death age increases.

Initial consumption c_0 will increase if variance of death age T increases given life expectancy e_0 is fixed.[10] This can be seen as follows. Note that as v_0 increases, wage w increases and the steady interest rate r decreases.[11] To make lifetime consumption equal to lifetime labor income, we must have that initial consumption c_0 increases. This is also clear from the settings of the model. As variance v_0 increases, early death rate increases.

be uniquely decided by the steady interest rate in the simplest case. We can also derive the response of the steady interest rate to a change in the life expectancy for the parameters we have studied when the variance of death age is fixed. As shown in Appendix C, we find $dr/de_0 < 0$

[9]Total wealth may decrease when life expectancy is small and variance of death age is high. Under the case that life expectancy is low and variance of death age is high, the wealth of old people is higher than young people but the difference becomes smaller. As variance of death age increases at high level, the loss at young dominates the gain from old people. Moreover, the normality assumption may not hold since the variance is too high relative to its low life expectancy.

[10]Only for the case that life expectancy is not small or variance of death age is not too high.

[11]In Appendix C, we show that the derivative of the steady interest rate with respect to the variance of death age for the parameters values as we have studied when the life expectancy is fixed is negative.

The income increases for young people by both earnings from insurance company and wage. Therefore, initial consumption level increases when variance increases.

Total consumption C is an increasing function of v_0 given e_0 fixed. Similar to the former argument, as variance v_0 increases, there are more old people and less young people. Although the consumption from young people decreases, the total consumption level of old people increases a lot and is larger than the loss.

4.3 Joint effects from changing in life expectancy and uncertainty of death age

Historical data and expected futures show that both life expectancy and variance of death age change in the process of demographic aging. It is important to study the joint effects from these two aspects. In this section, we show that it is exactly this joint effect which accounts for the differences among constant death rate, fixed death age and normal distribution of death age cases.

We first study the steady interest rate, which, as we have shown, is a decreasing function of life expectancy and variance of death age respectively. A constant age-independent death rate implies $v_0 = e_0^2$. As e_0 increases, v_0 increases. So increases in e_0 and v_0 will affect the steady interest rate r in the same direction. However, the U.S. historical and expected future data exhibits increasing e_0 and decreasing v_0 as in figure 4. The joint effect of these opposite trends is a much higher steady interest rate. For the fixed death age, the variance v_0 is zero. Accordingly, wee find the highest steady interest rate r because the steady interest rate is a decreasing function of v_0.

Similar arguments apply to initial consumption c_0. Note that the joint effect of life expectancy and the variance of death age not only decreases the level of c_0, it might also make c_0 decrease when e_0 increases in some range.

The trend of aggregate consumption on life expectancy can be understood as follows. In equilibrium, individual consumption $c(t) = c_0 e^{k(t-s)}$ where $k = (r - \theta)/\gamma$. Under typical calibration, $r > \theta$ in equilibrium. Therefore, as e_0 increases, the consumption of old age population increases, and the aggregate consumption always increases. The effect of variance of death age on aggregate consumption is similar. Under the assumption that age at death T is normally distributed, as v_0 increases, there will be more people live longer and more people die earlier. Again, since the older will consume much more than the younger, the aggregate consumption will increase as v_0 increases. In the U.S., the effect on total consumption from increasing life expectancy is more significant than the effect from decreasing variance of death age and leads to increasing in total consumption.

4.4 Effects of Assumption 1

In this section, we compare the results under Assumption 1 with alternatives: constant death rate, fixed death age, and fitted distribution of death age. Figure 6 plots the comparative statics of steady interest rate r, total wealth K, individual initial consumption c_0, aggregate consumption C, wage w when life expectancy e_0 changes under the assumption of constant death rate, fixed death age, normally distributed death age and fitted death rate. We compare the exact values in table 1.

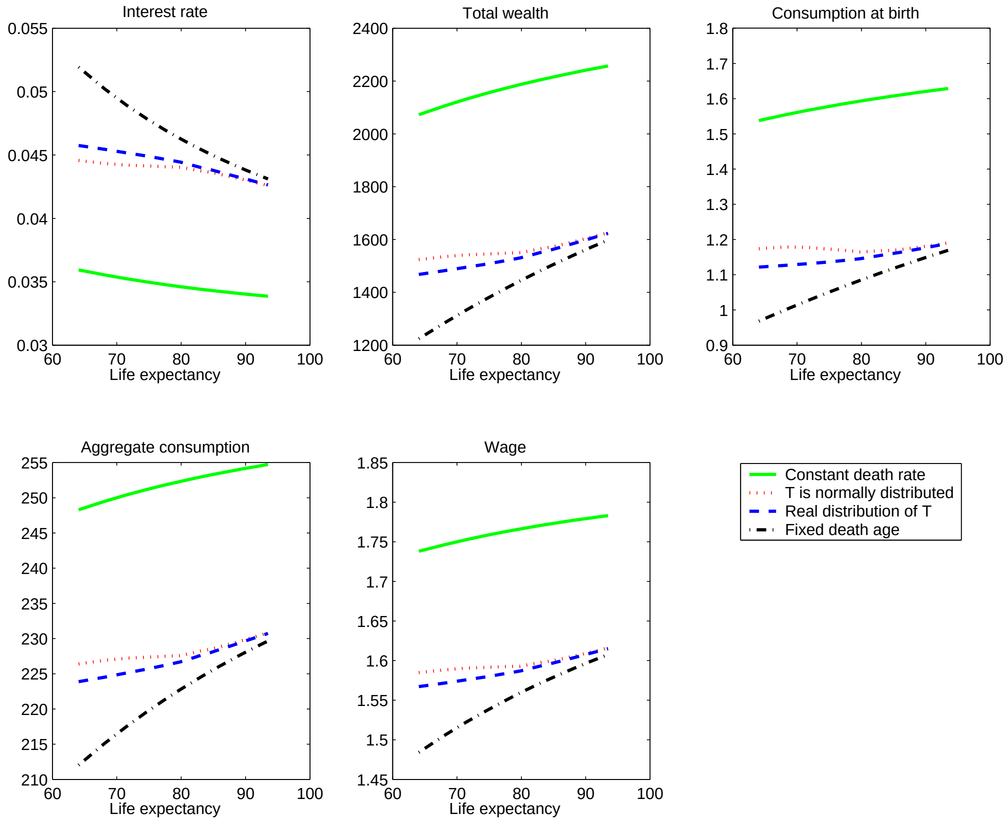

Figure 6: The simplest case

In figure 6, the levels among constant death rate, fixed death age and fitted death rate are quite different which is due to the different assumptions on variance of death age. It is clear that normality assumption is closest to the fitted curve. As life expectancy becomes larger, the survival curve becomes more rectangular. Fixed death age is also a good approximation for large life expectancy.

Although the assumption that death age T is normally distributed is a good approximation of fitted distribution, some differences remain. In particular, the mean of death age or so-called life expectancy is in fact not the peak of hump shape in the fitted distribution and the distribution of death age has long left tail. Moreover, we truncate at both the high end (ages > 120) and the low end (ages < 0) of distribution curve in our calculations. It can be shown that the error due to this truncation is very small. To measure the calculation errors due to these approximations, define adjustment coefficient α to be

$$\alpha = 1/\Phi(0, 120)$$

Name	*Constant Death Rate*	*Fixed Death Age*	*Normal Distribution*	*Fitted Distribution*
H	100	100	100	100
r	0.0346	0.0463	0.044	0.0444
w	1.7662	1.5592	1.593	1.5868
K	2186.6	1443.4	1550.3	1530.3
$c(0)$	1.5932	1.0841	1.1648	1.1456
C	252.308	222.7485	227.5745	226.6899

Table 1: The simplest case, $\gamma = 1$, $e_0 = 79.8339$

When $e_0 = 80$ and $v_0 = 400$, we find $\alpha = 1.0233$ which is very close to 1. Similarly, when $e_0 = 85$ and $v_0 = 300$, we have $\alpha = 1.0221$ which is also close enough.[12]

In figure 6, it is consistent with our expectations that the level of steady values using fitted distribution is very close to the case of normal distribution assumption of death age T. This can also be seen easily in table 1 for life expectancy 79.8339. In table 1, the difference of steady interest rate between normal distribution case and fitted distribution case is negligent. So are the other variables. Constant death rate case overstates the variance of death age which leads to a much lower steady interest rate. Fixed death age case completely ignores the variance of death age with a highest calibrated interest rate. In figure 6, the shapes of steady interest rate, total wealth, aggregated assumption and wage between normal distribution case and fitted distribution case are very close except consumption at birth. Although the consumption at births looks different, both of them capture a change from convexity to concavity around age 70.

In Assumption 1, $\gamma = 1$ is equivalent to an assumption of a Log utility function. It is interesting to relax this assumption to examine the effects of changing relative risk aversion γ using a constant relative risk aversion utility function.

Figures 7 and 8 show that more risk averse (higher γ) implies less total consumption, less total wealth and higher steady interest rate. As relative risk aversion coefficient γ increases, people become more risk averse and are more unlikely to borrow and consume less when they are young. The effect of γ on steady values is significant. From $\gamma = 1$ to $\gamma = 2$, steady interest rate increases about twenty percent.

Under the normality assumption of death age, the steady interest rate is much higher than that of constant age independent death rate. Therefore, the "risk free rate puzzle" by Weil (1989) becomes more severe if we include age dependent mortality into the continuous overlapping-generations model.

[12] We called that this is fitted distribution since the data after 1996 is projected. Our analysis cover from 1900 to 2060. Data is real between 1900 and 1996. For the other part, it is forecasted using Lee-Carter model.

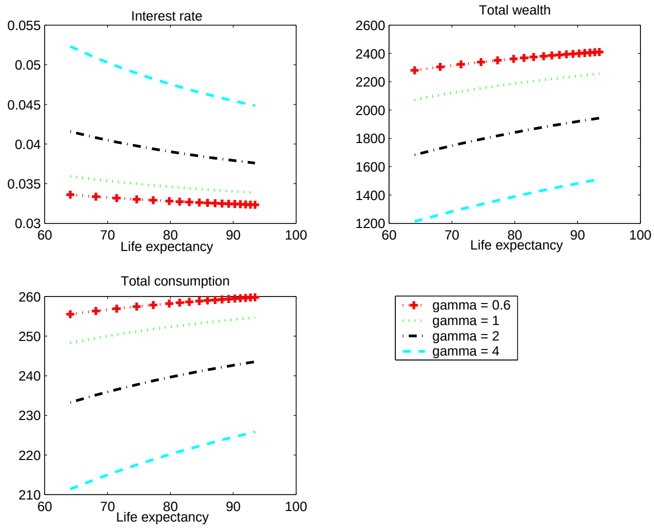

Figure 7: The simplest case under constant death rate

5 Comparative statics: schooling and retirement

Kalemli-Ozcan, Ryder and Weil (2000) examine the relationship between schooling and increased life expectancy. They find that schooling years increases as life expectancy increases. Their conclusion is based on the stylized assumption of constant death rate. They also do not include retirement into their model. In this section, we conduct our study using more realistic mortality rate. We also take account of retirement in our analysis.

5.1 Schooling without retirement

To compare our results with that of Kalemli-Ozcan, Ryder and Weil (2000), we first study the schooling case without considering retirement. We also adopt the same assumption on income function as in their study. In particular, the function $f(a_s)$ in equation (8) is defined as

$$f(a_s) = \frac{\Theta}{1-\Psi} a_s^{1-\Psi}$$

where $\Psi = 0.58$ and $\theta = 0.32$.

Under this assumption, steady values can be solved by two equations (16) and (20). After steady interest rate and schooling age are found, we can calculate all the other aggregate steady variables. The calibration results are plotted in figure 9 and summerized in table 2.

Figure 9 compares two cases under Assumption 1 with and without schooling. In the figure, total human capital H increases, wage w decreases, and interest r increases after schooling is introduced. In the schooling case, aggregate wealth K, aggregate consumption C and initial consumption c_0 are higher.

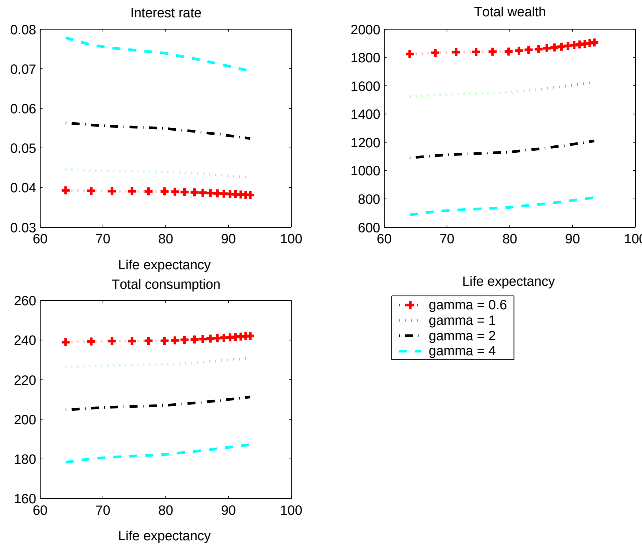

Figure 8: The simplest case under normally distributed death age

Name	Constant Death Rate	Normal Distribution
H	1278.3	875.2
a_s	22.8203	15.8077
r	0.0396	0.044
w	1.6666	1.3766
K	23039	8339.4
$c(0)$	7.0312	2.9932
C	3043.6	1721.1

Table 2: Schooling Case, $\gamma = 1$, $e_0 = 79.8339$

Different from the simplest case, total human capital is not constant anymore in the schooling case. Wage will change in the same direction as the ratio of total wealth and total human capital (K/H) instead of total wealth alone in the simplest case. Interest rate still has the opposite trend of wage.

To understand the schooling case, it is important to understand the changes of total human capital. On the one hand, the changes of schooling directly affect the lifetime working length. On the other hand, the higher the schooling, the higher the function $f(a_s)$ which implies high efficient labor. Since retirement is not considered in these comparison, the total human capital is given by

$$H(t) = bNE_T[\int_{a_s \wedge T}^{T} e^{f(a_s)} dx]$$

As schooling a_s increases, there are two effects: $e^{f(a_s)}$ increases and the integral lower

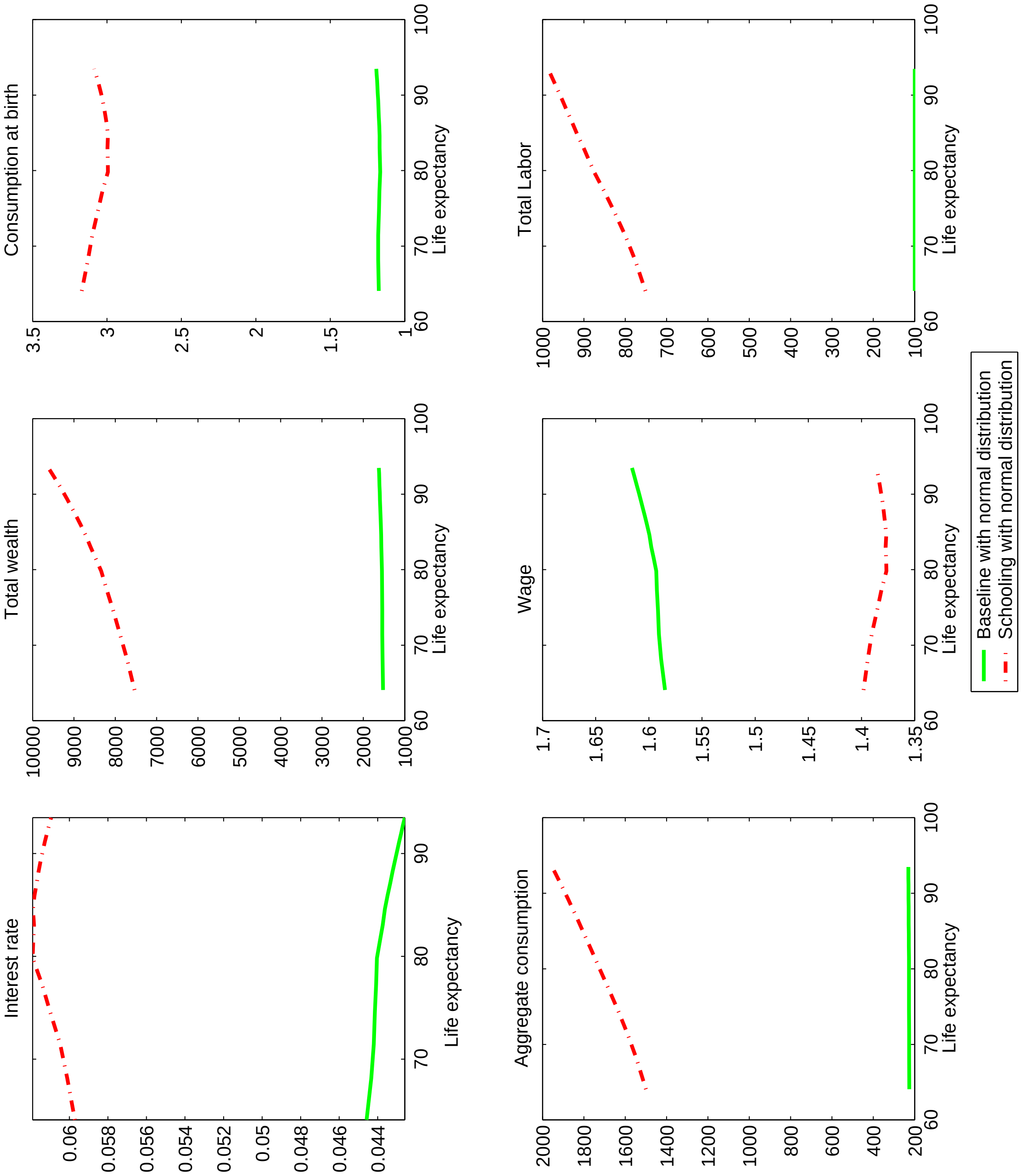

Figure 9: Schooling case

bounder $a_s \wedge T$ also increases. It is generally hard to say which effect is more significant. As we know, a_s is usually small relative to $e_0 - a_s$. In our calibration, the labor profile is fitted with historical data which has the property that total human capital H increases as schooling a_s is introduced. In other words, the effect from $e^{f(a_s)}$ dominates.

As in the simplest case, total wealth K is an increasing function of life expectancy. However, the ratio K/H does not monotonically increase. Although the ratio of total wealth and total human capital does not change a lot, it decreases first and then increases. Wage w and interest rate r change correspondingly. In particular, the steady interest rate increases first and then decreases as life expectancy increases.

One major difference between the simplest case and the schooling case is that individual chooses schooling length to optimize initial consumption c_0. However, changes of c_0 depends on w/r, $e^{f(a_s)}$ and $P(-r) - g(-r)$ as shown in equation (14). It is generally hard to conclude a definite trend of consumption at birth.

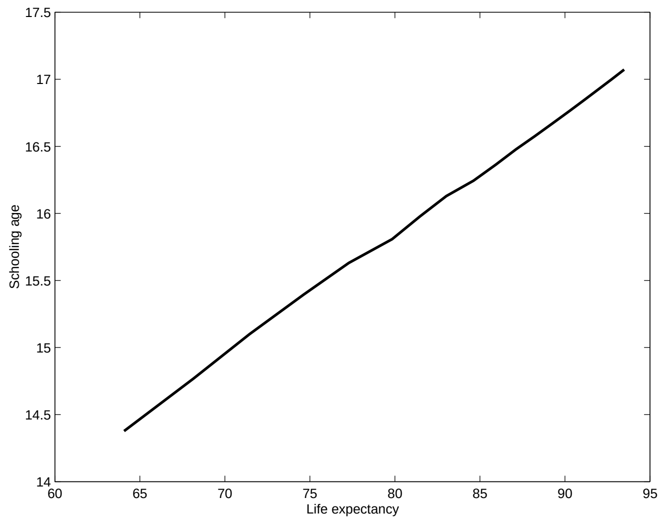

Figure 10: Schooling case: schooling age

One major goal in this analysis is to answer the question: what is the relationship between schooling age and life expectancy? We find that the schooling age is an increasing function of life expectancy e_0 as in figure 10. This is consistent with the results of Boucekkine, de la Croix and Licandro (2002), Kalemli-Ozcan, Ryder and Weil (2000) and our common knowledge. People want more education if they can live longer. However, again, since there exists negative effects from the variance of death age, steady schooling years in our case is about 6-7 years shorter than that of Kalemli-Ozcan, Ryder and Weil (2000). Note that the optimal schooling age in our case is much lower than that of Kalemli-Ozcan, Ryder and Weil (2000) as shown in table 2.

Another important difference between the simplest case and schooling is the steady interest rate r. The steady interest rate r increases at median life expectancy and finally decreases at high life expectancy. This is totally different from the simplest case. The increasing of the steady interest rate for median life expectancy is the period when the variance of death age decreases dramatically. It seems that schooling increases the effect of the variance of death age. Note that the total consumption C still always increases. This is again due to the assumption that people can work until they die.

Although it is not plotted, the level difference between fitted distribution and normal distribution is much closer than that between fitted distribution and constant death rate.

5.2 Schooling with retirement

In this section, we include retirement into the model. Different from Boucekkine, Croix and Licandro (2002), retirement is exogenous instead of endogenous. This assumption is reasonable since a specific retirement age is specified by law in pension system. Individual has only highly limited control on retirement age. Since schooling affects labor income, individual need to decide the optimal schooling age at birth under a certain pre-specified retirement age.

To well understand schooling with retirement, we need to understand the pure retirement effects in which schooling age is fixed. Then it will be possible to explain what we find in schooling with retirement through both schooling and retirement effects. To understand pure retirement effects, we choose the schooling age to be 14 and study the pure effects from retirement.

Based on our calibrations, the case in which the retirement with fixed schooling is shown in figure 11. When retirement age a_r decreases, figure 11 shows that total human captial H decreases. This is easy to understand since the total labor decreases and the schooling is fixed. Accordingly, wage w increases and total wealth K decreases which are also consistent with our intuition. Since the steady interest rate and wage always change in the opposite direction, the steady interest rate r decreases. As the total wealth decreases, it is reasonable that total consumption C decreases. When a_r is large,[13], as a_r decreases, initial consumption c_0 increases.

Although it is not shown in figure 11, the main surprise is that for small a_r and large life expectancy, c_0 may decrease.[14] We briefly discuss this here. The argument can also be applied to the pure schooling case with minor changes.

To simplify notation and analysis, assume $\gamma = 1$. Thus $k = r - \theta$.

$$c_0 = \frac{-\theta \cdot we^{f(a_s)}}{r} \cdot \frac{P(-r) - Q(-r)}{g(-\theta) - 1}$$

First note that $g(-\theta) - 1 < 0$ and a_s is fixed, then c_0 is proportional to $w/r \cdot [P(-r) - Q(-r)]$. Note also that $P(-r) > Q(-r)$. As a_r decreases, w/r increases and $P(-r) = E_T[e^{-r(T \wedge a_s)}]$ increases. The key is the change of $Q(-r) = E_T[e^{-r(T \wedge a_r)}]$. For large a_r, $Q(-r)$ increases but is highly limited since the change is mainly from r. However, when a_r is small and life

[13]If we set $a_s = 14$, a large retirement age can be the age > 45.

[14]This is not shown in the figure 11. However, it can be found if we specify low retirement age, such as at age 40.

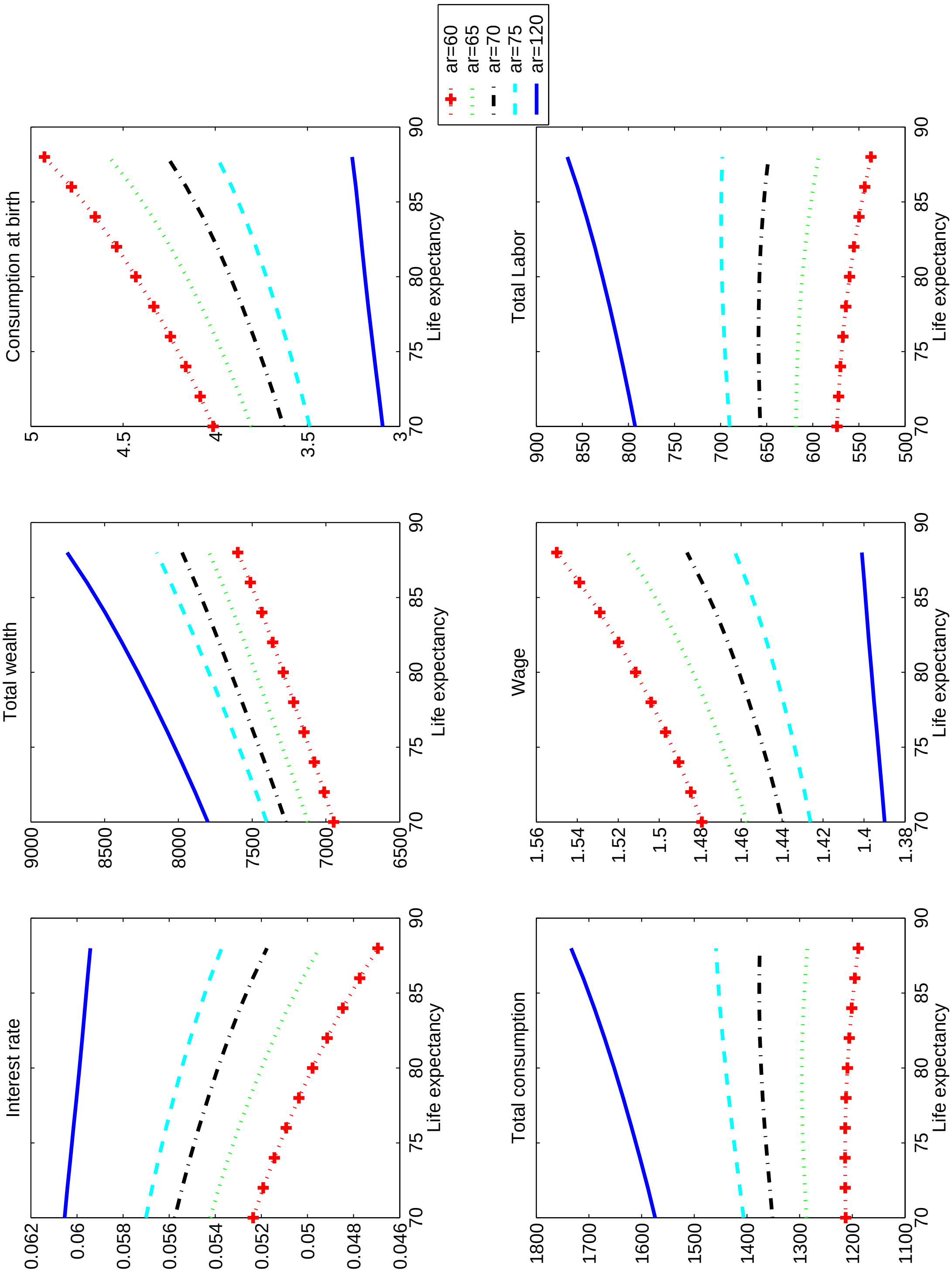

Figure 11: Retirement with fixed schooling years at 14

expectancy e_0 is large, $Q(-r)$ is approximately e^{-ra_r}. Therefore, the increases from $Q(-r)$ may be larger than that of $P(-r)$ which implies $P(-r) - Q(-r)$ may decrease. At some points, this loss dominates the gain from K/r.

In the case of schooling with retirement, figure 12 shows the changes of the steady interest rate, schooling and other aggregate variables with respect to changes in life expectancy and retirement age. As life expectancy becomes larger, the effect of retirement age also becomes more significant. This is consistent with our expectations. As life expectancy becomes larger, more people live longer. Their living will be affected more seriously by exogenous retirement age.

What are the effects of changing retirement age a_r on other variables? For example, how does schooling change? Figure 12 shows that schooling age is a decreasing function of retirement age. One simple explanation is that people need more education for higher salary so that they can pay for longer retirement life. A more detail analysis is given later.

Figure 12 also shows that $dc_0/da_r < 0$ and $dC/da_r > 0$. As retirement age a_r increases, the initial consumption c_0 will decrease. This result is counter intuitive since the partial derivative of c_0 with respect to a_r is positive. However, notice from figure 12, the steady interest rate r will increase as retirement a_r increases. Again, there are negative effects between r and c_0 and the substitution effect and human capital effect seem dominate.[15] The reason that $dC/da_r > 0$ is again due to the exponential increase of individual consumption. Although the initial start point of individual consumption c_0 decreases, the rate of exponential increasing consumption increases which will eventually increase the level of consumption of old cohorts. The aggregate effects of increasing retirement a_r on consumption is still positive.

We can also note that wage w in figure 12 decreases in a_r which make senses. Similarly, both total wealth K and total human capital H are increasing functions of retirement age. Although it is not plotted, we also compare the case between normal distribution assumption and fitted death rate. All the level, order and shape of curves between fitted distribution and normal distribution are very close.

The relationship between schooling and life expectancy has been addressed in the schooling case without considering retirement. In the rest of this section, we focus on the relationship between retirement age and schooling.

As we have observed, when $a_r - a_s$ is large, as a_r increases, schooling a_s decreases. Equivalently, as a_r decreases, schooling a_s increases. To understand this, for simplicity and without loss of generality, we assume $\gamma = 1$. Since retirement is considered in the comparison, the total human capital is given by

$$H(t) = bNE_T[\int_{a_s \wedge T}^{a_r \wedge T} e^{f(a_s)} dx]$$

and consumption at birth is

[15] See Deaton (1992) for more discussions on interest rate and individual consumption.

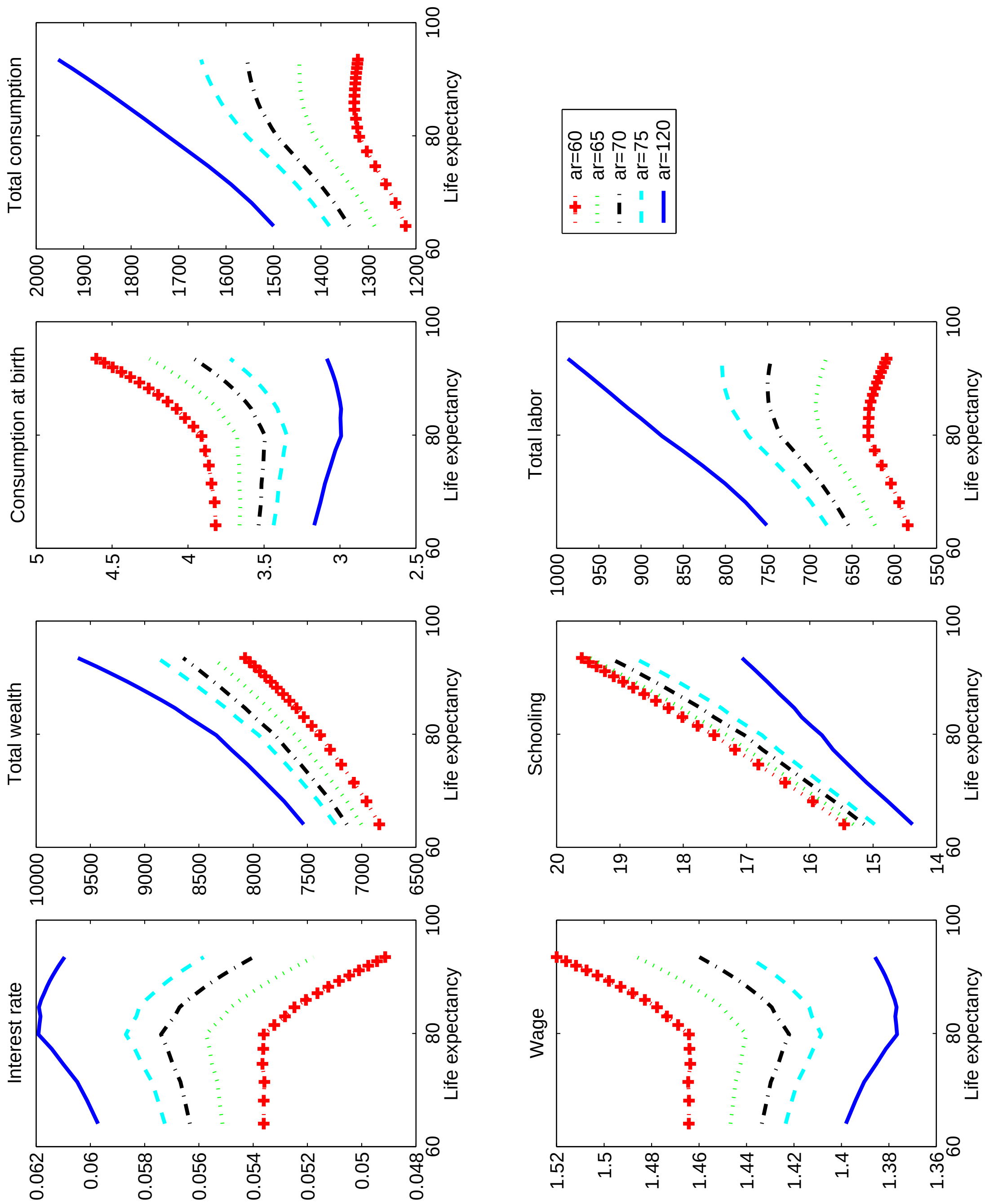

Figure 12: Complete case

$$c_0 = \frac{-\theta w e^{f(a_s)}}{r} \cdot \frac{P(-r) - Q(-r)}{g(-\theta) - 1}$$

As a_r decreases, the total labor trends to decrease. As schooling a_s increases, there are two effects. On the one hand, $e^{f(a_s)}$ increases. Since the interest rate r increases, $P(-r)$ decreases and $Q(-r)$ decreases. However, since a_r is quite larger than a_s, $P(-r) - Q(-r)$ increases. On the other hand, wage w decreases and the interest rates increases. w/r decreases. It is generally hard to say which effect is more significant.

When individuals maximize their consumption at birth, they take the interest rate and wage as given. Therefore, by assuming very low schooling age a_s, large retirement age a_r and high life expectancy e_0, the optimization problem can be simplified to maximize

$$e^{f(a_s)} \cdot e^{-ra_s} = e^{f(a_s) - ra_s}$$

The first order condition implies that optimal schooling age $a_s = (0.32/r)^{\frac{1}{0.58}}$. In the case that a_r decreases, without changing schooling, interest rate r decreases[16] which implies a higher schooling age a_s. In the case when a_r increases, by first fixing schooling age, the interest rate r increases which leads to a lower schooling age.

How to understand this intuitively? Although an individual wants to maximize its initial consumption c_0, it is unnecessarily that c_0 increases due to the changes of interest rate and wage. It is also true that increasing or decreasing of schooling does not imply any definite changes of total wealth K and initial consumption c_0. This is a general equilibrium effect instead of a partial equilibrium effect. First, individuals face budget constraints. Second, their consumption are decided by either initial consumption c_0 or interest rate r since $c(t) = c_0 e^{(r-\theta)t}$. The change of retirement age first affects labor. It seems that we can understand the changes of schooling by thinking that schooling is used to offset the human capital changes due to changes of retirement age. In particular, without changing schooling, the change of the human capital due to changes in retirement age makes schooling age not optimal. The other variables overreact to the change of human capital. The whole system can be optimized by readjusting the changes of human capital through schooling.

However, this trend is not true for low retirement age. Although it is not plotted here, when a_r is small, as a_r decreases, a_s decreases. This can be explained as follows.

Since retirement age a_r is very small, total human capital $H(t)$ is proportional to $e^{f(a_s)(a_r - a_s)}$. It can be verified that this term will increase when a_s decreases. Therefore, when retirement age a_r decreases, if schooling is fixed, H decreases. To offset these changes, schooling a_s need to decrease. Alternatively, we can also analyze as follows. When individuals maximize their initial consumption, they take interest rate and wage as given. Therefore, by assuming very low a_s, low a_r, the optimization problem becomes to maximize

$$e^{f(a_s)} \cdot [e^{-ra_s} - e^{-ra_r}] \approx r e^{f(a_s)} [a_r - a_s]$$

The first order condition is

$$0.32 \cdot \frac{a_r - a_s}{a_s^{0.58}} = 1$$

[16]See Figure 11.

It is clear that retirement age a_r is greater than schooling age a_s. Then as a_r decreases, schooling age a_s must decrease to satisfy this first order condition. Similarly, as a_r increases, a_s increases.

6 Conclusion

Our model extends previous research on overlapping-generations models of aging by including realistic demographic mortality. We derive analytical solutions for our model in the steady values. We can analyze other models such as those assuming constant death rate or fixed death age directly from our results.

Analysis of the U.S. population history and forecasts suggest our assumption that the distribution of death age is normally distributed. This assumption is not only more realistic but also analytically solvable. In the calibrations we find that this age dependent death rate does make dramatic different implications in many perspectives.

First we find that the steady interest rate r is always much higher than constant death rate in all the cases. This significant difference is exactly due to the more realistic assumption on the distribution of death age. The level differences also appear in all other macroeconomic variables. In particular, we give intuitions on the effects of two important aspects of demographic aging: life expectancy and variance of death age.

In the case of schooling, our results is very different from results in Kalemli-Ozcan, Ryder and Weil (2000). The steady interest rate r does not monotonically decrease when e_0 increases. This results in dramatically differences in many other variables.

Finally our paper studies the effect of endogenous schooling and exogenous retirement. Several counter intuitive relations have been found. For example, the initial consumption decreases as age of retirement increases. Another finding is that the schooling age will decrease as age of retirement increases in general equilibrium.

Above all, the relationship between life expectancy and other macroeconomic variables is complicated. We must take into account the variance of death age along with life expectancy. Although our analysis has incorporated more realistic age structure and shown significant different results, there is much more work to be done. One is to study a dynamic population structure. Another is to make retirement age endogenous. Eventually, we would like to introduce social security system and other public finance instruments into our model.

Appendix

A Closed forms of P(z), Q(z) and $\lambda(a)$

In this part, we derive the closed forms of P(z), Q(z) and $\lambda(a)$ under the assumption that the distribution of age at death T is normal.

$$P(z) \equiv E_T[e^{z(T \wedge a_s)}]$$

Now we want to solve $E_T[e^{z(T \wedge a_s)}]$.

$$E_T[e^{z(T \wedge a_s)}] = E_T[e^{zT} 1_{T<a_s}] + E_T[e^{za_s} 1_{T \geq a_s}]$$

We calculate them separately. The first term is

$$E_T[e^{za_s} 1_{T \geq a_s}] \approxeq e^{za_s} \cdot [\Phi(\frac{T_{max} - e_0}{\sigma}) - \Phi(\frac{a_s - e_0}{\sigma})]$$

The second term is

$$E_T[e^{zT} 1_{T<a_s}] \approxeq \int_0^{T_{max}} e^{zx} 1_{x<a_s} \frac{1}{\sqrt{2\pi}\sigma} \cdot e^{-\frac{(x-e_0)^2}{2\sigma^2}} dx$$

Finally,

$$E_T[e^{zT} 1_{T<a_s}] \approxeq e^{\frac{z^2\sigma^2}{2} + ze_0} \cdot [\Phi(-z\sigma + \frac{a_s - e_0}{\sigma}) - \Phi(-z\sigma - \frac{e_0}{\sigma})]$$

Above all, we have

$$P(z) = E_T[e^{z(T \wedge a_s)}] \approxeq e^{\frac{z^2\sigma^2}{2} + ze_0} \cdot [\Phi(-z\sigma + \frac{a_s - e_0}{\sigma}) - \Phi(-z\sigma - \frac{e_0}{\sigma})] + e^{za_s} \cdot [\Phi(\frac{T_{max} - e_0}{\sigma}) - \Phi(\frac{a_s - e_0}{\sigma})] \quad (24)$$

Take derivative, it becomes

$$\frac{dP(z)}{da_s} = \frac{g(z)}{\sqrt{2\pi}\sigma} e^{-\frac{(a_s - e_0 - z\sigma^2)^2}{2\sigma^2}} + ze^{za_s} \cdot [\Phi(\frac{T_{max} - e_0}{\sigma}) - \Phi(\frac{a_s - e_0}{\sigma})] - \frac{e^{za_s}}{\sqrt{2\pi}\sigma} e^{-\frac{(a_s - e_0)^2}{2\sigma^2}} \quad (25)$$

A.1 The simplest case

As we know, e_0 is around 80 and σ_0 is around 20. Therefore, we have

$$P(z) = e^{\frac{z^2\sigma^2}{2} + ze_0} \cdot \Phi(-z\sigma + \frac{a_s - e_0}{\sigma}) + e^{za_s} \cdot [1 - \Phi(\frac{a_s - e_0}{\sigma})] \quad (26)$$

That is,

$$P(z) = g(z) \cdot \Phi(-z\sigma + \frac{a_s - e_0}{\sigma}) + e^{za_s} \cdot [1 - \Phi(\frac{a_s - e_0}{\sigma})] \quad (27)$$

Take derivative, it becomes

$$\frac{dP(z)}{da_s} = \frac{g(z)}{\sqrt{2\pi}\sigma} e^{-\frac{(a_s - e_0 - z\sigma^2)^2}{2\sigma^2}} + ze^{za_s} \cdot [1 - \Phi(\frac{a_s - e_0}{\sigma})] - \frac{e^{za_s}}{\sqrt{2\pi}\sigma} e^{-\frac{(a_s - e_0)^2}{2\sigma^2}} \quad (28)$$

A.2 $Q(z)$

$$Q(z) \equiv E_T[e^{z(T\wedge a_r)}]$$

Since death age T is normally distributed, it becomes

$$Q(z) = e^{\frac{z^2\sigma^2}{2}+ze_0}\cdot[\Phi(-z\sigma+\frac{a_r-e_0}{\sigma})-\Phi(-z\sigma-\frac{e_0}{\sigma})]]+e^{za_r}\cdot[\Phi(\frac{T_{max}-e_0}{\sigma})-\Phi(\frac{a_r-e_0}{\sigma})] \quad (29)$$

A.3 $\lambda(a)$

$$\lambda(a) \equiv E_T[T\wedge a]$$

where $a = a_r$ or $a = a_s$.

$$\lambda(a) = E_T[T1_{T<a}] + E_T[a1_{T>a}]$$

Since

$$E_T[a1_{T>a}] = a[\Phi(\frac{T_{max}-e_0}{\sigma}) - \Phi(\frac{a-e_0}{\sigma})]$$

and

$$E_T[T1_{T<a}] = \int_0^a \frac{x}{\sqrt{2\pi}\sigma}e^{-\frac{(x-e_0)^2}{2\sigma^2}}dx$$

Therefore,

$$E_T[T1_{T<a}] = e_0[\Phi(\frac{a-e_0}{\sigma}) - \Phi(\frac{-e_0}{\sigma})] - \frac{\sigma}{\sqrt{2\pi}}[e^{-\frac{(a-e_0)^2}{2\sigma^2}} - e^{-\frac{e_0^2}{2\sigma^2}}]$$

Above all,

$$\lambda(a) = a[\Phi(\frac{T_{max}-e_0}{\sigma})-\Phi(\frac{a-e_0}{\sigma})]+e_0[\Phi(\frac{a-e_0}{\sigma})-\Phi(\frac{-e_0}{\sigma})]-\frac{\sigma}{\sqrt{2\pi}}[e^{-\frac{(a-e_0)^2}{2\sigma^2}}-e^{-\frac{e_0^2}{2\sigma^2}}] \quad (30)$$

where $a = a_r$ or $a = a_s$.

B Calculations on $K(t)$

In this part, we derive the equations for aggregate wealth $K(t)$. Note that the population structure is stationary. Let $v(a)$ be the net asset for a consumer at age a. By Yaari and Blanchard, $v(a)$ satisfies

$$\frac{dv(a)}{da} = [r+\mu(a)]v(a) + y(a) - c(a)$$

Under the boundary condition v(0)=0. Note that $c(a) = c_0 e^{ka}$, and $y(a) = wh(a) = we^{f(a_s)}, a_s < a < a_r$. We can solve this first order differential equation.

$$v(a)e^{\int_0^a -(r+\mu(m))dm} = \int_0^a [y(x) - c(x)]e^{\int_0^x -(r+\mu(m))dm}dx$$

$$v(a) = \frac{e^{ra}}{l(a)} \int_0^a [y(x) - c(x)]e^{-rx}l(x)dx \tag{31}$$

which is a function of age a.

Therefore,

$$K(t) = bNE_T[\int_0^T v(x)dx] = bNE_T[\int_0^T \frac{e^{rx}}{l(x)} \int_0^x [y(a) - c(a)]e^{-ra}l(a)dadx]$$

By the definition of expectation, it becomes

$$K(t) = bN \int_0^{T_{max}} e^{rx} \int_0^x [y(a) - c(a)]e^{-ra}l(a)dadx$$

Exchange the order of integration,

$$K(t) = bN \int_0^{T_{max}} [y(a) - c(a)]e^{-ra}l(a) \int_a^{T_{max}} e^{rx}dxda$$

$$K(t) = \frac{bN}{r} \int_0^{T_{max}} [y(a) - c(a)]l(a)[e^{r(T_{max}-a)} - 1]da \tag{32}$$

We can also write it in expectation form,

$$K(t) = \frac{bN}{r} E_T\{\int_0^T [y(a) - c(a)][e^{r(T_{max}-a)} - 1]da\} \tag{33}$$

Define

$$\phi(r, e_0, \sigma_0) \equiv \frac{1}{w} E_T\{\int_0^T [y(a) - c(a)][e^{r(T_{max}-a)} - 1]da\}$$

Substitute the income $y(a)$, it becomes

$$\begin{aligned}\phi(r, e_0, \sigma_0) = \frac{1}{w}[E_T\{\int_0^{T\wedge a_s} [-c(a)][e^{r(T_{max}-a)} - 1]da\}] \\ + \frac{1}{w}[E_T\{\int_{T\wedge a_s}^{T\wedge a_r} [we^{f(a_s)} - c(a)][e^{r(T_{max}-a)} - 1]da\}] \\ + \frac{1}{w}[E_T\{\int_{T\wedge a_r}^{T} [-c(a)][e^{r(T_{max}-a)} - 1]da\}]\end{aligned}$$

Note that

$$c_0 = \frac{(k-r)we^{f(a_s)}}{r} \cdot \frac{P(-r) - Q(-r)}{g(k-r) - 1} = we^{f(a_s)}\beta$$

where

$$\beta = \frac{(k-r)}{r} \cdot \frac{P(-r) - Q(-r)}{g(k-r) - 1}$$

Then

$$I = \frac{1}{w}[E_T\{\int_0^{T\wedge a_s} [-c(a)][e^{r(T_{max}-a)} - 1]da\}] = -e^{f(a_s)}\beta E_T\{\int_0^{T\wedge a_s} e^{ka}[e^{r(T_{max}-a)} - 1]da\}$$

By the definition of $P(k-r)$, it becomes

$$I = -e^{f(a_s)}\beta\{\frac{e^{rT_{max}}}{k-r}[P(k-r) - 1] - \frac{1}{k}[P(k) - 1]\} \tag{34}$$

For the second term,

$$II = E_T\{\int_{T\wedge a_s}^{T\wedge a_r} [e^{f(a_s)} - e^{f(a_s)}\beta e^{ka}][e^{r(T_{max}-a)} - 1]da\}$$

By the definition of $P(-r)$, $P(k-r)$, $P(k)$, $Q(-r)$, $Q(k-r)$, and $Q(k)$, it becomes

$$II = e^{f(a_s)}\{\frac{-e^{rT_{max}}}{r}[Q(-r) - P(-r)] - [\lambda(a_r) - \lambda(a_s)] - \frac{\beta e^{rT_{max}}}{k-r}[Q(k-r) - P(k-r)] + \frac{\beta}{k}[Q(k) - P(k)]\}$$

For the Third term,

$$III = \frac{1}{w}[E_T\{\int_{T\wedge a_r}^{T} [-c(a)][e^{r(T_{max}-a)} - 1]da\}] = -e^{f(a_s)}\beta E_T\{\int_{T\wedge a_r}^{T} e^{ka}[e^{r(T_{max}-a)} - 1]da\}$$

Thus

$$III = -e^{f(a_s)}\beta\{\frac{e^{rT_{max}}}{k-r}[g(k-r) - Q(k-r)] - \frac{1}{k}[g(k) - Q(k-r)]\} \tag{35}$$

Finally,

$$\phi(r, e_0, \sigma_0) = e^{f(a_s)} \cdot \varphi(r, e_0, \sigma_0) \tag{36}$$

where

$$\varphi(r, e_0, \sigma_0) = \frac{r-k}{kr}\frac{(Q(-r) - P(-r))(g(k) - 1)}{g(k-r) - 1} - \lambda(a_r) + \lambda(a_s)$$

C Analytical results

The steady interest rate must satisfy equations

$$f'(a_s)[P(-r) - Q(-r)] + \frac{dP(-r)}{da_s} = 0 \tag{37}$$

and

$$\frac{r-k}{kr}\frac{(Q(-r) - P(-r))(g(k) - 1)}{g(k-r) - 1} = \frac{\lambda(a_r) - \lambda(a_s)}{1-\alpha} \tag{38}$$

In the following part, we first want to show that the steady interest rate is not unique if it exists in the simplest case. Then we study the response of the steady interest rate to a change in the life expectancy and the variance of death age respectively.

In the simplest case, $a_s = 0$, $a_r = \infty$, $\gamma = 1$ and T is normally distributed for $r \in [\theta, \infty)$. The equation (20) can be written as

$$\frac{-\theta}{g(-\theta) - 1} \cdot \frac{g(-r) - 1}{-r} \cdot \frac{g(r-\theta) - 1}{r - \theta} = \frac{e_0}{1-\alpha} \tag{39}$$

Note that $\frac{g(r-\theta)-1}{r-\theta} \to e_0$ as $r \to \theta$. Therefore, the left hand side of equation (39) is e_0. Since $\alpha < 1$, we have $LHS < RHS$.

Moreover, let r_1^* be the value $\{r : g(-r) - 1 = 0\}$. Then, in the range $r \in (\theta, r_1^*)$, $\frac{-\theta}{g(-\theta)-1} > 0$, $\frac{g(-r)-1}{-r} > 0$ and $\frac{g(r-\theta)-1}{r-\theta} > 0$. Since $LHS = 0 < RHS$ when $r = r_1^*$. It is clear that there will be at least two steady interest rate if there exists a steady interest rate in the range (θ, r_1^*).

In our calibration, we always choose the steady interest rate to be the one which is greater than θ with the smallest value.

Now we study the response of the steady interest rate to a change in the life expectancy when the variance of death age is fixed. Define

$$G(r, e_0, v_0) = \frac{r-k}{kr}\frac{(g(k) - 1)(g(-r) - 1)}{g(k-r) - 1} - \frac{e_0}{1-\alpha}$$

For the steady interest rate, it is always true that $G(r, e_0, v_0) = 0$. For the reasonable range of parameters, for example, $e_0 \in (64, 94)$, $v_0 \in (140, 900)$. We find that $\partial G(r, e_0, v_0)/\partial r > 0$. Similarly,

$$\frac{\partial G}{\partial e_0} = -\frac{1}{1-\alpha} +$$
$$\frac{r-k}{kr}\frac{[-rg(-r)(g(k) - 1) + kg(k)(g(-r) - 1)](g(k-r) - 1) - (g(-r) - 1)(g(k) - 1)(k-r)g(k-r)}{(g(k-r) - 1)^2}$$

In the same range, $\partial G(r, e_0, v_0/\partial e_0 > 0$. Therefore,

$$\frac{dr}{de_0} = -\frac{\frac{\partial G}{\partial e_0}}{\frac{\partial G}{\partial r}} < 0$$

The response of the steady interest rate to a change in the variance of death age when the life expectancy is fixed can be studied similarly. We find that

$$\frac{dr}{dv_0} = -\frac{\frac{\partial G}{\partial v_0}}{\frac{\partial G}{\partial r}} < 0$$

References

[1] Bils, M., Klenow, J.P., 2000. Does schooling cause growth?, American Economic Review, 1160-1183.

[2] Blanchard, O.J., 1985. Debt, deficits and finite horizons, Journal of Political Economy, Vol. 93, 223-247.

[3] Blanchard, O.J., Fischer, S., 1989. Lectures on macroeconomics, The MIT Press.

[4] Bommier, A., Lee, R., 2002. Overlapping generations models with realistic demography: Statics and dynamics, UC Berkeley working paper.

[5] Boucekkine, R., Croix, D., Licandro, O., 2002. Vintage human capital, demographic trends, and endogenous growth, Journal of Economic Theory 104, 340-375.

[6] Deaton, A., 1992. Understanding consumption, Oxford University Press.

[7] Futagami, K., Nakajima, T., 2001. Population aging and economic growth, Journal of Macroeconomics, Vol. 23, 31-44.

[8] Kalemli-Ozcan, S., Ryder, E.H., Weil, D.N., 2000. Mortality decline, human capital investment, and economic growth, Journal of Development Economics, Vol. 62, 1-23.

[9] Lee, R., Carter, L., 1992. Modeling & forecasting the time series of U.S. mortality, Journal of the American Statistical Association, 659-671.

[10] Lee, R., Tuljapurkar, S., 1994. Stochastic population forecasts for the United States: Beyond high, medium, and low, Journal of the American Statistical Association, 1175-1189.

[11] Wilmoth, J.R., Deegan, L.J., Lundstrom, H., Horiuchi, S., 2000. Increase of maximum life-span in Sweden, 1861-1999, Science, 2366-2368.

[12] Wilmoth, J.R., Horiuchi, S., 1999. Rectangularization revisited: variability of age at death within human populations, Demography 36(4): 475-495.

[13] Weil, P., 1989. The equity premium puzzle and the risk-free rate puzzle, Journal of Monetary Economics 24, 401-421.

[14] Yaari, M.E., 1965. Uncertain lifetime, life insurance, and the theory of the consumer, Review of Economic Studies, Vol. 22, 137-150.