Forecasting Irregularly Spaced UHF Financial Data: Realized Volatility vs UHF-GARCH Models

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Abstract:
A very promising literature has been recently devoted to the modeling of ultra-high-frequency (UHF) data. Our first aim is to develop an empirical application of Autoregressive Conditional Duration GARCH models and the realized volatility to forecast future volatilities on irregularly spaced data. We also compare the out sample performances of ACD GARCH models with the realized volatility method. We propose a procedure to take into account the time deformation and show how to use these models for computing daily VaR.

Résumé :
Dans cet article, nous comparons les prévisions de variance obtenues à partir de modèles à très haute fréquence. Nous analysons la performance des modèles ACD-GARCH, ACD-GARCH augmenté et celui de la variance réalisée. Pour ce faire, nous prenons en compte le phénomène de la déformation temporelle, un problème souvent négligé, et nous agrégeons les résultats de façon uniforme d'un modèle à l'autre. Nos résultats montrent que la technique de la variance réalisée tend à surpasser les autres modèles d'analyse à très haute fréquence. Cette étude peut se révéler utile pour le calcul de la VaR sur un horizon très rapproché.

Keywords: Realized volatility, Ultra High Frequency GARCH, time deformation, financial markets, Daily VaR.

JEL classification: C22;C53;G14

1. Introduction[1]

The very recent implementation of electronic order-matching systems on financial markets has entailed increasing numbers and frequencies of trades. While data on prices and volumes were registered daily two decades ago, transactions (and especially those due to electronic systems) are now recorded instantaneously with an accuracy of a fraction of one second. The growing interest devoted to intra-daily models in the financial literature is a direct consequence of the availability of higher frequency measurements. This phenomenon stylized by increasing frequencies of observations is at the origin of the concept of ultra-high frequency. In this context, the development of econometric methods for the analysis of ultra-high frequency data seems to be promising. But the other side of the coin is the problem induced by the irregularity at which the observations arrive. For example, when we estimate a simple GARCH process on the S&P500, we usually use the returns observed every day or every week. In this case, the interval between each observation is fixed: one day or one week. But when analyzing intra-day observations, the information arrives sometimes in clusters and at different time intervals. This problem is called time deformation because time is not the same as calendar time. The fact that the arrival of information is irregularly spaced is a salient feature of ultra-high frequency data. Aggregates of these data up to fixed intervals of time entail an important loss of information. To avoid this loss, Engle and Russell (1998) and Engle (2000) have recently developed methods that are directly tailored to irregular spacing of the data. The basic model is the autoregressive conditional duration (ACD) model which is a type of

[1]This paper is based on previous research done by Racicot (2003).

dependent Poisson process. The ACD model applied to IBM transactions arrival times by Engle (2000) in GARCH framework has produced ultra-high frequency measures of volatility. The results observed by Engle (2000) are promising and indicate that this theoretical specification seems to be accurate to estimate models for ultra-high frequency data or transaction data. The ACD GARCH and extended ACD GARCH volatility models proposed by Engle (2000) warrant indeed very large gains in forecast error accuracy from a theoretical perspective. Considering this result, it would be very interesting to apply these models to ultra-high frequency data.

Therefore we propose two main contributions. Our first aim in this paper is to develop an empirical application of ACD GARCH models in forecasting future volatilities. Then we propose another contribution in comparing the performance of ACD GARCH models to a new and simple way of modeling financial market volatility using high-frequency data recently developed by Bollerslev and Wright (2001): the integrated volatility or recently called, the realized volatility (Dacarogna et al. (2001), Barndorff-Nielsen et Shephard (2002a), Andersen et al. (2003)). According to Bollerslev and Wright (2001), volatility dynamics may be modeled by fitting a long autoregressive (AR) representation to ultra-high frequency data. The main interrogation in their approach is that they ignore the fact that data arrive at irregular intervals. Thus, we have to make an adjustment to take into account these fundamental features of ultra-high frequency data.

The plan of the rest of the paper is as follows. Section 2 is devoted to the presentation of the models and their necessary adjustments. We also show how to use volatility forecasts to compute daily VaR. Section 3 contains a discussion of the ultra-high frequency data and the adjustment procedures employed. Section 4 details the volatility calculations and volatility forecasts, followed by a comparison of the results and a short discussion. Section 5 concludes with some suggestions for future research.

2. Ultra-high-frequency variance models

2.1 UHF GARCH(1,1) model

Ultra-high frequency GARCH(1,1) model allows taking into account the irregular character of market transactions even if the current durations of these transactions are not explicitly considered as additional sources of information. In this sense, it is the simplest model of conditional variance defined at ultra-high-frequency. This model may be written as follows:

$$\sigma_i^2 = \omega + \alpha\varepsilon_{i-1}^2 + \beta\sigma_{i-1}^2 \quad (1)$$

with σ_i^2, the conditional variance and ε_i, the innovation.

2.2 The ACD-GARCH model

The autoregressive conditional duration (ACD) model was firstly developed by Engle and Russel (1998). Later, Jasiak (1999), Gouriéroux et al. (1999), Gouriéroux and Jasiak (2001), and Engle (2000) refined and applied the model in a similar context.

The basic formulation of the ACD is specified in terms of the conditional density of the durations. The duration is the interval between two arrival times denoted by $x_i = t_i - t_{i-1}$.

The expectation of the ith duration is given by the following function:

$$E\left(x_i \middle| x_{i-1}, ..., x_1\right) = \theta\left(x_{i-1}, ..., x_1; \psi\right) \equiv \theta_i \qquad (2)$$

under the assumption that

$$x_i = \theta_i e_i \qquad (3)$$

where $\{e_i\}$~ i.i.d., and ψ is a set of parameters to be estimated. By definition, the conditional expectation of the duration depends on past durations.

A more general formulation of the model, called the ACD(p, q), may be given by the following specification:

$$\theta_I = w + \sum_{j=0}^{p} \alpha_i x_{i-j} + \sum_{j=0}^{q} \beta_j \theta_{i-j} \qquad (4)$$

where p and q are the orders of the lags. This specification may be used to study the marks associated with the arrival times so that hypothesis from the market microstructure theories may be tested.

Engle (2000) proposed a non-linear generalization of this model to define a measure of price volatility using transaction data and to analyse how the arrival time influences this volatility. Assuming that r_i is the return from transaction *i-1* to *i*, the conditional variance per transaction is defined as

$$V_{i-1}\left(r_i \middle| x_i\right) = h_i \qquad (5)$$

where x_i is defined as previously. The conditional variance is dependent on current and past returns and durations. As mentioned by Engle (2000), volatility is always measured over a fixed interval and is frequently reported in annualized terms. Therefore, the conditional volatility per unit of time is the most interesting measure to be evaluated. It is given by:

$$V_{i-1}\left(\frac{r_i}{\sqrt{x_i}}\middle| x_i\right) = \sigma_i^2 \qquad (6)$$

These two variances imply:

$$h_i = x_i \sigma_i^2 \qquad (7)$$

In this case, the forecasted conditional transaction variance may be defined as:

$$E_{i-1}(h_i) = E_{i-1}(x_i \sigma_i^2) \qquad (8)$$

The simple GARCH(1, 1) of Bollerslev (1986) may be extended to compute σ_i^2 with the following specification:

$$\sigma_i^2 = w + \alpha e_{i-1}^2 + \beta \sigma_{i-1}^2 + \gamma x_i^{-1} \qquad (9)$$

where x_i^{-1} is the reciprocal of duration. This model is called ACD-GARCH. In this specification durations are directly introduced into the conditional variance. It should be noted that the standard GARCH(1,1) wihtout ajustment for duration, while it is certainly not the best model for UHF data, is also used in the UHF literature for computing volatility forecasts. For example, daily VaR using this model are very simple to obtain. One simply have estimate the model using standard econometric package like EViews and then to compute a one step-a-head forecast. The VaR[2] number is then given by : $VaR = 1.65 \times \text{amount} \times \sigma_{i+1}$. Section 4.2 shows how it is simple to obtain forecast from GARCH(1,1), this might explain the popularity of the model.

[2] For more details on VaR computations using GARCH models, see Tsay (2005).

To give another financial illustration to this econometric model we can use the results obtained by Easley and O'Hara (1992). According to the market microstructure model of Easley and O'Hara (1992), a fraction of the investors is informed and knows if there are news concerning their assets. When it is time for them to do transactions, they will buy if the news are favourable, sell on bad news and will not trade if no news. In this model, long intervals (x_i) will be interpreted as no news. This implies that we expect a positive value for γ in our extended ACD GARCH model. Long durations indicate indeed that there are no news and consequently a lower volatility[3].

2.3 The extended ACD-GARCH

Engle (2000) suggests promising extensions and proposes a richer formulation allowing both observed and expected durations to enter the model. He also introduces a long run volatility variable defined by the following equation:

$$\sigma_i^2 = \alpha_0 + \alpha_1 e_{i-1}^2 + \beta\sigma_{i-1}^2 + \gamma_1 x_i^{-1} + \gamma_2 \frac{x_i}{\theta_i} + \gamma_3 \xi_{i-1} + \gamma_4 \theta_i^{-1} \tag{10}$$

where ξ_i is the long run volatility, θ_i is the conditional duration and might be defined by the parsimonious ACD(1,1) model. Engle (2000) proposes to compute the long run volatility by a Exponential Weighted Moving Average (EWMA) model on r^2 / x as

$$\xi_i = \lambda\xi_{i-1} + (1-\lambda)\frac{r_{i-1}^2}{x_{i-1}} \tag{11}$$

In this extended model for computing volatility using high frequency data, the influences of durations on volatility have been incorporated in three parameters. They measure the

[3] We can note that long durations cannot induce the conditional variance to be negative with this formulation.

effect of surprise in duration, the reciprocal duration and the expected reciprocal duration[4] respectively. As in any other GARCH models, forecasting volatility can be found simply by computing the conditional expectation and is given by:

$$E_{i-1}\left(\sigma_i^2\right) = \alpha_0 + \alpha_1 e_{i-1}^2 + \beta\sigma_{i-1}^2 + \gamma_1 E_{i-1}\left(x_i^{-1}\right) + \gamma_2 + \gamma_3\xi_{i-1} + \gamma_4\theta_i^{-1} \qquad (12)$$

This calculation led by Engle (2000) reveals us that parameter γ_2 is not persistent. However, parameters γ_1 and γ_4 indicate a long run influence on future volatilities because of the persistence[5] of durations.

2.4 A more parsimonious approach: realized volatility

While Engle (2000) approach for modeling and computing volatility using high-frequency data seems promising on the theoretical side of the coin, this approach is complicated by the fact that there is a lot of data manipulations that must be done before having an estimate of volatility.

The concept of *realized volatility* has been firstly developed by Andersen and Bollerslev (1998) and later applied for computing daily volatility forecasts of exchange rates and S&P 500 Index-Futures, respectively, by Bollerslev and Wright (2001) and Martens (2002) using the appellation integrated volatility. The relation between realized volatility and integrated volatility is well explained in Barndorff-Nielsen et Shephard (2002a, 2002b) and Meddahi (2002, 2003). Simply put, the realized volatility is measured

[4] The expected reciprocal duration is the expected rate of arrivals of transactions.
[5] As mentioned by Engle (2000) these models might be estimated by QMLE (quasi-maximum likelihood estimator) without specifying the density of the disturbances and using Bollerslev-Wooldridge (1992) robust standard errors.

by the squared value of intra-daily returns. This measure is also considered to be a more accurate measure of ex-post volatility. As Anderson *et al.* (2001) point out, assuming that the returns follow a special semi-martingale process, "the quadratic variation of this process constitutes a natural measure of ex-post realized volatility". It also corresponds directly to the theoretical definition of volatility used in diffusion and stochastic volatility models[6]. Taking into account time deformation, we can give a mathematical definition of realized volatility as follows:

$$\sigma_I^2(m) = \frac{1}{n}\sum_{n=1}^{N} r_{m,n}^2 \tag{13}$$

where $r_{m,n}^2$ is the *n*th squared return on day *m*. Because the returns are not observed at a constant interval, the numbers of observations N will vary from day to day. Compared to the UHF-GARCH model, we can easily observe the simplicity of the calculations required for obtaining an estimate of the volatility. As in the GARCH framework, it is possible to obtain a forecast of the integrated volatility. The method might be described as follows. The forecasts are based on a long memory autoregressive model where the lag p of the autoregressive process must approach infinity. The coefficients obtained from this autoregression are then used to construct a forecast function that takes the following form:

$$\hat{\sigma}_I^2(m) = \frac{1}{n}\sum_{n=1}^{N}\left(\hat{\mu} + \hat{v}_{N(m-1)+n|N(m-1)}\right) \tag{14}$$

where

$$\hat{v}_{t+1|t} = \sum_{j=1}^{\infty}\hat{\alpha}_j v_{t-j} \tag{15}$$

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[6] See Barndorff-Neilson and Shephard (2001), and Hull and White (1987) for an example of such uses.

$$\hat{v}_{t+k|t} = \sum_{j=1}^{k-1} \hat{\alpha}_j v_{t+k-j|t} + \sum_{j=k}^{\infty} \hat{\alpha}_j v_{t+k-j} \qquad (16)$$

and where the coefficients α_j might be estimated in the time domain by a long order autoregression[7], $v_t = \log(r_t^2) - \hat{\mu}$ and, where $\hat{\mu}$ is the sample mean of $\log(r_t^2)$. It should be noted that these coefficients might be estimated in a frequency domain using a Wiener-Kolmogorov filter. The results from using these techniques appear to be similar (Bollerslev and Wright, 2001). So in the following application, we use the long order autoregression on the log-squared returns[8] which we assume to be a martingale difference. More precisely,

$\alpha(L)\left(\log(r_t^2) - \mu\right) = e_t$ where $\alpha(L) = 1 - \alpha_1 L - \alpha_2 L^2 - \alpha_3 L^3 - \ldots$, $e_t \sim WN(0,\sigma^2)$ and the lagged polynomial is assumed to converge. So to implement the forecasting formula represented by equation (14), we simply have to fit a long order autoregression to the log-squared returns and use this estimated equation to compute our forecasts. This point is made clearer in the following section. Because the log-squared returns may yield large negative numbers for returns close to zero, we have applied the following transformation:

$$r_t^* = \log(r_t^2 + \tau s^2) - \frac{\tau s^2}{r_t^2 + \tau s^2} \qquad (17)$$

where s^2 is the sample variance of r_t and τ is chosen to be equal to 0,02 as in Fuller (1996) and Breidt and Carriquiry (1996).

[7] More precisely, we make the assumption that the time series of volatilities may be represented by an appropriate proxy such as the log-squared returns which has an autoregressive representation.
[8] As alternative hypothesis, we might specify that the squared or absolute returns has an autoregressive representation (see Bollerslev and Wright, 2001).

3. Data and measurements

We have to remind that our first aim is to compare Engle's UHF-GARCH model (2000) with the realized volatility, which is also called the integrated volatility concept by Bollerslev and Wright (2001). Therefore, we use the sample of observations used by Engle (2000). The irregularly spaced ultra-high frequency data are the transaction quotes for IBM stocks. The data were abstracted from the Trades, Orders Reports, and Quotes (TORQ) data set constructed by Joel Hasbrouck and NYSE. Two types of random variables compose the transaction data: the time of transactions and the marks at this time[9]. In our application, we define a point process as the time at which a transaction occurred. The marks are volumes of shares, prices, bid and ask prices of the traded contract at the transaction time. Our data set includes around 60000 transactions made on the NYSE from November 1, 1990 through January 31, 1991. Only the trades occurring between 9:30 AM and 4:00 PM are used for calculations[10]. Following Engle (2000), we delete transactions on Thanksgiving Friday and the day before Christmas and New Years, as well as all transactions without a reported set of quotes. According to Engle (2000), this procedure leaves 52146 unique transaction times. We consider only the unique times and remove all zero durations. As justified by Engle (2000): "This is consistent with

[9] The information about bid and ask quote movements, the volume associated with the transactions, the transaction prices, and a time stamp measured in seconds after, reflecting the time at which the transaction occurred, are considered in the data set.

[10] We have to mention that two days have been deleted from the 63 trading days in the 3 month sample. As explained by Engle (2000) : "A halt in IBM trading of just over an hour and 15 minutes occurred on Friday, November 23. On December 27th there was a one and a half hour delay in the opening".

interpreting a trade as a transfer of ownership from one ore more sellers to one or more buyers at a point in time".

We may add that the average volume corresponding to each time stamp is 1861 shares. Moreover, the minimum time between events is 1 second and the maximum duration is 561 seconds (or 9 minutes and 21 seconds). A simple description of the data used by Engle (2000) shows that the average duration between successive events for IBM is 28.38 seconds with a standard deviation of 38.41.

As mentioned by Engle and Russell (1998), we have to seasonally adjust the data to take out the time of day effect. An important literature has been devoted to this effect. It induces indeed a higher frequency of transactions near the opening and the closing of the market. The procedure we use is called by Engle and Russell a "diurnally" adjustment. Therefore, we define an adjusted duration given by the following equation:

$$\tilde{x}_i = \frac{x_i}{\varphi(t_{i-1};\beta)}$$

where $x_i = t_i - t_{i-1}$ is the duration between trades and $\varphi(.)$ is a piecewise linear spline function used to seasonally adjust the durations. Exhibit 1 gives an illustration of a linear spline. The knots are the points where the linear pieces of the splines join together. They appear at 9:30, 10:00, 11:00, 12:00, 1:00, 2:00, 3:00, 3:30, 4:00.

[Please insert exhibit 1]

Specifically, the seasonal adjustment is done by regressing the durations on the time using a linear spline[11] specification that takes the following form:

$$x = c + \beta_1 t_1 + \beta_2 t_2 + \beta_3 t_3 + \beta_4 t_4 + \beta_5 t_5 + \beta_6 t_6 + \beta_7 t_7 + \beta_8 t_8 + e$$

where t_{i-1} for i=2,…,9 are vectors of time variables constructed from the knots. From this regression we obtain $\hat{x}_i = \varphi(t_{i-1};\hat{\beta})$. The resulting variable $\tilde{x}_i$, which is free of the typical time of day effect, represents fractions of durations below or above normal.

4. Analysis and results

4.1 Volatility calculations: a comparison[12]

We have to notice that the two methods discussed to compute the daily volatility present important differences. Thus an adjustment is necessary before we can make a comparison. The UHF-GARCH specification gives volatility calculations per seconds while the integrated volatility gives a volatility estimate for a day. We have to transform the models to obtain comparable units. The UHF-GARCH calculation is transformed on a daily basis[13]. We introduce a new and obvious procedure which has not been explored in the literature. We proceed by analogy with the integrated volatility calculations. More precisely, we suggest that we average out the intra-day volatilities to define a daily volatility as:

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[11] Note that we used a linear spline. We might have used a k*th*-order spline which is a piecewise polynomial approximation, with polynomials of degree k differentiable k-1 times everywhere.

[12] The maximum likelihood estimator is used to estimate the parameters of all our UHF-GARCH models.

[13] We can mention that this transformation would be accurate to evaluate one-day options.

$$\sigma_d^2 = \frac{1}{N}\sum_{i=1}^{N}\sigma_i^2$$

where σ_i^2 is obtained by estimating high-frequency GARCH models. In table 1, we summarize the results observed for the four different methods used to compute the daily volatility: integrated volatility, GARCH(1, 1), ACD GARCH and Extended ACD GARCH. We compute volatilities for five consecutive days using our intra-daily transactions on IBM stock for the first week of our sample[14]. A first glance at the results confirms immediately the accuracy of our proposition made to compare volatility calculations.

[Please insert table 1]

We observe indeed that all the GARCH calculations follow quite closely the realized volatility methodology. This result is reassuring and confirms our first intuition. As explained in Bollerslev & Wright (2001) and as shown in table 2, the simple GARCH (i.e. GARCH(1, 1)) has the worst performance compared to the realized volatility for high-frequency data. This observation is not surprising.

[Please insert table 2]

The Extended ACD GARCH seems to have the best performance among the GARCH models in our comparison. However, this first glance at volatility calculations needs to be completed by a more detailed analysis. To have a better idea on the performance of these volatility models, we compute standard measures such as the R-

[14] The first week begins on a Thursday in November 1990. There are no particular reasons that explain why we have chosen this specific segment of time to make our calculations. It is simply for comparisons of calculations.

squared of the Mincer-Zarnowitz (1969) regression. Bollerslev and Wright (2001) and Martens (2002) underline the accuracy of this procedure to evaluate the performance of models of time-varying conditional heteroskedasticity. They use ultra-high frequency data, but they make the assumption that the data arrive at constant intervals[15], which is of course not the case in reality. As acknowledged by Engle (2000) ultra-high frequency data arrive at irregular intervals. Considering these unsatisfactory approaches, we propose to use the concept of autoregressive conditional duration introduced by Engle and Russell (1998) and improved by Engle (2000).

In the next section we implement our approach. Then we present the forecasting performances of our four models.

4.2 Volatility forecasts: a comparison

Our aim is to make volatility forecasts based on our four models and then to compare their performances. As suggested by Bollerslev and Wright (2001), we use the Mincer-Zarnowitz R-squared[16]. For each model we proceed as follows to compute forecasts. The

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[15] Bollerslev and Wright (2001) consider a data set with five-minute return series. For 24-hour markets, "there are 288 five-minute observations in a day". An other application of this method may be found in Bollerslev and Zhang (2003).

[16] Other popular measures might be used such as the mean absolute error (MAE), the root mean squared error (RMSE), the heteroskedasticy adjusted mean absolute error (HMAE) or the heteroskedasticy adjusted root mean squared error (HRMSE). HMAE is defined as $\frac{1}{T}\sum_{t=1}^{T}\left|\left(1-\frac{\text{Realized}_t}{\text{Forecast}_t}\right)\right|$ and HRMSE = $\sqrt{\frac{1}{T}\sum_{t=1}^{T}\left(1-\frac{\text{Realized}_t}{\text{Forecast}_t}\right)^2}$ where the forecasted errors are adjusted for heteroskedasticity. See Andersen et al.(1999) and Martens (2002) for an application of the last two measures.

forecasts induced by the realized volatility[17] are given by our equation (16). We know that an appropriate proxy for the time series of volatilities has an autoregressive representation. One can easily demonstrate that this specification may be approximated by a simple ARMA(1, 1)[18]. Our forecasts are based on the estimation of that process.

The computation of forecasts from a simple GARCH(1, 1), commonly used in finance, can be done by using the following formula[19]:

$$E_t {\sigma_{t+n}}^2 = \frac{1-(\alpha_1+\beta)^n}{1-\alpha_1-\beta}\alpha_0 \quad + \quad {\sigma_t}^2(\alpha_1+\beta)^n$$

$$= \frac{\alpha_0}{1-\alpha_1-\beta} - \frac{\alpha_0(\alpha_1+\beta)^n}{1-\alpha_1-\beta} \quad + \quad {\sigma_t}^2(\alpha_1+\beta)^n \qquad (18)$$

$$= \frac{\alpha_0}{1-\alpha_1-\beta} \quad + \quad (\alpha_1+\beta)^n\left({\sigma_t}^2 - \frac{\alpha_0}{1-\alpha_1-\beta}\right).$$

We have to mention that equation (18) might be used for computing forecasts at any horizon simply by replacing *n* by a value of interest[20].

The computation of forecasts based on the ACD and extended ACD GARCH models seems much more complicated. We need indeed expected values of durations. This

[17] For a recent application of this method to forecasting and a discussion of the problem of time aggregation, see Gosier, Madhavan, Serbin and Yang (2005).
[18] Bollerslev and Wright (2001) have used an AR (2050) to fit their ultra-high frequency data observed at fixed intervals. It is well known in the econometric literature that high order autoregressive models may be approximated by parsimonious ARMA models. For example see Mills (1990), Hamilton (1994) or Racicot and Théoret (2001).
[19] The index n represents the number of steps.
[20] When $\alpha_1+\beta=1$, the conditional expectation of volatility n periods ahead is instead $E_t({\sigma_{t+n}}^2) = {\sigma_t}^2 + n\alpha_0$.

problem may be solved easily. We know that x_i is related to θ_i. So, we assume that the expectation of the durations may be represented by a simple ARMA(1, 1) process[21]. Then, we use the forecasted values of the durations induced to include them in the ACD GARCH and in the extended ACD GARCH models. The main disadvantage of this procedure is the increasing number of computations we have to perform to get a forecast[22]. For our purpose we use a simple ARMA(1, 1) specification. Thus the process is defined by the following equation:

$$x_i = v + \alpha x_{i-1} + \beta \varepsilon_{i-1} + \varepsilon_i \qquad (19)$$

where $\varepsilon_i \equiv x_i - \theta_i$ is a martingale difference (i.e. $\varepsilon_i = x_i - E_{i-1}(x_i)$). Forecasted values of x_i can be obtained from (19) and then included in equation (12).

Table 3, presented below, shows the results of the performance of the GARCH models compared with the realized volatility.

[Please insert table 3]

As we can see if we compare the RMSE, MAE or the R^2 of the Mincer-Zarnowitz[23] (1969) regression, the realized volatility method outperforms all the GARCH models. However, we have to moderate this result. It must be noted that none of the numbers presented in this

[21] We observe that the expectation of the duration has the same type of representation as the conditional duration. As suggested by Engle and Russell (1998), we use an ARMA process.

[22] Another approach is to assume that the durations may be represented by log linear regression models.

[23]To obtain the Mincer-Zarnowitz regressions we regress the ex post realized values of the variable under scrutiny on the forecasted values of this variable plus a constant term. For example, in our case, we forecast the IBM prices for different sample sizes: 700, 1400 and 2100, and then we run the regressions (i.e. $y_t^* = c + \beta y_{ft}^* + \varepsilon_t$). The resulting R^2 are shown in table 3.

table is significant. Nevertheless, for the realized volatility, the t statistics of Mincer-Zarnowitz are near significance level[24].

The poor performances of the four models are not so surprising and the results are consistent with previous studies devoted to forecasting models[25]. As mentioned by Andersen, Bollerslev and Lange (1999), the standard GARCH volatility models tend to perform very poorly when they are applied directly to UHF data. But it is deceiving that the ACD GARCH models do not perform better in comparison with the simple realized volatility method. In fact, it would be possible to conclude that the gains in forecast error accuracy from ACD GARCH and extended ACD GARCH models remain large from a pure theoretical perspective.

The simple way of modeling financial markets volatility using ultra-high frequency data introduced by Bollerslev and Wright (2001), the realized volatility, tends to perform better than the standard and autoregressive conditional duration GARCH models. However, improvements have to be made to take into account the fact that data arrive at irregular intervals.

5. Conclusions

In this paper we have proposed an application of the autoregressive conditional durations GARCH models (ACD GARCH) recently developed by Engle and Russel (1998) and

[24] More precisely, the t statistics are 1.75 (0.08), 1.82 (0.06), and 1.80 (0.07) for sample sizes of 700, 1400, and 2100 respectively. Their corresponding p-values are in brackets.
[25] See Mills (1999).

Engle (2000). We have used them to make volatility forecasts before comparing their performances with the recent concept of realized volatility introduced by Bollerslev and Wright (2001). To take into account the time deformation induced by the fact that ultra-high frequency data arrive at irregular intervals, we have made some assumptions and proposed adjustments.

Our results show that the realized volatility seems to be better, in terms of RMSE, MAE, and Mincer-Zarnowitz (1969) criterions, than any of UHF-GARCH models. Although none of the tested models has well performed on the IBM stock data used for the empirical analysis, it is quite deceiving that the ACD GARCH has not performed better than the realized volatility. It is well known in the literature that when using GARCH models to forecast higher frequency data, they perform very poorly[26]. Nevertheless, the theoretical improvement developed by Engle to take into account time deformation seems to show poor performances when using to forecast volatilities.

As suggested in an other framework by Donaldson and Kamstra (1997), we think it would be very interesting to improve the forecasting power by adding an Artificial Neural Network component in the ACD GARCH model. Moreover, as in Engle (2000) we have used linear splines to adjust the data, using non-linear splines may improve the results. We leave all these issues for future research.

[26] See Andersen et al. (1998) and (2001).

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Exhibit 1.
Representation of the linear spline

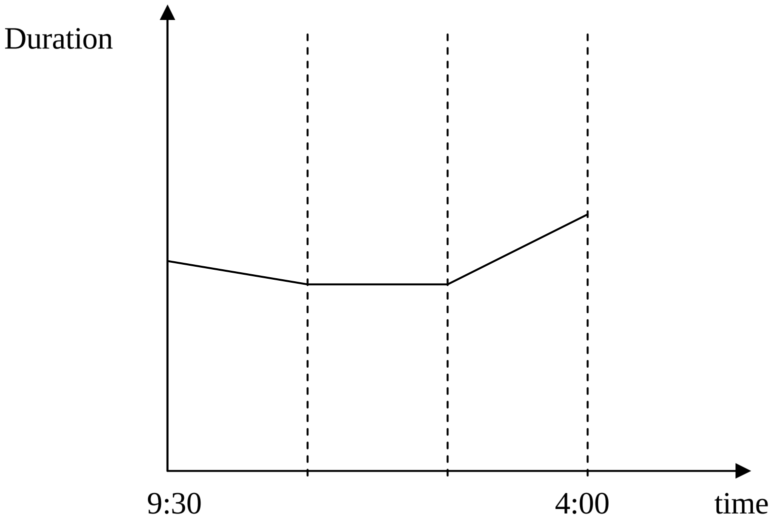

Table 1 Volatility computations (in sample)*

Day	*Realized volatility*	*Simple GARCH*	*ACD GARCH*	*Extended ACD GARCH*	*Number of observations*
Thurs.	3.18	3.68	3.69	3.24	688
Friday	9.13	12.04	12.17	7.49	792
Monday	2.59	3.23	3.04	3.15	671
Tuesday	6.16	6.97	6.96	5.76	732
Wednes.	3.58	3.62	3.62	3.27	649

* The simple GARCH(1,1) model is estimated by using equation (1), the ACD GARCH, by using equation (9) and the extended ACD GARCH, by using equation (12). Realized volatility is given by equation (13).

Table 2

Average absolute percentage changes (in sample)*

Day	*Simple GARCH*	*ACD GARCH*	*Extended ACD GARCH*
Thursday	15.72%	16.04%	1.89%
Friday	31.87%	33.30%	17.96%
Monday	24.71%	17.37%	21.62%
Tuesday	13.15%	12.99%	6.49%
Wednesday	1.12%	1.12%	8.66%
Average	**17.31%**	**16.16%**	**11.32%**

* Table 2 is simply a recast of table 1. As realized variance is the benchmark, we express the data of table 1 as percentage deviations from this benchmark. This table is used to compare the three GARCH models to the model of realized variance, which is the simplest to compute.

Table 3

Forecasts evaluation of GARCH models and realized volatility*

Number of observations	*Simple GARCH*		*ACD GARCH*		*Extended ACD GARCH*		*Realized Volatility*	
700	RMSE :	13.12	RMSE :	13.11	RMSE :	10.94	RMSE :	2.26
	MAE :	3.84	MAE :	3.84	MAE :	3.91	MAE :	2.03
	R^2 :	0.0002	R^2 :	0.0001	R^2 :	0.0006	R^2 :	0.0044
1400	RMSE :	14.83	RMSE :	14.83	RMSE :	12.47	RMSE :	2.23
	MAE :	4.35	MAE :	4.35	MAE :	4.34	MAE :	1.99
	R^2 :	0.0001	R^2 :	0.0001	R^2 :	0.0003	R^2 :	0.0024
2100	RMSE :	15.55	RMSE :	15.54	RMSE :	13.02	RMSE :	2.17
	MAE :	4.31	MAE :	4.31	MAE :	4.32	MAE :	1.97
	R^2 :	0.00008	R^2 :	0.00008	R^2 :	0.0002	R^2 :	0.0015

* The evaluation of forecasts is performed for forecasting samples of 700, 1400 and 2100 observations, which are supposed to represent the number of possible transactions in one day. These numbers are based on table 1. To compare these samples, we aggregate the observations of everyone by using the formula: $\sigma_d^2 = \frac{1}{N}\sum_{i=1}^{N}\sigma_i^2$, with N the number of transactions in one "day" and where σ_i^2 is obtained by estimating the ACD GARCH models.