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Abstract *The present value of growth opportunities with stable long run value and decreasing investment cost is addressed in a real options perspective. The model is solved in terms of closed form solutions, and a duality between elementary real options of waiting to invest is conjectured to be a fundamental structure of a forthcoming theory of real options. A pure capital budgeting perspective is pursued. Natural lines for future research are accounted for.*

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1 Introduction

Real Options (RO) represent a major tool of analysis in modern capital budgeting and corporate strategy. Standard models recognize basic drivers of the value of a RO on the *asset* side of projects. The present contribution sets forth a perspective in which the value drivers of a RO are recognized on an equal footing on both the asset and liability sides of the investment opportunity. A fundamental *duality* is thus assessed between standard models and a RO model in which it is the diminishing pattern of the investment expenditure which drives the value of the RO. The conjecture set forth by the present contribution is that such a duality might contribute to sharpen the understanding of the RO approach up to the point of building a RO *theory*, bounded by well defined conditions for the existence of a value embedded into managerial flexibility, and tailored along simple and elegant lines.

Myers (1977) introduced the expression 'real options' to denote the firm's growth opportunities associated with assets *not* in place. The 'real' specification was meant to distinguish RO from *financial* options, which a few years before had been given a foundational status by Black and Scholes (1973) and Merton (1973) by means of the BSM model for European options. The seminal analysis by Myers had an effect in driving capital budgeting towards the recognition of the bundle of RO embedded within the firm as the key issue in a forward looking perspective on the firm. It is by now undisputed that RO provide coherent quantitative representations of the value of the firm as *going concern* under uncertainty, a value not (yet) embedded in book values of assets in place, and yet reflected in the expectations of investors, which aggregate in the market value of the firm. The present value of growth opportunities (PVGO) is meant to measure the gap.

In first instance the value of the firm must encompass the net present value (NPV) of expected cash flows. That may be the whole story. The expression *cash cows* denotes firms which guarantee stable and huge cash flow prospects, generated by assets already in place. On the other hand, there may be more to the value of the firm, provided a bundle of growth opportunities can be conceived, and a suitable plan of sequential investment expenditures devised, so as to capitalize on *nested* investment opportunities; "strategy involves choosing among and committing to long-term paths or trajectories of competence development" (Smit and Trigeorgis, 2004, p. 91). The typical case is a firm expanding its capacity in order to enhance competitive advantage in market shares and experiencing. RO methods are meant to provide quantitative valuations in such respects, thereby contributing a representation of corporate strategy in which investments decisions lie at the core of the quest for market power.

A RO resembles to some extent a perpetual *call* option, the right, without the duty, to exercise an investment opportunity, i.e. to buy a definite asset at a specified price, the cost of the *real* investment expenditure. Myers (1977, p. 149) points out that the value of such RO "depends on the rule for deciding whether the options are to be exercised", i.e. the complete investment plan contingent on the realization of definite states of nature. That is what RO models are supposed to yield. As is well known, the monopolistic nature of the investment opportunity inherently underpins basic RO models; competition can be introduced as a further step of analysis (see for instance Grenadier, 2002 for the role of RO in a Cournot competition setting). At a rapid pace, a quantitative RO approach has developed and grown into a *comprehensive* line of research, whose conceptual foundations enable to represent the required returns on *irreversibility* of the investment expenditure, as well as the value of operational *flexibility* (Trigeorgis, 1996), if any. A whole 'taxonomy' of RO has been tailored, ranging through expanding, mothballing, abandoning, switching between different mixes of inputs or outputs, and much more. The generality of the approach has reverberated RO in both the professional direction and towards foundational perspectives.

On the one hand, RO analysis is now a firm pillar of applied corporate finance (see for instance Arnold and Shockley, 2002, Copeland and Antikarov, 2003, Bulan, 2005). RO investment triggers represent a step forward with respect to discounted cash flow (DCF) analysis, in that, in first instance, they exploit the role of the *volatility* of profit prospects in business judgement rules: "the firm requires a higher return to invest when volatility is higher, but it does so exactly because it is more likely to encounter periods of very low returns" (Dixit and Pindyck, 1994, p. 422). In fact, "DCF analysis does not reflect the value of management" (Brealey and Myers, 2003, p. 617). On the other hand, scholars have been led to argue about the extent to which RO should be made to fit *fundamental* theoretical perspectives, like competitive equilibrium analysis (Dixit and Pindyck, 1994, ch. 8). Such top-down and bottom-up research lines seem to point at a generality of the *structure* of the RO approach, which is worth attention in and by itself. The aim of this paper is to set forth a RO model in continuous time (original, to the author's knowledge) which admits both a transparent interpretation and a closed form solution, thereby contributing, in first instance, to the conceptual understanding of the structure of the RO approach. In fact, the structure of the RO approach is currently investigated on computational grounds, see for instance Nagae and Akamatsu (2008).

The option of waiting to invest (McDonald and Siegel, 1986) can be considered the *elementary* RO in continuous time, in that the minimal stylized fact is accounted for, namely, the choice of the investment trigger as a realization of a geometric brownian motion[1] (GBM) with positive drift, with respect to a constant lump sum expenditure. The 'inverse' problem is optimal disinvestment once the expected value of the asset is supposed to decrease at a definite rate (see for instance Lambrecht and Myers, 2007). Evidently, not all investment opportunities are encompassed in such schemes: investment opportunities exist on growth processes which approach a constant asymptotic value, for instance the level

[1] We refer to Karatzas and Shreve (1998) for a thorough account of the Markov and martingale properties represented in brownian motions.

at which a market saturates. The saturation of markets represents a key issue in the global economy, in which competition is worldwide and *fast*, focused on the time horizon which entrants will take to satiate an unsatiated demand, which typically corresponds to a barely constant level in the long run. The case of LG Semicon and Hyundai Electronics entering too late the DRAM market in the nineties is by far a textbook (Hill, 2007, ch. 11) case.

In this paper we set forth a RO of waiting to invest for which the dynamics of the value of the investment opportunity approaches a deterministic limit, and the investment expenditure is a decreasing lump sum. We leave it to a broader audience to establish the applicative relevance of the present model, which may depend on the role of the project in a broader investment strategy, and/or on employing definite financial architectures. Recall that diminishing costs are related to technological progress and to the trend of *learning curves* (Spence, 1981), a typical exponential damping. Thus, a *service* may be offered to intercept a non satiated demand, with costs diminishing as the enterprise gains experience. The role of diminishing costs in strategic investment has long been recognized. “A cost leadership position is more likely to be attractive when demand’s sensitivity to price is highly elastic, when the product is more commodity-like, and when customer services are hard to differentiate” (Smit and Trigeorgis, 2004, p. 59). Our model is meant to assess a basic theoretical role for such an issue, at the same time, admittedly, leaving somewhat unspecified the interpretations of the model. A typical pattern of deterministic monotonic growth with asymptotic stabilization is the logistic one. In fact, RO models on stochastic underlying processes which generalize such a pattern suffer from encompassing special functions (see for instance Alvarez, 2000) which make it rather cumbersome to grasp economic insights. The point of the present contribution is to represent the dynamics of the value of assets not in place by means of a GBM process, therefore paving the way to standard computational recipes. The simplicity of the present model aligns with seminal contributions, in first instance in the *multiplicative* representation of the relation between the NPV and RO triggers, which disentangles the effects of value drivers, as discussed in sections 3 and 5.

RO have reshaped the *foundations* of the science of investment under uncertainty, allowing a natural coupling to strategic analysis (see for instance Thijssen, 2008, Roques and Savva, 2009, and references therein), thereby finding a way in the Industrial Organization perspective. Among the various forms of competition, one can embed the bundle of growth options which characterize the firm in a growth phase. Furthermore, *incentive* scheme have been coupled to RO problems, for instance by Grenadier and Wang (2005) and Schianchi and Mantovi (2007), therefore establishing a quantitative representation of the effect of agency issues in shifting to suboptimal investment policy. On the other hand, Roemer (2004) tailors a parallel between the RO approach and Transaction Costs Economics so that the irreversibility of the investment expenditure corresponds to the specificity of the investment in the institutional framework with respect to which transaction costs are assessed. Recall that in first instance the firm and the pricing of corporate liabilities is the perspective in which financial option emerged (Black and Scholes, 1973).

Such a wide perspective in the background, the present analysis sets forth a model meant to sharpen the conditions for the applicability of RO methods as well as the intelligibility of the embedded insights. We address a pure capital budgeting problem in continuous time. A risk neutral agent faces an exogenous stochastic process representing the dynamics of the value of the investment opportunity; such a process approaches a finite asymptotic limit, which sets the fundamental scale of out problem. A constant risk free discount rate defines the value of time and the benchmark for returns on risk free investments. We then postulate a perfect frictionless capital market in which *spanning* holds, so that contingent claims analysis represents the no-arbitrage condition yielding the differential equation for the value of the RO. The model does not encompass the financing side of the investment, i.e. the cost and the structure of the capital of the firm represented by the risk neutral agent. Needless to say, our model can be taken as disentangled from the financial policy to the extent that the expectation about costs and revenues are not influenced by the liability side of the balance sheet. As already stated, our choice of addressing a pure capital budgeting problem is a methodological one. Such being the rationale of the analysis, we shall confine the mathematical details to a bare minimum, make our model fit the orthodox RO perspective and point out the original insights.

As is well known, the differential equations of RO represent the Bellman equations of a dynamic programming approach to the optimality of the investment. The present analysis focuses on the contingent claims approach as a *perfection* hypothesis on the elementary stylized fact. Dynamic programming is well known to be the natural setting in which complex problems, for instance incremental investment, can be formulated. The coherence of our contingent claims approach is not spoiled by the irreversibility of the *real* investment expenditure, as Brealey and Myers (2003) point out. Correspondingly, Merton (1977) points out that contingent claims reflect the properties of the market embedded in the value of the underlying, such as, for instance, the price of risk. Along the same lines, Brennan and Schwartz (1985, p. 154) point out the properties of the market embedded in the self replicating portfolio, “on the assumption that such portfolios may be formed by trading in future contracts in the output commodity”. The minimal model we set forth is a full fledged consistent *pure* investment problem, in which essential *ideal* properties of the market can be embedded in the value of the underlying. “Unless there are bankruptcy costs, the Modigliani-Miller theorem holds, so the firm’s real investment decisions are independent of its financial structure” (Dixit and Pindyck, 1994, p. 174). The plan of the paper is as follows. In section 2 we sketch the problem of the optimal deterministic investment problem. In section 3 we set forth the RO model. In section 4 we argue about a duality between our model and standard RO models. In section 5 the comparative statics for the crucial value drivers exploits the basic insights. Finally, section 6 tailors major lines of relevance for the model.

2 Deterministic trigger

The focus of traditional capital budgeting was on break-even conditions and present values computed via risk-adjusted discount rates. DCF analysis is meant to assess a *spread* over the risk free rate which accounts for the effect of uncertainty on expected returns (possibly, utilities). The improvement brought in by RO is to let operational flexibility enlarge the computational perspective upon profit prospects. As a typical fact, operational flexibility may enable one to choose the time of exercising an investment opportunity (a freedom which is far from guaranteed in a competitive environment). Dixit and Pindyck (1994, section 5.1.A) start their analysis by establishing the optimal investment rule for the opportunity to purchase the property rights of an asset whose value grows at a constant rate, and whose cost, a lump sum expenditure, is constant. We shall refer to such a problem as DO_1, and to the corresponding RO problem in which the underlying is a GBM as RO_1. Along similar lines, let us consider the problem of purchasing the property rights on an asset whose value *saturates* at a constant rate, and whose cost, again a lump sum expenditure, decreases at the same rate. The model is as follows.

Let r be the risk free discount rate. Let the asset side of the problem be defined by the deterministic dynamics of the value $S(t)$ of the investment opportunity, assumed to be

$$S(t) = \Xi\,(1 - e^{-at}) \equiv \Xi\, x(t) \equiv \Xi\,(1 - y(t)) \quad , \tag{2.1}$$

a monotone growth pattern which exponentially converges to the limit Ξ. Such a parameter defines the fundamental scale of the problem, which is naturally interpreted as the target long run NPV of the project. Initial time is set at the vanishing of the state variable S. The normalized dimensionless state variables x and y are meant to factor out the scale Ξ. The dynamics of the state variables $S(t)$ and $y(t)$ is governed by the ordinary differential equations (ODE)

$$dS = -\alpha(\Xi - S)dt \quad , \qquad dy = -\alpha\; y\; dt \tag{2.2}$$

which represent the same dynamic content; we shall take advantage of the possibility of switching between the two equivalent descriptions. Notice that the growth pattern (2.1) may represent an approximation to a logistic growth which has already reached the asymptotic phase.

Let the liability side of the problem be defined by a lump sum cost which guarantees the property rights upon the asset. Let such a lump sum cost decrease with time as given by

$$C(t) = I\; e^{-at} = I\; y(t) \tag{2.3}$$

with the *same*[2] rate α as in (2.1). The initial amount I of the investment cost is the second scale of our model, which only represents cost at the initial time, when the project has vanishing value. The profitability (NPV of the asset *once* in place) of the project being the present value of the difference between such quantities, the recognition of the very existence of a value in waiting to invest is our first issue.

The break even time

$$t_{\text{break even}} = \frac{1}{\alpha}\ln\frac{\Xi + I}{\Xi} \equiv \frac{1}{\alpha}\ln\xi \quad , \qquad \xi \equiv \frac{\Xi + I}{\Xi} \tag{2.4}$$

establishes the instant at which the project is becoming worth. Notice that $t_{\text{break even}}$ has the correct dimension of time and that the argument of the logarithm is dimensionless. In fact, the parameter ξ sets a natural benchmark for the investment problem at hand. The monotonicity represented in (2.1,3) enables us to state the break even condition in terms of state variables, namely

$$S_{\text{break even}} = \frac{I}{\xi} = \frac{\Xi\, I}{\Xi + I} \quad , \quad y_{\text{break even}} = \frac{1}{\xi} = \frac{\Xi}{\Xi + I} \tag{2.5}$$

which represent a deterministic benchmark for RO approaches. Formulas (2.5) embody the relative effects of the scales Ξ and I in determining the break even point.

Define the optimization problem DO_2 as

$$t^* = \underset{t \in [0,\infty)}{\arg\max}\left(-Ie^{-\alpha t} + \Xi(1 - e^{-\alpha t})\right)e^{-rt} \quad , \tag{2.6}$$

[2] Such an exact identity leads to simple closed form solutions. In section 4 we set forth a profound significance for the identity. It is a natural conjecture that the qualitative picture resulting is *stable* with respect to the small differences in the two rates.

that is, the optimality of the exercise is defined so as to maximise the discounted profitability of the investment opportunity. The unique stationary point

$$t_0 = \frac{1}{\alpha}\ln\left(\frac{r+\alpha}{r}\xi\right) = \frac{1}{\alpha}\ln\left(\frac{b+1}{b}\xi\right), \quad b \equiv \frac{r}{\alpha} \tag{2.7}$$

is an interior maximum: our model encompasses a value in waiting to invest, since revenues from exercising increase and the expenditure decreases. Evidently, the existence of a value in waiting depends on the domain of the optimization problem: had we posed the problem as starting at a later time, we may not have found a value in deferring. In fact, as stated in the introduction, the study of the conditions for the existence of RO is the bottom line of the present contribution. At the point (2.7), marginal increase in discounted revenues equals marginal decrease in discounted expenditure.

Let us compare our triggers with those of the problem DO_1. The break even value I in DO_1 represents the value of the asset which just offsets the cost; in our model such a condition is represented by (2.4), at which the balance of the processes (2.1) and (2.3) holds. Then, the trigger

$$\frac{r}{r-\alpha}I \tag{2.8}$$

(Dixit and Pindyck, 1994, section. 5.1.A) of the problem DO_1 represents the value of the asset at which the optimality condition is fulfilled. Recall that such a problem is well defined once $\alpha < r$ [3]. In our problem we do not need such a condition to be satisfied since the value S of the projects approaches asymptotic stability. Thus, b can be any positive number; in the limit $b \to 0$ the value of time expires, i.e. the agent is indifferent between the same flows even if scheduled at different times.

Then, corresponding to (2.8), the expression for the investment trigger y^* for our DO_2 problem reads

$$y^* = \frac{b}{b+1}\frac{1}{\xi} = \frac{b}{b+1}y_{\text{break even}} \ , \tag{2.9}$$

a dimensionless number ranging between 0 and $y_{\text{break even}}$ which characterizes the fraction of long run NPV at which, once realized, investment is optimal. The multiplier $\frac{b}{b+1}$ embodies the value in waiting. RO models typically represent the required returns on irreversibility in such a form, an occurrence related to the analytic properties of the dynamics of the profitability. A natural correspondence between the optimal investment rules is set by comparing the *cost* benchmark I in the problem DO_1 with the dimensionless benchmark ξ in the problem DO_2.

The next step is the expression of the value V of the investment opportunity as parametrized by the optimal trigger (2.9), namely

$$V(y) = \frac{\Xi}{b+1}\left(\frac{y}{y^*}\right)^{-b} \ , \tag{2.10}$$

an expression which parallels the one in Dixit and Pindyck (1994, p. 139). Notice that the argument of the power law is dimensionless, and that V is proportional to the scale Ξ. As pointed out by Myers (1977), the value of the investment opportunity depends on the *recognition* of the optimality of the investment policy, which (2.10) is meant to represent explicitly: there is a value in owning the possibility to delay investment, a value which grows according to the power law (2.10). Such a formula does not encompass any uncertainty inherent to the investment opportunity, the issue which RO models are meant to account for, once well defined stochastic processes model the expectations about profitability. The solution of the problem DO_2 is an explicit representation of the reason for the *existence* of a value in waiting to invest: the value of the project is growing at a decreasing speed, but one finds it profitable to wait since the investment expenditure is decreasing too. It is not difficult to convince oneself that a constant lump sum expenditure for the acquisition of the asset (2.1) does *not* leave room for a value in waiting to invest.

The functional dependence represented in (2.10) points at the RO valuation perspective. Formula (2.10) defines the value of the investment opportunity as a function of the actual state of the underlying: we can get rid of the dynamics in time and represent the optimization with respect to the state variable. Along this line we take the next step, by computing the basic building block of the RO approach, i.e. the expression of profitability as a function of the state

[3] Evidently, an investment opportunity with $\alpha \geq r$ should not be disregarded, it is only optimization problems which need to be tailored on a rationale different from the one represented in (2.6).

variables. Such a representation supports the definition of the free boundary problem for the RO which generalize the deterministic model. Again, the monotonicity represented in (2.1,3) enables us to get rid of the time dependence in the profitability P, and write it as a function of the state variables, namely

$$P(S) = \xi\, S - I \ , \quad P(y) = \Xi\,(1-\xi\; y) \quad . \tag{2.11}$$

With a slight abuse of notation, we employ the same symbol P for the two functions, which represent the same content. Being $\Xi > 1$, our setting departs from standard recipes built on the problem DO_1, in that the slope of the straight lines in (2.11) differs from 45°, since the cost of the project is not constant. In fact, as such a costs vanishes asymptotically, the slope of $P(S)$ exceeds 45%. Then the following pictures

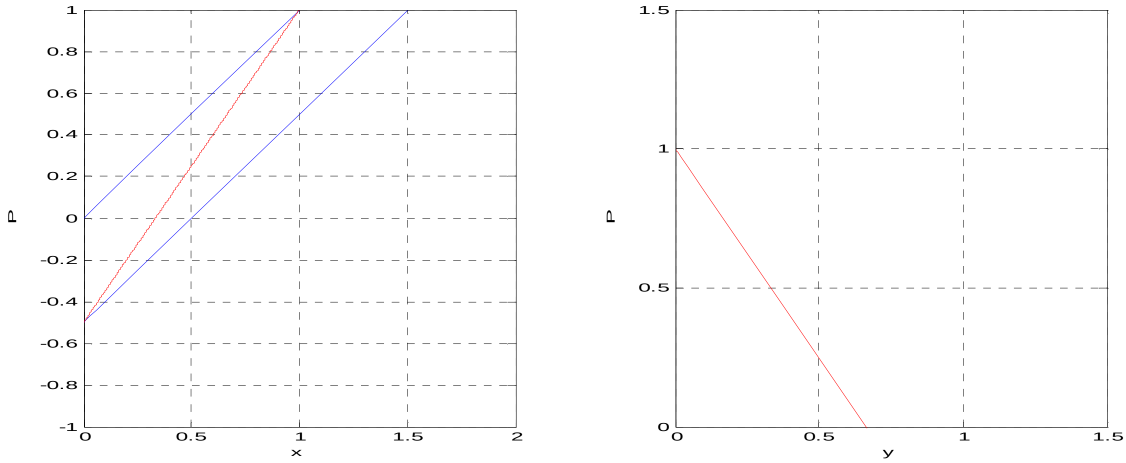

Figure 1 *Left*: profitability as a function of the normalized asset value *x*; *Right*: profitability as a function of the normalized variable *y*. The investment opportunity is characterized by $\Xi = 1$ and $I = 0.5$.

represent the benchmark for the RO approach: the profitability of the project is plotted as a function of the state variable, with respect to which a RO model is in charge of optimizing an investment schedule, which will select a point along such lines. Then, a convex function is meant to represent the value of the RO to invest, a function matching such lines at the investment trigger. Evidently, the red line in the left plot does not enter the half plane above the 45° line since such a region corresponds to profit exceeding revenues. and is bounded by the value of the asymptotic profitability of the project, which coincides with the maximum value Ξ attained by S.

Our understanding of RO enables us to predict the effect of stochasticity in the dynamics of profits. We expect volatility to increase the value of waiting, and therefore to push forward in time the investment trigger. Then, we conjecture the functional form of the RO_2 value to approach the power law (2.10) in the deterministic limit. We shall confirm such expectations in the following section. The deterministic optimization we have gone through, not only supports the definition of the RO model, but defines the ideal benchmark of perfect information with respect to which the RO defines a smooth departure.

Let us turn to numbers, so as to come to grips with the effective value of waiting. We take advantage of the factorization (2.9) to leave unspecified the scales Ξ and I, which establish the break even condition, and rather focus on the driver α of the value of flexibility. Being r = 3% , we find the optimal investment schedule

$\alpha =$	0.01	0.02	0.03	0.10
$\frac{b+1}{b} =$	1.33	1.67	2	4.33

which quantifies the value in waiting: as expected, the lag shrinks as α decreases, already for α = 1% the departure from the limit $\frac{b}{b+1} \to 1$ of no value in flexibility is not that large.

3 Real options

As a generalization of the ordinary differential equations (2.2), let us consider the stochastic differential equations

$$dy = -\alpha\, y\, dt + \sigma\, y\, dz \quad , \quad dS = -\alpha\, (\Xi - S)\, dt + \sigma\, (\Xi - S)\, dz \tag{3.1}$$

the first one representing a GBM with negative drift. The insight underlying (3.1) is growth with slowing trend which approaches a stable deterministic asymptotic limit, so that the stochastic effect diminishes correspondingly. The analytic rationale for such a choice follows the line of the standard literature, which has long recognized the conceptual and analytical relevance of such processes (Chang, 2004). In principle, any stochastic differential equation may be employed as underlying a RO model. Due to their natural economic interpretation, arithmetic brownian motions, mean reverting process and Poisson jumps have been extensively employed (Dixit and Pindyck, 1994) in the literature. Yet, the properties of GBM processes have a unique analytical and conceptual grip on fundamental models. The relevance of such models parallels the foundational role of the *exponential* form of the discounting factor in continuous time; in a sense, GBM processes define a *reference* stochastic growth.

As is well known, there exist closed form solutions for the time evolution of GBM sample paths, and the time dependent log-normal distribution of the state variable is a standard tool in stochastic finance. Furthermore, for a GBM x with negative drift, the barrier $x = 0$ is well known to absorb sample paths within a finite expected time. We shall not come to grips with such explicit expressions, and rather employ Ito's lemma we take advantage of the smoothness of the RO differential problem, as compared to the non differentiability of GBM sample paths. We can dispense with the non differentiability of sample paths just because we address an elementary stylized fact. Recall that an absorbing barrier determines a transition from a stochastic growth to a deterministic stability which is *beyond* the RO trigger at which optimal investment is supposed to be exercised. As already stated, the recognition of such optimality drives the value of the investment opportunity; our stylized fact resembles the RO approach to disinvestment, as discussed in section 5.

Following the standard recipe, the value of a RO is represented by a free boundary problem for a second order ODE. Provided definite ideal assumptions are satisfied by the financial 'environment', the ODE represents the smooth variation of a contingent claim, and the free boundary conditions define the optimality of the investment exercise strategy. Let us simply sketch the recipe, and refer to Dixit and Pindyck (1994) and references therein for a comprehensive analysis. Consider a portfolio made of a long position in the option V and a short position in the asset S with size $\frac{dV}{dS}$. Contingent on the state S, the stochastic return (differential) of the finely tuned *hedged* portfolio reads $dV - \frac{dV}{dS} dS$. Applying Ito's lemma to the expansion

$$dV = \frac{dV}{dS} dS + \frac{1}{2} \frac{d^2 V}{dS^2} (dS)^2$$

of the stochastic differential dV we confirm that the fine tuning results in a risk free return on the portfolio. Then, standard no-arbitrage argument applies for the coincidence of such return with the risk free rate r. The resulting smooth condition leads us to a linear homogenous second order ODE, which we write for the normalized variable x as

$$\frac{1}{2} \sigma^2 (1-x)^2 \frac{d^2 V}{dx^2} + \alpha\, (1-x) \frac{dV}{dx} - rV = 0 \quad . \tag{3.2}$$

Notice that the scale Ξ does not appear in the equation. It is natural to conjecture V as proportional to the scale Ξ; in fact, the differential problem is completed by the initial condition and the free boundary problem, in which the scales Ξ and I play a role. Upon switching to the variable y we face the differential problem defined by the ODE

$$\frac{1}{2} \sigma^2 y^2 \frac{d^2 W}{dy^2} - \alpha\, y \frac{dW}{dy} - rW = 0 \tag{3.3}$$

for the function $W(y) \equiv V(x(y))$, defined on $[y^*, +\infty)$, being y^* the solution of the free boundary problem. Equation (3.3) is of the Euler type, and takes to the familiar landscape of basic RO models. Notice that in switching to equation (3.3) the second term has reversed its sign.

The two dimensional linear space $\mathcal{L}$ of solutions to (3.3) is spanned by the linear combinations

$$W(y) = A_1 y^{\beta_1} + A_2 y^{\beta_2} \ ,$$

in which the power laws are determined as the roots of the *fundamental quadratic* $\frac{1}{2}\sigma^2\beta(\beta-1)-\alpha\beta-r=0$, which admits real distinct roots with opposite signs. As is well known, the essence of the RO investment problem lies in such exponents and in their deterministic limit: the multiplicative representation for the trigger (see below) disentangles the effects of the scales Ξ and *I* from that of the key value drivers α and σ, as we discuss at length in section 5. The elements in $\mathcal{L}$ are positive functions in the relevant domain provided the coefficients $A_{1,2}$ are positive, a consistency requirement for a RO which our model must fulfil.

Building on the intuition tailored in the preceding section, we expect the boundary problem to select the negative root

$$\beta_1 = \frac{\alpha+\frac{\sigma^2}{2}-\sqrt{\left(\alpha+\frac{\sigma^2}{2}\right)^2+2r\sigma^2}}{\sigma^2} \tag{3.4}$$

of the fundamental quadratic, which approaches $-\frac{r}{\alpha}=-b$ in the deterministic limit in which σ vanishes, as can be checked by means of a Taylor expansion of the square root about σ = 0. It is not difficult to convince oneself that $\frac{\partial\beta_1}{\partial\alpha}>0$ and $\frac{\partial\beta_1}{\partial r},\frac{\partial\beta_1}{\partial\sigma}<0$, whose significance we shall exploit in section 5.

The solution of our problem is in fact characterized by A_2 = 0, as the *negative* power law β_1 is uniquely determined the boundary value condition

$$W(y)\underset{y\to+\infty}{\to}0 \quad , \tag{3.5}$$

which represents the vanishing of the option value at the 'initial state' of the underlying *y*, i.e. the infinite past: as intuition suggests, the farther the trigger, the lower the PVGO. As is well known, this property characterizes the RO perspective on disinvestment, as issue to which we turn in section 5. On the contrary, the standard investment problem RO_1 is characterized by a *positive* power law, as the boundary condition poses the vanishing of the RO value at the zero value of the asset.

At the opposite extreme of the interval, the free boundary problem is defined by a pair of boundary conditions. The value matching condition

$$W(y^*)=A_1 y^{*\beta_1}=\Xi\left(1-\xi\, y^*\right) \tag{3.6}$$

establishes the intersection of RO value plot with the line which represents the profitability of the investment once exercised, in our model, the straight line in Figure 1 Right. Then, the smooth pasting condition.

$$\frac{dW}{dy}(y^*)=A_1\beta_1 y^{*\beta_1-1}=-\Xi\xi \tag{3.7}$$

establishes the continuity of the slope of the RO graph at the trigger, a requirement of the *optimality* of the trigger (see for instance Dixit and Pindyck, 1994, and references therein). As in standard settings, we expect conditions (3.6,7) to select among the elements of functional space $\mathcal{L}$ a unique solution for the option value. As is standard procedure, the trigger is established by the ratio of (3.6) and (3.7), resulting in

$$y^*=\frac{\beta_1}{\beta_1-1}\frac{1}{\xi} \tag{3.8}$$

which, as expected, entails a *multiplier* for representing the RO optimality of the exercise with respect to the break even benchmark. Notice that (3.8) is consistent with the deterministic trigger (2.9), since $\beta_1 \to -b$ in the deterministic limit. Such a value can be employed to fix the value of the constant

$$A_1=-\Xi\,\beta_1^{-\beta_1}(1-\beta_1)^{1-\beta_1} \quad ; \tag{3.9}$$

notice the minus sign which guarantees the positive sign of A_1 . Then, the solution is uniquely determined as

$$W_{RO_2}(y) = A_1 y^{\beta_1} = -\frac{\Xi}{\beta_1 - 1}\left(\frac{y}{y^*}\right)^{\beta_1} \tag{3.10}$$

of the model. Notice the correspondence with the deterministic solution: as the power law β_1 approaches the deterministic limit $-b$, the value of the RO approaches the solution (2.10) which represents the value in flexibility in the absence of uncertainty in profit prospects. Figure 2 represents a typical behaviour of a solution of the type (3.10).

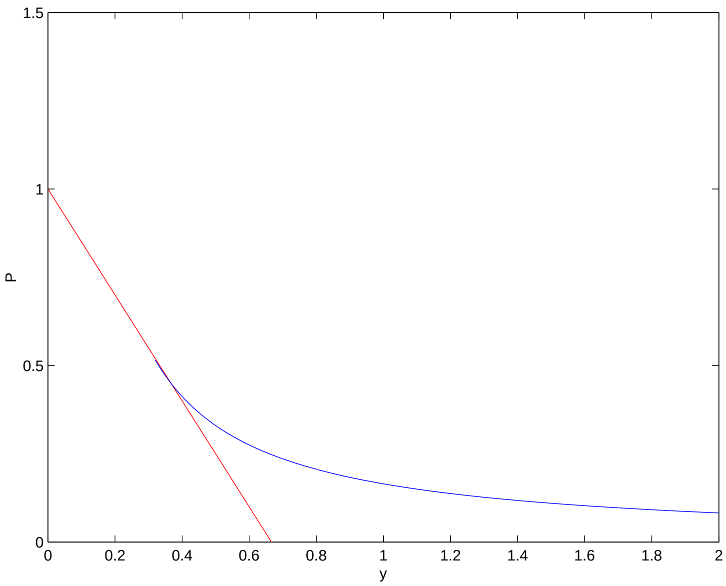

Figure 2 The value of a typical real option (blue curve) as a function of *y* for a profitability (red line) defined by $\Xi = 1$ and $I = 0.5$.

The closed form solution for the RO value in terms of the variable establishes the main difference with respect to the model; the power law is a negative one. Yet, the convexity property remains. Working with a variable *y* ranging in [0,1] makes intuition easily grasped. As *y* approaches zero, the NPV of the project approaches its asymptotic limit, so that the smaller the trigger y^*, the more value in waiting. As a first check of the consistency of the analysis, let us establish the comparative statics for the investment trigger with respect to the parameters Ξ, *I*, *r*, whose effect as value drivers of investment opportunities rests on general grounds, not specific to the RO approach. The factorization which underpins basic RO analysis tailors the lines of the analysis. The scales Ξ and *I* enter the break even benchmark ξ, whereas the rate *r* is embedded in the multiplier.

It is quite obvious that the initial size *I* of the expenditure increases the value in waiting, as represented in the following comparative statics. The scales Ξ and *I* of the problem have been factored out in the homogeneous differential equations (3.1-2); they plays a role in the boundary conditions. As the trigger y^* is proportional to the break even scale ξ, the effects of Ξ and *I* are embodied in the comparative statics

$$\frac{dy^*}{d\Xi} = \frac{I\ y^*}{(\Xi + I)} > 0 \ , \quad \frac{dy^*}{dI} = -\frac{y^*}{(\Xi + I)} < 0 \tag{3.11}$$

which exploits the natural fact that the asymptotic value Ξ of the asset not in place is a positive value driver, whereas the opposite holds for the amount *I* of the investment expenditure.

Then, the rate *r* is embodied in the multiplier, as a coefficient of the fundamental quadratic. As a general capital budgeting issue, we expect that the lower *r*, the more value in waiting. Such an effect is represented in the comparative statics

$$\frac{\partial y^*}{\partial r} = \frac{1}{\beta(\beta - 1)} y^* \frac{\partial \beta_1}{\partial r} > 0 \quad , \tag{3.12}$$

which sets the smooth sensitivity of the investment trigger with respect to the risk free rate which measures absolute time preference.

The next step is the comparative statics with respect to the value drivers α and σ, which characterize the RO approach. Yet, before that we need to take a major step towards the profound significance of our model. We have already pointed out the key role of the hypothesis that the same rate α drives the dynamics of the asset S as well as that of the liability side C. Such a 'measure zero' hypothesis calls for a comparison with models in which a finite difference in the rates is allowed. We do not enter such an issue; we are interested in the *elementary* insight built into the exponential damping of the investment expenditure, meant to represent an ideal *benchmark*, much like the perfect information hypotheses underpinning benchmark general equilibrium models. Yet, the RO approach inherits *stability* properties which may result in a deep theoretical relevance, in first instance, the smooth deterministic limit (see section 5).

4 The value of delaying investment

In the previous section we have solved an original RO model facing a long run stabilization of the profitability of the investment opportunity. The closed form solution shares fundamental analytical properties with the standard RO model, and represents *quantitatively* the basic tradeoff which generates a value in delaying investment in the problem RO_2: it is profitable *not* to exercise immediately the investment opportunity due to the ongoing increase in the value of the underlying asset as well as to the diminishing trend of the investment expenditure. Both facts more than compensate the opportunity cost not to exercise immediately an investment opportunity with positive NPV[4]. Such an intuitive stylized fact takes us to the foundations of the RO approach, whereupon we can take advantage of the transparency of the solution of model RO_2 to raise a general issue about the *conditions* for the very existence of a value in waiting to invest. As previously stated, a constant lump sum cost for the acquisition of the asset S described by (2.1) or (3.1) does not leave room for a value in deferring investment.

In a nutshell, RO exist once three conditions are fulfilled. First, there exists a definite flexibility in the exercise of an investment opportunity. Second, a well defined optimization problem can be set up. Third, the ongoing realization of the stochastic underlying can be observed to bring about the information upon which the investment schedule is tailored. The investment trigger (3.8) represents the elementary schedule for the model RO_2, in a perfect parallel to the corresponding trigger for the problem RO_1. In such a parallel we can recognize that the value in waiting to invest in the model RO_1 stems exclusively from the increase in value of the underlying asset. It is natural to look at the *dual* problem in which the value in waiting stems exclusively from the diminishing trend in expenditures. Such a duality might represent a fundamental structure in a RO theory, much like the asset-liability dichotomy is the fundamental structure of immunization theory.

Such a conjecture about a fundamental structure in RO theory is supported by the observation that the model RO_2 can easily be recognized as formally *equivalent* to a model (call it RO_3) in which the value of the asset not yet in place is at a constant level Ξ, whereas its lump sum cost is represented by the stochastic process $(I + \Xi)\, y(t)$ upon which to set up the free boundary problem. Notice that, in fact, that is the perspective we have pursued in the previous section when working with the variable y. The conjecture that such a duality might represents a line of force along which a *theory* of RO might build is then a natural step. The profitability of the investment opportunity can be modelled via stochastic dynamics on both the asset and the liability sides of the problem. Our analysis seems to point at the two elementary facts which entail a value in waiting to invest. The first occurs for a GBM with positive drift, representing the dynamics of the asset value, coupled to a constant lump sum investment. The second occurs once the cost of acquiring an asset with constant value is a decreasing lump sum represented by a GBM with negative drift. Such a duality appears at an elementary enough level to suggest that no further structure can be recognized at a more elementary level. Then, with respect to such 'coordinates' for the space of possible RO, one can envisage a structure for a theory built along such lines. Just as RO_1 turned out to be a stylized fact which can be dressed which great complexity, the same, in line of principle, can be thought of the problem RO_3. As already stated in the introduction, the role of diminishing expenditure in the modelling of complex RO has long been recognized. Our point is to argue about the relevance of introducing the dichotomy RO_1 - RO_3 as a conceptual framework for organizing the insights embedded into complex models.

The key issue in the RO_1 - RO_3 duality is the difference between the asymptotic behaviour of the growth opportunities built in the models. It is quite obvious than, at any instant, one can trade an increase in the value of an asset with a corresponding decrease in the expenditure required for acquiring the property of such an asset. Such an algebraic identity is not evidently our point. It is the different asymptotic behaviour which calls for different relation between the asset and liability for a value in waiting to exist. Figure 1 represents such a difference: in the left plot, the 45° slope of the profitability of the project in a model DO_1 (blue line) represents a constant expenditure, which does not influence

[4] Notice that the explicit functional form for the solutions of RO problem are related to the explicit functional form of discounting, an issue with which Myers (1977) does not come to grips, therefore maintaining a greater generality.

the 45% slope of the straight line $P = x$. On the other hand, the slope of the profitability in the model DO_2 (the red line) is larger than 45°, as a result of the diminishing of the cost of acquisition. Evidently, such a slope cannot be maintained indefinitely, so as not to enter the region of the plane in which the NPV of the project exceeds the PV of the asset. The difference in the asymptotic behaviour characterizes the diversity of the models, which cannot be mixed: one cannot combine the asset side of model RO_1 with the liability side of model RO_3 and obtain a consistent RO model. Combining an asset with exponential expected growth with a liability exponentially damped leads to a profitable investment opportunity, which, nonetheless, cannot be address in terms of a RO of waiting to invest. A forthcoming RO theory should be bounded in such a direction, and, possibly, envisage generalized criterions of optimality.

5 Value drivers in comparison

The conceptual significance of the duality set forth appears quite natural. Thus, we expect the analytic properties of our model to be correspondingly transparent. Let us assess the comparative statics for the pair of key drivers of the value in flexibility, namely, the damping rate α, at which the size of the liability side decreases, and the volatility σ, which enhances the profitability of capitalizing on upward shocks. The comparative statics confirm the coherency of the two models, in which the value drivers play, mutatis mutandis, the same role. Then, a comparison which standard reference approaches assesses the relative significance of each model, therefore contributing to shaping the boundaries of the RO approach. Thus, the somewhat 'heterodox' slopes of the red lines in Figure 1 can be blessed by a natural fitting of the model in the overall RO framework.

As a first step, let us come to grips with numbers, generalizing the schedule of section 2, which represents the limit of vanishing σ. Again, we leave unspecified the scales Ξ and I, and focus on the value drivers embedded in the multiplier. For $\sigma = 0.10$ and 0.20 respectively, we find the optimal schedules

$\alpha =$	0.01	0.02	0.03	0.10
$\frac{\beta-1}{\beta} =$	1.729	2	2.295	4.547

$\alpha =$	0.01	0.02	0.03	0.10
$\frac{\beta-1}{\beta} =$	2.457	2.721	3	5.160

As expected, the effect of volatility is to enhance the value of flexibility; such an effect superposes to the monotone trend driven by α, resulting in an amplification of the value of flexibility. Let us turn to the smooth setting of comparative statics.

The growth rate α characterizes the duality under inspection: in the model RO_1 it is the driver of the growth of the value of the asset, in the model RO_3 it is the driver of the damping of expenditures. In both models, the higher α the less value in delaying exercise. Such an expectation is confirmed by the comparative statics

$$\frac{\partial y^*}{\partial \alpha} = \frac{1}{\beta(\beta-1)} y^* \frac{\partial \beta_1}{\partial \alpha} > 0 \ ; \tag{5.1}$$

recall that the state variable y has an expected value which decreases with time, so that a positive $\frac{\partial y^*}{\partial \alpha}$ entails accelerating the exercise of the option. Such an effect is coherent with that of the rate of growth of the asset value in the model RO_1, in which the profitability grows accordingly to the value of the asset.

Then, the role of the volatility σ of the GBM underlying is what characterizes the RO and the profitability of waiting for the optimal trigger to exercise the investment opportunity. The smooth enhancement of the value in delaying for increasing is represented by

$$\frac{\partial y^*}{\partial \sigma} = \frac{1}{\beta(\beta-1)} y^* \frac{\partial \beta_1}{\partial \sigma} < 0 \ , \tag{5.2}$$

as long established, the higher the volatility, the more profitable shocks may turn out to be, and the farther the investment trigger poses the optimality of exercise. Recall that the *size* of the shocks is proportional to the actual size $y(t)$ of the state variable, thus shrinking on average as time elapses.

Out of such comparative statics we get the explicit expression of the shifts of the RO value caused by changes in α and σ. The insights have been well established: we expect an increase in α to induce a downward shift in W, as a consequence of the trigger being pulled back in time, whereas an increase in σ determines an upward shift in W, as a consequence of the trigger being pushed forward in time. The corresponding comparative statics for the value of the RO as represented in (3.10) stems from application of the chain rule, namely, for any y in the relevant domain,

$$\frac{\partial W}{\partial \alpha} = \frac{\partial W}{\partial y^*}\frac{\partial y^*}{\partial \alpha} > 0$$
$$\frac{\partial W}{\partial \sigma} = \frac{\partial W}{\partial y^*}\frac{\partial y^*}{\partial \sigma} < 0 \quad (5.3)$$

Notice that the analytical structure of the problem is tailored by the multiplicative form (3.8): both the effects of the drift and stochasticity are embodied in the root β_1 of the fundamental quadratic. The analytical relevance of the elementary models RO_1 - RO_3 can be traced back to such neat expressions. The RO approach has grown naturally factorizing the value in waiting in the multiplier, a pattern into which our model fits naturally.

As already stated, the multiplier built out of β_1 embodies both the effect of the *trend* of the dynamics of profitability, as measured by α, and the effect of shocks, as measure by σ. The resulting expressions exploit the increase in σ as a result of the enhanced profitability of capitalizing on the freedom of waiting for positive shocks. The two opposite effects get balanced once, for any y in the relevant domain,

$$dW = \frac{\partial W}{\partial \alpha}d\alpha + \frac{\partial W}{\partial \sigma}d\sigma = 0 \quad (5.4)$$

the vanishing of a differential form, which can be represented as a flow in the α, σ plane.

Then, delicately switching off uncertainty lead us to the deterministic limit discussed in section 2. Notice that the very *existence* of a smooth deterministic limit is of great relevance, a stability property of the RO approach which may represent a handle upon which general economic issues may rely for addressing the issue of efficiency under uncertainty, as Dixit and Pindyck (1994, ch. 8,9) set forth.

The insights embodied in the previous relations do fit the standard RO framework, as we further confirm in a comparison with the basic RO perspectives on investment and disinvestment. Basic models encompass GBM underlying with, respectively, positive and negative drift. In a sense, our model does fit 'in between' such recipes in well defined analytical sense, which we sketch in comparison with a pair of representative papers. RO_3 is a model of investment, thus focused on the *increase* in the value of the underlying. Yet, as the underlying approaches an asymptotic limit, the model RO_3 shares the analytical form of the solution with the disinvestment perspective.

Consider the strategic investment problem set forth by Grenadier (2002), which models Cournot competition facing a stochastic market: exogenous uncertainty in demand is driven by a GBM with positive drift, a multiplicative shock on demand. The Author employs a definite coupling for the differential problems of the competitors, which builds on the model RO_1 and entails a *positive* power law (Grenadier, 2002, p. 703) for the value of the RO, whose effect is entangled with the complex dynamics of the stochastic game. Our model, on the other hand, represents a market which saturates, and which therefore entails a *negative* power law.

The negative power law characterizes the RO perspective on disinvestment, for which we take Lambrecht and Myers (2007) as representative analysis. The Authors employ a GBM with negative drift to model the stochastic evolution of the value of the firm. They partition the value of the RO to disinvest into the stylized claimholders and potential acquirers. Their RO model for the firm value entails a *negative* power law (Lambrecht and Myers, 2007, p. 815), which is selected by an initial condition which parallels (3.5). Their theory, among other things, exploits the role of flexibility and volatility into the value maximizing strategies of players with different objectives. Recall that the RO perspective on disinvestment differs from the one embedded in the European option perspective on corporate liabilities (Black and Sholes, 1973), and specifically in the put-call parity (see for instance Miller, 1988), in which a pre-established time of expiry of the option represents debt maturity. The strike price of the European option naturally corresponds to the exercise price of the RO, which is, on the other hand, triggered by the realizing sample path of the stochastic underlying. In both models shareholders may choose to replenish bondholders and retain all claims on the firm's cash flows; the different analytical properties of the models set the difference in the economic perspective, in first instance, the absence of the drift of the GBM underlying in the BSM equation.

6 Perspectives

Among the many types of RO, the option to delay investment plays a key role, in that the operational flexibility is stripped down to its essentials. We have gone through a RO model in continuous time[5] which is *dual* to the standard RO model of delaying investment. In standard settings, the value drivers of delay are, in first instance, the expected rate of growth of the value of assets not in place. On the opposite, our model encompasses a value in waiting to invest which can be driven solely by the decrease in the investment expenditure. Thus, natural *pairings* are established: the exponential growth of the expected value of the asset can be paired to a constant lump sum expenditure, whereas the

[5] A corresponding analysis in discrete time may provide further insight.

exponential decrease of expected expenditures can be paired to a constant asset value. This is the point of the present contribution, whose rationale is a *symmetry* requirement to envisage the asset and liability sides on a equal footing with respect to the search for value drivers. As already stated, the role of diminishing costs as value drivers for general RO models has long been recognized. Our point is to posit the RO_1 – RO_3 duality as a *basic* framework for building analytical models as well as for organizing insights. We conjecture that such a duality encompasses a *complete* characterization of the conditions for the existence of value in waiting to invest. The rationale behind the conjecture is the *elementary* duality of the stylized facts. Should the conjecture turn out to be falsified in some sense, a major step towards the building of a RO *theory* would anyway come about.

As already stated, we envisage a natural application of the present model to real business in services. In fact, the market for energy supply may represent a natural application, in which the diminishing costs of producing energy from renewable sources entail a value in waiting to invest. In fact, regardless definite interpretations of the model, the duality RO_1 – RO_3 may represent a means by which to enlarge the insights built into the analysis of incremental investment, as represented in Dixit and Pindyck (1994), which build on a demand driven by a multiplicative GBM shift variable with positive drift, therefore committing the analysis to the RO_1 side.

Once we expand our risk neutral agent into more fundamental structure, the net of contracts inside the firm comes into view, establishing the boundaries of the firm as well as the coordination and incentive mechanisms therein. The RO model set forth by Lambrecht and Myers (2007) enlightens the divergence of objectives between the stylized claimholders induced by the flexibility in the disinvestment opportunity. Such incentives drive the play according to the contractual arrangements taken as exogenously given. As is well known, the opposite perspective tailors the Principal-Agent model (Bolton and Dewatripoint, 2005), which sets the general framework for addressing optimal incentive mechanisms and contract arrangements, *given* the preferences of players and models for informational asymmetries. Grenadier and Wang (2005) provide a seminal approach to represent the effect of agency problems in causing a departure from the first best RO trigger: once the exercise of the investment opportunity is delegated, the agent may capitalize on asymmetric information. The Authors employ the model RO_1 to exploit the effect of adverse selection and moral hazard in shifting the first best optimal investment trigger to the second best correspondent. Our model represents the natural dual perspective on the investment sector.

Then, switching from the inside to the outside of the firm, we turn to the competitive issues addressed by Industrial Organization. Upon relaxing the hypothesis of monopolistic investment opportunities, one is led to consider the whole range of competitive frameworks, from imperfect to perfect competition, for which Smit and Trigeorgis (2004) tailor natural lines for taking into account the PVGO in the payoff structure of strategic interactions. For instance, pre-emption becomes an issue which our model can be made to represent, along the lines of Grenadier (2002), possibly providing a 'structural' organization for the insights about value drivers. In first instance, the issue about the conditions for the very existence of a value in flexibility may be sharpened, along with the recognition of the nested structure of growth option which the firm can pursue in the quest for market power. The point is to envisage the role of investment opportunities on both the asset and the liability side. As a typical fact, the asset side is characterized by exogenous uncertainty, for instance market demand, upon which the firm may have little control. On the other hand, the liability side may provide handles (for instance, learning) by means of which the firm can drive PVGO and investment strategies. As it has become clear in the last decade, strategic (game theoretic) investment models represent the real 'battlefield' upon which RO methods can demonstrate their power. "Value creation suggests that the investment strategy should be focused explicitly on the relevant value drivers" (Smit and Trigeorgis, p. 32). "Developing growth options and developing a strategic position to acquire their benefits are enabled in part because of policies pursued and experiences and efficiencies acquired in earlier periods" (Smit and Trigeorgis, 2004, p. 91). The learning effect on diminishing costs embedded in our model comes to the fore.

As a final remark, our model may provide further insights with respect to the role which RO may play in a regulatory[6] framework (see for instance Roques and Savva, 2009). It is tempting to envisage RO as a major scientific contribution to the assessment of *objective* measures with respect to which regulators can pivot the design of *rules*, and, possibly, contribute to the building of a world "in which there is not only greater prosperity, but also social justice", as Stiglitz (2001) posits.

[6] The italian market for electric energy suppliers has been recently liberalized.

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