JEL Classification: D72, H41

Keywords: public goods, redistributive policies, electoral competition, all-pay auctions.

Acknowledgements: I would like to thank Paolo Bertoletti for his useful comments and suggestions. The usual disclaimers obviously apply.

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Serie di Economia e Politica Economica WP 13/2006

December 2006

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Electoral Competition and Incentives to Local Public Good Provision.

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July 18, 2006

Abstract

Local public good provision from different government levels is subject to many bias coming from the political process; incentives indeed, vary with the size of the beneficiaries’ set and costs may affect the results of political competition by reducing total resources available for redistribution. Present work represents a first attempt to look at these issues together; indeed, it considers the situation where politicians have a finite budget to use both for redistributive policies and for the provision of a public good that affects the utility of a fraction of the electorate. In this setting it is not enough that benefits balance costs, in order for the public good to be implemented; the required level of efficiency moreover, is influenced by benefits concentration. If those interested in the public good are less than half of the electorate, concentration increases the efficiency threshold; on the contrary if they amount for more, benefits concentration decreases the required level of efficiency.

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1 Introduction

The political process deals mainly with problems of resource redistribution; almost every government intervention in the economy has some redistributive effects and modifies the welfare of different sectors of the population. In a context where resources are limited, as it is the case if the state is concerned at least with the viability of public debt, funds allocation plays a relevant role in the formation of social consensus.

This last issue becomes absolutely crucial anytime in a democratic system, elections are held; redistributive programs are in this context, the main tool to form and increase social consensus and ultimately to pursue the politicians’ aims (no matter how they are defined) through electoral competition.
Candidates though, face an electorate where different groups, sharing some characteristics or interests, have different preferences with respect to the allocation of the public budget; it may be the case for instance, that some of them are interested in the provision of local public goods and constitute a special interest groups within the electorate. Political competition thus, is influenced by the behavior of people whose voting decision depends on whether or not politicians includes these goods in their electoral platforms.

Local public good provision from different government levels is subject to many bias coming from the political process since incentives vary with the electoral system; Lizzeri and Persico (2001) in particular, show that proportional systems provide more often public goods with respect to majority systems.

Public goods costs moreover, affect the result of political competition by reducing funds available for redistribution. Sahuguet and Persico (2006) analyze how parties with different levels of resources compete in a proportional electoral system when electoral platforms include only a redistribution program for the public budget; they find that in a two parties system, the candidate with the lowest perceived ability sells out to some minorities in the electorate.

Present work represents a first attempt to look at these issues together by analyzing the case where politicians have a finite budget to use both for redistributive policies and for the implementation of a public good that affects the utility of a fraction of the electorate. In this setting it is not enough that benefits balance costs, in order for the public good to be implemented; some degrees of efficiency are required that depend on the level of benefits concentration.

The analysis that follows is a generalization of the model by Lizzeri and Persico (2006) if a proportional system is considered; indeed public good expenses do not absorb the whole budget nor its benefits accrue to all voters in the same measure, as it is the case for those authors.

Dropping these assumptions allows to disentangle the effects of benefits concentration on the level of efficiency required for the implementation of the public good; in this setting if the special interest group represents less than half of the voters, benefits concentration increases the efficiency threshold. On the contrary if those taking advantage from the public good are more than half of the electorate, benefits concentration decreases the required level of efficiency.

A first interesting implication of the above mentioned results deals with lobbying activity. Indeed, if the analysis on pork barrels policies conducted on the "demand side", has found that small groups are more effective in promoting their special interest¹, changing the point of view and considering the "supply" side gives opposite outcomes; interventions that favor small groups require in general higher degrees of efficiency or higher side payments from the lobby. In this sense present work complements this strain of the literature by considering the effects of pork-barrel policies on electoral competition.

A second observation concerns the effects of decentralization over the provision of public goods; in this framework decentralization is not neutral since varying the size of the electorate changes also the size of the beneficiaries' set.

¹See Olson (1965).
An optimal choice of the government level entrusted of public goods provision then, must take into account that different incentives derive to politicians, depending on the fraction of the electorate that support a specific intervention.

This represents an original contribution with respect to current literature; instead of focusing on how these policies form after the electoral competition\textsuperscript{2}, the situation where politicians make binding promises during the campaign is considered. The choice on public good provision then, is the result of the strategic interactions among competing candidates.

2 The Model

I consider a three-stage game where parties compete in a proportional electoral system.

In the first two stages party 1 and party 2 present their platforms to voters; these platforms include a decision over the implementation of a local public good, \( g \) and a redistribution plan for the public budget.

The public good is a fixed size one and benefits a fraction of the population that forms a special interest group within the electorate.

In the second stage elections are held.

2.1 Parties

Both party 1 and party 2, try to maximize vote shares \( S_i \) (\( i = \{1, 2\} \)); these quantities define also the probability that, after the elections, party \( i \) platform is implemented. In other words, a probabilistic compromise is the outcome of the political process\textsuperscript{3}.

The government uses money to buy \( g \) and to finance the redistribution program; funds allocation among these interventions is defined by party \( i \) platform, with probability \( S_i \).

The notion of public budget is crucial in the kind of redistributive issues considered; therefore, it is useful to make clear some of its characteristics before proceeding.

As in Myerson (1993), the public budget comes from taxation; once in office indeed, the government taxes each voter, \( n \), for 1 dollar.

The electorate is a continuum described by the interval \([0, 1]\); this allows to avoid dealing with the problems posed by a large finite number of voters.

The amount of funds collected thus, is infinite; however this infinite electorate can be considered as an approximation for a large finite population so

\textsuperscript{2}See for instance Weingast, Shively and Johnsen (1981), Baron and Ferejohn (1987 and 1989), and Besley and Coate (2003).

\textsuperscript{3}See Sahuguet and Persico (2006) for further details on the probabilistic compromise outcome.
that also the public budget is in facts, as a large finite amount of resources with a monetary value proportional to the number of voters.

The resources that the government receives from each voter form the initial per-capita budget; its size corresponds to the taxes paid by a single voter and is 1.

Notice that whenever the public good is implemented the budget is reduced by the cost of \( g \); this poses some technical problems. The cost of the public good indeed, is finite and must be subtracted from an infinite budget so that its impact over it would be null.

The same problem arises further in the definition of the benefits generated by \( g \); since the special interest group is a fraction of an infinite electorate and counts an infinite number of members, the summation of individual benefits gives an infinite quantity.

It is possible though to extend the interpretation adopted for the public budget to both these variables and define total costs and benefits of \( g \) as finite quantities; in particular the cost of the public good is proportional to the sum paid by each voter for \( g \). Overall benefits instead are proportional to the size of the lobby and to the utility that a single member of the group gets from the public good.

In order to make the analysis simple and get rid of these problems, it is useful to express all the relevant variables in per-capita terms.

The per-capita share of the benefit generated by the public good is fixed and is equal to \( U \); only a fraction \( \lambda \leq 1 \) of the electorate though, gets utility from \( g \). This means that each of those voters receives in facts \( U \lambda \).

Total benefits received by the special interest group then, increase as \( \lambda \) decreases; when \( \lambda = 1 \) the whole electorate is a special interest group and aggregate benefit \( U \), coincides with the per-capita share accruing to each voter.

The cost of the public good instead, is always bared by all voters; its per-capita share amounts to \( U (1 - \xi) \) where \( \xi \in (0, 1) \) is a parameter that measures the degree of efficiency of the project. In order to disentangle the effects of benefits concentration, the return rate from the investment in \( g \) is assumed to be constant and equal to \( \frac{1}{1 - \xi} \).

The size of party \( i \) redistributable budget is:

\[
 b_i^r = 1 - g_i
\]

where \( g_i \in \{0; U (1 - \xi)\} \) and is \( g_i = U (1 - \xi) \) if the public good is implemented and \( g_i = 0 \) if is not.

Since the parties cannot present platforms that require public debt to be financed it must also be the case that:

\[
 U (1 - \xi) \leq 1
\]

Consider now the composition of the electoral platform that party \( i \) submit to voters; they consist of two main elements:

1. Public good expenses, \( g_i \), that take only two values: either 0 or \( U (1 - \xi) \), i.e. \( g_i \in \{0; U (1 - \xi)\} \).
2. A redistribution plan for the available budget, \( b'_i = 1 - g_i \) that specifies for a generic voter, \( n \), the ratio, \( x^n_i \in [0, +\infty) \), among the sum that party \( i \) platform transfers her and the per-capita budget \( b'_i \).

The parties must fulfill a balanced budget condition in the definition of the redistribution program:

\[
\int_0^1 x^n_i \cdot dn = 1
\]

with \( i = 1, 2 \), \( x^n_i \in [0; +\infty] \) and \( n \in [0; 1] \).

Notice that \( x^n_i \) can take any value among the positive real numbers since the budget constraint requires only that party \( i \) does not make promises which average more than \( b'_i \) over the whole electorate; given that there is a continuum of voters the law of large numbers applies and guarantees that the above condition holds with equality.

Also the redistribution program cannot be financed through public debt.

### 2.2 Voters

The electorate is a continuum of voters described by the interval \([0, 1]\).

There are two main types of voters: standard voters and members of the special interest group; I denote the first type with the letter \( k \) and the second one with \( l \). When I do not refer to a specific type, the previous notation, \( n \) is used.

#### 2.2.1 Standard voter

A standard voter \( k \) derives utility only from money and does not get any benefit from the public good; the utility function for the \( k \)-th voter, when party \( i \) platform is implemented, includes only the cash transfer received through the fiscal system i.e.:

\[
u^k = x^k_i \cdot b'_i
\]

where \( x^k_i \in [0, +\infty) \) is the ratio between the promise that a standard voter receives through the redistribution plan from party \( i \) and the per-capita public budget.

Given the assumption of probabilistic compromise, a standard voter gets the following expected utility:

\[
E[u^k] = S_1 \cdot x^k_1 \cdot b'_1 + S_2 \cdot x^k_2 \cdot b'_2
\]
2.2.2 Special Interest Group

Members of the special interest group amount for a fraction $\lambda$ of the electorate; the size and location of the group are common knowledge among the parties.

Utility of special interest group members is described by the composite function:

$$u_l = m + q(g_l)$$

where $m$ is money, and $q(g_l)$ is a function defining the benefit derived from the public good such that $q[U(1 - \xi)] = \frac{U}{\lambda}$ and $q(0) = 0$.

A member of the group, $l$, receives the following utility if party $i$ platform is implemented:

$$u_l = x^l_i \cdot b^i_1 + q(g_i)$$

where $x^l_i \in [0, +\infty)$ is the ratio among the sum promised to a lobbyist through the redistribution plan and party $i$ per-capita budget.

Expected utility amounts to:

$$E[u_l] = S_1 \cdot [x^l_i \cdot b^i_1 + q(g_1)] + S_2 \cdot [x^l_i \cdot b^i_2 + q(g_2)]$$

2.3 Timing and Structure of the Game

The game is divided into three stages, played sequentially with the timing described below:

1. Parties announce their decision over the public good.
2. Parties present the redistribution programs.
3. Elections

The first two stages define the process of platforms presentation. Each stage is described in the following sections.

2.3.1 Platforms presentation.

Platform presentations includes two stages.

Initially the parties decide simultaneously wether or not to buy the public good; then both observe the opponent’s choice and simultaneously present their redistribution plans for $b^i_1$ and $b^i_2$. For the sake of simplicity, parties are assumed to be credible when commit to their platforms.

Parties actions in the first stage, consist only of the choice over the public good; given that the project is a fixed size one, $g_i$ must be either 0 or $U(1 - \xi)$. The strategy space, $\Sigma^B_i$, is a function of the cost and the benefits of the public good:
\[ \Sigma_i^B (U, \xi, \lambda) : [0, +\infty) \times [0, 1] \times [0, 1] \rightarrow \{0; U (1 - \xi)\} \]

The redistribution programs are presented simultaneously after the first step. A strategy for party \( i \) \((i = \{1, 2\})\), consists in the specification of a function that for each voter \( n \) (either lobbyist or not) defines the ratio \( x_i^n \), among the sum that she receives if party \( i \) platform is implemented and the available per-capita budget \( (b_i = 1 - g_i) \).

A strategy for party \( i \), \( \Sigma_i^P \) is a function of its own choice over \( g \) and of that of the opponent; it is defined as:

\[ \Sigma_i^P (g_i ; g_j) : \{0, U (1 - \xi)\} \times \{0, U (1 - \xi)\} \rightarrow \{x_i (n)\} \]

where

\[ x_i (n) : [0, 1] \rightarrow [0, +\infty) \]

such that \( n = k, l \) and

\[ \lambda \int_0^1 x_i^1 \cdot dt + (1 - \lambda) \int_0^1 x_i^k \cdot dk = 1 \]

2.3.2 Elections

After platforms presentation, elections are held and votes are cast simultaneously; a strategy for a standard voter \( k \) is the vote itself and depends on the offers received and on the size of parties budgets. This last element in particular, is determined by the choice over the public good so that:

\[ \Sigma_k (x^k_i \cdot b^k_i ; x^k_j \cdot b^k_j) = [0, +\infty) \times [0, +\infty) \rightarrow \{1, 2\} \]

A strategy for a lobbyist is again the vote; it depends on the definition of parties redistribution plan and on the decision over the public good:

\[ \Sigma_l (g_i ; g_j ; x^k_i \cdot b^k_i ; x^k_j \cdot b^k_j) = \{0, U (1 - \xi)\} \times \{0, U (1 - \xi)\} \times [0, +\infty) \times [0, +\infty) \rightarrow \{1, 2\} \]

3 Equilibrium Analysis

The game is sequential and requires backward induction to define an equilibrium; the analysis starts then, from the last stage: the elections.
3.1 Elections

When a generic voter n turns to vote, she compares parties platforms consider which one is better for her.

A standard voter, \( k \), chooses party \( i \) whenever it is the case that:
\[
x_k^i b^r_i \geq x_k^j b^r_j
\]

a member of the special interest group instead, chooses party \( i \) whenever:
\[
x_k^i b^r_i + q(g_i) \geq x_k^j b^r_j + q(g_j)
\]

3.2 Platforms Presentation

Before the elections take place, both parties present their redistribution programs.

The definition of the program is affected by the choice over \( g \); indeed the implementation of the public good changes the amount of resources available for redistribution because its costs must be subtracted from the public budget.

Different scenarios emerge in this framework depending on the strategies adopted in the previous stage of the game and on the characteristics of the public good.

3.2.1 Neutral Public Good

Assume initially that the public good is neutral and generates benefits equal to its costs i.e. \( \xi = 0 \); consider then, the case where only one party chooses \( g \) and assume without loss of generality that party 1 is the one to provide the public good.

Since the presentation of the redistribution program happens after the decision over \( g \), the corresponding subgame is considered first. No pure strategies equilibria exist for it\(^4\); this is a standard result in the literature concerning redistributive politics (see Myerson (1993)) when the whole bunch of resources is freely disposable.

Present case though, differs from the standard one because some money is bound to a specific location and to a specific distribution; in particular the benefits deriving from the public good are equivalent to an amount of money

\(^4\)A pure strategy for this sub-game is a plan that specifies for each voter (lobbyist or standard voter) the fraction of \( b^r_i \) that she is going to receive. If party 1 adopts a pure strategy, a best response for the challenger is to identify the voters that have been promised the highest fractions of \( b^r_i \) and offer nothing to them; the resources diverted from the favored group can be used to increase the promises to the remaining voters up to the point where they are slightly bigger than those of the incumbent. Eventually party 2 will obtain all the votes of the electors who have been promised a positive amount; of course this cannot be optimal for party 1 and thus no equilibria in pure strategies survive.
of size $U$ and accrue to special interest group members in the fixed measure of $\lambda$ each. Thus what turns out to be crucial is the amount of freely disposable resources.

Indeed given that is $b_2^* \geq b_1^*$, party 2 can always identify a fraction of the special interest group, $\phi$, such that:

$$b_2^* \geq b_1^* + \phi \cdot \lambda \frac{U}{\lambda}$$

Party 2 then, plays on this subspace as the favored party in the setting of Sahuguet and Persico (2006) and simply ignores the remaining voters. This implies that only equilibria in mixed strategies survive.

A mixed strategy defines a probability for every feasible pure strategy, i.e. attaches a probability to each specific redistribution of the budget over the voters space.

Considering a continuum of voters widens a lot the space of pure strategies that consequently become particularly complicated objects to handle; the same approach used by Myerson (1993) is adopted to deal with this problem.

The analysis focus then on a key element of parties strategies: the probability that party $i$ redistribution program awards a generic voter less than $x$; in particular, $F^i_n(x)$ ($n = l, k$) denotes the cumulative probability function defining the expected fraction of type $n$ voters that receive less than $x$ of $b_i^*$. In the definition of the redistribution program, party $i$ chooses two random variables, one for each type of voter; every $n$ type voter receives, then, an offer $x$ that is an independent draw from $F^i_n$. Present analysis thus, considers only equilibria where the offers to each voter type are realization of the same random variable\(^5\).

In this subgame party 1 has an advantage over the members of the special interest group; nonetheless party 2 can sort those receiving the benefit $\frac{U}{\lambda}$, given that voters types are known.

It is possible then, to order the electorate space and split it in two different subspaces: the lobby subspace where special interest group members are placed, and the non-lobby subspace that includes only standard voters. Party 1 counts on an advantage $\frac{U}{\lambda}$ on each individual belonging to the first subspace.

Consider now the size of the budget available to each party for the redistribution plan.

Party 1 total amount of resources derives from the summation of the initial budget minus the cost of the public good i.e.:

$$b_1^* = 1 - U$$

Party 2 instead can count on

---

\(^5\) The fact that mixed strategies are considered requires to discuss some issues relative to the independence assumption over the draws that define parties promises to each voter; given that the electorate is infinite there are also infinite independent draws and this poses some technical problems discussed by Myerson (1993) and Alos-Ferrer (2002). Though if we consider the continuum of voters as an approximation for a large finite number, the independence assumptions does not cause any problem.
so that is:

$$b_i^2 \geq b_i^r$$

for $U \geq 0$, i.e. party 2 has always at least as much money as its opponent to redistribute.

The problem of the parties is studied using the approach introduced by Sahuguet and Persico (2006); in particular, the strategic equivalence between the problem where two parties try to maximize their votes by redistributing a given budget to the electorate, and the problem of two players that maximize their expected utility trying to win some objects in an all-pay auction is exploited.

Present framework though, is slightly different from that studied by those authors given that there is an unhomogeneous electorate where one party has a specific advantage on some voters; the following proposition thus, is required:

**Proposition 1** The problem faced by the parties in the definition of the redistribution plan is strategically equivalent to that of two players competing simultaneously on two types of independent first-price all-pay auctions where the same object is auctioned and where players’ valuations are known.

**Proof.** Define $F_i^l (x_i^l)$ and $F_i^k (x_i^k)$ as the cumulative distribution functions for party $i$ promises respectively to a member of the special interest group and to a standard voter; each function specifies the probability that the ratio among the sum promised to a member of the group or to a standard voter, and party $i$ redistributable budget is smaller than $x_i^l$ and $x_i^k$. The maximization problem of party 1 is:

$$\begin{align*}
\max_{F_1^l, F_1^r} & \quad \lambda \int_0^\infty F_2^l \left( x_1^l \cdot \frac{b_r^l}{b_2^l} + \frac{U}{\lambda \cdot b_2^l} \right) dF_1^l (x_1^l) + \\
& \quad + \left( 1 - \lambda \right) \int_0^\infty F_2^k \left( x_1^k \cdot \frac{b_r^k}{b_2^k} \right) dF_1^k (x_1^k) + \\
& \quad + \mu_1 \left[ 1 - \lambda \right] \int_0^\infty x_1^l dF_1^l (x_1^l) - \left( 1 - \lambda \right) \int_0^\infty x_1^k dF_1^k (x_1^k) \end{align*}$$

Define now:

$$\begin{align*}
x_1^l \cdot \frac{b_r^l}{b_2^l} &= y_1^l \\
x_1^k \cdot \frac{b_r^k}{b_2^k} &= y_1^k \\
F_1^l \left( y_1^l \cdot \frac{b_r^l}{b_2^l} \right) &= \hat{F}_1^l (y_1^l) \\
F_1^k \left( y_1^k \cdot \frac{b_r^k}{b_2^k} \right) &= \hat{F}_1^k (y_1^k) \end{align*}$$
and 

\[ \frac{b_1}{b_2} = \rho \]

Party 1 problem can be rewritten as:

\[
\begin{align*}
\max_{\hat{F}_1, F_1^k} & \int_0^\infty F_2 (y_1') \, d\hat{F}_1 (y_1') + \\
& + (1 - \lambda) \int_0^\infty F_2^k (y_1^k) \, d\hat{F}_1^k (y_1^k) + \\
& + \mu_1 \left[ \rho - \lambda \int_0^\infty y_1 d\hat{F}_1 (y_1') - (1 - \lambda) \int_0^\infty y_1^k d\hat{F}_1^k (y_1^k) \right]
\end{align*}
\]

or equivalently as:

\[
(\lambda \cdot \mu_1) \left\{ \max_{\hat{F}_1, F_2} \int_0^\infty \left[ \frac{1}{\mu_1} F_2 (y_1') + \frac{U}{\lambda \cdot b_2} \right] \, d\hat{F}_1 (y_1') + \right\}
\]

Up to a linear transformation then, this setting is analogous to the problem of a risk-neutral agent that maximizes his expected utility competing on two types of independent all-pay auctions where two identical objects are auctioned.

The first type of auction is played with probability \( \lambda \) and the auctioneer awards player 1 an advantage by increasing his bids of the amount \( \frac{U}{\lambda \cdot b_2} \); the second type of auction shows up with complementary probability, \( (1 - \lambda) \), and players are symmetric. At the time when he submits his bid, player 1 recognizes which type of auction is played.

Consider now party 2 maximization problem:

\[
\begin{align*}
\max_{\hat{F}_2, F_2^k} & \int_0^\infty F_1^2 (x_2') \, d\hat{F}_2^2 (x_2') + \\
& + (1 - \lambda) \int_0^\infty F_1^k (x_2^k) \, d\hat{F}_2^k (x_2^k) + \\
& + \mu_2 \left[ 1 - \lambda \int_0^\infty x_2^2 d\hat{F}_2^2 (x_2') - (1 - \lambda) \int_0^\infty x_2^k d\hat{F}_2^k (x_2^k) \right]
\end{align*}
\]

Since on the lobby subspace, party 1 has a fixed advantage over each voter, there are some actions in the set of party 2 strategies that are strictly dominated; namely every promise made to a member of the group such that \( 0 < x_2^2 \leq \frac{U}{b_2} \) is strictly dominated by \( x_2^2 = 0 \). Any promise smaller than party 1 advantage is not useful to obtain votes from the members of the special interest group and results in a mere reduction of party 2 budget. It is possible thus, to consider an equivalent setting where these strictly dominated strategies are not considered.
and rewrite the problem as follows:

\[
\max_{F_1^2, F_2^2, \lambda} \int_{\frac{U}{b_2}}^{\infty} F_1^2 \left( x_2^l \cdot \frac{b_2^l}{b_1^l} - \frac{U}{\lambda \cdot b_1^l} \right) dF_2^2 (x_2^l) + \\
+ (1 - \lambda) \int_{0}^{\infty} F_1^2 \left( x_2^k \cdot \frac{b_2^k}{b_1^k} \right) dF_2^k (x_2^k) + \\
+ \mu_2 \left[ 1 - \lambda \int_{\frac{U}{b_2}}^{\infty} x_2^l dF_2^l (x_2^l) - (1 - \lambda) \int_{0}^{\infty} x_2^k dF_2^k (x_2^k) \right]
\]

or

\[
\max_{F_1^2, F_2^2, \lambda} \int_{\frac{U}{b_2}}^{\infty} F_1^2 \left( x_2^l - \frac{U}{\lambda \cdot b_2} \right) dF_2^l (x_2^l) + \\
+ (1 - \lambda) \int_{0}^{\infty} F_1^k (x_2^k) dF_2^k (x_2^k) + \\
+ \mu_2 \left[ 1 - \lambda \int_{\frac{U}{b_2}}^{\infty} x_2^l dF_2^l (x_2^l) - (1 - \lambda) \int_{0}^{\infty} x_2^k dF_2^k (x_2^k) \right]
\]

Following the same steps described in the previous case, we get:

\[
(\lambda \cdot \mu_2) \left\{ \max_{F_1^2} \int_{\frac{U}{b_2}}^{\infty} \left[ \frac{1}{\mu_2} F_1^2 \left( x_2^l - \frac{U}{\lambda \cdot b_2} \right) - x_2^l \right] dF_2^l (x_2^l) \right\} + (4)
\]

Again, up to a linear transformation, this problem is equivalent to that of a risk neutral player that bids on two independent all-pay auctions of the type described above.

In this alternative setting the players don’t have a binding budget constraint while the parties do; the inverse of the Lagrangian multipliers attached to the budget constraints in the original framework defines, moreover the value of the prize. This implies that players value the object differently when parties have different budgets and thus, face different shadow costs of money; in particular, the player corresponding to the party with the biggest budget, has the highest valuation of the object while that corresponding to the party with less funds available, has the lowest one.

From these observations some implications for the analysis of the equilibrium derive.

Consider initially that given that the auctions where players bid are independent, each of them can be considered separately.

The auction corresponding to the non-lobby subspace is a standard all-pay one whose equilibrium strategies are known in the literature; the analysis thus

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focus on the auction equivalent to the redistribution program on the lobby subspace.

In this context, different cases emerge depending on the relative size of parties budgets and on \( \frac{1}{\lambda} \); all possible situations are analyzed considering the equivalent problem where two types of independent all-pay auctions, with identical objects, are played.

The description of the equilibrium for an all-pay auction where one bidder has an advantage over the opponent is skipped; a wider discussion of this argument can be found in the appendix.

Total resources available for redistribution represent the crucial variable in order to define how the parties are going to compete in the two subspaces. As showed before party 1 has always the smallest redistributable budget; in the non-lobby subspace thus, party 2 plays as the favored party and gets the biggest fraction of those votes\(^6\). Indeed it can always decide to concentrate all its resources in that subspace.

In the auction setting this is equivalent to say that player 2’s valuation of the object is bigger than that of his opponent; in other words it is:

\[
\frac{1}{\mu_2} > \frac{1}{\mu_1}
\]

In the lobby subspace instead, the crucial quantity is \( \frac{1}{\mu_2} - \frac{1}{\mu_1} - \frac{U}{\lambda b_2} \); i.e. the difference between players’ valuations of the object including also the effects of the advantage \( \frac{U}{\lambda} \). In particular, whenever is:

\[
\frac{1}{\mu_2} - \frac{1}{\mu_1} - \frac{U}{\lambda b_2} \geq 0
\]

party 2 is leader on both subspaces i.e. its available resources permits to obtain the main fraction of votes on both subspaces, even if party 1 counts on the advantage \( \frac{U}{\lambda} \).

\[
\frac{1}{\mu_2} - \frac{1}{\mu_1} - \frac{U}{\lambda b_2} < 0
\]

party 2 is follower on the lobby subspace and leader on the non-lobby one.

A first result is the following:

**Proposition 2** Party 2 is never leader on the lobby subspace, i.e. it is the case that:

\[
\frac{1}{\mu_2} - \frac{1}{\mu_1} - \frac{U}{\lambda b_2} \leq 0
\]

**Proof.** Suppose is \( \frac{1}{\mu_2} - \frac{1}{\mu_1} - \frac{U}{\lambda b_2} > 0 \).

In the equivalent setting described above, this happens when player 2 values the auctioned object more than the opponent’s valuation increased for his advantage on the lobby-type auction.

\(^6\)A formal proof for this result can be found in Sahughet and Persico (2006).
Consider the all-pay auction corresponding to the lobby subspace; in equilibrium player 2 can always obtain a positive payoff amounting to \( \frac{1}{\mu_2} - \frac{1}{\mu_1} - \frac{U}{\lambda \cdot b_2} > 0 \) while player 1 cannot get more than zero.

From the previous observations the following equilibrium conditions can be derived:

- **Player 1**
  \[
  \left( \frac{1}{\mu_1} - y_1 \right) F_2 \left( y_1 + \frac{U}{\lambda \cdot b_2} \right) - y_1 \left[ 1 - F_2 \left( y_1 + \frac{U}{\lambda \cdot b_2} \right) \right] = 0 \quad (5)
  \]
  so that is:
  \[
  F_2 \left( y_1 + \frac{U}{\lambda \cdot b_2} \right) = \left( y_1 + \frac{U}{\lambda \cdot b_2} \right) \cdot \mu_1 - \frac{U}{\lambda \cdot b_2} \cdot \mu_1
  \]
  moreover since every bid \( y_1 \) of player 1 is equivalent to a bid of player 2 amounting to \( y_1 + \frac{U}{\lambda \cdot b_2} \), is also:
  \[
  F_2 \left( x_2 \right) = \left( x_2 - \frac{U}{\lambda \cdot b_2} \right) \cdot \mu_1
  \]

Player 2 then, randomizes according to a uniform distribution over the support \( \left[ \frac{U}{\lambda \cdot b_2}, \frac{1}{\mu_1} + \frac{U}{\lambda \cdot b_2} \right] \)

- **Player 2**
  \[
  \left( \frac{1}{\mu_2} - x_2 \right) F_1 \left( x_2 - \frac{U}{\lambda \cdot b_2} \right) - x_2 \left[ 1 - F_1 \left( x_2 - \frac{U}{\lambda \cdot b_2} \right) \right] = \frac{1}{\mu_2} \left( \frac{1}{\mu_1} \right) - \frac{U}{\lambda \cdot b_2} \quad (6)
  \]
  so that is:
  \[
  F_1 \left( x_2 - \frac{U}{\lambda \cdot b_2} \right) = 1 - \frac{\mu_2}{\mu_1} + \left( x_2 - \frac{U}{\lambda \cdot b_2} \right) \cdot \mu_2
  \]
  finally, since every bid \( x_2 \) of player 2 is equivalent to a bid of player 1 amounting to \( x_2 - \frac{U}{\lambda \cdot b_2} \), we get:
  \[
  F_1 \left( y_1 \right) = 1 - \frac{\mu_2}{\mu_1} + y_1 \cdot \mu_2
  \]

Player 1 randomizes according to a uniform distribution over the support \( \left[ 0; \frac{1}{\mu_1} \right] \) with an atom of probability amounting to \( \left( 1 - \frac{\mu_2}{\mu_1} \right) \) in zero.
In the non-lobby subspace the equivalent problem boils down to a standard all-pay auction where the players have different valuations of the auctioned object. As a consequence, in equilibrium, player 1 bids zero with probability \( 1 - \frac{\mu_2}{\mu_1} \) and randomizes over the support \( \left[ 0, \frac{1}{\mu_1} \right] \), according to a uniform distribution; player 2 instead, randomizes according to a uniform distribution on the support \( \left[ 0, \frac{1}{\mu_1} \right] \).  

Consider now the problem of the parties and notice that since is:

\[
x_1^i = y_1^i \cdot \frac{b_2^i}{b_1^i}
\]

is also:

\[
F_1^i (x_1^i) = 1 - \frac{\mu_2}{\mu_1} + x_1^i \cdot \rho \cdot \mu_2
\]

Party 1 thus promises zero with probability \( 1 - \frac{\mu_2}{\mu_1} \) and with probability \( \frac{\mu_2}{\mu_1} \) randomizes according to a uniform distribution over the support \( \left[ 0; \frac{1}{\mu_1} \cdot \frac{1}{\rho} \right] \).

Moreover in the non-lobby auction it is the case that:

\[
F_1^k (x_1^k) = 1 - \frac{\mu_2}{\mu_1} + x_1^k \cdot \rho \cdot \mu_2
\]

therefore party 1 promises zero with probability \( 1 - \frac{\mu_2}{\mu_1} \) and with complementary probability randomizes according to a uniform distribution over the support \( \left[ 0; \frac{1}{\mu_1} \cdot \frac{1}{\rho} \right] \).

The distribution of party 2 promises to both types of voters coincide with the distribution of player 2’s bids in the all-pay auctions setting.

Consider the budget constraints in the original problem; look first at party 2:

\[
1 = \lambda \int_{\frac{b_1^2}{b_2^2}}^{\frac{b_1^1}{b_2^1}} x_2^i \cdot \mu_1 + (1 - \lambda) \int_{\frac{b_1^2}{b_2^2}}^{\frac{b_1^0}{b_2^0}} x_2^k \cdot \mu_1
\]

or

\[
1 - \frac{U}{b_2^2} = \int_{\frac{b_1^1}{b_2^1}}^{\frac{b_1^0}{b_2^0}} x_2^i \cdot \mu_1
\]

For party 1 instead holds:

\[
1 = \lambda \int_{\frac{b_1^1}{b_2^1}}^{\frac{b_1^0}{b_2^0}} x_1^i \cdot \mu_2 + (1 - \lambda) \int_{\frac{b_1^1}{b_2^1}}^{\frac{b_1^0}{b_2^0}} x_1^k \cdot \mu_2
\]

or:

\[7\text{Further details about this result can be found in Sahuguet and Persico (2006).}\]
\[ \rho = \int_0^{\frac{1}{\mu_1}} x_1^k \cdot \mu_2 \]

then it is the case that:

\[ \frac{\mu_2}{\mu_1} = \frac{\rho}{1 - \frac{U}{\lambda b_2}} = \frac{1 - U}{1 - U} = 1 \]

The values for the Lagrangians attached to the parties budget constraints must now be computed.

In particular, there can be only a pair of values, \( \mu_1 \) and \( \mu_2 \), deriving from the equilibrium strategies for the all-pay auctions that coincides also with the shadow cost of money for the parties; this guarantees the uniqueness of the equilibrium in the original setting.

Solving the integral in party 2 budget constraint gives:

\[ \frac{1}{\mu_1} = \frac{1}{\mu_2} = 2(1 - U) \quad (9) \]

The initial condition: \( \frac{1}{\mu_2} - \frac{1}{\mu_1} - \frac{U}{\lambda b_2} > 0 \) thus, can never be true and must be the case that:

\[ \frac{1}{\mu_1} + \frac{U}{\lambda b_2} - \frac{1}{\mu_2} \geq 0 \]

Party 1 then, is leader on the lobby subspace; this implies that party 2 promises a positive amount to special interest group members only when is:

\[ \frac{1}{\mu_2} \geq \frac{U}{\lambda b_2} \quad (10) \]

The proof descends straightforwardly from the fact that in the modified setting when is \( \frac{1}{\mu_2} < \frac{U}{\lambda b_2} \) any strictly positive bid smaller than \( \frac{U}{\lambda b_2} \) has probability zero to win the object and gives a negative payoff; thus player 2 will either not participate to the auction or bid more than \( \frac{U}{\lambda b_2} \).

In terms of the original problem this means that party 2 offers zero with probability one to the members of the special interest group and competes only for standard voters; when this happens an optimal response for party 1 is to offer zero to the members of the special interest group as well and concentrate the available budget on the rest of the electorate.

The problem then, boils down to a single all-pay auction, corresponding to the non-lobby subspace, where parties payoffs depend on their available budgets; the equilibrium strategies are analogous to those described in Sahuguet and Persico (2006). In particular, party 1, in equilibrium, promises zero with probability \( \left(1 - \frac{\mu_2}{\mu_1}\right) \) and with complementary probability randomizes according to a
uniform distribution over the support \([0; \frac{1}{\mu_1} \cdot \frac{1}{1}]\); party 2 promises instead, are distributed according to a uniform distribution over the support \([0; \frac{1}{\mu_1}]\).

The equilibrium values for the Lagrangians derive from the budget constraint equations reported below:

\[
\rho = (1 - \lambda) \int_0^{\frac{\mu}{\mu_1}} x_1^k \cdot \mu_2
\]

for party 1 and

\[
1 = (1 - \lambda) \int_0^{\frac{\mu}{\mu_1}} x_2^k \cdot \mu_1
\]

for party 2.

Solving the integrals gives:

\[
1 \frac{\mu_1}{\mu} = \frac{2}{1 - \lambda}
\]

and

\[
1 \frac{\mu_2}{\mu} = \frac{2}{1 - \lambda} \cdot \frac{1}{\rho}
\]

From the initial condition the benefit transferred to a member of the special interest group must be such that:

\[
\frac{2}{1 - \lambda} \cdot \frac{1}{\rho} < \frac{U}{\lambda}
\]

Since holds \(b_2^* \geq b_1^*\), \((\rho \leq 1)\) parties payoffs are:

\[
S_1 = (1 - \lambda) \frac{\rho}{2} + \lambda = S
\]

and

\[
S_2 = (1 - \lambda) \left(1 - \frac{\rho}{2}\right) = 1 - S
\]

**Proposition 3** When \(\frac{2}{(1 - \lambda)} \cdot \frac{1}{\rho} < \frac{U}{\lambda}\) party 1 payoff is always smaller than \(\frac{1}{2}\).

**Proof.** In order to have \(S_1 \geq \frac{1}{2}\) must hold:

\[
U \leq \frac{\lambda}{1 - \lambda}
\]

The initial condition can be rewritten as:

\[
2 \cdot \frac{\lambda}{1 - \lambda} < (1 - U) U
\]

Consider now the first derivative of the right hand side of the previous inequality with respect to \(U\):
\[
\frac{\delta (1-U) U}{\delta U} = 1 - 2 \cdot U
\]

When \( U \geq \frac{1}{2} \) the right hand side decreases in \( U \); \((1-U)U\) reaches its maximum in 0 and it can never be the case that \( 2 \cdot \frac{\lambda}{1-\lambda} < 0 \).

If instead \( U < \frac{1}{2} \) and \( U \leq \frac{\lambda}{\lambda-1} \), \((1-U)U\) reaches its maximum at \( U = \frac{\lambda}{\lambda-1} \); this means that again it cannot be \( 2 \cdot \frac{\lambda}{1-\lambda} < \left( 1 - \frac{\lambda}{\lambda-1} \right) \frac{\lambda}{1-\lambda} \) since \( \lambda \leq 1 \). □

Consider now the case when is \( \frac{1}{\mu_2} \geq \frac{U}{\lambda b_2} \); the following proposition hold.

**Proposition 4** In the unique equilibrium for this subgame, party 1 promises to special interest group members are distributed according to a uniform distribution over the support \( h_0; \frac{1}{\mu_1} \); in the standard voters subspace moreover, it promises zero with probability \( 1 - \frac{\mu_2}{\mu_1} \) and with complementary probability randomizes according to a uniform distribution over the support \( \left[ 0; \frac{1}{\mu_1} \right] \) for the standard voters subspace.

Party 2 randomizes according to a uniform distribution over the support \( \left[ \frac{U}{\lambda b_2}; \frac{1}{\mu_2} \right] \) in the lobby subspace and uses a uniform distribution on the support \( h_0, \frac{1}{\mu_2} \) in the lobby subspace and uses a uniform distribution on the support \( h_0, \frac{1}{\mu_2} \) for the standard voters subspace.

Party 1 payoff, \( S_1 \), is smaller than \( \frac{1}{2} \).

**Proof.** If \( \frac{1}{\mu_2} \geq \frac{U}{\lambda b_2} \) and \( \frac{1}{\mu_2} < \frac{1}{\mu_1} + \frac{U}{\lambda b_2} \), player 2 gets at most zero in the lobby auction; indeed player 1 can always obtain a positive payoff by bidding \( \frac{1}{\mu_2} - \frac{U}{\lambda b_2} \) with probability one.

The equilibrium conditions for this case are:

- **Player 1**

\[
\left( \frac{1}{\mu_1} - y_1^l \right) F_2\left( y_1^l + \frac{U}{\lambda \cdot b_2} \right) - y_1^l \left[ 1 - F_2\left( y_1^l + \frac{U}{\lambda \cdot b_2} \right) \right] = \frac{1}{\mu_1} + \frac{U}{\lambda \cdot b_2} - \frac{1}{\mu_2}
\]

so that is:

\[
F_2\left( y_1^l + \frac{U}{\lambda \cdot b_2} \right) = 1 - \frac{\mu_1}{\mu_2} + \left( y_1^l + \frac{U}{\lambda \cdot b_2} \right) \mu_1
\]

moreover, since every bid \( y_1^l \) of player 1 is equivalent to a bid of player 2 amounting to \( y_1^l + \frac{U}{\lambda b_2} \), is also:

\[
F_2\left( x_2^l \right) = 1 - \frac{\mu_1}{\mu_2} + x_2^l \cdot \mu_1
\]

Player 2 randomizes according to a uniform distribution over the support \( \left[ \frac{U}{\lambda b_2}; \frac{1}{\mu_2} \right] \), given that every bid smaller than \( \frac{U}{\lambda b_2} \) is a strictly dominated action.
• Player 2

\[
\left(\frac{1}{\mu_2} - x_2^l\right) \bar{F}_1^l \left(x_2^r - \frac{U}{\lambda \cdot b_2^r}\right) - x_2^l \left[1 - \bar{F}_1^l \left(x_2^r - \frac{U}{\lambda \cdot b_2^r}\right)\right] = 0
\]  

(14)

so that is:

\[
\bar{F}_1^l \left(x_2^r - \frac{U}{\lambda \cdot b_2^r}\right) = \left(x_2^r - \frac{U}{\lambda \cdot b_2^r}\right) \cdot \mu_2 + \frac{U}{\lambda \cdot b_2^r} \cdot \mu_2
\]

finally, since every bid \(x_2^l\) of player 2 is equivalent to a bid of player 1 amounting to \(x_2^l - \frac{U}{\lambda \cdot b_2^r}\), is also:

\[
\bar{F}_1^l (y_1) = \left(y_1 + \frac{U}{\lambda \cdot b_2^r}\right) \cdot \mu_2
\]

Player 1 randomizes according to a uniform distribution over the support \([0; \frac{1}{\mu_2} - \frac{U}{\lambda \cdot b_2^r}]\) with an atom of probability amounting to \(\frac{U}{\lambda \cdot b_2^r} \cdot \mu_2\) in zero.

In the non-lobby subspace the equilibrium strategies do not change: player 1 bids zero with probability \(\left(1 - \frac{\mu_2}{\mu_1}\right)\) and randomizes over the support \([0, \frac{1}{\mu_1}]\), according to a uniform distribution; player 2 instead, randomizes according to a uniform distribution on the support \([0, \frac{1}{\mu_1}]\).

Consider now the problem of the parties and notice that since is:

\[
x_1^l = y_1^l \cdot \frac{b_2^r}{b_1^r}
\]

is also:

\[
\bar{F}_1^l (x_1^l) = \left(x_1^l + \frac{U}{\lambda \cdot b_1^r}\right) \cdot \rho \cdot \mu_2
\]

Party 1 randomizes according to a uniform distribution over the support \([0; \frac{1}{\mu_2} \cdot \frac{1}{\rho} - \frac{U}{\lambda \cdot b_1^r}]\) with an atom in zero amounting to \(\frac{U}{\lambda \cdot b_2^r} \cdot \mu_2\).

Moreover in the non-lobby auction it is the case that:

\[
\bar{F}_1^k (x_1^k) = 1 - \frac{\mu_2}{\mu_1} + x_1^k \cdot \rho \cdot \mu_2
\]

and party 1 promises zero with probability \(\left(1 - \frac{\mu_2}{\mu_1}\right)\) and with complementary probability randomizes according to a uniform distribution over the support \([0; \frac{1}{\mu_1} \cdot \frac{1}{\rho}]\).

The distribution of party 2 promises coincides with the distribution of player 2’s bids.

Consider now the budget constraint in the original problem; look first at party 2:
\[ 1 = \lambda \int_0^{\frac{1}{\lambda}} x_1^\nu \cdot \mu_1 + (1 - \lambda) \int_0^{\frac{1}{\lambda}} x_2^k \cdot \mu_1 \]  \hspace{1cm} (15)

For party 1 holds:

\[ \rho = \lambda \int_0^{\frac{1}{\lambda}} x_1^\nu \cdot \mu_2 + (1 - \lambda) \int_0^{\frac{1}{\lambda}} x_1^k \cdot \mu_2 \]  \hspace{1cm} (16)

then it is the case that:

\[ \frac{1}{\mu_1} = \frac{\lambda}{2} \left[ \left( \frac{1}{\mu_2} \right)^2 - \left( \frac{U}{\lambda} \right)^2 \right] + \frac{1 - \lambda}{2} \left( \frac{1}{\mu_1} \right)^2 \]

and

\[ \frac{1}{\mu_2} = \frac{\lambda}{2} \left[ \left( \frac{1}{\mu_2} \right)^2 + \left( \frac{U}{\lambda} \right)^2 \right] + \frac{1 - \lambda}{2} \left( \frac{1}{\mu_1} \right)^2 \]

so that:

\[ \frac{1}{\mu_2} - \frac{1}{\mu_1} = \lambda \left( \frac{U}{\lambda} \right)^2 \]

The initial condition then, is always verified; indeed holds:

\[ \frac{1}{\mu_1} + \frac{U}{\lambda} \cdot \nu_2 - \frac{1}{\mu_2} = \frac{U}{\lambda} (1 - U) \geq 0 \]

To prove uniqueness of the equilibrium the values of the Lagrangians must be computed; to this aim look at party 2 budget constraint and solve the integral to get:

\[ \frac{1}{\mu_1} = 1 - \lambda^2 \left( \frac{U}{\lambda} \right)^2 \pm \sqrt{\left[ 1 - \lambda^2 \left( \frac{U}{\lambda} \right)^2 \right] \left[ 1 - \lambda^2 \left( \frac{U}{\lambda} \right)^2 + \lambda \left( \frac{U}{\lambda} \right)^2 \right]} \]

Notice that when is \( \frac{U}{\lambda} = 0 \), is also \( \frac{1}{\mu_1} = 2 \); therefore it must be the case that:

\[ \frac{1}{\mu_1} = 1 - \lambda^2 \left( \frac{U}{\lambda} \right)^2 \pm \sqrt{\left[ 1 - \lambda^2 \left( \frac{U}{\lambda} \right)^2 \right] \left[ 1 - \lambda^2 \left( \frac{U}{\lambda} \right)^2 + \lambda \left( \frac{U}{\lambda} \right)^2 \right]} \]

and

\[ \text{8 See Sahuguet and Persico (2006) for a formal proof.} \]
\[
\frac{1}{\mu_2} = 1 - \lambda^2 \left( \frac{U}{\lambda} \right)^2 + \lambda \left( \frac{U}{\lambda} \right)^2 + \sqrt{\left[ 1 - \lambda^2 \left( \frac{U}{\lambda} \right)^2 \right] \left[ 1 - \lambda^2 \left( \frac{U}{\lambda} \right)^2 + \lambda \left( \frac{U}{\lambda} \right)^2 \right]}
\]

Consider now parties payoffs.

In the standard voters subspace party 1 gets half of the votes of those who receive a promise \( x_k^1 > 0 \); in the special interest group subspace instead, party 1 gets all the votes of those who receive a zero promise from party 2 and half of those who receive a positive promise from both parties.

Party 1 then, loses the votes of the member of the special interest group that receive a positive promise from party 2 and get at the same time, only the fixed amount \( \frac{U}{\lambda} \) from its redistribution plan; therefore it must be:

\[
S_1 = (1 - \lambda) \cdot \frac{1}{2} \cdot \frac{\mu_2}{\mu_1} + \lambda \cdot \left[ 1 - \frac{\mu_1}{\mu_2} + \frac{U}{\lambda} \cdot \mu_1 + \frac{1}{2} \left( \frac{\mu_1}{\mu_2} - \frac{U}{\lambda} \cdot \mu_1 \right) \left( 1 - \frac{U}{\lambda} \cdot \mu_2 \right) \right]
\]

or

\[
S_1 = (1 - \lambda) \cdot \frac{1}{2} \cdot \frac{\mu_2}{\mu_1} + \lambda \cdot \left[ 1 - \frac{1}{2} \left( \frac{\mu_1}{\mu_2} - \left( \frac{U}{\lambda} \right)^2 \cdot \mu_2 \cdot \mu_1 \right) \right] = S
\] (17)

and also

\[
S_2 = (1 - \lambda) \left[ 1 - \frac{1}{2} \cdot \frac{\mu_1}{\mu_2} \right] + \lambda \cdot \frac{1}{2} \left( \frac{\mu_1}{\mu_2} - \left( \frac{U}{\lambda} \right)^2 \cdot \mu_2 \cdot \mu_1 \right) = 1 - S
\] (18)

Notice that \( S_1 \leq \frac{1}{2} \) since substituting for \( \frac{1}{\mu_1} \) in the equation defining party 1 payoff gives:

\[
\frac{1}{2} \cdot \frac{1 - \lambda^2 \left( \frac{U}{\lambda} \right)^2}{1 - \lambda^2 \left( \frac{U}{\lambda} \right)^2 + \lambda \left( \frac{U}{\lambda} \right)^2 + 2 \sqrt{1 - \lambda^2 \left( \frac{U}{\lambda} \right)^2} \left[ 1 - \lambda^2 \left( \frac{U}{\lambda} \right)^2 + \lambda \left( \frac{U}{\lambda} \right)^2 \right]} \leq \frac{1}{2}
\]

and this is always true for \( \lambda \geq 0 \) and \( U \geq 0 \).

In order to define the equilibrium for the subgame where a decision on public good provision is taken, some other situations must be considered and in particular those where the parties make the same choice with respect to \( g \).

Look initially at party 1 maximization problem when both decide to implement the public good:
Max\(F_l, F_k\lambda \int_0^\infty F_j^l \left[ x_l \cdot \frac{b_l^r}{b_j^r} + \frac{q(g_l)}{b_l^r} - \frac{q(g_j)}{b_j^r} \right] dF_i^l (x_l^i) + \\
+ (1 - \lambda) \int_0^\infty F_j^k \left( x_k \cdot \frac{b_k^r}{b_j^r} \right) dF_i^k (x_k^i) + \\
+ \mu_i \left[ 1 - \lambda \int_0^\infty x_l^i dF_i^l (x_l^i) - (1 - \lambda) \int_0^\infty x_k^i dF_i^k (x_k^i) \right].
\]

The maximization problem above can be rewritten as:

Max\(F_l, F_k\lambda \int_0^\infty F_j^l (x_l^i) dF_i^l (x_l^i) + \\
+ (1 - \lambda) \int_0^\infty F_j^k (x_k^i) dF_i^k (x_k^i) + \\
+ \mu_i \left[ 1 - \lambda \int_0^\infty x_l^i dF_i^l (x_l^i) - (1 - \lambda) \int_0^\infty x_k^i dF_i^k (x_k^i) \right].
\]

that is party \(i\) maximization problem when the parties choose not to buy \(g\).

In the modified setting, this is equivalent to:

\((\lambda \cdot \mu_2) \left\{ \text{Max} F_j^l \int_0^\infty \left[ \frac{1}{\mu_2} F_i^l (x_l^i) - x_l^i \right] dF_i^l (x_l^i) + \\
+ (1 - \lambda) \mu_2 \left\{ \text{Max} F_j^l \int_0^\infty \left[ \frac{1}{\mu_2} F_i^k (x_k^i) - x_k^i \right] dF_i^k (x_k^i) \right\} + \mu_2 \right\}
\]

The players thus, bid on two standard all-pay auctions where the same object is auctioned; in terms of the problem where we started from this means that the parties face an homogeneous electorate and the division in subspaces has no reason to be.

Both these cases then, are strategically equivalent and the definition of the equilibrium for the subgame boils down into the problem studied by Sahuguet and Persico (2006); the same unique mixed strategies equilibrium arises defined by the following elements:

\* \(\frac{1}{\mu_2} = \frac{1}{\mu_2} = 2;\)

\* Each voter receives from party \(i\) \((i = 1, 2)\) the promise of a fraction of the unitarian per-capita redistributable budget that depends upon the realization of a random variable distributed as a uniform on the support \(0; \frac{1}{\mu_i};\)

\* \(S_i = S_j = \frac{1}{2}\).
It is possible now to characterize the equilibrium for the first stage of the platform presentation process i.e. the choice over the public good. This subgame can be described loosely as a standard prisoner’s dilemma whose representation in normal form is:

<table>
<thead>
<tr>
<th>Party 1</th>
<th>0</th>
<th>( \frac{S}{2} )</th>
<th>( g )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Party 2</td>
<td>0</td>
<td>( \frac{1 - S}{2} )</td>
<td>1 - S; S</td>
</tr>
<tr>
<td></td>
<td>( S; 1 - S )</td>
<td>( \frac{1 - S}{2} )</td>
<td></td>
</tr>
</tbody>
</table>

Since is \( S < \frac{1}{2} \) in all the considered cases, the only Nash equilibrium is the choice not to implement the public good for both parties.

### 3.2.2 Efficient Public Good

Consider now the situation where the public good generates benefits exceeding costs and \( \xi > 0 \).

Given the results of previous analysis, this section focuses on the level of efficiency required for \( g \) to be implemented; moreover it considers how this level varies when \( \lambda \) changes in order to disentangle the effects on \( \xi \) of different degrees of concentration in the benefits generated by the public good.

Notice that under the assumption of \( \xi > 0 \), nothing changes in the equilibria if parties are symmetric; differences arise when only one party includes \( g \) in its platform.

Assume again that party 1 chooses the public good. An increase in the efficiency of \( g \) affects mainly the budget constraint of the party choosing it; it is the case then, that party 1 redistributable budget increases and amounts now to:

\[
\tilde{b}_1^* = 1 - U (1 - \xi)
\]

In the modified setting, players’ valuations of the object change; in particular, the fact that party 1 budget constraint is eased increases player 1’s valuation. Indeed party 1 shadow cost of money decreases and:

\[
\tilde{\mu}_1 < \mu_1 \quad (19)
\]

where \( \tilde{\mu}_1 \) defines the new value of the Lagrangian attached to party 1 budget constraint.

Moreover since the shadow cost of money is defined with respect to the opponent’s budget it is also:

\[
\tilde{\mu}_2 > \mu_2 \quad (20)
\]

where \( \tilde{\mu}_2 \) defines the new value of the Lagrangian attached to party 2 budget constraint.

As in the previous case, party 2 does not promise any positive amount to special interest group members if \( \frac{1}{\tilde{\mu}_2} < \frac{U}{h_2} \).
Consider first the case where \( \frac{1}{\mu_2} \geq \frac{\lambda}{\mu_2} \) and notice that the variation in the values of the Lagrangians is neutral with respect to the form of players’ bids distribution; only the supports change.

Party 1 promises moreover, need to be rescaled for the factor:

\[
\tilde{\rho} = \frac{\tilde{\mu}_1}{\tilde{\rho}_2} = 1 - U (1 - \xi)
\]

Given these observations, in equilibrium, party 1 randomizes in the standard voters subspace according to the following cumulate probability function:

\[
F_1^l (x_1^l) = \left( x_1^l + \frac{U}{\lambda \cdot b_2} \right) \cdot \tilde{\rho} \cdot \tilde{\mu}_2
\]

over the support \([0; \frac{1}{\mu_2}, \frac{1}{\mu_2} - \frac{U}{\lambda b_2}]\).

In the special interest group subspace, instead the cumulate probability function is:

\[
F_1^k (x_1^k) = 1 - \frac{\tilde{\mu}_1}{\tilde{\mu}_2} + x_1^k \cdot \tilde{\rho} \cdot \tilde{\mu}_2
\]

party 1 promises zero with probability \(1 - \frac{\tilde{\mu}_2}{\tilde{\mu}_1}\) and with complementary probability randomizes according to a uniform distribution over the support \([0; \frac{1}{\mu_2}, \frac{1}{\mu_2} - \frac{U}{\lambda b_2}]\).

Party 2 equilibrium strategy for the lobby subspace is described by:

\[
F_2^l (x_2^l) = 1 - \frac{\tilde{\mu}_1}{\tilde{\mu}_2} + x_2^l \cdot \tilde{\mu}_1
\]

defined over the support \(\left[ \frac{\tilde{\mu}_1}{\lambda b_2}, \frac{1}{\mu_2} \right]\) with an atom in zero amounting to \(1 - \frac{\tilde{\mu}_1}{\tilde{\mu}_2} + \frac{U}{\lambda b_2} \cdot \tilde{\mu}_1\).

Promises to standard voters finally, are distributed according to the following cumulate probability function

\[
F_2^k (x_2^k) = x_2^k \cdot \tilde{\mu}_1
\]

over the support \([0, \frac{1}{\mu_2}]\).

Full characterization of the equilibrium requires to compute the values of \(\tilde{\mu}_1\) and \(\tilde{\mu}_2\).

Consider the budget constraint in the original problem; look first at party 2:

\[
1 = \lambda \int_{\frac{\tilde{\mu}_1}{\lambda b_2}}^{\frac{x_2^k}{\lambda b_2}} x_2^k \cdot \tilde{\mu}_1 + (1 - \lambda) \int_{0}^{\frac{\tilde{\mu}_1}{\lambda b_2}} x_2^k \cdot \tilde{\mu}_1
\]

For party 1 instead, holds:
\[
\bar{\rho} = \lambda \int_{0}^{\frac{1}{2}} \frac{\mu_2}{\mu_1} x_1 \cdot \bar{\mu}_2 + (1 - \lambda) \int_{0}^{\frac{1}{2}} x_{1}^{k} \cdot \bar{\mu}_2
\]

Notice further that in order for the public good to be chosen must be the case that \( S_1 \geq \frac{1}{2} \) i.e.: 

\[
(1 - \lambda) \cdot \frac{1}{2} \cdot \frac{\mu_2}{\mu_1} + \lambda \cdot \left\{ 1 - \frac{1}{2} \left[ \frac{\mu_1}{\mu_2} - \left( \frac{U}{\lambda} \right)^2 \cdot \mu_2 \cdot \mu_1 \right] \right\} \geq \frac{1}{2}
\]

An equilibrium for the game where the public good is implemented is characterized then, by a system of three equations in three unknowns \((\frac{1}{\mu_1}, \frac{1}{\mu_2}, \xi)\) defined as follows:

\[
\begin{cases}
\frac{1}{\mu_1} = \frac{1}{2} \left[ \left( \frac{1}{\mu_2} \right)^2 - \left( \frac{U}{\lambda} \right)^2 \right] + \frac{1 - \lambda}{2} \left( \frac{1}{\mu_1} \right)^2 \\
\frac{1}{\mu_2} (1 + U \cdot \xi) = \frac{1}{2} \left[ \left( \frac{1}{\mu_2} \right)^2 + U^2 \right] + \frac{1 - \lambda}{2} \left( \frac{1}{\mu_1} \right)^2 \\
(1 - \lambda) \cdot \frac{\mu_2}{\mu_1} + \lambda \cdot \left[ 2 - \frac{\mu_2}{\mu_1} + \left( \frac{U}{\lambda} \right)^2 \cdot \mu_2 \cdot \mu_1 \right] = 1
\end{cases}
\]

\( (21) \)

Solve initially for \( \frac{1}{\mu_1} \) as a function of \( \frac{1}{\mu_2} \) in the first equation to get:

\[
\frac{1}{\mu_1} = \frac{1}{1 - \lambda} \cdot \left\{ 1 + \sqrt{1 - \lambda (1 - \lambda) \left[ \left( \frac{1}{\mu_2} \right)^2 - \left( \frac{U}{\lambda} \right)^2 \right]} \right\}
\]

Notice now that when \( \lambda = 0 \) present framework reduces to the problem studied in Sahuguet and Persico (2006); in this case is \( \frac{1}{\mu_1} = \frac{1}{\mu_2} = 2 \) and therefore must be:

\[
\frac{1}{\mu_1} = \frac{1}{1 - \lambda} \cdot \left\{ 1 + \sqrt{1 - 4 \cdot \lambda \left( 1 - \lambda \right) \left[ 1 + \frac{1 - \lambda}{\lambda} \cdot U^2 - \left( \frac{U}{2 \cdot \lambda} \right)^2 \right]} \right\}
\]

Substitute the above value in the third equation and solve for \( \frac{1}{\mu_2} \) to get:

\[
\frac{1}{\mu_2} = 2 \sqrt{1 + \frac{1 - \lambda}{\lambda} \cdot U^2}
\]

Solve then, for \( \frac{1}{\mu_1} \) as a function of \( \lambda \) and \( U \):

\[
\frac{1}{\mu_1} = \frac{1}{1 - \lambda} \cdot \left\{ 1 + \sqrt{1 - 4 \cdot \lambda \left( 1 - \lambda \right) \left[ 1 + \frac{1 - \lambda}{\lambda} \cdot U^2 - \left( \frac{U}{2 \cdot \lambda} \right)^2 \right]} \right\}
\]

and substitute for \( \frac{1}{\mu_2} \) and \( \frac{1}{\mu_1} \) in the second equation to obtain the value of \( \xi \):
\[ \xi(U, \lambda) = \frac{1}{2 \cdot U} \cdot \frac{1}{1 - \lambda} \left[ \sqrt{1 + \frac{1 - \lambda}{\lambda} \cdot U^2 \pm (1 - 2 \cdot \lambda)} \right] - \frac{1}{U} \quad (22) \]

Two cases must be distinguished:

- \( \lambda \geq \frac{1}{2} \)

  When this happens and

  \[ \frac{1}{\hat{\mu}_2} = 2 \sqrt{1 + \frac{1 - \lambda}{\lambda} \cdot U^2} \geq \frac{U}{\lambda} \]

  or

  \[ U \leq \frac{2 \cdot \lambda}{2 \cdot \lambda - 1} \]

  the minimum level of efficiency required for public good implementation is:

  \[ \xi(U, \lambda) = \frac{1}{2 \cdot U} \cdot \frac{1}{1 - \lambda} \left[ \sqrt{1 + \frac{1 - \lambda}{\lambda} \cdot U^2 + 4 \cdot \lambda - 3} \right] \]

Consider now what happens to the above quantity, if the degree of concentration of public good benefits increases; this requires to look at \( \frac{\xi(U, \lambda)}{\lambda} \), i.e. at the first derivative of public good efficiency with respect to \( \lambda \):

\[ \frac{(1 + \frac{1 - \lambda}{\lambda} \cdot U^2)^{\frac{1}{2}}}{2 \cdot (1 - \lambda)} \cdot \left( \frac{1}{1 - \lambda} \cdot \frac{1}{U} + U \cdot \frac{2 \cdot \lambda - 1}{2 \cdot \lambda^2} \right) + \frac{1}{2 \cdot U} \cdot \frac{1}{(1 - \lambda)^2} \geq 0 \]

As the number of voters benefitting from the public good increases, also the degree of efficiency required for the implementation of \( g \) increases; in this case thus, benefits concentration pays. Public goods that produce large advantages for a small fraction of the electorate are more easily chosen.

Notice finally that in order for \( g \) be implemented it is not enough that the incentive constraint is satisfied, i.e. \( S_1 \geq \frac{1}{\lambda} \) but it must be the case that also the budget constraint holds so that:

\[ 1 - U \cdot (1 - \xi) \geq 0 \]

and thus:

\[ \xi(U) \geq \frac{U - 1}{U} \]
In other words $g$ is included in party 1 platform if:

$$
\xi \geq \max \left\{ \frac{1}{2 \cdot U} \cdot \frac{1}{1 - \lambda} \left[ \sqrt{1 + \frac{1 - \lambda}{\lambda} \cdot U^2 + 4 \cdot \lambda - 3} \right] ; \frac{U - 1}{U} \right\}
$$

(23)

- $\lambda < \frac{1}{2}$

When this is the case and

$$
U \leq \frac{2 \cdot \lambda}{1 - 2 \cdot \lambda}
$$

holds also:

$$
\xi (U, \lambda) = \frac{1}{2 \cdot U} \cdot \frac{1}{1 - \lambda} \left[ \sqrt{\left(1 + \frac{1 - \lambda}{\lambda} \cdot U^2 - 1\right)} \right]
$$

Look now at the effects of an increase in the fraction of voters that derive utility from $g$, and analyze $\frac{\delta \xi (U, \lambda)}{\delta \lambda}$:

$$
- \frac{1 - (1 + \frac{1 - \lambda}{\lambda} \cdot U^2)^{\frac{1}{2}}}{(2 \cdot U) (1 - \lambda)} - \frac{(1 + \frac{1 - \lambda}{\lambda} \cdot U^2)^{\frac{1}{2}}}{4 \cdot \lambda^2} \cdot \frac{1 - 2 \cdot \lambda}{1 - \lambda} < 0
$$

(24)

As $\lambda$ increases the level of efficiency required for the public good to be implemented decreases; concentration in this case does not pay and makes it more difficult for a politician to include the public good in his platform.

Also in this case the budget constraint requires that $g$ is implemented if:

$$
\xi \geq \max \left\{ \frac{1}{2 \cdot U} \cdot \frac{1}{1 - \lambda} \left[ \sqrt{1 + \frac{1 - \lambda}{\lambda} \cdot U^2 - 1} \right] ; \frac{U - 1}{U} \right\}
$$

(25)

The above statements though, hold when both parties compete over the whole electorate; a different scenario emerges if $\frac{1}{\mu_2} < \frac{U}{\lambda}$.

When this is the case party 2 gives up competing for the votes of those who are benefitted by the public good; the voting game then, boils down to a standard electoral competition of the type described in Saluguet and Persico (2006), where the electorate size is $(1 - \lambda)$. Parties problem is equivalent to that of two players bidding on a single all-pay auction where no advantages are awarded to any player.

In equilibrium party 1 promises zero with probability $\left(1 - \frac{\mu_2}{\mu_1}\right)$ and with complementary probability randomizes according to a uniform distribution over the support $\left[0; \frac{1}{\mu_1} \cdot \frac{1}{\rho}\right]$; party 2 promises instead, are distributed according to a uniform distribution over the support $\left[0; \frac{1}{\mu_2}\right]$. 27
 Parties payoffs are:

\[ S_1 = \frac{1 - \lambda}{2} \cdot \frac{\bar{\mu}_2}{\bar{\mu}_1} + \lambda = \bar{S} \]

and

\[ S_2 = (1 - \lambda) \left( 1 - \frac{1}{2} \cdot \frac{\bar{\mu}_2}{\bar{\mu}_1} \right) = 1 - \bar{S} \]

The system of equations corresponding to the above equilibrium strategies is:

\[
\begin{aligned}
1 &= (1 - \lambda) \int_0^l x^2 \cdot \bar{\mu}_1 \\
\bar{\rho} &= (1 - \lambda) \int_0^l x^1 \cdot \bar{\mu}_2 \\
\frac{1}{\bar{\mu}_1} \cdot \frac{\bar{\mu}_2}{\bar{\mu}_1} + \lambda &\geq \frac{1}{2} \tag{26}
\end{aligned}
\]

Solve first for \( \frac{1}{\bar{\mu}_1} \) in the first equation to get:

\[ \frac{1}{\bar{\mu}_1} = \frac{2}{1 - \lambda} \]

Substitute now this result in the third equation and solve for \( \frac{1}{\bar{\mu}_2} \):

\[ \frac{1}{\bar{\mu}_2} = \frac{2}{1 - \lambda} \cdot \frac{1}{1 - U (1 - \xi)} \]

Consider the last equation when the above values for \( \frac{1}{\bar{\mu}_1} \) and \( \frac{1}{\bar{\mu}_2} \) are used; solving for \( \xi \) gives:

\[ \xi (U, \lambda) \geq 1 - \frac{1}{U} \cdot \frac{\lambda}{1 - \lambda} \tag{27} \]

Solve finally for \( \frac{1}{\bar{\mu}_2} \) to have:

\[ \frac{1}{\bar{\mu}_2} = \frac{2}{1 - 2 \cdot \lambda} \]

and notice that the present case is possible only if \( \lambda < \frac{1}{2} \).

The initial condition \( \frac{1}{\bar{\mu}_2} < U \), can be rewritten now as:

\[ \frac{2 \cdot \lambda}{1 - 2 \cdot \lambda} < U \]

Look now at the first derivative of \( \xi (U, \lambda) \) with respect to \( \lambda \) to see what happens as the degree of concentration of the benefits deriving from \( g \) increases:

\[ \frac{\delta \cdot \xi (U, \lambda)}{\delta \cdot \lambda} = -\frac{1}{U} \cdot \frac{1}{1 - \lambda} \cdot \frac{\lambda}{(1 - \lambda)^2} < 0 \tag{28} \]

Also in this different setting, if \( \lambda < \frac{1}{2} \), benefits concentration does not pay; widely distributed benefits favor the implementation of the public good.
In order for the inclusion of $g$ in party 1 platform to be feasible it is required that:

$$\xi \geq \max \left\{ 1 - \frac{1}{U}; \frac{\lambda}{1 - \lambda}; \frac{U - 1}{U} \right\}$$  \hspace{1cm} (29)$$

The function $\xi(U, \lambda)$, then, has a point of minimum in $\lambda = \frac{1}{2}$; indeed for a given $U$, the level of efficiency required for the implementation of the public good decreases as the fraction of beneficiaries approaches half of the electorate and then start to increase.

Given the previous observations, the game in normal form can be described as follows:

<table>
<thead>
<tr>
<th>Party 1</th>
<th>$0$</th>
<th>$\bar{g}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Party 2</td>
<td>$\bar{g}$</td>
<td>$\frac{1}{2}; \frac{1}{2}$</td>
</tr>
<tr>
<td></td>
<td>$0; 1 - S$</td>
<td>$\frac{1}{2}; \frac{1}{2}$</td>
</tr>
</tbody>
</table>

where $\bar{S} \geq \frac{1}{2}$.

As a consequence buying the public good is a weakly dominant strategy when both the budget and the incentive constraint hold.

## 4 Final Remarks

The choice to implement a local public good that provides some benefits to a fraction of the electorate is subject to some constraints if available resources are limited and redistribution includes also direct cash transfers; efficiency in particular, is a major issue and it is not enough to have benefits balancing costs.

The fact that some resources are bounded to a specific location indeed, represents a strategic disadvantage that needs to be compensated; benefits then, must exceed costs.

Present analysis shows that the required degree of efficiency varies with the ratio among the number of those getting positive utility from the public good and total voters.

In particular when the beneficiaries amounts for less than half of the electorate, and total benefits are fixed, the level of efficiency that permits to implement the public good is inversely related to the fraction of voters taking advantage from it; in other words public goods that provide small benefits to many people are more easily chosen since require a lower level of efficiency.

On the contrary if beneficiaries amount for more than half the electorate, the level of efficiency required for the implementation of the public good increases as the fraction of voters taking advantage from it increases; it is the case then, that interventions that generates concentrated benefits, are more easily chosen.

From previous observations descends further, that there is an optimal size for the set of beneficiaries of a public good, and it coincide with half of the electorate; in that point indeed, the function that defines the level of efficiency...
required for the implementation reaches its minimum for each amount of benefits generated.

5 References

References


6 Appendix

6.1 Two Players All-pay Auctions where a Bidder has an Advantage over the Opponent

Consider the case where two risk-neutral players, 1 and 2, compete in a first-price all-pay auction with complete information and one player has an advantage over its opponent; without loss of generality, suppose that every bid of player 1 is increased of a fixed quantity $\alpha_1 > 0$. This additional amount is paid by the auctioneer.

Players do not have budget constraints and can bid any amount $x_i \geq 0$.

Player $i$’s valuation of the auctioned item is:

$$V_i > 0 \quad \text{with} \quad i = 1, 2.$$ 

Different equilibria emerge depending on players’ valuations of the auctioned item and on the size of player 1’s advantage; the crucial element is the comparison between $V_1 + \alpha_1$ and $V_2$.

**Proposition 5** When is:

$$V_2 < \alpha_1$$

player 1 and player 2 bid zero with probability one. Player 1 gets a payoff amounting to $V_1$ while player 2 gets zero.

**Proof.** The proof descends straightforwardly from the fact that any strictly positive bid smaller than $\alpha_1$ has zero probability to win and gives player 2 a strictly negative payoff; any bid $x_2 \geq \alpha_1$ on the other hand, requires player 2 to pay a sum that exceeds his valuation of the auctioned item and again gives a negative expected payoff.

A zero bid is a dominant strategy for player 2 and gives a null payoff.

Since player 2 does not submit positive bids, player 1 can always get the object by bidding zero with probability one. Any offer strictly bigger than zero cannot increase the probability to win and decreases player 1’s payoff.

A zero bid is a dominant strategy also for player 1 and gives a payoff equal to the valuation of the object $V_1$. ■

If is $V_2 \geq \alpha_1$, two main situations are possible:

- $V_2 > V_1 + \alpha_1$
- $V_2 \leq V_1 + \alpha_1$

Consider the first case; the following proposition holds:
Proposition 6  If \( V_2 > V_1 + \alpha_1 \) then player 1’s equilibrium distribution has an atom in zero amounting to 
\[
\left( 1 - \frac{V_1}{V_2} \right)
\] 
and is a uniform distribution over the support \([0, V_1]\) with probability \( \frac{V_1}{V_2} \); expected payoff is zero.

Player 2 instead, randomizes over the support \([\alpha_1, V_1 + \alpha_1]\) according to a uniform distribution; his equilibrium payoff is:
\[
V_2 - V_1 - \alpha_1 > 0
\]

Proof. Successive rounds of elimination of strictly dominated strategies allow to restrict the intervals where players randomize with positive probability to \([0, V_1]\) for player 1 and to \([\alpha_1, V_1 + \alpha_1]\) for player 2.

Player 1 indeed, never bids more than his own valuation of the object since this would give a negative payoff; bidding zero then, strictly dominates \( x_1 > V_1 \).

Consider now player 2.

Every strictly positive bid smaller than \( \alpha_1 \) is strictly dominated by bidding zero; any of such offer in facts, has zero probability to win and gives a negative payoff.

Since player 1 never bids more than \( V_1 \), player 2 can get the auctioned item with probability one by bidding \( x_2 \geq V_1 + \alpha_1 \); every bid strictly greater than \( V_1 + \alpha_1 \) though, is strictly dominated by \( x_2 = V_1 + \alpha_1 \). In both cases indeed, player 2 wins with probability one but choosing \( x_2 = V_1 + \alpha_1 \) involves a smaller payment to the auctioneer and results in a higher payoff.

Player 2 then, can always secure himself a payoff of \( V_2 - V_1 - \alpha_1 > 0 \).

The argument developed by Hillman and Riley (1987) allows to exclude that in equilibrium, player 1 submits a strictly positive offer, \( \kappa \), in the set \([0, V_1]\), with some strictly positive probability; when this happen indeed, player 2’s probability to win the auction rises discontinuously at \( x_2 = \kappa \); then there is some \( \varepsilon > 0 \) such that player 2 bids on the interval \([\kappa - \varepsilon, \kappa]\) with zero probability. This means that player 1 is better off by moving the mass of probability down from \( \kappa \) to \( \kappa - \varepsilon \).

Discontinuities in player 1’s mixed strategies at other points than zero are ruled out; an analogous argument excludes also discontinuities in player 2’s mixed strategies at other points than \( \alpha_1 \).

Notice that it cannot be the case that player 1 places an atom of probability in zero and that, at the same time, player 2’s distribution has an atom of probability in \( \alpha_1 \); if this happens indeed, each player can increase his probability to get the object by a finite amount, by bidding slightly more than \( 0 \) for player 1 and than \( \alpha_1 \) for player 2. There is then, a profitable deviation for both players and this cannot be in equilibrium.

Given that player 1 and player 2 never place at the same time an atom of probability respectively in zero and \( \alpha_1 \), and given that players randomize continuously on the sets of pure undominated strategies, the probability of ties is

\[\text{Notice that when player 2 bids } \frac{V_1}{V_2} + \alpha_1, \text{ player 1 is indifferent between submitting an identical bid or zero with probability one; I assume that the second choice is always preferred.} \]

\[\text{See also Baye, Kovenock and de Vries (1996), Bertoletti (2005), Che and Gale (1998) and Ellingsen (1991).} \]
null.

Since only equilibria in mixed strategies survive, it is useful then, to focus on the function denoted by \( F_i(x_i) \) (\( i = 1, 2 \)) that defines the probability that player \( i \) bids less than \( x_i \); in equilibrium both players must obtain from each pure strategy over which they randomize with positive probability, the same expected payoff.

Consider then, players’ expected payoffs starting from player 1:

\[
E[U_1(x_1, x_2)] = (V_1 - x_1) F_2(x_1 + \alpha_1) - x_1 [1 - F_2(x_1 + \alpha_1)] \quad (A1)
\]

Player 2’s payoff instead, is:

\[
E[U_2(x_2, x_1)] = (V_2 - x_2) F_1(x_2 - \alpha_1) - x_2 [1 - F_1(x_2 - \alpha_1)] \quad (A2)
\]

Since player 2 can always secure himself a positive payoff, must be \( E[U_2(x_2, x_1)] > 0 \); setting \( x_2 = \alpha_1 \) in A2 requires that \( F_2(0) > 0 \).

Player 1 instead can get at most zero; if indeed were \( E[U_1(x_1, x_2)] > 0 \), setting \( x_1 = 0 \) in A1, would require \( F_1(\alpha_1) > 0 \).

In equilibrium though, it cannot be the case that \( F_1(0) > 0 \) and \( F_2(\alpha_1) > 0 \) at the same time; thus it must be \( F_2(\alpha_1) = 0 \) and \( E[U_1(x_1, x_2)] = 0 \).

Since his equilibrium payoff is zero, player 1 never spends more than \( V_1 \); this requires \( F_1(V_1) = 1 \). Setting \( x_2 = V_1 + \alpha_1 \) in A2 gives:

\[
(V_2 - V_1 - \alpha_1) F_1(V_1) - (V_1 + \alpha_1) [1 - F_1(V_1)] = V_2 - V_1 - \alpha_1
\]

This leads to the following equilibrium condition:

**Player 1:**

\[
(V_1 - x_1) F_2(x_1 + \alpha_1) - x_1 [1 - F_2(x_1 + \alpha_1)] = 0 \quad (A3)
\]

or

\[
F_2(x_1 + \alpha_1) = \frac{x_1 + \alpha_1}{V_1} - \frac{\alpha_1}{V_1}
\]

Since every bid \( x_1 \) is equivalent to a bid of player 2 amounting to \( x_1 + \alpha_1 \), is also:

\[
F_2(x_2) = \frac{x_2 - \alpha_1}{V_1} \quad (A4)
\]

**Player 2** then, randomizes in equilibrium according to a uniform distribution over the interval \( [\alpha_1; V_1 + \alpha_1] \).

**Player 2**

\[
(V_2 - x_2) F_1(x_2 - \alpha_1) - x_2 [1 - F_1(x_2 - \alpha_1)] = V_2 - V_1 - \alpha_1 \quad (A5)
\]

or

\[
F_1(x_2 - \alpha_1) = 1 - \frac{V_1}{V_2} + \frac{x_2 - \alpha_1}{V_2}
\]
Since every bid $x_2$ is equivalent to a bid of player 1 amounting to $x_2 - \alpha_1$, must be:

$$F_1(x_1) = 1 - \frac{V_1}{V_2} + \frac{x_1}{V_2}$$ \hspace{1cm} (A6)

Player 1’s equilibrium distribution has an atom of probability amounting to 

$$(1 - \frac{V_1}{V_2})$$

that is placed in zero.

Player 1 thus, randomizes according to a uniform over the interval $[0, V_1]$ with probability $\frac{V_1}{V_2}$ and bids zero with probability $\left(1 - \frac{V_1}{V_2}\right)$.

Consider now the case where:

$$V_2 \leq V_1 + \alpha_1.$$ 

Proposition 7 If is:

$$V_2 \geq \alpha_1$$

and

$$V_2 \leq V_1 + \alpha_1$$

player 1 randomizes in equilibrium, over the support $[0, V_2 - \alpha_1]$ according to the following cumulative distribution function:

$$F_1(x_1) = \begin{cases} \frac{\alpha_1}{V_2} & \text{if } x_1 = 0 \\ \frac{x_1}{V_2} + \frac{\alpha_1}{V_2} & \text{if } x_1 > 0 \end{cases}$$

His equilibrium payoff amounts to:

$$V_1 + \alpha_1 - V_2 \geq 0.$$ 

Player 2 instead, randomizes over the support $[\alpha_1, V_2] \cup \{0\}$ according to the cumulative distribution function defined below:

$$F_2(x_2) = \begin{cases} 1 - \frac{V_2}{V_1} + \frac{x_2}{V_1} & \text{if } x_2 = 0 \\ 1 - \frac{V_2}{V_1} + \frac{\alpha_1}{V_1} & \text{if } x_1 \geq \alpha_1 \end{cases}$$

He gets in equilibrium, a null payoff.

Proof. Successive rounds of elimination of strictly dominated strategies, restrict the intervals where players randomizes with positive probability to $[0, V_2 - \alpha_1]$ for player 1 and to $[\alpha_1, V_2] \cup \{0\}$ for player 2.

Indeed player 2 never submits strictly positive bids smaller than $\alpha_1$, because this would give him a negative expected payoff; the same happens for bids exceeding $V_2$. These actions are strictly dominated by bidding zero.

Player 1 instead, bids $x_1 > V_2 - \alpha_1$ with probability zero since player 2’s maximum bid is $V_2$; any of such actions is strictly dominated by $x_1 = V_2 - \alpha_1$. In both cases indeed, player 1 wins the item with probability one but bidding $x_2 = V_1 + \alpha_1$ involves a smaller payment to the auctioneer and provides a higher payoff.

Player 1 then, can always secure himself a payoff of $V_1 + \alpha_1 - V_2 > 0$.

The same arguments used in the previous case rules out discontinuities, at other points than zero for player 1 and at other points than zero or $\alpha_1$ for player 2.

Consider now players’ expected payoffs; player 1 always gets $E[U_1(x_1, x_2)] > 0$; this requires setting $x_1 = 0$ in (2) that $F_2(\alpha_1) > 0$. 

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Player 2 cannot get in equilibrium, more than zero; indeed setting $E[U_2(x_2, x_1)] > 0$ requires $F_1(-\alpha_1) > 0$ when $x_2 = 0$, but this cannot be since no negative bids are allowed.

Setting instead $x_2 = \alpha_1$ in (2) requires $F_1(0) > 0$; this implies that is $F_1(0) > 0$ and $F_2(\alpha_1) > 0$ at the same time. Player 1 and player 2 then, could increase of a finite amount their probability of winning by bidding respectively slightly more than 0 and slightly more than $\alpha_1$; this though, is not possible in equilibrium. As a consequence must be $E[U_2(x_2, x_1)] = 0$.

Given that his equilibrium payoff is zero, player 2 never bids more than $V_2$ so that $F_2(V_2) = 1$; setting now $x_1 = V_2 - \alpha_1$ in A1 gives:

$$(V_1 - V_2 + \alpha_1) F_2(V_2) - (V_2 - \alpha_1) [1 - F_2(V_2)] = V_1 - V_2 + \alpha_1$$

The equilibrium conditions in this setting change as follows:

Player 2:

$$(V_2 - x_2) F_1(x_2 - \alpha_1) - x_2' F_1(x_2 - \alpha_1) = 0 \quad (A7)$$

or

$$F_1(x_2 - \alpha_1) = \frac{x_2 - \alpha_1}{V_2} + \frac{\alpha_1}{V_2}$$

since every bid $x_2$ is equivalent to a bid of player 1 amounting to $x_2 - \alpha_1$ is also:

$$F_1(x_1) = \frac{x_1 + \alpha_1}{V_2} \quad (A8)$$

No negative bids are allowed; therefore player 1’s equilibrium distribution has an atom in zero amounting to $\frac{\alpha_1 V_1}{V_2}$; for $x_1 \in (0, V_2 - \alpha_1]$ instead, the cumulative distribution function is $F_1(x_1) = \frac{\alpha_1 V_1 + x_1}{V_2}$.

Player 1:

$$(V_1 - x_1) F_2(x_1 + \alpha_1) - x_1 \left[1 - F_2(x_1 + \alpha_1)\right] = V_1 + \alpha_1 - V_2 \quad (A9)$$

or

$$F_2(x_1 + \alpha_1) = 1 - \frac{V_2}{V_1} + \frac{x_1 + \alpha_1}{V_1}$$

Since every bid $x_1$ is equivalent to a bid of player 2 amounting to $x_1 + \alpha_1$, holds finally:

$$F_2(x_2) = 1 - \frac{V_2}{V_1} + \frac{x_2}{V_1} \quad (A10)$$

Player 2’s equilibrium distribution has an atom of probability in zero amounting to $1 - \frac{V_2}{V_1} + \frac{\alpha_1}{V_1} \geq 0$; Player 2 moreover, randomizes according to the cumulative distribution function $\frac{x_2}{V_1}$ over the support $[\alpha_1, V_2]$. ■