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Covariance estimation via Fourier method in the presence of asynchronous trading and microstructure noise

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Abstract

We analyze the effects of market microstructure noise on the Fourier estimator of multivariate volatilities. We prove that the estimator is consistent in the case of asynchronous data and robust in the presence of microstructure noise. This result is obtained through an analytical computation of the bias and the mean squared error of the Fourier estimator and confirmed by Monte Carlo experiments.

JEL: C14, C32, G1

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1 Introduction

Computation of the covariance of financial asset returns plays a central role for many issues in finance. Recent papers have shown the potential of using high frequency data for the computation of covariances, see [Andersen and al., 2003, Bollerslev and Zhang, 2003, Fleming et al., 2003]. Nevertheless, when sampling high frequency returns, two difficulties arise. The first one refers to the so called Epps effect (see [Epps, 1979]): the non-synchronicity of the arrival times of trades across markets leads to a bias towards zero in correlations among stocks as the sampling frequency increases. The second one is the distortion from efficient prices due to the market microstructure contamination.

Motivated by the consequences of the effect of asynchronous trading, a number of alternative covariance estimators have been proposed in the literature; nevertheless most of them rely upon the *quadratic covariation* formula, a classical result essentially due to Wiener. Following the study in [Martens, 2004] the different approaches to estimate covariances can be split in two groups. The first group uses interpolation of data, in order to obtain synchronous returns which are necessary to construct the *realized covariance-quadratic variation* estimator; for instance [Scholes and Williams, 1977] modify the standard covariance estimator by adding the first lead and lag of the sample auto-covariance, [Dimson, 1979, Cohen et al., 1983] generalize this estimator to include k leads and lags;

[Zhang, 2006] provides an analytical study of the realized covariance estimator in a general framework which includes asynchronous trading. The second group utilizes all transaction data, [Harris et al., 1995, De Jong and Nijman, 1997, Brandt and Diebold, 2006]. Among the latter [De Jong and Nijman, 1997, Hayashi and Yoshida, 2005] propose an alternative to the realized covariance estimator which uses all data and does not rely on any synchronization methods. We refer to this estimator as the *All-overlapping (returns)* estimator, as suggested by [Corsi and Audrino, 2007]. The All-overlapping estimator is unbiased and consistent under the assumption that the observations are uncontaminated by noise. [Sheppard, 2005] introduces the concept of scrambling to describe the link between the price generating process and the sampling process.

The impact of microstructure noise has been studied extensively in the context of univariate volatility measurement, see [Aït-Sahalia and al., 2005a, Hansen and Lunde, 2006, Bandi and Russel, 2006, Barndorff-Nielsen and al., 2005, Zhang and al., 2005]. For the multivariate case [Bandi and Russel, 2005b] provide an analytical study of realized covariance in the presence of noise, but they do not address the non-synchronicity issue. [Voev and Lunde, 2007] study the properties of the All-overlapping estimator in the presence of noise. They prove that the realized covariation and the All-overlapping estimator are not biased by i.i.d noise. Nevertheless both realized covariation and the All-overlapping estimator are inconsistent under i.i.d noise, because the mean squared error (MSE) diverges as the number of observations increases; then they propose a bias correction for the All-overlapping estimator, which is still inconsistent, anyway. The authors compare through Monte Carlo simulation different realized covariance type estimators, specifically realized covariance, realized covariance plus one lead and lag, the estimator proposed by [Bandi and Russel, 2005b], the All-overlapping and the bias corrected All-overlapping estimator. The conclusion is that the last one is the winner in all the asynchronous trading scenarios contaminated by microstructure noise. Also [Griffin and Oomen, 2006] investigate the properties and the efficiency of three covariance estimators, namely realized covariance, realized covariance plus lead- and lag-adjustments, and All-overlapping covariance estimator, when the price observations are subject to non-synchronicity and contaminated by (i.i.d) microstructure noise and Poisson arrival times. They find that the ordering of covariance estimators in terms of efficiency depends crucially on the level of microstructure noise, in particular for high level of noise the All-overlapping estimator can be less efficient than the standard realized covariance estimator.

In this paper we consider the multivariate volatility estimation methodology proposed in [Malliavin and Mancino, 2002], which does not rely on any data synchronization methods but employs all data at disposal. We refer to this estimator as the *Fourier* estimator. The estimator's construction exploits a general identity, obtained in [Malliavin and Mancino, 2005], which relates the Fourier transform of the co-volatility function with the Fourier transform of the log-returns. The peculiarity of Fourier estimator is that it uses all the available observations and avoids any synchronization of the original data, because it is based on the integration of the time series of returns rather than on its differentiation. Therefore from the practitioner's point of view it is easy to implement as it does not rely on any choice of synchronization methods or sampling schemes.

We proceed in two directions: we first analyze the efficiency of Fourier estimator in comparison with realized covariance, realized covariance plus lead- and lag-adjustments, and All-overlapping covariance estimator, when the price observations are uniquely subject to non-synchronicity. This emphasizes the impact of the non-synchronicity. Secondly we compare them in the presence of i.i.d. microstructure noise. In both cases, we derive analytical expressions for the bias and the MSE of the Fourier estimator, which can be applied to real data. We obtain that the Fourier estimator is asymptotically unbiased and consistent under asynchronous observations, in the absence of microstructure noise, under the condition that $\rho(n)N \to 0$, where N is the number of frequencies to be included in the estimator and $\rho(n)$ is the mesh of the data partition. In the presence of i.i.d. microstructure noise the computation of the bias shows that the Fourier covariance estimator is unaffected by i.i.d. noise in terms of bias, as it happens for the realized covariance estimator. Therefore even in the presence of i.i.d. noise contamination the Fourier estimator is asymptotically unbiased under the same condition $\rho(n)N \to 0$. More interestingly and in contrast with the behavior of the realized covariance and the All-overlapping estimator, the MSE of Fourier estimator does not diverge as the number of observations increases, under the same condition $\rho(n)N \to 0$. This result is due to the following property of the Fourier estimator: the high-frequency noise is ignored by the Fourier estimator by cutting the highest frequencies. Finally we consider a dependent noise setting. The computation of the bias of the Fourier estimator shows that even in this case the Fourier estimator is asymptotically unbiased. In summary our analysis shows that Fourier estimator of covariance is robust to all the different scenarios considered without requiring any ad hoc adjustment.

Our theoretical results are confirmed by a simulation study. By reproducing the regular non-synchronous trading scenario of [Voev and Lunde, 2007], we evaluate the impact of different noise and sampling specifications in order to validate our theoretical analysis. We show that the analytical expressions for the MSE of the Fourier estimator provided in the paper can be effectively used to build optimal MSE-based estimators under regular non-synchronous trading. Secondly, under more general trading scenarios, the performance of the Fourier estimator of the integrated covariance is compared with the behavior of the realized covariance, the realized covariance plus leads and lags and the All-overlapping estimator.

The paper is organized as follows. In Section 2 we resume the multivariate Fourier estimation methodology developed in [Malliavin and Mancino, 2005]. In Section 3 we prove the consistency result for the Fourier estimator under general asynchronous observations. Section 4 analyzes the bias of the Fourier estimator under asynchronous observations and microstructure noise. In Section 5 we analytically compute the MSE for the Fourier covariance estimator in the presence of i.i.d. microstructure noise and asynchronous observations. The analysis is extended to dependent noise specification in Section 6. In Section 7, we test our theoretical findings by means of Monte Carlo simulations. The technical proofs are contained in the Appendix.

2 Fourier estimation method

A Fourier analysis based method to estimate multivariate volatility has been proposed in [Malliavin and Mancino, 2002] to overcome the difficulties arising by applying the quadratic covariation theorem to the true returns data, due to the non-synchronicity of observed prices for different assets. In fact the quadratic covariation formula is not well suited to provide a good estimate of cross-volatilities, because it requires synchronous observations, while in reality they are not available.

The non-synchronicity trading problem has been studied for quite a long time in empirical finance, e.g. [Lo and MacKinlay, 1990]. The bias (Epps effect) caused by non-synchronicity and random sampling for the cross-correlations estimation has been recently highlighted in [Hayashi and Yoshida, 2005, Zhang, 2006]. The Fourier methodology is immune from these difficulties due to its own definition since, being based on the integration of "all" data, it does not need any adjustment to fit to asynchronous observations.

We briefly recall the methodology. Assume that $p(t) = (p^1(t), \ldots, p^n(t))$ are Brownian semi-martingales satisfying Itô stochastic differential equations

$$dp^j(t) = \sum_{i=1}^{d} \sigma_i^j(t)\ dW^i + b^j(t)\ dt, \qquad j = 1, \ldots, n, \tag{1}$$

where $W = (W^1, \ldots, W^d)$ are independent Brownian motions, and σ_*^* and b^* are adapted random processes satisfying hypothesis (**H**):

$$(\mathbf{H}) \qquad E[\int_0^{2\pi} (b^i(t))^2 dt] < \infty, \quad E[\int_0^{2\pi} (\sigma_i^j(t))^4 dt] < \infty \quad i = 1, \ldots, d,\ j = 1, \ldots, n.$$

¿From the representation (1) define the *volatility matrix*, which in our hypothesis *depends upon time*:

$$\Sigma^{jk}(t) = \sum_{i=1}^{d} \sigma_i^j(t)\sigma_i^k(t). \tag{2}$$

The Fourier method reconstructs $\Sigma^{*,*}(t)$ on a fixed time window (which can be always reduced to $[0, 2\pi]$ by a change of the origin and rescaling) using the Fourier transform of $dp^*(t)$. The main result in [Malliavin and Mancino, 2005] relates the Fourier transform of $\Sigma^{*,*}(t)$ to the Fourier transforms of the log-returns $dp^*(t)$. More precisely the procedure is the following: compute the Fourier transform of dp^j for $j = 1, \ldots, n$, defined for any integer k by

$$\mathcal{F}(dp^j)(k) := \frac{1}{2\pi} \int_{]0,2\pi[} e^{-\mathrm{i}kt} dp^j(t) \tag{3}$$

and consider the Fourier transform of the cross-volatility function defined for any integer k by

$$\mathcal{F}(\Sigma^{ij})(k) := \frac{1}{2\pi} \int_0^{2\pi} e^{-\mathrm{i}kt}\ \Sigma^{ij}(t) dt.$$

Then (see [Malliavin and Mancino, 2005]) the following convergence in probability holds for any integer k

$$\mathcal{F}(\Sigma^{ij})(k) = \lim_{N\to\infty} \frac{2\pi}{2N+1} \sum_{s=-N}^{N} \mathcal{F}(dp^i)(s)\mathcal{F}(dp^j)(k-s).$$

As a particular case, by choosing $k = 0$, we can compute the integrated covariance given the log-returns of stocks. More precisely it results:

Theorem 2.1 *Under the hypothesis* (**H**) *the following convergence in probability holds for any* $i, j = 1, 2$

$$\int_{]0,2\pi[} \Sigma^{ij}(t)\ dt = \lim_{N\to\infty} \frac{(2\pi)^2}{2N+1} \sum_{s=-N}^{N} \mathcal{F}(dp^i)(s)\mathcal{F}(dp^j)(-s). \tag{4}$$

3 Consistency of Fourier covariance estimator under asynchronous observations

For notational simplicity we consider two assets whose log-prices (p^1, p^2) satisfy the semimartingale model (1). We are interested in estimating the integrated covariance defined by:

$$\int_0^{2\pi} \Sigma^{12}(t)dt = \int_0^{2\pi} \sum_{i=1}^{2} \sigma_i^1(t)\sigma_i^2(t)dt.$$

In this section we analyze a setting, where we account for asynchronous observations, but we do not include microstructure effects, which will be discussed in the following sections.

First we recall the definition of some covariance estimators proposed in the recent literature. Suppose that the processes are observed at a discrete unevenly spaced grid $\{0 \leq t_1^l \leq t_2^l \leq \cdots \leq t_{n_l}^l \leq 2\pi\}$ for any $l = 1, 2$. The following estimators are based on the choice of a synchronization procedure, such as interpolation or imputation, which yields the observations times $\{0 \leq \tau_1 \leq \tau_2 \leq \cdots \leq \tau_n \leq 2\pi\}$ for both assets.

The quadratic covariation-realized covariance estimator is defined by

$$RC_{1,2} := \sum_{i=1}^{n-1} \delta_i(p^1)\delta_i(p^2),$$

where $\delta_i(p^*) = p^*(\tau_{i+1}) - p^*(\tau_i)$. The realized covariance estimator is not consistent under asynchronous trading. The bias due to the synchronization procedure is analyzed in [Hayashi and Yoshida, 2005, Zhang, 2006].

The following modifications of realized covariance have been proposed: the realized covariance plus leads and lags estimator

$$RCLL_{1,2} := \sum_i \sum_{h=-l}^{L} \delta_{i+h}(p^1)\delta_i(p^2), \tag{5}$$

and the realized covariance kernels estimator

$$RCLLW_{1,2} := \sum_i \sum_{h=-l}^{L} w(h)\delta_{i+h}(p^1)\delta_i(p^2), \tag{6}$$

where $w(h)$ is a kernel, see [Griffin and Oomen, 2006].

The estimators (5) and (6) have good properties under microstructure noise contaminations of the prices, but they are still not consistent for asynchronous observations. This is due to the fact that all the realized covariance type estimators need a data synchronization procedure.

In view of this intrinsic limitation of the realized covariance type estimators [Malliavin and Mancino, proposed an alternative estimation method based on Fourier analysis, which relies on the formula (4). Denote by $\{t_i^1, i=1,\ldots,n_1\}$ and $\{t_j^2, j=1,\ldots,n_2\}$ the trading times for the asset 1 and 2 respectively. For simplicity suppose that both assets trade at $t_1^1 = t_1^2 = 0$ and $t_{n_1}^1 = t_{n_2}^2 = 2\pi$. Let $\rho(n_1) := \max_{1\le i\le n_1-1} |t_i^1 - t_{i+1}^1|$ and $\rho(n_2) := \max_{1\le j\le n_2-1} |t_j^2 - t_{j+1}^2|$ and $\rho(n) := \rho(n_1) \vee \rho(n_2)$. We will denote $I_i^1 = [t_i^1, t_{i+1}^1[$ and $J_j^2 = [t_j^2, t_{j+1}^2[$. The Fourier estimator of the integrated covariance $\int_0^{2\pi} \Sigma^{12}(t)dt$ is

$$\Sigma^{12}_{N,n_1,n_2} := \frac{(2\pi)^2}{2N+1} \sum_{|s|\le N} \mathcal{F}(dp^1_{n_1})(s)\mathcal{F}(dp^2_{n_2})(-s), \tag{7}$$

where

$$\mathcal{F}(dp^1_{n_1})(s) := \frac{1}{2\pi}\sum_{i=1}^{n_1-1} \exp(-\mathrm{i}st_i^1)\delta_{I_i^1}(p^1), \quad \mathcal{F}(dp^2_{n_2})(s) := \frac{1}{2\pi}\sum_{j=1}^{n_2-1} \exp(-\mathrm{i}st_j^2)\delta_{J_j^2}(p^2)$$

and $\delta_{I_i^1}(p^1) := p^1(t_{i+1}^1) - p^1(t_i^1)$, $\delta_{I_j^2}(p^2) := p^2(t_{j+1}^2) - p^2(t_j^2)$. The Fourier covariance estimator is consistent under asynchronous observations as it does not require any synchronization procedure. This result is stated by the following

Theorem 3.1 *Let Σ^{12}_{N,n_1,n_2} be defined in (7). If $\rho(n)N \to 0$, the following convergence in probability holds:*

$$\lim_{n_1,n_2,N\to\infty} \Sigma^{12}_{N,n_1,n_2} = \int_0^{2\pi} \Sigma^{12}(t)dt. \tag{8}$$

The proof is a particular case of Theorem 3.4 in [Malliavin and Mancino, 2005].

Now we compute the Fourier estimator bias for a fixed number of data observations and for a fixed number of Fourier coefficients N.

Proposition 3.2 *The bias of the Fourier estimator is given by*

$$E[\Sigma^{12}_{N,n_1,n_2} - \int_0^{2\pi} \Sigma^{12}(t)dt] = \sum_{i=1}^{n_1-1}\sum_{j=1}^{n_2-1} \left(D_N(t_i^1 - t_j^2) - 1\right) E[\int_{I_i^1\cap J_j^2} \Sigma^{12}(t)dt], \tag{9}$$

where $D_N(t)$ denotes the rescaled Dirichlet kernel defined by

$$D_N(t) := \frac{1}{2N+1}\sum_{|s|\le N} e^{\mathrm{i}st} = \frac{1}{2N+1}\frac{\sin[(N+\frac{1}{2})t]}{\sin\frac{t}{2}}. \tag{10}$$

Therefore the Fourier estimator is asymptotically unbiased under the condition $\rho(n)N \to 0$ as $n, N \to \infty$.

Recently [Hayashi and Yoshida, 2005] have faced the non-synchronicity problem, proposing an estimator which is consistent under asynchronous observations of the prices. The Hayashi-Yoshida estimator is

$$AO^{12}_{n_1,n_2} := \sum_{i,j} \delta_{I^1_i}(p^1)\delta_{J^2_j}(p^2) I_{(I^1_i \cap J^2_j \neq \emptyset)}, \tag{11}$$

where $I_{(P)} = 1$ if proposition P is true and $I_{(P)} = 0$ if proposition P is false. We will refer to estimator (11) as the All-overlapping (AO) estimator. The AO estimator is unbiased in the absence of noise. In [Griffin and Oomen, 2006, Voev and Lunde, 2007]) the AO-estimator is found out to be not efficient in the presence of microstructure noise.

We remark that the Fourier estimator can be written as

$$\Sigma^{12}_{N,n_1,n_2} = \sum_{i,j} D_N(t^1_i - t^2_j)\delta_{I^1_i}(p^1)\delta_{J^2_j}(p^2) I_{(I^1_i \cap J^2_j \neq \emptyset)} \tag{12}$$

$$+ \sum_{i,j} D_N(t^1_i - t^2_j)\delta_{I^1_i}(p^1)\delta_{J^2_j}(p^2) I_{(I^1_i \cap J^2_j = \emptyset)}.$$

The first addend in (12) generalizes the All-overlapping estimator, the second one provides a lead-lag adjustment which takes into account cross dependence of non-overlapping returns. A peculiarity of the Fourier estimator is the weight D_N, the Dirichlet kernel which depends on the number of frequencies N, besides on the delay between two trading. Note that the Fourier estimator is not of the realized kernels type (6) (see also [Mancino and Sanfelici, 2008] in an univariate context for a discussion on this issue).

4 Bias of Fourier covariance estimator under microstructure noise

In this section we analyze the behavior of the Fourier estimator of the integrated covariance under asynchronous observations and microstructure noise.

Consider the following model for the observed log-returns

$$\widetilde{p}^i(t) := p^i(t) + \eta^i(t) \quad \text{for } i = 1, 2, \tag{13}$$

where

$$dp^i(t) = \sum_{k=1}^{2} \sigma^i_k(t) dW^k(t) \tag{14}$$

and hypothesis **(H)** holds. Moreover the following assumptions hold:

(M1). $p := (p^1, p^2)$ and $\eta := (\eta^1, \eta^2)$ are independent processes, moreover $\eta(t)$ and $\eta(s)$ are independent for $s \neq t$ and $E[\eta(t)] = 0$ for any t.

(M2). $E[\eta^i(t)\eta^j(t)] = \omega_{ij} < \infty$ for any t, $i, j = 1, 2$.

We fix the following notation

$$\delta_{I_i^1}(\tilde{p}^1) := \tilde{p}^1(t_{i+1}^1) - \tilde{p}^1(t_i^1) \quad \delta_{J_j^2}(\tilde{p}^2) := \tilde{p}^2(t_{j+1}^2) - \tilde{p}^2(t_j^2), \tag{15}$$

$$\varepsilon_{I_i^1}^1 := \eta^1(t_{i+1}^1) - \eta^1(t_i^1) \quad \varepsilon_{J_j^2}^2 := \eta^2(t_{j+1}^2) - \eta^2(t_j^2). \tag{16}$$

¿From (7) the Fourier estimator of the covariance between asset 1 and 2 is given by

$$\widehat{\Sigma}_{N,n_1,n_2}^{12} = \frac{1}{2N+1} \sum_{|s| \leq N} \sum_{i,j} e^{\mathrm{i}st_i^1} \delta_{I_i^1}(\tilde{p}^1) e^{-\mathrm{i}st_j^2} \delta_{J_j^2}(\tilde{p}^2). \tag{17}$$

Observe that (17) is equal to

$$\sum_{ij} D_N(t_i^1 - t_j^2) \left(\delta_{I_i^1}(p^1)\delta_{J_j^2}(p^2) + \varepsilon_{I_i^1}^1 \varepsilon_{J_j^2}^2 + \delta_{I_i^1}(p^1)\varepsilon_{J_j^2}^2 + \varepsilon_{I_i^1}^1 \delta_{J_j^2}(p^2) \right).$$

In the sequel we consider the case of *regular asynchronous trading* analyzed in [Voev and Lunde, 2007]. The asset 1 trades at regular points: $\Pi^1 = \{t_i^1 : i = 1, \ldots, n_1$ and $t_{i+1}^1 - t_i^1 = \frac{2\pi}{n_1 - 1}\}$. Also asset 2 trades at regular points: $\Pi^2 = \{t_j^2 : j = 1, \ldots, n_2$ and $t_{j+1}^2 - t_j^2 = \frac{4\pi}{n_1 - 1}\}$, where $n_2 = n_1/2$, but no trade of asset 1 occurs at the same time of a trade of asset 2. Specifically, the link between the trading times of the two assets is the following: $t_j^2 = t_{2(j-1)+1}^1 + \frac{\pi}{n_1 - 1}$ for $j = 1, \ldots, n_2$. Moreover, suppose $t_1^1 = 0$ and $t_{n_1}^1 = 2\pi$. For simplicity denote $n := n_1$ and assume n is even. We will denote by $\widehat{\Sigma}_{N,n}^{12}$ the Fourier estimator in this setting.

The following theorem provides the bias of the Fourier covariance estimator under i.i.d. microstructure noise, neglecting minor end-effects.

Theorem 4.1 *Under the asynchronous trading model above specified and if the i.i.d microstructure noise satisfies* **(M1)-(M2)**, *the bias of the Fourier covariance estimator is*

$$E[\widehat{\Sigma}_{N,n}^{12} - \int_0^{2\pi} \Sigma^{12}(t)dt] = \sum_{j=1}^{\frac{n}{2}-1} \sum_{i=2(j-1)+1}^{2(j-1)+3} (D_N(t_i^1 - t_j^2) - 1) E[\int_{t_i^1}^{t_{i+1}^1} \Sigma^{12}(t)dt]. \tag{18}$$

Therefore the Fourier covariance estimator is asymptotically unbiased in the presence of i.i.d. microstructure noise, under the condition $\rho(n)N \to 0$ as $n, N \to \infty$.

A comparison between (9) and (18) shows that when the noise satisfies the i.i.d assumption the bias of the Fourier estimator is not affected by the presence of microstructure noise. Therefore the Fourier estimator remains asymptotically unbiased in the presence of i.i.d microstructure. Note that the univariate Fourier estimator has the same property under microstructure noise, see [Mancino and Sanfelici, 2008]. On the other hand, we can observe that for the realized covariation and realized covariation with leads and lags the situation is different from the corresponding univariate estimator: in fact in the univariate case the i.i.d noise renders the realized volatility biased, while the realized covariation is not biased by i.i.d noise under synchronous trading. The AO estimator is not biased by i.i.d noise, nevertheless both realized covariation and the AO estimator are inconsistent under i.i.d noise, because the MSE diverges as the number of observations increases (as it is proved in [Voev and Lunde, 2007]). In view of these considerations our next step will be the computation of the Fourier estimator's MSE in the presence of i.i.d. microstructure noise.

5 MSE of Fourier estimator under microstructure noise and asynchronous trading

The first result of this section contains the computation of the Fourier estimator's MSE in the asynchronous setting considered in Theorem 4.1, without including the microstructure component.

Proposition 5.1 *Under the specified asynchronous trading setting, it holds:*

$$E[(\sum_{i,j} D_N(t_i^1 - t_j^2)\delta_{I_i^1}(p^1)\delta_{J_j^2}(p^2) - \int_0^{2\pi} \Sigma^{12}(t)dt)^2] \leq \frac{1}{4}\rho(n)^2(N+1)^2 E[(\int_0^{2\pi} \Sigma^{12}(t)dt)^2]$$

$$+4\sqrt{2\pi}\rho(n)^{\frac{1}{2}} E[\int_0^{2\pi} (\Sigma^{11}(t))^2 dt]^{\frac{1}{2}} E[\int_0^{2\pi} (\Sigma^{22}(t))^2 dt]^{\frac{1}{2}}$$

$$+2\rho(n)^{\frac{1}{2}} E[\int_0^{2\pi} (\Sigma^{11}(t))^2 dt]^{\frac{1}{2}} E[\int_0^{2\pi} \Sigma^{22}(t)dt] + \sqrt{2\pi}\rho(n)^{\frac{1}{2}} E[\int_0^{2\pi} (\Sigma^{12}(t))^2 dt]$$

$$+\frac{16\sqrt{2\pi}}{\sqrt{2N+1}} E[\int_0^{2\pi} (\Sigma^{11}(t))^2 dt]^{\frac{1}{2}} E[\int_0^{2\pi} (\Sigma^{22}(t))^2 dt]^{\frac{1}{2}}.$$

Under the condition $\rho(n)N \to 0$, the r.h.s of the inequality obtained in Proposition 5.1 converges to 0 as $n, N \to \infty$, thus confirming the result in Theorem 3.1.

Theorem 5.2 *Under the above specified asynchronous trading setting and noise satisfying assumptions* (**M1**) − (**M2**)*, if* $\rho(n)N \to 0$ *as* $n, N \to \infty$ *then it holds:*

$$E[(\widehat{\Sigma}^{12}_{N,n} - \int_0^{2\pi} \Sigma^{12}(t)dt)^2] = o(1) + 2\omega_{22} \sum_{i=1}^{n-1} D_N^2(t_i^1 - t^2_{\frac{n}{2}-1}) E[\int_{t_i^1}^{t_{i+1}^1} \Sigma^{11}(t)dt] \tag{19}$$

$$+2\omega_{11} \sum_{j=1}^{\frac{n}{2}-1} D_N^2(t^1_{n-1} - t_j^2) E[\int_{t_j^2}^{t^2_{j+1}} \Sigma^{22}(t)dt] + 4\,\omega_{22}\omega_{11}\, D_N^2(t^1_{n-1} - t^2_{\frac{n}{2}-1}),$$

where $o(1)$ *is a term which goes to zero in probability.*

Remark 5.3 *The* $o(1)$ *term in (19) has been computed in Proposition 5.1. The other terms arise from the corrections due to microstructure effects.*

The previous results allow a comparison between the behavior of the AO estimator and the Fourier estimator in the presence of microstructure noise and asynchronous observations. First consider the MSE of the AO estimator without microstructure terms (MSE_{AO}) and in the presence of noise effects (MSE_{AOm}):

$$MSE_{AO} = o(1),$$

$$MSE_{AOm} = o(1) +$$

$$+2\omega_{11} \sum_{j=1}^{\frac{n}{2}-1} E[\int_{t_j^2}^{t^2_{j+1}} \Sigma^{22}(t)dt] + 2\omega_{22} \sum_{i=1}^{n-1} E[\int_{t_i^1}^{t^1_{i+1}} \Sigma^{11}(t)dt] + 2(n-1)\omega_{11}\omega_{22}.$$

MSE_{AO} converges to zero because the estimator is consistent. As for MSE_{AOm}, it increases with n, i.e. the number of the most frequently traded asset, and therefore it diverges for very high frequency observations, due to the term $2(n-1)\omega_{11}\omega_{22}$. Note that the other two terms in the MSE_{AOm} are constant for increasing n, because I_i^1 and J_j^2 are partitions of $[0, 2\pi]$. This result is found in [Voev and Lunde, 2007].

Secondly consider the MSE of the Fourier estimator without microstructure terms (MSE_F) and in the presence of noise effects (MSE_{Fm}):

$$MSE_F = o(1),$$

$$MSE_{Fm} = o(1) + 2\omega_{11} \sum_{j=1}^{\frac{n}{2}-1} D_N^2(t^1_{n-1} - t_j^2) E[\int_{t_j^2}^{t^2_{j+1}} \Sigma^{22}(t)dt]$$

$$+2\omega_{22} \sum_{i=1}^{n-1} D_N^2(t_i^1 - t^2_{\frac{n}{2}-1}) E[\int_{t_i^1}^{t^1_{i+1}} \Sigma^{11}(t)dt] + 4\omega_{11}\omega_{22} D_N^2(t^1_{n-1} - t^2_{\frac{n}{2}-1}).$$

MSE_F converges to zero as the estimator is consistent, see Proposition 5.1. Consider now the MSE of the Fourier estimator in the presence of microstructure noise. The constant terms have similar behavior to the corresponding ones in MSE_{AOm}, but the term $4\omega_{22}\omega_{11}D_N^2(t^1_{n-1} - t^2_{\frac{n}{2}-1})$ converges to the constant $4\omega_{22}\omega_{11}$ as n, N increase at the proper rate $\rho(n)N \to 0$. In summary the Fourier estimator of multivariate volatility is consistent under asynchronous observations and it is robust in the presence of i.i.d. microstructure noise, i.e. the MSE does not diverge at the highest frequencies.

6 Robustness of Fourier covariance estimator under dependent microstructure noise

In this section we suppose that the microstructure noise is correlated with the price process and there is also a temporal dependence in the noise components. Precisely we follow [Bandi and Russel, 2005b, Voev and Lunde, 2007] by considering a noise specification which allows dependence for a limited amount of time. We investigate here the behavior of the Fourier estimator in this respect.

The general noise specification satisfies the following assumption:

(MD1). $\eta^i(t+s)$ and $p^j(t)$ are independent if $s > \theta_0$ for some finite $\theta_0 \geq 0$ and for any $t > 0$ and $i, j = 1, 2$;

(MD2). $E[\eta^i(t)\eta^j(t+s)] = 0$ if $s > \theta_0$ for some finite $\theta_0 \geq 0$ and for any $t > 0$ and $i, j = 1, 2$.

Consider the regular asynchronous trading setting introduced in section 4. Under assumptions **(MD1)-(MD2)** it is possible to choose a positive integer b such that for any t^2_j it holds:

$$|t^2_j - t^1_{2(j-1)-b+1}| > \theta_0 \quad \text{and } |t^2_{j+1} - t^1_{2(j-1)+4+b}| > \theta_0.$$

The following result holds.

Theorem 6.1 *Under the asynchronous trading model specified in section 4 and the dependent microstructure noise satisfying* **(MD1)-(MD2)***, the Fourier covariance estimator is asymptotically unbiased, that is*

$$\lim_{n,N\to\infty} E[\widehat{\Sigma}^{12}_{N,n} - \int_0^{2\pi} \Sigma^{12}(t)dt] = 0 \tag{20}$$

under the condition $N\rho(n) \to 0$ as $n, N \to \infty$.

More precisely the proof of the theorem shows that under the specified dependent noise

$$E[\widehat{\Sigma}^{12}_{N,n} - \int_0^{2\pi} \Sigma^{12}(t)dt] = \sum_{j=1}^{\frac{n}{2}-1} \sum_{i=2(j-1)+1}^{2(j-1)+3} E[\int_{t_i^1}^{t_{i+1}^1} \Sigma^{12}(t)dt](D_N(t_i^1 - t_j^2) - 1) + N\rho(n)C,$$

where the constant C depends on the noise variance ω_{ii}, on $E[\int_0^{2\pi} \Sigma^{ii}(t)dt]$ for $i = 1, 2$ and on the integer b which is a measure of the dependence.

Concerning the MSE of the Fourier estimator in this dependent noise setting, the simulation results in the next section indicate that the behavior in the presence of dependent microstructure noise is very close to that in the presence of i.i.d. noise obtained in Theorem 5.2. In summary we conclude that the Fourier estimator is robust even to the presence of dependent microstructure noise.

7 Monte Carlo simulations

The aim of this section is twofold: by reproducing the regular non-synchronous trading scenario of [Voev and Lunde, 2007], we evaluate the impact of different noise and sampling specifications in order to validate our theoretical analysis. Secondly, under more general trading scenarios, the performance of the Fourier estimator of the integrated covariance is compared to the behavior of the realized covariance $RC_{1,2}$, the realized covariance plus leads and lags $RCLL_{1,2}$ and the All-overlapping estimator $AO_{1,2}$.

We simulate discrete data from the continuous time bivariate GARCH model [Hoshikawa and al., 20

$$\begin{bmatrix} dp^1(t) \\ dp^2(t) \end{bmatrix} = \begin{bmatrix} \beta_1 \sigma_1^2(t) \\ \beta_2 \sigma_4^2(t) \end{bmatrix} dt + \begin{bmatrix} \sigma_1(t) & \sigma_2(t) \\ \sigma_3(t) & \sigma_4(t) \end{bmatrix} \begin{bmatrix} dW_5(t) \\ dW_6(t) \end{bmatrix}$$

$$d\sigma_i^2(t) = (\omega_i - \theta_i \sigma_i^2(t))dt + \alpha_i \sigma_i^2(t) dW_i(t), \quad i = 1, \ldots, 4,$$

where $\{W_i(t)\}_{i=1}^6$ are independent Wiener processes. Moreover, we assume that the logarithmic noises $\eta^1(t), \eta^2(t)$ are i.i.d. Gaussian, possibly contemporaneously correlated and independent from p. We also consider the case of dependent noise, assuming for simplicity $\eta_i^j = \alpha(p^j(t_i^j) - p^j(t_{i-1}^j)) + \bar{\eta}_i^j$, for $j = 1, 2$ and $\bar{\eta}_i^j$ i.i.d. Gaussian. [Voev and Lunde, 2007] define the noise variance to be 90% of the total variance for 1 second returns, which is in fact quite moderate. For instance, [Aït-Sahalia and al., 2005a] report a study of 274 NYSE stocks in which the noise is twelve times this amount. Therefore, in our simulations we consider both the case of 90% noise variance and ten times such an amount, which we call *increased noise term*. Finally, in order to compare our results with the ones by [Griffin and Oomen, 2006], we further increase the level of noise in such a way that the quantity they define as *noise ratio* is around 7. As already found by [Griffin and Oomen, 2006], at this level, the All-overlapping estimator can be even less efficient than the standard realized covariance estimator while the Fourier estimator always provides a valid alternative. When the noise correlation matrix is not diagonal, the correlation is set to 0.5. ¿From the simulated data, integrated covariance estimates can be compared to the value of the true variance quantities.

We generate (through simple Euler Monte Carlo discretization) high frequency evenly sampled true and observed returns by simulating second-by-second return and variance paths over a daily trading period of $h = 6$ hours, for a total of 21600 observation per day. Then we sample the observations according to different trading scenarios: *regular synchronous trading* with duration $\rho_1 = \rho(n_1)$ between trades for the first asset and $\rho_2 = 2\rho_1$ for the second, i.e. the second asset trades each second time the first asset trades; *regular non-synchronous trading* with duration ρ_1 between trades for the first asset and $\rho_2 = 2\rho_1$ for the second and displacement $\delta \cdot \rho_1$ between the two, i.e. the second asset starts trading $\delta \cdot \rho_1$ seconds later; *Poisson trading* with durations between observations drawn from an exponential distribution with means λ_1 and λ_2 for the two assets respectively. The other parameters of the model are: $\alpha_1 = 0.1$ $\alpha_2 = 0.1$, $\alpha_3 = 0.2$, $\alpha_4 = 0.2$, $\beta_1 = 0.02$, $\beta_2 = 0.01$, $\omega_1 = 0.1$, $\omega_2 = 0.1$, $\omega_3 = 0.2$, $\omega_4 = 0.2$, $\theta_1 = 0.1$, $\theta_2 = 0.1$, $\theta_3 = 0.1$, $\theta_4 = 0.1$, $\alpha = 0.1$. The simulations are run for 500 daily replications, using the computer language Matlab.

In implementing the Fourier estimator $\hat{\Sigma}^{12}_{N,n_1,n_2}$, the smallest wavelength that can be evaluated in order to avoid aliasing effects is twice the smallest distance between two consecutive prices, which under uniform sampling yields $N \leq \min((n_1 - 1)/2, (n_2 - 1)/2)$ (*Nyquist frequency*). Nevertheless, as pointed out in the univariate case by [Mancino and Sanfelici, 2008] and confirmed by our theoretical study in the present paper, smaller values of N may provide better variance/covariance measures.

Fig. 1 shows the effect of the truncation of the Fourier expansion in terms of the MSE and bias of the Fourier estimator, by choosing different values of the cutting frequency N_{cut} in three different trading scenarios. The MSE and bias curves are plotted versus the sampling interval of the first asset ρ_1 and correspond to different choices of the cutting frequency: $N_{cut} = 720, 360, 180, 90$. In order to separate the Epps effect from other microstructure effects, we split non-synchronicity from microstructure noise. The plots at the top refer to the regular non-synchronous trading setting with displacement $\delta = 2/3$ and no noise. Notice that, as ρ_1 increases from 2 seconds to 2 minutes, the level of non-synchronicity increases as well since it is proportional to ρ_1. It is evident that, for any fixed ρ_1, the cutting procedure can reduce both the MSE and the bias for the estimated covariance, thus contrasting the Epps effect according to Proposition 5.1. More precisely, the MSE is generally decreased when the number of the Fourier coefficient is reduced, except for the highest sampling frequencies where the MSE is first reduced by choosing $N_{cut} = 720$ and $N_{cut} = 360$ and then increased if the number of Fourier coefficients is too small with respect to the number of observations. The corresponding bias is reduced in absolute value for any sampling frequencies as N_{cut} is reduced. In the case that N_{cut} was set equal to the Nyquist frequency, we would observe an explosion of the MSE for high sampling frequencies, while the bias would be constant and negative over the different ρ_1 values.

The plots in the middle refer to the case of synchronous trading ($\delta = 0$) with $\rho_2 = 2\rho_1$ and uncorrelated noise. The reduction of N_{cut} has large benefit on the bias, while the effect on the MSE has the same characteristics as before, which suggests for each value of n_1 the existence of an optimal value for N_{cut} minimizing the MSE. Such a value increases with n_1, i.e. at the highest frequencies, and must be such that $N/n_1 \to 0$ according to our theoretical analysis. Now, let us consider the combination of non-synchronicity with noise effects. The plots at the bottom refer to regular non-synchronous trading with uncorrelated i.i.d. noise. The reduction of N_{cut} can reduce the negative bias of the Fourier estimator. More precisely, the choice of a suitable N_{cut} in the range $[360, 720]$ should yield a strong reduction of the MSE at the highest frequencies and provide an almost unbiased covariance estimate.

Fig. 2 shows the MSE and the bias of $\hat{\Sigma}^{12}_{N,n_1,n_2}$ as a function of the number of the Fourier coefficients included in the expansion, in the regular non-synchronous trading setting with uncorrelated i.i.d. noise (Panels A and B), in the Poisson trading setting with contemporaneously correlated i.i.d. noise (Panels C and D) and in the same setting with an increased noise term (Panels E and F). The MSE curves are convex and their minima are attained at suitable cutting frequencies N_{cut} which can be used to build optimal MSE-based covariance estimates. In particular, it is evident from Figures 1 and 2

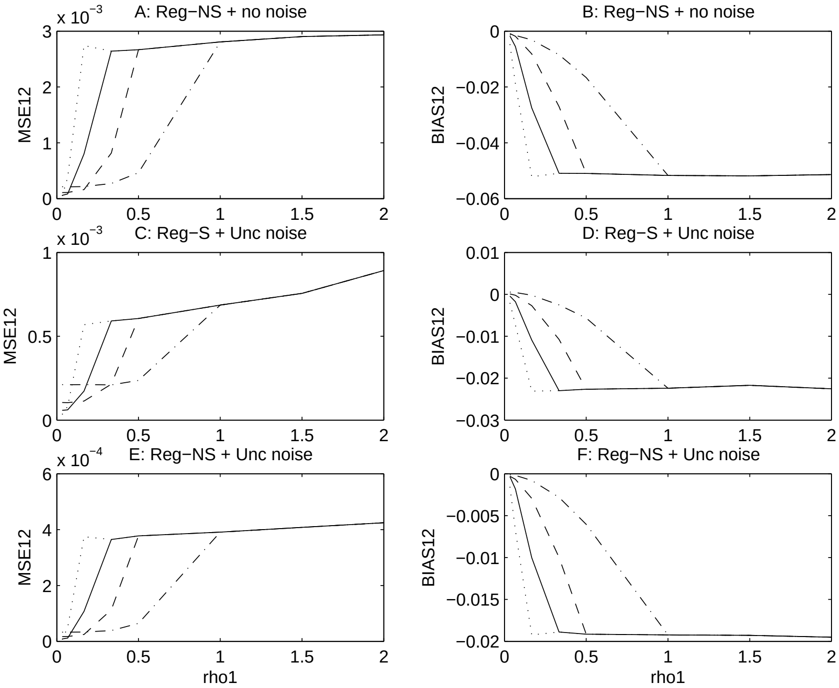

Figure 1: MSE and bias for $\hat{\Sigma}^{12}_{N,n_1,n_2}$ as a function of the sampling period ρ_1 in the regular non-synchronous trading setting and no noise (Panels A,B), in the regular synchronous trading setting and uncorrelated noise (Panels C,D) and in the regular non-synchronous trading setting and uncorrelated noise (Panels E,F). ':' for $N_{cut} = 720$; '-' for $N_{cut} = 360$; '- -' for $N_{cut} = 180$; '-.' for $N_{cut} = 90$. $\rho_1 = 2, 4, 10, 30, 60, 90, 120$ sec.

that the smallest MSE is obtained at the highest sampling frequencies for suitable values of N_{cut} while, on the contrary, a naïve choice of N_{cut} would result in an explosion of MSE. Moreover, the Fourier estimator turns out to be asymptotically unbiased, as pointed out by the theoretical results.

In [Hoshikawa and al., 2008] a purely empirical comparison between realized covariance, the All-overlapping estimator and the Fourier method is conducted under no market microstructure noise . Nevertheless the analysis is conducted by allowing the frequency N go to infinity without establishing any criterion for the optimal choice of N. The present paper fills this gap, while the importance of the choice of N_{cut} for the diagonal elements of the covariance matrix in the presence of market microstructure effects is analyzed in [Mancino and Sanfelici, 2008] and will not be discussed here any longer. A different approach is proposed by [Oya, 2005], who applies the subsampling bias correction method of [Zhang and al., 2005] to the Fourier estimator of the univariate integrated volatility and obtains smaller MSE's than with other bias-corrected estimators. In

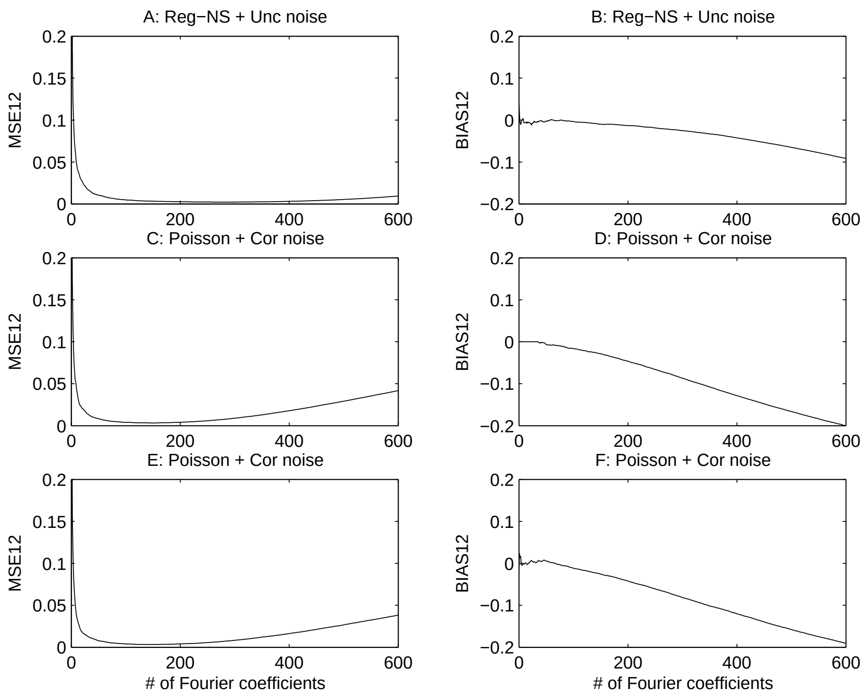

Figure 2: MSE and bias for $\hat{\Sigma}^{12}_{N,n_1,n_2}$ as a function of the cutting frequency N_{cut}. Panels A,B: regular non-synchronous trading setting, with $\rho_1 = 5$ sec, $\rho_2 = 10$ sec, $\delta = 2/3$ and uncorrelated i.i.d. noise. Panels C,D: Poisson trading setting, with $\lambda_1 = 5$ sec, $\lambda_2 = 10$ sec and correlated i.i.d. noise, with $\rho = 0.5$. Panels E,F: Poisson trading setting, with $\lambda_1 = 5$ sec, $\lambda_2 = 10$ sec and increased correlated i.i.d. noise, with $\rho = 0.5$.

[Precup and Iori, 2007] two interpolation based methods (the traditional Pearson coefficient and the Co-volatility weighted method proposed by [Dacorogna et al., 2001]) have been compared with the Fourier method. The authors show that the Fourier method generates more accurate results than the other two; in particular the other methods generate correlation estimates which are inferior to the Fourier method in terms of magnitude and smoothness.

Proposition 5.1 and Theorem 5.2 provide an operative tool to choose the optimal N_{cut} value in the theoretical setting considered in this paper. The practical calculation hinges on the estimation of the relevant noise moments as well as on the preliminary identification of $E[\int_0^{2\pi} \Sigma^{ii}(t)dt]$ and $E[\int_0^{2\pi} (\Sigma^{ij}(t))^2 dt]$. Since the noise moments do not vary across frequencies in our context, in computing the MSE estimates we use sample moments constructed by interpolating quote-to-quote return data on a high frequency uniform grid in order to estimate the relevant population moments of the noise components based on the following relation [Bandi and Russel, 2005b]:

$$\text{plim}_{M\to\infty}\frac{1}{M}\sum_{k=1}^{M}\delta_k(\tilde{p}^i)\delta_k(\tilde{p}^j) = E[\varepsilon^i\varepsilon^j] = 2\omega_{ij},$$

where M is the number of high frequency sampling intervals. Preliminary estimates of $E[\int_0^{2\pi}\Sigma^{ii}(t)dt]$ and $E[\int_0^{2\pi}(\Sigma^{ij}(t))^2dt]$ are obtained by computing

$$\int_0^{2\pi}\Sigma^{ii}(t)dt \cong \sum_{k=1}^{\bar{M}}\delta_k(\tilde{p}^i)\delta_k(\tilde{p}^i), \tag{21}$$

$$\int_0^{2\pi}(\Sigma^{ij}(t))^2dt \le \int_0^{2\pi}[\Sigma^{ii}(t)\Sigma^{jj}(t) + (\Sigma^{ij}(t))^2]dt \cong$$

$$\cong \frac{\bar{M}}{2\pi}\sum_{k=1}^{\bar{M}}\delta_k^2(\tilde{p}^i)\delta_k^2(\tilde{p}^j) - \frac{\bar{M}}{2\pi}\sum_{k=1}^{\bar{M}-1}\delta_k(\tilde{p}^i)\delta_k(\tilde{p}^j)\delta_{k+1}(\tilde{p}^i)\delta_{k+1}(\tilde{p}^j) \quad \text{for } i \neq j \tag{22}$$

and in particular

$$\int_0^{2\pi}(\Sigma^{ii}(t))^2dt \cong \frac{\bar{M}}{4\pi}\left[\sum_{k=1}^{\bar{M}}\delta_k^4(\tilde{p}^i) - \sum_{k=1}^{\bar{M}-1}\delta_k^2(\tilde{p}^i)\delta_{k+1}^2(\tilde{p}^i)\right],$$

using $\bar{M}$ 2-min or, equivalently, 15-min returns. We remark that at this stage the integrated volatility estimate (21) can be substituted by a Fourier estimate using returns sampled at 2-min frequency and choosing N equal to the Nyquist frequency. This approach, although more coherent with our Fourier analysis, does not yield any significant difference in the results. Concerning the computation of the quarticity by means of the estimator (22), the corresponding formula in the Fourier framework is not available at the moment and will be the object of future work. The quantities given by (21)-(22), together with the estimates of Proposition 5.1 and Theorem 5.2 allow to measure the MSE of the co-volatility estimator also in the case of empirical market quote data, where the efficient price and volatility and the noise contaminations are not available, assuming that our theoretical framework holds. Therefore, they can be used to build optimal MSE-based estimators by choosing the cutting frequency N_{cut} which minimizes the estimated MSE instead of the true one.

In Figure 3 we show the true (dotted line) and estimated (solid line) MSE for the Fourier estimator as a function of the cutting frequency N, obtained according to Proposition 5.1 and Theorem 5.2 in the case of uncorrelated i.i.d noise (Panel A) and of correlated i.i.d noise (Panel B) under regular non-synchronous trading. The estimated MSE provides an upper bound of the actual one, which nevertheless can be used to find out an optimal

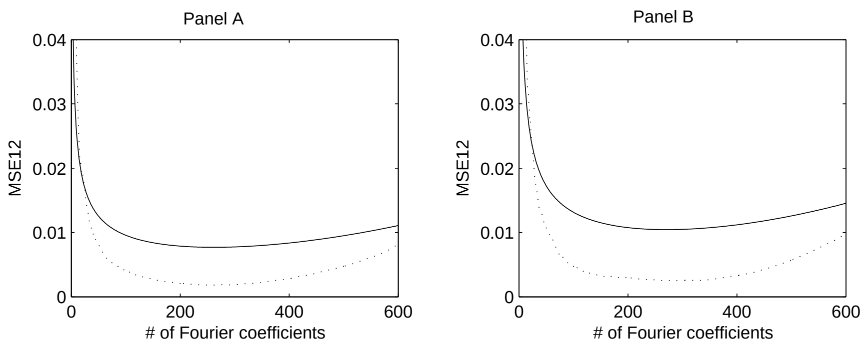

Figure 3: Real (:) and estimated (-) MSE for $\hat{\Sigma}^{12}_{N,n_1,n_2}$ as a function of the cutting frequency N_{cut}. Panels A: regular non-synchronous trading setting, with $\rho_1 = 5$ sec, $\rho_2 = 10$ sec, $\delta = 2/3$ and uncorrelated i.i.d. noise. Panels B: regular non-synchronous trading setting, with $\rho_1 = 5$ sec, $\rho_2 = 10$ sec, $\delta = 2/3$ and correlated i.i.d. noise.

cutting frequency N_{cut}. In fact, in the uncorrelated case, the minimum of the true MSE is 0.0018 and is attained at $N = 252$. The covariance estimate is 0.6062. The minimum of the estimated MSE is attained at $N = 261$ and the corresponding true MSE value is 0.0018. The covariance estimated for this choice of the cutting frequency is 0.6054, while the true covariance is 0.6251. In the correlated case, the minimum of the true MSE is 0.0025 and is attained at $N = 278$. The covariance estimate is 0.6677. The minimum of the estimated MSE is attained at $N = 271$ and the corresponding true MSE value is 0.0025. The covariance estimated for this choice of the cutting frequency is 0.6687, while the true covariance is 0.6897.

This optimal MSE-based covariance estimator is compared to the behavior of the realized covariance $RC^{0.5min}_{1,2}$ based on half a minute returns, the realized covariance $RC^{1min}_{1,2}$ based on 1 minute returns, the realized covariance $RC^{5min}_{1,2}$ based on 5 minute returns and the corresponding realized covariance plus leads and lags $RCLL^{0.5min}_{1,2}$, $RCLL^{1min}_{1,2}$ and $RCLL^{5min}_{1,2}$, with $l = L = 1$. The low frequency returns are obtained by imputation on a uniform grid. As any estimator based on interpolated prices, these methods suffer from the Epps effect when trading is non-synchronous, but the lead-lag correction reduces such an effect. The optimal MSE-based Fourier estimator is obtained by minimizing the true MSE with respect to N. For the regular non-synchronous trading setting, the true MSE and bias of the optimal estimator based on the minimization of the estimated MSE (as given by the upper bound in Theorem 5.2) are given in parenthesis. Finally, in our analysis we consider the All-overlapping estimator $AO_{1,2}$ as well.

The results are reported in Tables 1 and 2. Within each table entries are the values of the MSE and bias, using 500 Monte Carlo replications. In the first day, the initial values for p_i's and σ_i's are extracted from independent standard half normal distributions and are the same for all the trading setting; then, in the next days, they are set equal to the closing value of the previous day. Rows correspond to the trading scenarios and columns to different estimators. The trading settings are denoted by the terms Reg-S, Reg-NS, Poisson, while the second term (Unc, Cor, Dep) refers to the type of noise, namely contemporaneously uncorrelated ($\omega_{ij} = 0$ for $i \neq j$), contemporaneously correlated and dependent on the price process, respectively.

When we consider covariance estimates, the most important effect to deal with is the Epps effect. The presence of other microstructure effects represents a minor aspect in this respect. In fact, from Table 1 we see that in the Reg-NS setting without noise the effects imputable to non-synchronicity are evident and spoil all the covariance estimates except the AO estimator, which shows the best performance. Nevertheless, the optimal MSE-based Fourier estimator achieves a very low MSE, but a larger negative bias. The 0.5 minute return lead-lag correction offers a good alternative as it mitigates the bias induced by non-synchronicity by adding one lead and one lag of the empirical autocovariance function of returns to the realized covariance measure. Notice, however, that the level of non-synchronicity is very low in this setting and that the assets are quite active (they trade each 2 and 4 seconds respectively, with a displacement of 1 second) so that interpolation of data is not needed actually. The addition of a moderate amount of independent and uncorrelated noise does not have great effect on the estimates. On the contrary, it may in some sense even compensate the effects due to non-synchronicity. In general, the AO estimator provides the best results, followed by the Fourier estimator, which outperforms $RC_{1,2}^{0.5min}$ in terms of MSE to the disadvantage of a slightly larger bias. An exception to this ranking is provided by the Poisson trading setting with correlated noise and by the trading settings with dependent noise. In theses cases, the Fourier estimator and the realized covariance plus lead and lag $RCLL_{1,2}^{0.5min}$ slightly outperform the AO estimator. We remark that under regular non-synchronous trading the optimal Fourier estimator based on the minimization of the estimated MSE (whose true bias and MSE are given in parenthesis) is comparable to the Fourier estimator based on the minimization of the true MSE, thus supporting our theoretical results. Finally, note that the lead/lag correction for the realized covariance estimator contrasts the Epps effect, thus producing occasionally positive biases.

In Table 2, the noise term and the level of non-synchronicity are both increased, by taking $\rho_1 = 5$ sec, $\rho_2 = 10$ sec with a displacement of 2 seconds, $\lambda_1 = 5$ sec and $\lambda_2 = 10$ sec. Again, we see that in some trading scenarios the Fourier estimator outperforms the AO estimator. Indeed, the AO estimator can sometimes become less effective, as can be seen in the Poisson trading scheme with correlated noise and in the trading settings with dependent noise. In fact, the AO remains unbiased under independent noise whenever the probability of trades occurring at the same time is zero which is not the case for Poisson arrivals. In particular, in the trading scenarios with noise dependent on the price process the Fourier estimator remains a robust alternative which outperforms all the other methods, included the bias corrected realized covariance.

Finally, we further increase the level of noise in such a way that the noise ratio defined by [Griffin and Oomen, 2006] is around 7. The results are shown in Table 3.

As already found by [Griffin and Oomen, 2006], at this noise level, the All-overlapping estimator can be even less efficient than the standard realized covariance estimator while the Fourier estimator always provides a valid alternative. The lead/lag bias correction comes with a noise accumulation and the balancing of this trade-off determines the relative efficiency of the estimators. It is clear from the above tables that the Fourier estimator provides an efficient balance which is robust to any level of noise and non-synchronicity.

Therefore, we can conclude that, in agreement with our theoretical analysis, the Fourier covariance estimator is not much affected by the presence of noise, so that it becomes a very interesting alternative especially when microstructure effects are particularly relevant in the available data.

8 Conclusions

In this paper we analyzed the properties of Fourier estimator of multivariate volatilities in the presence of asynchronous trading and microstructure noise. We have proved that the Fourier estimator of covariance is: (i) consistent under asynchronous trading, (ii) asymptotically unbiased in the presence of i.i.d. microstructure noise, (iii) "nearly" consistent in the presence of i.i.d. microstructure noise, in the sense that the MSE of the Fourier estimator converges to a constant as the number of observations increases and it does not diverge as it happens for the realized covariance or the All-overlapping estimator. Finally the results have been extended to some dependent microstructure noise. Our theoretical results are supported by several Monte Carlo simulations.

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9 Appendix: Proofs

Proof of Proposition 3.2 Note that

$$\int_0^{2\pi} \Sigma^{12}(t)dt = \sum_{i=1}^{n_1-1} \int_{I_i^1} \Sigma^{12}(t)dt = \sum_{i=1}^{n_1-1} \sum_{j=1}^{n_2-1} \int_{\{I_i^1 \cap J_j^2\}} \Sigma^{12}(t)dt, \tag{23}$$

because I_i^1 and J_j^2 are partition of $[0, 2\pi]$. By Itô energy identity we have

$$E[D_N(t_i^1 - t_j^2)\delta_{I_i^1}(p^1)\delta_{J_j^2}(p^2)I_{\{I_i^1 \cap J_j^2 \neq \emptyset\}}] = D_N(t_i^1 - t_j^2)E[\int_{\{I_i^1 \cap J_j^2 \neq \emptyset\}} \Sigma^{12}(t)dt].$$

We prove now the asymptotical unbiasedness of Fourier estimator. Remark that

$$E[D_N(t_i^1 - t_j^2)\delta_{I_i^1}(p^1)\delta_{J_j^2}(p^2)I_{\{I_i^1 \cap J_j^2 = \emptyset\}}] = D_N(t_i^1 - t_j^2)E[\int_{\{I_i^1 \cap J_j^2 = \emptyset\}} \Sigma^{12}(t)dt] = 0,$$

therefore only the sum over the intervals such that $\{I_i^1 \cap J_j^2 \neq \emptyset\}$ gives contribution in (9), which follows straightforwardly. Moreover, for $t_i^1, t_j^2 \in I_i^1 \cap J_j^2 \neq \emptyset$, we have that $|t_i^1 - t_j^2| \leq \rho(n) := \rho(n_1) \vee \rho(n_2)$. Observe that

$$\sum_{i=1}^{n_1-1} \sum_{j=1}^{n_2-1} \left(D_N(t_i^1 - t_j^2) - 1\right) E[\int_{I_i^1 \cap J_j^2} \Sigma^{12}(t)dt] \leq \rho(n)N \sum_{i=1}^{n_1-1} \sum_{j=1}^{n_2-1} E[\int_{I_i^1 \cap J_j^2} \Sigma^{12}(t)dt]$$

$$= \rho(n)N \; E[\int_0^{2\pi} \Sigma^{12}(t)dt],$$

which goes to zero under the condition $\rho(n)N \to 0$. •

Proof of Theorem (4.1)

Under the data specification of Section 4 we write (17) as

$$\sum_{j=1}^{\frac{n}{2}-1} \sum_{i=1}^{n} D_N(t_i^1 - t_j^2)\delta_{I_i^1}(\tilde{p}^1)\delta_{J_j^2}(\tilde{p}^2)I_{\{I_i^1 \cap J_j^2 \neq \emptyset\}} + \sum_{j=1}^{\frac{n}{2}-1} \sum_{i=1}^{n} D_N(t_i^1 - t_j^2)\delta_{I_i^1}(\tilde{p}^1)\delta_{J_j^2}(\tilde{p}^2)I_{\{I_i^1 \cap J_j^2 = \emptyset\}}$$

$$= \sum_{j=1}^{\frac{n}{2}-1} \sum_{i=2(j-1)+1}^{2(j-1)+3} D_N(t_i^1 - t_j^2)\delta_{I_i^1}(\tilde{p}^1)\delta_{J_j^2}(\tilde{p}^2) \tag{24}$$

$$+ \sum_{j=1}^{\frac{n}{2}-1} \left(\sum_{i=1}^{2(j-1)} D_N(t_i^1 - t_j^2)\delta_{I_i^1}(\tilde{p}^1)\delta_{J_j^2}(\tilde{p}^2) + \sum_{i=2(j-1)+4}^{n-1} D_N(t_i^1 - t_j^2)\delta_{I_i^1}(\tilde{p}^1)\delta_{J_j^2}(\tilde{p}^2) \right). \tag{25}$$

First consider (24). This is equal to

$$E[\sum_{j=1}^{\frac{n}{2}-1} \sum_{i=2(j-1)+1}^{2(j-1)+3} D_N(t_i^1 - t_j^2)\delta_{I_i^1}(p^1)\delta_{J_j^2}(p^2)] \tag{26}$$

$$+E[\sum_{j=1}^{\frac{n}{2}-1}\sum_{i=2(j-1)+1}^{2(j-1)+3} D_N(t_i^1-t_j^2)\varepsilon^1_{I_i^1}\delta_{J_j^2}(p^2)] \tag{27}$$

$$+E[\sum_{j=1}^{\frac{n}{2}-1}\sum_{i=2(j-1)+1}^{2(j-1)+3} D_N(t_i^1-t_j^2)\delta_{I_i^1}(p^1)\varepsilon^2_{J_j^2}] \tag{28}$$

$$+E[\sum_{j=1}^{\frac{n}{2}-1}\sum_{i=2(j-1)+1}^{2(j-1)+3} D_N(t_i^1-t_j^2)\varepsilon^1_{I_i^1}\varepsilon^2_{J_j^2}]. \tag{29}$$

The terms (27) and (28) are zero for the independence between the asset prices and the noise process. Concerning the term (29) we have

$$E[\sum_{j=1}^{\frac{n}{2}-1}\sum_{i=2(j-1)+1}^{2(j-1)+3} D_N(t_i^1-t_j^2)\varepsilon^1_{I_i^1}\varepsilon^2_{J_j^2}] =$$

$$=\sum_{j=1}^{\frac{n}{2}-1} E[(\eta^2_{t^2_{j+1}}-\eta^2_{t^2_j})\{D_N(t^1_{2(j-1)+1}-t^2_j)(\eta^1_{t^1_{2(j-1)+2}}-\eta^1_{t^1_{2(j-1)+1}})$$
$$+D_N(t^1_{2(j-1)+2}-t^2_j)(\eta^1_{t^1_{2(j-1)+3}}-\eta^1_{t^1_{2(j-1)+2}})+D_N(t^1_{2(j-1)+3}-t^2_j)(\eta^1_{t^1_{2(j-1)+4}}-\eta^1_{t^1_{2(j-1)+3}})\}].$$

Each of these terms is zero due to non-synchronicity. Thus, we have to evaluate the bias

$$E[\sum_{j=1}^{\frac{n}{2}-1}\sum_{i=2(j-1)+1}^{2(j-1)+3} D_N(t_i^1-t_j^2)\delta_{I_i^1}(p^1)\delta_{J_j^2}(p^2)-\int_0^{2\pi}\Sigma^{12}(t)dt].$$

This can be obtained using the result in absence of microstructure noise, that is

$$\sum_{j=1}^{\frac{n}{2}-1}\sum_{i=2(j-1)+1}^{2(j-1)+3} E[\int_{t_i^1}^{t_{i+1}^1}\Sigma^{12}(t)dt](D_N(t_i^1-t_j^2)-1).$$

The term (25) is equal to

$$E[\sum_{j=1}^{\frac{n}{2}-1}(\sum_{i=1}^{2(j-1)} D_N(t_i^1-t_j^2)\delta_{I_i^1}(p^1)\delta_{J_j^2}(p^2)+\sum_{i=2(j-1)+4}^{n-1} D_N(t_i^1-t_j^2)\delta_{I_i^1}(p^1)\delta_{J_j^2}(p^2)\ )] \tag{30}$$

$$+E[\sum_{j=1}^{\frac{n}{2}-1}(\sum_{i=1}^{2(j-1)} D_N(t_i^1-t_j^2)\varepsilon^1_{I_i^1}\delta_{J_j^2}(p^2)+\sum_{i=2(j-1)+4}^{n-1} D_N(t_i^1-t_j^2)\varepsilon^1_{I_i^1}\delta_{J_j^2}(p^2)\ )] \tag{31}$$

$$+E[\sum_{j=1}^{\frac{n}{2}-1}(D_N(t_i^1-t_j^2)\delta_{I_i^1}(p^1)\varepsilon^2_{J_j^2}+\sum_{i=2(j-1)+4}^{n-1} D_N(t_i^1-t_j^2)\delta_{I_i^1}(p^1)\varepsilon^2_{J_j^2}\ )] \tag{32}$$

$$+E[\sum_{j=1}^{\frac{n}{2}-1}(\sum_{i=1}^{2(j-1)} D_N(t_i^1-t_j^2)\varepsilon^1_{I_i^1}\varepsilon^2_{J_j^2}+\sum_{i=2(j-1)+4}^{n-1} D_N(t_i^1-t_j^2)\varepsilon^1_{I_i^1}\varepsilon^2_{J_j^2}\ )]. \tag{33}$$

Observe that (30) is zero because $I_i^1 \cap J_j^2 = \emptyset$. Moreover (31), (32), (33) are zero due to the independence between price and noise and the non-synchronicity. •

Proof of Proposition 5.1

$$E[(\sum_{i,j} D_N(t_i^1 - t_j^2)\delta_{I_i^1}(p^1)\delta_{J_j^2}(p^2) - \int_0^{2\pi} \Sigma^{12}(t)dt)^2] \tag{34}$$

$$= E[(\sum_{j=1}^{\frac{n}{2}-1} \sum_{i=2(j-1)+1}^{2(j-1)+3} D_N(t_i^1 - t_j^2)\delta_{I_i^1}(p^1)\delta_{J_j^2}(p^2) - \int_0^{2\pi} \Sigma^{12}(t)dt)^2] \tag{35}$$

$$+E[\Big(\sum_{j=1}^{\frac{n}{2}-1}(\sum_{i=1}^{2(j-1)} D_N(t_i^1 - t_j^2)\delta_{I_i^1}(p^1)\delta_{J_j^2}(p^2) + \sum_{i=2(j-1)+4}^{n-1} D_N(t_i^1 - t_j^2)\delta_{I_i^1}(p^1)\delta_{J_j^2}(p^2))\Big)^2]. \tag{36}$$

Firstly consider (35). Remark that

$$\int_0^{2\pi} \Sigma^{12}(t)dt = \sum_{i,j} \int_{\{I_i^1 \cap J_j^2\}} \Sigma^{12}(t)dt,$$

then we split (35) in two parts:

$$\sum_{j=1}^{\frac{n}{2}-1} \sum_{i=2(j-1)+1}^{2(j-1)+3} D_N(t_i^1 - t_j^2)\delta_{I_i^1}(p^1)\delta_{J_j^2}(p^2) - \int_0^{2\pi} \Sigma^{12}(t)dt$$

$$= \frac{1}{2N+1} \sum_{|s|\le N} \sum_{j=1}^{\frac{n}{2}-1} \sum_{i=2(j-1)+1}^{2(j-1)+3} (B_{ij} + C_{ij}),$$

where

$$B_{ij} = e^{\mathrm{i}s(t_j^2 - t_i^1)}\Big(\delta_{I_i^1}(p^1)\delta_{J_j^2}(p^2) - \int_{\{I_i^1 \cap J_j^2\}} \Sigma^{12}(t)dt\Big), \tag{37}$$

$$C_{ij} = (e^{\mathrm{i}s(t_j^2 - t_i^1)} - 1)\int_{\{I_i^1 \cap J_j^2\}} \Sigma^{12}(t)dt. \tag{38}$$

Concerning the term containing (38) we have:

$$E[(\frac{1}{2N+1} \sum_{|s|\le N} \sum_{j=1}^{\frac{n}{2}-1} \sum_{i=2(j-1)+1}^{2(j-1)+3} C_{ij})^2] \le E[\frac{1}{2N+1}(\sum_{|s|\le N} |s|\rho(n) \sum_{j=1}^{\frac{n}{2}-1} \sum_{i=2(j-1)+1}^{2(j-1)+3} \int_{\{I_i^1 \cap J_j^2\}} \Sigma^{12}(t)dt)^2]$$

$$= \rho(n)^2 E[|\frac{1}{2N+1}(N(N+1))\int_0^{2\pi} \Sigma^{12}(t)dt|^2] \le \frac{1}{4}\rho(n)^2(N+1)^2 E[(\int_0^{2\pi} \Sigma^{12}(t)dt)^2].$$

Consider now the term containing (37). We have

$$E[(\frac{1}{2N+1} \sum_{|s|\le N} \sum_{j=1}^{\frac{n}{2}-1} \sum_{i=2(j-1)+1}^{2(j-1)+3} B_{ij})^2] \le E[(\sum_{j=1}^{\frac{n}{2}-1} \sum_{i=2(j-1)+1}^{2(j-1)+3} \delta_{I_i^1}(p^1)\delta_{J_j^2}(p^2) - \int_{\{I_i^1 \cap J_j^2\}} \Sigma^{12}(t)dt)^2].$$

For any fixed j we have:

$$\delta_{J^2_j}(p^2)\sum_{i=2(j-1)+1}^{2(j-1)+3}\delta_{I^1_i}(p^1)=\int_{t^2_j}^{t^2_{j+1}}dp^2(t)\int_{t^2_j}^{t^2_{j+1}}dp^1(t)$$

$$+\int_{t^2_j}^{t^2_{j+1}}dp^2(t)(\int_{t^1_{2(j-1)+1}}^{t^2_j}dp^1(t)+\int_{t^2_{j+1}}^{t^1_{2(j-1)+4}}dp^1(t)).$$

Taking into account the identity (23), first of all we consider

$$\int_{t^2_j}^{t^2_{j+1}}dp^1(t)\int_{t^2_j}^{t^2_{j+1}}dp^2(t)-\int_{t^2_j}^{t^2_{j+1}}\Sigma^{12}(t)dt. \tag{39}$$

Let $X(t)=[p^1(t)-p^1(t^2_j)]$ and $Y(t)=[p^2(t)-p^2(t^2_j)]$. By Itô formula

$$d(XY)=(p^1(t)-p^1(t^2_j))dp^2(t)+(p^2(t)-p^2(t^2_j))dp^1(t)+\Sigma^{12}(t)dt.$$

Then (39) is equal to

$$\int_{t^2_j}^{t^2_{j+1}}(p^1(t)-p^1(t^2_j))dp^2(t)+\int_{t^2_j}^{t^2_{j+1}}(p^2(t)-p^2(t^2_j))dp^1(t). \tag{40}$$

Consider the first addend in (40); the second term is analogous. Observe that

$$\sum_{j=1}^{\frac{n}{2}-1}\int_{t^2_j}^{t^2_{j+1}}(p^1(t)-p^1(t^2_j))dp^2(t)=\sum_j\int_{J^2_j}X(t)dp^2(t)=\int_0^{2\pi}X(t)dp^2(t).$$

By Itô energy identity and the orthogonality of W^1 and W^2

$$E[(\int_0^{2\pi}X(t)dp^2(t))^2]=E[\int_0^{2\pi}(X(t))^2\Sigma^{22}(t)dt]$$

and, by Cauchy-Schwartz inequality

$$\leq E[\int_0^{2\pi}(X(t))^4dt]^{\frac{1}{2}}E[\int_0^{2\pi}(\Sigma^{22}(t))^2dt]^{\frac{1}{2}}. \tag{41}$$

As by hypothesis **(H)**

$$E[\int_0^{2\pi}(\Sigma^{22}(t))^2dt]<\infty,$$

let us consider

$$E[\int_0^{2\pi}(X(t))^4dt]=\sum_j E[\int_{J^2_j}(\int_{t^2_j}^t(\sigma^1_1(r)dW^1(r)+\sigma^1_2(r)dW^2(r)))^4dt].$$

For j fixed and applying Burkholder-Davis-Gundy inequality

$$E[\int_{J^2_j}(\int_{t^2_j}^t(\sigma^1_1(r)dW^1(r)+\sigma^1_2(r)dW^2(r)))^4dt]$$

$$\leq E[\int_{J_j^2} \max_{t_j^2 \leq t \leq t_{j+1}^2} (\int_{t_j^2}^t \sigma_1^1(r)dW^1(r) + \int_{t_j^2}^t \sigma_2^1(r)dW^2(r))^4 dt]$$

$$\leq 4E[\int_{J_j^2} (\int_{t_j^2}^{t_{j+1}^2} \Sigma^{11}(r)dr)^2 dt] \leq 4|J_j^2|^2 E[\int_{J_j^2} (\Sigma^{11}(t))^2 dt].$$

Now we have

$$\sum_j |J_j^2|^2 E[\int_{J_j^2} (\Sigma^{11}(t))^2 dt] \leq 2\pi\rho(n_2) E[\int_0^{2\pi} (\Sigma^{11}(t))^2 dt].$$

Then (41) is less or equal to

$$2\sqrt{2\pi}\, \rho(n_2)^{\frac{1}{2}} E[\int_0^{2\pi} (\Sigma^{11}(t))^2 dt]^{\frac{1}{2}} E[\int_0^{2\pi} (\Sigma^{22}(t))^2 dt]^{\frac{1}{2}}.$$

The second term can be treated similarly. Therefore

$$E[(\sum_j \int_{t_j^2}^{t_{j+1}^2} dp^1(t) \int_{t_j^2}^{t_{j+1}^2} dp^2(t) - \int_{t_j^2}^{t_{j+1}^2} \Sigma^{12}(t)dt)^2]$$

$$\leq 4\sqrt{2\pi}\, \rho(n_2)^{\frac{1}{2}} E[\int_0^{2\pi} (\Sigma^{11}(t))^2 dt]^{\frac{1}{2}} E[\int_0^{2\pi} (\Sigma^{22}(t))^2 dt]^{\frac{1}{2}}.$$

We analyze now the other terms. Fix j

$$E[(\int_{t_{2(j-1)+1}^1}^{t_j^2} dp^1(t) \int_{t_j^2}^{t_{j+1}^2} dp^2(t) + \int_{t_{j+1}^2}^{t_{2(j-1)+4}^1} dp^1(t) \int_{t_j^2}^{t_{j+1}^2} dp^2(t))^2]$$

$$= \Big(E[\int_{t_{2(j-1)+1}^1}^{t_j^2} \Sigma^{11}(t)dt] + E[\int_{t_{j+1}^2}^{t_{2(j-1)+4}^1} \Sigma^{11}(t)dt]\Big) E[\int_{t_j^2}^{t_{j+1}^2} \Sigma^{22}(t)dt].$$

Therefore, by Cauchy-Schwartz inequality

$$\sum_j E[(\int_{t_{2(j-1)+1}^1}^{t_j^2} dp^1(t) \int_{t_j^2}^{t_{j+1}^2} dp^2(t) + \int_{t_{j+1}^2}^{t_{2(j-1)+4}^1} dp^1(t) \int_{t_j^2}^{t_{j+1}^2} dp^2(t))^2]$$

$$\leq \rho(n_1)^{\frac{1}{2}} \sum_j E[\int_{t_j^2}^{t_{j+1}^2} \Sigma^{22}(t)dt](E[\int_{t_{2(j-1)+1}^1}^{t_j^2} (\Sigma^{11}(t))^2 dt]^{\frac{1}{2}} + E[\int_{t_{j+1}^2}^{t_{2(j-1)+4}^1} (\Sigma^{11}(t))^2 dt]^{\frac{1}{2}})$$

$$\leq 2\rho(n_1)^{\frac{1}{2}} E[\int_0^{2\pi} \Sigma^{22}(t)dt] E[\int_0^{2\pi} (\Sigma^{11}(t))^2 dt]^{\frac{1}{2}}.$$

Now by considering the summation in j we observe that in the mixing terms only first order autocorrelations remains: in fact

$$\sum_j \sum_{j'>j} E[\delta_{J_j^2}(p^2) \sum_{i=2(j-1)+1}^{2(j-1)+3} \delta_{I_i^1}(p^1) \delta_{J_{j'}^2}(p^2) \sum_{i'=2(j'-1)+1}^{2(j'-1)+3} \delta_{I_{i'}^1}(p^1)]$$

$$= \sum_{j=1}^{\frac{n}{2}-1} E[\delta_{J_j^2}(p^2) \sum_{i=2(j-1)+1}^{2(j-1)+3} \delta_{I_i^1}(p^1) \delta_{J_{j+1}^2}(p^2) \sum_{i=2j+1}^{2j+3} \delta_{I_i^1}(p^1)]$$

$$= \sum_{j=1}^{\frac{n}{2}-1} E[\int_{t^1_{2(j-1)+3}}^{t^2_{j+1}} \Sigma^{12}(t)dt]E[\int_{t^2_{j+1}}^{t^1_{2(j-1)+4}} \Sigma^{12}(t)dt].$$

Finally we have:

$$E[(\sum_{j=1}^{\frac{n}{2}-1} \int_{t^1_{2(j-1)+1}}^{t^2_j} dp^1(t) \int_{t^2_j}^{t^2_{j+1}} dp^2(t) + \int_{t^2_{j+1}}^{t^1_{2(j-1)+4}} dp^1(t) \int_{t^2_j}^{t^2_{j+1}} dp^2(t))^2]$$

$$\leq 2\rho(n_1)^{\frac{1}{2}} E[\int_0^{2\pi} \Sigma^{22}(t)dt]E[\int_0^{2\pi} (\Sigma^{11}(t))^2 dt]^{\frac{1}{2}} + \rho(n_1)^{\frac{1}{2}} E[\int_0^{2\pi} (\Sigma^{12}(t))^2 dt].$$

Finally it remains to consider the lead-lag components (36):

$$E[\Big(\sum_{j=1}^{\frac{n}{2}-1} (\sum_{i=1}^{2(j-1)} D_N(t^1_i - t^2_j)\delta_{I^1_i}(p^1)\delta_{J^2_j}(p^2) + \sum_{i=2(j-1)+4}^{n-1} D_N(t^1_i - t^2_j)\delta_{I^1_i}(p^1)\delta_{J^2_j}(p^2))\Big)^2].$$

Define

$$U(\phi_1, \phi_2) := \sum_i \sum_j D_N(t^1_i - t^2_j)\chi_{I^1_i}(\phi_1)\chi_{J^2_j}(\phi_2),$$

where $D_N(s)$ is the rescaled Dirichlet kernel. Then

$$\sum_{i,j} D_N(t^1_i - t^2_j)\delta_{I^1_i}(p^1)\delta_{J^2_j}(p^2) I_{\{I^1_i \cap J^2_j = \emptyset\}} =$$

$$= \int\int_{\phi_1<\phi_2} U(\phi_1, \phi_2)dp^1(\phi_1)dp^2(\phi_2) + \int\int_{\phi_2<\phi_1} U(\phi_1, \phi_2)dp^1(\phi_1)dp^2(\phi_2).$$

Consider the first term. The second one is analogous. By Itô energy identity and Cauchy-Schwartz inequality, we have

$$E[(\int\int_{\phi_1<\phi_2} U(\phi_1, \phi_2)dp^1(\phi_1)dp^2(\phi_2))^2]$$

$$= E[\int_0^{2\pi} (\int_0^{\phi_2} U(\phi_1, \phi_2)dp^1(\phi_1))^2 \Sigma^{22}(\phi_2)d\phi_2]$$

$$\leq E[\int_0^{2\pi} (\int_0^{\phi_2} U(\phi_1, \phi_2)dp^1(\phi_1))^4 d\phi_2]^{\frac{1}{2}} E[\int_0^{2\pi} (\Sigma^{22}(\phi_2))^2 d\phi_2]^{\frac{1}{2}}.$$

By hypothesis **(H)**

$$E[\int_0^{2\pi} (\Sigma^{22}(\phi_2))^2 d\phi_2] < \infty,$$

then it is enough to consider the first term. The Burkoholder-Davis-Gundy inequality allows to estimate

$$E[\int_0^{2\pi} (\int_0^{\phi_2} U(\phi_1, \phi_2)dp^1(\phi_1))^4 d\phi_2] \leq E[\int_0^{2\pi} \max_{0\leq\phi_2\leq 2\pi} \left(\int_0^{\phi_2} U(\phi_1, \phi_2)dp^1(\phi_1) \right)^4 \ d\phi_2]$$

$$\leq 4E[\int_0^{2\pi} (\int_0^{2\pi} U^2(\phi_1, \phi_2)\Sigma^{11}(\phi_1)d\phi_1)^2 \ d\phi_2]$$

$$= 4E[\int_0^{2\pi}\int_0^{2\pi} D_N^4(\phi_1-\phi_2)(\Sigma^{11}(\phi_1))^2 d\phi_1 d\phi_2] \leq 8\int_0^{2\pi} D_N^4(v)dv\ E[\int_0^{2\pi}(\Sigma^{11}(\phi_1))^2 d\phi_1].$$

Since $|D_N(v)| \leq 1$, then by Plancherel inequality

$$\int_0^{2\pi} D_N^4(v)dv \leq \int_0^{2\pi} D_N^2(v)dv = \frac{2\pi}{2N+1},$$

and we conclude

$$E[\int_0^{2\pi}(\int_0^{\phi_2} U(\phi_1,\phi_2)dp^1(\phi_1))^4 d\phi_2]^{\frac{1}{2}} \leq \frac{4\sqrt{2\pi}}{\sqrt{2N+1}} E[\int_0^{2\pi}(\Sigma^{22}(t))^2 dt]^{\frac{1}{2}} E[\int_0^{2\pi}(\Sigma^{11}(t))^2 dt]^{\frac{1}{2}}.$$

•

Proof of Theorem 5.2

$$E[(\hat{\Sigma}^{12}_{N,n} - \int_0^{2\pi}\Sigma^{12}(t)dt)^2]$$

$$= E[(\sum_{j=1}^{\frac{n}{2}-1}\sum_{i=1}^{n-1} D_N(t_i^1 - t_j^2)\delta_{I_i^1}(p^1)\delta_{J_j^2}(p^2) - \int_0^{2\pi}\Sigma^{12}(t)dt)^2] \tag{42}$$

$$+E[(\sum_{j=1}^{\frac{n}{2}-1}\sum_{i=1}^{n-1} D_N(t_i^1 - t_j^2)\{\delta_{I_i^1}(p^1)\varepsilon^2_{J_j^2} + \delta_j(p^2)\varepsilon^1_{I_i^1} + \varepsilon^1_{I_i^1}\varepsilon^2_{J_j^2}\})^2] \tag{43}$$

$$+2E[(\sum_{j=1}^{\frac{n}{2}-1}\sum_{i=1}^{n-1} D_N(t_i^1 - t_j^2)\delta_{I_i^1}(p^1)\delta_{J_j^2}(p^2) - \int_0^{2\pi}\Sigma^{12}(t)dt)\times \tag{44}$$

$$\times(\sum_{j=1}^{\frac{n}{2}-1}\sum_{i=1}^{n-1} D_N(t_i^1 - t_j^2)\{\delta_{I_i^1}(p^1)\varepsilon^2_{J_j^2} + \delta_{J_j^2}(p^2)\varepsilon^1_{I_i^1} + \varepsilon^1_{I_i^1}\varepsilon^2_{J_j^2}\})].$$

As for (42) we know that it is $o(1)$ for Proposition 5.1. Therefore consider (43).

$$E[(\sum_{j=1}^{\frac{n}{2}-1}\sum_{i=1}^{n-1} D_N(t_i^1 - t_j^2)\{\delta_{I_i^1}(p^1)\varepsilon^2_{J_j^2} + \delta_{J_j^2}(p^2)\varepsilon^1_{I_i^1} + \varepsilon^1_{I_i^1}\varepsilon^2_{J_j^2}\})^2]$$

$$= E[(\sum_{j=1}^{\frac{n}{2}-1}\sum_{i=1}^{n-1} D_N(t_i^1 - t_j^2)\delta_{I_i^1}(p^1)\varepsilon^2_{J_j^2})^2] \tag{45}$$

$$+E[(\sum_{j=1}^{\frac{n}{2}-1}\sum_{i=1}^{n-1} D_N(t_i^1 - t_j^2)\delta_{J_j^2}(p^2)\varepsilon^1_{I_i^1})^2] \tag{46}$$

$$+E[(\sum_{j=1}^{\frac{n}{2}-1}\sum_{i=1}^{n-1} D_N(t_i^1 - t_j^2)\varepsilon^1_{I_i^1}\varepsilon^2_{J_j^2})^2] \tag{47}$$

$$+E[\sum_{j=1}^{\frac{n}{2}-1}\sum_{i=1}^{n-1} D_N(t_i^1 - t_j^2)\delta_{I_i^1}(p^1)\varepsilon^2_{J_j^2}\sum_{j'\neq j}\sum_{i'\neq i} D_N(t_{i'}^1 - t_{j'}^2)\delta_{J_{j'}^2}(p^2)\varepsilon^1_{I_{i'}^1}] \tag{48}$$

$$+E[\sum_{j=1}^{\frac{n}{2}-1}\sum_{i=1}^{n-1} D_N(t_i^1 - t_j^2)\delta_{J_j^2}(p^2)\varepsilon^1_{I_i^1} \sum_{j'\neq j}\sum_{i'\neq i} D_N(t_{i'}^1 - t_{j'}^2)\varepsilon^1_{I_{i'}^1}\varepsilon^2_{J_{j'}^2}] \tag{49}$$

$$+E[\sum_{j=1}^{\frac{n}{2}-1}\sum_{i=1}^{n-1} D_N(t_i^1 - t_j^2)\delta_{I_i^1}(p^1)\varepsilon^2_{J_j^2} \sum_{j'\neq j}\sum_{i'\neq i} D_N(t_{i'}^1 - t_{j'}^2)\varepsilon^1_{I_{i'}^1}\varepsilon^2_{J_{j'}^2}]. \tag{50}$$

First of all observe that (48), (49), (50) are zero. In fact as for (48) it is enough to note that due to the independence between noise and prices

$$E[\delta_{I_i^1}(p^1)\varepsilon^2_{J_j^2}\delta_{J_{j'}^2}(p^2)\varepsilon^1_{I_{i'}^1}] = E[\delta_{I_i^1}(p^1)\delta_{J_{j'}^2}(p^2)]E[\varepsilon^1_{I_{i'}^1}\varepsilon^2_{J_j^2}],$$

which is zero, because $E[\varepsilon^1_{I_{i'}^1}\varepsilon^2_{J_j^2}] = 0$ due to the non-synchronicity. Consider (49): by the independence between price and noise

$$E[\delta_{J_j^2}(p^2)\varepsilon^1_{I_i^1}\varepsilon^1_{I_{i'}^1}\varepsilon^2_{J_{j'}^2}] = E[\delta_{J_j^2}(p^2)]E[\varepsilon^1_{I_i^1}\varepsilon^1_{I_{i'}^1}\varepsilon^2_{J_{j'}^2}] = 0.$$

Analogously the term (50) is zero. Now we compute the second moments (45), (46), (47). We split (45) as follows:

$$E[\sum_{j=1}^{\frac{n}{2}-1}\sum_{i=1}^{n-1} D_N^2(t_i^1 - t_j^2)(\delta_{I_i^1}(p^1))^2(\varepsilon^2_{J_j^2})^2]$$

$$+E[\sum_{j=1}^{\frac{n}{2}-1}\sum_{i=1}^{n-1}\sum_{i'\neq i} D_N(t_i^1 - t_j^2)D_N(t_{i'}^1 - t_j^2)\delta_{I_i^1}(p^1)(\varepsilon^2_{J_j^2})^2\delta_{I_{i'}^1}(p^1)]$$

$$+2E[\sum_{j=1}^{\frac{n}{2}-2}\sum_{i=1}^{n-1} D_N(t_i^1 - t_j^2)D_N(t_i^1 - t_{j+1}^2)\varepsilon^2_{J_j^2}\varepsilon^2_{J_{j+1}^2}(\delta_{I_i^1}(p^1))^2]$$

$$+E[\sum_{j=1}^{\frac{n}{2}-1}\sum_{i=1}^{n-1}\sum_{j'\neq j,j+1}\sum_{i'} D_N(t_i^1 - t_j^2)D_N(t_{i'}^1 - t_{j'}^2)\varepsilon^2_{J_j^2}\varepsilon^2_{J_{j'}^2}\delta_{I_i^1}(p^1)\delta_{I_{i'}^1}(p^1)].$$

Note that

$$E[(\delta_{I_i^1}(p^1))^2] = E[\int_{t_i^1}^{t_{i+1}^1}\Sigma^{11}(t)dt],$$

$$E[(\varepsilon^2_{J_j^2})^2] = 2E[\eta^2_{t_j^2}] = 2\omega_{22},$$

$$E[\delta_{I_i^1}(p^1)\delta_{I_{i'}^1}(p^1)] = E[\delta_{I_i^1}(p^1)]E[\delta_{I_{i'}^1}(p^1)] = 0, \quad \text{if } i \neq i'$$

$$E[\varepsilon^2_{J_j^2}\varepsilon^2_{J_{j'}^2}] = -\omega_{22} \ \text{ if } |j - j'| = 1\ .$$

Therefore (45) is equal to

$$2\omega_{22}\sum_{j=1}^{\frac{n}{2}-1}\sum_{i=1}^{n-1} D_N^2(t_i^1 - t_j^2)E[\int_{t_i^1}^{t_{i+1}^1}\Sigma^{11}(t)dt]$$

$$-2\omega_{22}\sum_{j=1}^{\frac{n}{2}-2}\sum_{i=1}^{n-1} D_N(t_i^1 - t_j^2)D_N(t_i^1 - t_{j+1}^2)E[\int_{t_i^1}^{t_{i+1}^1}\Sigma^{11}(t)dt].$$

Note that

$$|D_N(t_i^1 - t_j^2) - D_N(t_i^1 - t_{j+1}^2)| \leq C\rho(n)N,$$

for a constant C; therefore we conclude the computation of (45) as

$$2\omega_{22}\ \sum_{i=1}^{n-1} D_N^2(t_i^1 - t_{\frac{n}{2}-1}^2)E[\int_{t_i^1}^{t_{i+1}^1}\Sigma^{11}(t)dt] + o(1).$$

Similarly (46) splits as:

$$\sum_{j=1}^{\frac{n}{2}-1}\sum_{i=1}^{n-1} D_N^2(t_i^1 - t_j^2)E[(\delta_{J_j^2}(p^2))^2(\varepsilon_{I_i^1}^1)^2]$$

$$+2\sum_{j=1}^{\frac{n}{2}-1}\sum_{i=1}^{n-1} D_N(t_i^1 - t_j^2)D_N(t_{i+1}^1 - t_j^2)E[(\delta_{J_j^2}(p^2))^2\varepsilon_{I_i^1}^1\varepsilon_{I_{i+1}^1}^1]$$

$$+\sum_{j=1}^{\frac{n}{2}-1}\sum_{i=1}^{n-1}\sum_{i'\neq i,i+1} D_N(t_i^1 - t_j^2)D_N(t_{i'}^1 - t_j^2)E[\varepsilon_{I_i^1}^1\varepsilon_{I_{i'}^1}^1(\delta_{J_j^2}(p^2))^2]$$

$$+\sum_{j=1}^{\frac{n}{2}-1}\sum_{i=1}^{n-1}\sum_{j'\neq j}\sum_{i'=1}^{n-1} D_N(t_i^1 - t_j^2)D_N(t_{i'}^1 - t_{j'}^2)E[\varepsilon_{I_i^1}^1\varepsilon_{I_{i'}^1}^1\delta_{J_j^2}(p^2)\delta_{J_{j'}^2}(p^2)].$$

Using again the independence of noise and prices, this is equal to

$$2\omega_{11}\sum_{j=1}^{\frac{n}{2}-1}\sum_{i=1}^{n-1} D_N^2(t_i^1 - t_j^2)E[(\delta_{J_j^2}(p^2))^2] - 2\omega_{11}\sum_{j=1}^{\frac{n}{2}-1}\sum_{i=1}^{n-2} D_N(t_i^1 - t_j^2)D_N(t_{i+1}^1 - t_j^2)E[(\delta_{J_j^2}(p^2))^2].$$

As $|D_N^2(t_i^1 - t_j^2) - D_N(t_{i+1}^1 - t_j^2)| \leq C\rho(n)N$, for a constant C, we conclude the computation of (46) as

$$2\omega_{11}\sum_{j=1}^{\frac{n}{2}-1} D_N^2(t_{n-1}^1 - t_j^2)E[\int_{t_j^2}^{t_{j+1}^2}\Sigma^{22}(t)dt] + o(1).$$

Consider now (47):

$$E[(\sum_{j=1}^{\frac{n}{2}-1}\sum_{i=1}^{n-1} D_N(t_i^1 - t_j^2)\varepsilon_{I_i^1}^1\varepsilon_{J_j^2}^2)^2] = E[\sum_{j=1}^{\frac{n}{2}-1}\left(\sum_{i=1}^{n-1} D_N(t_i^1 - t_j^2)\varepsilon_{I_i^1}^1\right)^2(\varepsilon_{J_j^2}^2)^2]$$

$$+2E[\sum_{j=1}^{\frac{n}{2}-1}\sum_{j'>j}\varepsilon_{J_j^2}^2\varepsilon_{J_{j'}^2}^2\left(\sum_{i=1}^{n-1} D_N(t_i^1 - t_j^2)\varepsilon_{I_i^1}^1\right)\left(\sum_{i'=1}^{n-1} D_N(t_{i'}^1 - t_{j'}^2)\varepsilon_{I_{i'}^1}^1\right)]$$

$$= E[\sum_{j=1}^{\frac{n}{2}-1}(\varepsilon_{J_j^2}^2)^2\left(\sum_{i=1}^{n-1} D_N^2(t_i^1 - t_j^2)(\varepsilon_{I_i^1}^1)^2 + 2\sum_{i=1}^{n-1}\sum_{i'>i} D_N(t_i^1 - t_j^2)D_N(t_{i'}^1 - t_j^2)\varepsilon_{I_i^1}^1\varepsilon_{I_{i'}^1}^1\right)]$$

$$+2E[\sum_{j=1}^{\frac{n}{2}-1}\sum_{j'>j}\varepsilon^2_{J^2_j}\varepsilon^2_{J^2_{j'}}\left(\sum_{i=1}^{n-1}D_N(t^1_i-t^2_j)D_N(t^1_i-t^2_{j'})(\varepsilon^1_{I^1_i})^2+2\sum_{i=1}^{n-1}\sum_{i'>i}D_N(t^1_i-t^2_j)D_N(t^1_{i'}-t^2_{j'})\varepsilon^1_{I^1_i}\varepsilon^1_{I^1_{i'}}\right)]$$

$$=\sum_{j=1}^{\frac{n}{2}-1}E[(\varepsilon^2_{J^2_j})^2]\sum_{i=1}^{n-1}D^2_N(t^1_i-t^2_j)E[(\varepsilon^1_{I^1_i})^2]+2\sum_{j=1}^{\frac{n}{2}-1}E[(\varepsilon^2_{J^2_j})^2]\sum_{i=1}^{n-2}D_N(t^1_i-t^2_j)D_N(t^1_{i+1}-t^2_j)E[\varepsilon^1_{I^1_i}\varepsilon^1_{I^1_{i+1}}]$$

$$+2\sum_{j=1}^{\frac{n}{2}-2}E[\varepsilon^2_{J^2_j}\varepsilon^2_{J^2_{j+1}}]\left(\sum_{i=1}^{n-1}D_N(t^1_i-t^2_j)D_N(t^1_i-t^2_{j+1})E[(\varepsilon^1_{I^1_i})^2]+\right.$$

$$\left.+\,2\sum_{i=1}^{n-2}D_N(t^1_i-t^2_j)D_N(t^1_{i+1}-t^2_{j+1})E[\varepsilon^1_{I^1_i}\varepsilon^1_{I^1_{i+1}}]\right)$$

$$=4\omega_{22}\omega_{11}\sum_{j=1}^{\frac{n}{2}-1}\sum_{i=1}^{n-1}D^2_N(t^1_i-t^2_j)-4\omega_{22}\omega_{11}\sum_{j=1}^{\frac{n}{2}-1}\sum_{i=1}^{n-2}D_N(t^1_i-t^2_j)D_N(t^1_{i+1}-t^2_j)$$

$$-4\omega_{22}\omega_{11}\sum_{j=1}^{\frac{n}{2}-2}\sum_{i=1}^{n-1}D_N(t^1_i-t^2_j)D_N(t^1_i-t^2_{j+1})+4\omega_{22}\omega_{11}\sum_{j=1}^{\frac{n}{2}-2}\sum_{i=1}^{n-2}D_N(t^1_i-t^2_j)D_N(t^1_{i+1}-t^2_{j+1})$$

$$=4\omega_{22}\omega_{11}\left(\sum_{i=1}^{n-1}D^2_N(t^1_i-t^2_{\frac{n}{2}-1})-\sum_{i=1}^{n-2}D^2_N(t^1_i-t^2_{\frac{n}{2}-1})\right)+o(1)=4\omega_{22}\omega_{11}D^2_N(t^1_{n-1}-t^2_{\frac{n}{2}-1})+o(1).$$

Finally term (44) is zero due to non-synchronicity and the independence between noise and prices. •

Proof of Theorem (6.1)

The computation is the same as in Theorem 4.1 except for (27), (28), (29), (31), (32) and (33). In this case we have:

$$E[\widehat{\Sigma}^{12}_{N,n}-\int_0^{2\pi}\Sigma^{12}(t)dt]$$

$$=E[\sum_{j=1}^{\frac{n}{2}-1}\sum_{i=2(j-1)+1}^{2(j-1)+3}D_N(t^1_i-t^2_j)\delta_{I^1_i}(p^1)\delta_{J^2_j}(p^2)-\int_0^{2\pi}\Sigma^{12}(t)dt] \tag{51}$$

$$+E[\sum_{j=1}^{\frac{n}{2}-1}(\sum_{i=1}^{2(j-1)}D_N(t^1_i-t^2_j)\delta_{I^1_i}(p^1)\delta_{J^2_j}(p^2)+\sum_{i=2(j-1)+4}^{n-1}D_N(t^1_i-t^2_j)\delta_{I^1_i}(p^1)\delta_{J^2_j}(p^2)\ )] \tag{52}$$

$$+E[\sum_{j=1}^{\frac{n}{2}-1}\sum_{i=2(j-1)+1}^{2(j-1)+3}D_N(t^1_i-t^2_j)\varepsilon^1_{I^1_i}\delta_{J^2_j}(p^2)] \tag{53}$$

$$+E[\sum_{j=1}^{\frac{n}{2}-1}(\sum_{i=1}^{2(j-1)}D_N(t^1_i-t^2_j)\varepsilon^1_{I^1_i}\delta_{J^2_j}(p^2)+\sum_{i=2(j-1)+4}^{n-1}D_N(t^1_i-t^2_j)\varepsilon^1_{I^1_i}\delta_{J^2_j}(p^2)\ )] \tag{54}$$

$$+E[\sum_{j=1}^{\frac{n}{2}-1}\sum_{i=2(j-1)+1}^{2(j-1)+3}D_N(t^1_i-t^2_j)\delta_{I^1_i}(p^1)\varepsilon^2_{J^2_j}] \tag{55}$$

$$+E[\sum_{j=1}^{\frac{n}{2}-1}(\sum_{i=1}^{2(j-1)} D_N(t_i^1-t_j^2)\delta_{I_i^1}(p^1)\varepsilon^2_{J_j^2}+\sum_{i=2(j-1)+4}^{n-1} D_N(t_i^1-t_j^2)\delta_{I_i^1}(p^1)\varepsilon^2_{J_j^2})] \tag{56}$$

$$+E[\sum_{j=1}^{\frac{n}{2}-1}\sum_{i=2(j-1)+1}^{2(j-1)+3} D_N(t_i^1-t_j^2)\varepsilon^1_{I_i^1}\varepsilon^2_{J_j^2}] \tag{57}$$

$$+E[\sum_{j=1}^{\frac{n}{2}-1}(\sum_{i=1}^{2(j-1)} D_N(t_i^1-t_j^2)\varepsilon^1_{I_i^1}\varepsilon^2_{J_j^2}+\sum_{i=2(j-1)+4}^{n-1} D_N(t_i^1-t_j^2)\varepsilon^1_{I_i^1}\varepsilon^2_{J_j^2})]. \tag{58}$$

We have that (51) is equal to (18). Moreover (52) is zero as the returns of the efficient price are taken over disjoint intervals. Consider now (53) and (54). Then (55) and (56) are analogous. We can rewrite the sum of (53) and (54) as follows

$$E[\sum_{j=1}^{\frac{n}{2}-1}\delta_{J_j^2}(p^2)\{\sum_{i=2(j-1)+1}^{2(j-1)+3} D_N(t_i^1-t_j^2)\varepsilon^1_{I_i^1}+\sum_{h=0}^{b-1} D_N(t^1_{2(j-1)-h}-t_j^2)\varepsilon^1_{I^1_{2(j-1)-h}} \tag{59}$$

$$+\sum_{h=1}^{b} D_N(t^1_{2(j-1)+3+h}-t_j^2)\varepsilon^1_{I^1_{2(j-1)+3+h}}\}]$$

$$+E[\sum_{j=1}^{\frac{n}{2}-1}\delta_{J_j^2}(p^2)\{\sum_{i=1}^{2(j-1)-b} D_N(t_i^1-t_j^2)\varepsilon^1_{I_i^1}+\sum_{i=2(j-1)+4+b}^{n} D_N(t_i^1-t_j^2)\varepsilon^1_{I_i^1}\}]. \tag{60}$$

Note that (60) is zero, because the closest points in time have distance $|t^1_{2(j-1)-b+1}-t^2_j|$ and $|t^1_{2(j-1)+4+b}-t^2_{j+1}|$, which are both greater than θ_0, so that we can use assumption **(MD1)** and the dependence vanishes.

Regarding (59), we remark that for any fixed j:

$$E[\delta_{J_j^2}(p^2)\{\sum_{i=2(j-1)+1}^{2(j-1)+3} D_N(t_i^1-t_j^2)\varepsilon^1_{I_i^1}+\sum_{h=0}^{b-1} D_N(t^1_{2(j-1)-h}-t_j^2)\varepsilon^1_{I^1_{2(j-1)-h}}$$

$$+\sum_{h=1}^{b} D_N(t^1_{2(j-1)+3+h}-t_j^2)\varepsilon^1_{I^1_{2(j-1)+3+h}}\}]$$

$$=E[\delta_{J_j^2}(p^2)\{\sum_{i=2(j-1)+1}^{2(j-1)+3}\varepsilon^1_{I_i^1}+\sum_{h=0}^{b-1}\varepsilon^1_{I^1_{2(j-1)-h}}+\sum_{h=1}^{b}\varepsilon^1_{I^1_{2(j-1)+3+h}}\}]+R^j_{n,N},$$

where

$$R^j_{n,N}:=E[\delta_{J_j^2}(p^2)(\sum_{i=2(j-1)+1}^{2(j-1)+3}\varepsilon^1_{I_i^1}(D_N(t_i^1-t_j^2)-1)$$

$$+\sum_{h=0}^{b-1}(D_N(t^1_{2(j-1)-h}-t_j^2)-1)\varepsilon^1_{I^1_{2(j-1)-h}}+\sum_{h=1}^{b}(D_N(t^1_{2(j-1)+3+h}-t_j^2)-1)\varepsilon^1_{I^1_{2(j-1)+3+h}})].$$

For any fixed j

$$E[\delta_{J_j^2}(p^2)\{\sum_{i=2(j-1)+1}^{2(j-1)+3}\varepsilon^1_{I_i^1}+\sum_{h=0}^{b-1}\varepsilon^1_{I^1_{2(j-1)-h}}+\sum_{h=1}^{b}\varepsilon^1_{I^1_{2(j-1)+3+h}}\}]=E[\delta_{J_j^2}(p^2)(-\eta^1_{t^1_{2(j-1)-b+1}}+\eta^1_{t^1_{2(j-1)+3+b+1}})]$$

which is zero, as $|t^1_{2(j-1)-b+1}-t^2_j|>\theta_0$, $|t^1_{2(j-1)+3+b+1}-t^2_{j+1}|>\theta_0$.

We observe that $R^j_{n,N}$ is $o(1)$. In fact we have:

$$|D_N(t^1_{2(j-1)-h}-t^2_j)-1|\leq|t^1_{2(j-1)-h}-t^2_j|N\leq(h\rho^1(n)+\rho^2(n))N\leq(b\rho^1(n)+\rho^2(n))N\to 0,$$

in fact if θ_0 is fixed then also b is finite and fixed. The estimation is uniform in j. Finally

$$|\sum_{j=1}^{\frac{n}{2}-1}R^j_{n,N}|\leq C\rho(n)N\sum_{j=1}^{\frac{n}{2}-1}E[|\delta_{J_j^2}(p^2)(-\eta^1_{t^1_{2(j-1)-b+1}}+\eta^1_{t^1_{2(j-1)+3+b+1}})]$$

$$\leq C_1\ \rho(n)N\sum_{j=1}^{\frac{n}{2}-1}E[(\delta_{J_j^2}(p^2))^2]^{\frac{1}{2}}E[(-\eta^1_{t^1_{2(j-1)-b+1}}+\eta^1_{t^1_{2(j-1)+3+b+1}})^2]^{\frac{1}{2}}$$

$$\leq C_1\ \rho(n)NE[\int_0^{2\pi}\Sigma^{22}(t)dt]^{\frac{1}{2}}(2\omega_{11})^{\frac{1}{2}},$$

where C_1 is a constant. •

	$\hat{\Sigma}^{12}_{N,n_1,n_2}$		$RC^{0.5min}_{1,2}$		$RC^{1min}_{1,2}$		$RC^{5min}_{1,2}$	
	MSE	bias	MSE	bias	MSE	bias	MSE	bias
Reg-NS	8.67e-4	-1.36e-2	2.90e-3	-4.37e-2	3.18e-3	-3.25e-2	1.17e-2	-8.75e-3
	(8.81e-4)	(-1.30e-2)						
Reg-S + Unc	7.19e-4	-7.01e-3	1.50e-3	-2.03e-2	2.08e-3	2.70e-3	1.14e-2	5.00e-3
Reg-NS + Unc	6.48e-4	-1.05e-2	2.36e-3	-3.65e-2	2.78e-3	-2.94e-2	9.98e-3	-2.13e-3
	(6.99e-4)	(-7.13e-3)						
Reg-NS + Cor	9.38e-4	-1.18e-2	3.17e-3	-4.41e-2	3.37e-3	-3.14e-2	1.11e-2	-5.38e-3
	(9.50e-4)	(-1.27e-2)						
Reg-NS + Dep	1.01e-3	-7.83e-3	3.52e-3	-4.09e-2	4.46e-3	-3.24e-2	1.56e-2	-3.14e-3
Poisson + Unc	1.66e-3	-1.81e-2	6.32e-3	-7.11e-2	3.33e-3	-3.26e-2	1.43e-2	-3.15e-3
Poisson + Cor	1.92e-3	-2.03e-2	6.43e-3	-7.06e-2	4.06e-3	-3.65e-2	1.40e-2	-5.51e-3
Poisson + Dep	1.79e-3	-1.93e-2	5.95e-3	-6.63e-2	4.32e-3	-3.43e-2	1.40e-2	-1.13e-2
	$RCLL^{0.5min}_{1,2}$		$RCLL^{1min}_{1,2}$		$RCLL^{5min}_{1,2}$		$AO_{1,2}$	
	MSE	bias	MSE	bias	MSE	bias	MSE	bias
Reg-NS	3.20e-3	-2.49e-3	6.51e-3	-3.68e-3	3.41e-2	1.16e-2	2.00e-4	-3.47e-4
Reg-S + Unc	3.25e-3	2.75e-3	6.42e-3	5.04e-3	3.12e-2	3.15e-4	2.64e-4	-1.10e-3
Reg-NS + Unc	2.89e-3	5.72e-4	5.84e-3	-1.94e-3	3.26e-2	-1.79e-3	2.69e-4	2.31e-4
Reg-NS + Cor	3.46e-3	-9.42e-4	6.37e-3	3.79e-3	3.58e-2	7.43e-3	3.41e-4	9.01e-5
Reg-NS + Dep	5.08e-3	1.58e-3	9.24e-3	-4.81e-4	4.30e-2	-2.53e-3	7.11e-3	7.86e-2
Poisson + Unc	3.79e-3	4.32e-3	7.60e-3	4.61e-3	4.07e-2	1.24e-3	6.21e-4	1.53e-3
Poisson + Cor	4.06e-3	-1.99e-4	7.91e-3	1.86e-3	4.20e-2	1.51e-3	3.66e-3	5.47e-2
Poisson + Dep	4.31e-3	-2.93e-3	8.24e-3	-5.16e-4	4.18e-2	-6.98e-3	8.01e-3	8.31e-2

Table 1: Comparison of integrated volatility estimators. The noise variance is 90% of the total variance for 1 second returns. $\rho_1 = 2$ sec, $\rho_2 = 4$ sec with a displacement of 0 seconds for Reg-S and 1 second for Reg-NS trading; $\lambda_1 = 2$ sec and $\lambda_2 = 4$ sec for Poisson trading.

	$\hat{\Sigma}^{12}_{N,n_1,n_2}$		$RC^{0.5min}_{1,2}$		$RC^{1min}_{1,2}$		$RC^{5min}_{1,2}$	
	MSE	bias	MSE	bias	MSE	bias	MSE	bias
Reg-S + Unc	1.72e-3	-8.98e-3	2.35e-3	-1.63e-3	3.92e-3	-1.70e-3	1.32e-2	-3.20e-3
Reg-NS + Unc	1.95e-3	-1.42e-2	3.28e-2	-1.74e-1	1.01e-2	-8.42e-2	1.14e-2	-1.55e-2
	(1.99e-3)	(-1.52e-2)						
Reg-NS + Cor	1.83e-3	-1.46e-2	3.02e-2	-1.67e-1	9.32e-3	-8.07e-2	1.20e-2	-1.90e-2
	(1.85e-3)	(-1.53e-2)						
Reg-NS + Dep	6.13e-3	-3.97e-3	8.72e-2	-2.04e-1	3.89e-2	-9.76e-2	3.29e-2	-1.55e-2
Poisson + Unc	4.62e-3	-3.48e-2	3.57e-2	-1.79e-1	1.29e-2	-9.39e-2	1.45e-2	-1.46e-2
Poisson + Cor	3.52e-3	-3.07e-2	2.69e-2	-1.55e-1	9.84e-3	-8.20e-2	1.20e-2	-2.23e-2
Poisson + Dep	6.43e-3	-1.97e-2	4.67e-2	-1.54e-1	2.19e-2	-8.06e-2	2.13e-2	-1.21e-2

	$RCLL^{0.5min}_{1,2}$		$RCLL^{1min}_{1,2}$		$RCLL^{5min}_{1,2}$		$AO_{1,2}$	
	MSE	bias	MSE	bias	MSE	bias	MSE	bias
Reg-S + Unc	5.17e-3	-5.50e-3	8.74e-3	-3.99e-3	3.59e-2	1.20e-2	2.16e-3	-3.10e-3
Reg-NS + Unc	4.13e-3	1.72e-3	7.09e-3	1.64e-3	3.38e-2	5.25e-3	1.94e-3	-1.44e-3
Reg-NS + Cor	4.14e-3	4.93e-4	8.04e-3	-5.65e-4	3.51e-2	3.18e-3	1.82e-3	-3.25e-3
Reg-NS + Dep	3.41e-2	-2.05e-3	2.89-2	-8.53e-3	6.45e-2	-1.44e-2	7.71e-2	7.92e-2
Poisson + Unc	5.27e-3	-6.90e-3	9.46e-3	-5.72e-3	4.43e-2	7.82e-3	2.70e-3	1.12e-3
Poisson + Cor	4.08e-3	-9.31e-3	7.12e-3	-6.25e-3	3.36e-2	-3.99e-3	1.11e-2	9.35e-2
Poisson + Dep	1.90e-2	-1.84e-3	1.84e-2	-8.70e-3	4.57e-2	9.31e-3	3.56e-2	6.56e-2

Table 2: Comparison of integrated volatility estimators. The noise is ten times the one in Table 1. $\rho_1 = 5$ sec, $\rho_2 = 10$ sec with a displacement of 0 seconds for Reg-S and 2 seconds for Reg-NS trading; $\lambda_1 = 5$ sec and $\lambda_2 = 10$ sec for Poisson trading.

	$\hat{\Sigma}^{12}_{N,n_1,n_2}$		$RC^{0.5min}_{1,2}$		$RC^{1min}_{1,2}$		$RC^{5min}_{1,2}$	
	MSE	bias	MSE	bias	MSE	bias	MSE	bias
Reg-S + Unc	5.60e-3	-5.77e-3	4.97e-2	-7.22e-3	3.09e-2	-3.99e-3	2.65e-2	-1.34e-2
Reg-NS + Unc	4.25e-3	-8.06e-3	7.01e-2	-1.63e-1	3.05e-2	-8.55e-2	2.47e-2	-1.79e-2
Reg-NS + Cor	5.21e-3	-3.40e-3	9.16e-2	-2.05e-1	4.20e-2	-9.16e-2	2.82e-2	-2.03e-2
Reg-NS + Dep	1.58e-2	6.68e-3	1.88e+0	-4.52e-2	9.60e-1	3.87e-3	2.59e-1	-3.91e-2
Poisson + Unc	7.77e-3	-2.36e-2	7.95e-2	-1.68e-1	4.16e-2	-9.64e-2	2.70e-2	-2.16e-2
Poisson + Cor	9.14e-3	-3.38e-3	5.09e-2	-3.95e-2	4.23e-2	-2.07e-2	3.37e-2	3.09e-3
Poisson + Dep	1.36e-2	-2.56e-3	1.25e+0	-1.66e-1	6.93e-1	-4.43e-2	1.75e-1	-1.19e-2
	$RCLL^{0.5min}_{1,2}$		$RCLL^{1min}_{1,2}$		$RCLL^{5min}_{1,2}$		$AO_{1,2}$	
	MSE	bias	MSE	bias	MSE	bias	MSE	bias
Reg-S + Unc	4.23e-2	-8.06e-3	2.74e-2	-1.65e-2	5.47e-2	-2.75e-2	1.16e-1	-2.39e-2
Reg-NS + Unc	3.02e-2	1.99e-3	2.67e-2	4.86e-3	4.28e-2	-2.99e-3	7.32e-2	-2.80e-2
Reg-NS + Cor	3.91e-2	8.87e-3	3.45e-2	1.11e-3	5.50e-2	2.16e-2	9.97e-2	-1.12e-2
Reg-NS + Dep	1.37e+0	-6.30e-3	6.39-1	2.58e-2	1.83e-1	2.22e-2	3.66e+0	1.84e-1
Poisson + Unc	4.11e-2	-3.19e-2	2.85e-2	-1.04e-2	4.46e-2	-4.31e-3	7.79e-2	8.41e-3
Poisson + Cor	4.35e-2	-6.34e-3	3.14e-2	5.57e-3	6.81e-2	6.95e-3	1.62e+0	1.22e+0
Poisson + Dep	8.53e-1	3.32e-2	4.48e-1	3.83e-2	1.39e-1	-2.93e-2	2.02e+0	5.10e-2

Table 3: Comparison of integrated volatility estimators. Noise ratio $\gamma \simeq 7$. $\rho_1 = 5$ sec, $\rho_2 = 10$ sec with a displacement of 0 seconds for Reg-S and 2 seconds for Reg-NS trading; $\lambda_1 = 5$ sec and $\lambda_2 = 10$ sec for Poisson trading.