4. EMPIRICAL STUDY ON THE PERFORMANCES OF BLACK-SCHOLES MODEL FOR EVALUATING EUROPEAN OPTIONS

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Abstract

In this study we aim at analyzing the way the model Black-Scholes works in practice. The data used for analysis refer to European-type call options having as support-assets the CAC-40 money-market index. Our approach will be structured in two parts. The first will be dedicated to an estimate of daily implicit volatilities, which is of those values of volatilities which, once applied in the Black-Scholes evaluation formula, minimize the sum of square errors given by the model. Once this problem is solved, we will analyze the relationship existing between implicit volatility moneyness and due term of options, that is the so-called volatility smile. The second part of the study will have as core the analysis of errors provided by the Black-Scholes model, which will be studied, given moneyness and due-term of options.

Key-words: options; Black-Scholes model; implicit volatility; moneyness; due term; volatility smile; volatility smirk; evaluation error

JEL Classification: G12

I. Implicit Volatility. Volatility Smile and the Limits of the Black-Scholes Model

In the beginning we point that, according to the Black-Scholes model, the bonus **C** of an European-type call option having as support-assets a financial instrument that generates an annual income at a continuous rate **q** is thus determined:

$$C = Se^{-q\tau}N(d_1) - Ke^{-r\tau}N(d_2) \quad (1)$$

where: **S** = the market price of the support-assets at the moment of evaluating the option

τ = duration up to the option due-term

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K = exercising price

r = rate of interest without risk, annual and continuous

N(d$_1$), **N(d$_2$)** = values of the function of repartition of normal standard law for **d$_1$** and **d$_2$** parameters

d$_1$ şi **d$_2$** parameters are calculated by using the formulas:

$$d_1 = \frac{\ln\frac{S}{K} + \left(r - q + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}} \quad (2)$$

$$d_2 = \frac{\ln\frac{S}{K} + \left(r - q - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}} = d_1 - \sigma\sqrt{\tau} \quad (3)$$

where: σ represents the option support-assets volatility. The main hypotheses of Black-Scholes model are:

a) Trading of short sales is authorised on the market.
b) There are no trading costs or other taxes and the traded securities are perfectly dividable.
c) Support-asset of the option is a share that does not generate dividends.
d) There are no arbitrage opportunities on the market.
e) Trading of securities is a continuous process.
f) Non-risk profitability and the securities volatility are considered to be constant.

One of the main problems we are dealing with in evaluating options is represented by **the selection of the volatility that is to be applied in the Black-Scholes model**. If the exercising price, the option due-term, the market price of the support-assets and rate of interest represent publicly available information, the volatility of the support is the single parameter of the Black-Scholes model which is not directly noticeable on the market, its estimating being necessary. Thus the so-called implicit volatility is obtained.

In the present study we determined the daily implicit volatilities by minimizing the sums of the error squares given by the Black-Scholes model:

$$\sigma_{imp.}^{(t)} = \arg\min \sum_{i=1}^{N_t} \left[C_{BS}^{(i)}\left(\sigma_{imp.}\right) - C_{OBS.}^{(i)}\right]^2 \quad (4)$$

where: $\sigma_{imp.}^{(t)}$ = the implicit volatility for **t** day;

N_t = number of options dealt in **t** day;

$C_{BS}^{(i)}\left(\sigma_{imp.}\right)$ = the theoretic Black-Scholes price of **i** option (calculated for implicit volatility $\sigma_{imp.}$);

$C_{OBS.}^{(i)}$ = market price for **i** option

Of course, the minimizing of function (4) cannot be achieved in a classical way, as in its composition $N(d_1)$ and $N(d_2)$ appear, having the expressions:

$$N(d_i) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_i} e^{-\frac{x^2}{2}} dx, \qquad i = \overline{1,2} \tag{5}$$

Thus, it is necessary to use a numerical procedure for estimating implicit volatilities. One of the most used methods is the Newton-Raphson method, which we will be briefly presenting further on. We do not know the option implicit volatility, yet we may try to guess a value close to reality. Be that value σ_0. Based on it, we calculate:

$$\sigma_1 = \sigma_0 - \frac{C_{BS}(\sigma_0)}{\frac{\partial C_{BS}}{\partial \sigma}(\sigma_0)} = \sigma_0 - \frac{C_{BS}(\sigma_0)}{\vartheta_{call}} = \sigma_0 - \frac{C_{BS}(\sigma_0)}{S\sqrt{\tau}N'(d_1)} \tag{6}$$

where we took into account that the differential for the function of the option price given support-assets volatility (risk) is the sensitivity index **vega** which shows the reaction of option price to the modification of support-assets volatility, having the expression:

$$\vartheta_{call} = S\sqrt{\tau}N'(d_1) \tag{7}$$

As a result of this calculation σ_1 is determined. We repeat the process and obtain:

$$\sigma_2 = \sigma_1 - \frac{C_{BS}(\sigma_1)}{\frac{\partial C_{BS}}{\partial \sigma}(\sigma_1)} = \sigma_1 - \frac{C_{BS}(\sigma_1)}{S\sqrt{\tau}N'(d_1)}$$

$$\sigma_3 = \sigma_2 - \frac{C_{BS}(\sigma_2)}{\frac{\partial C_{BS}}{\partial \sigma}(\sigma_2)} = \sigma_2 - \frac{C_{BS}(\sigma_2)}{S\sqrt{\tau}N'(d_1)}$$

$$\sigma_n = \sigma_{n-1} - \frac{C_{BS}(\sigma_{n-1})}{\frac{\partial C_{BS}}{\partial \sigma}(\sigma_{n-1})} = \sigma_{n-1} - \frac{C_{BS}(\sigma_{n-1})}{S\sqrt{\tau}N'(d_1)}$$

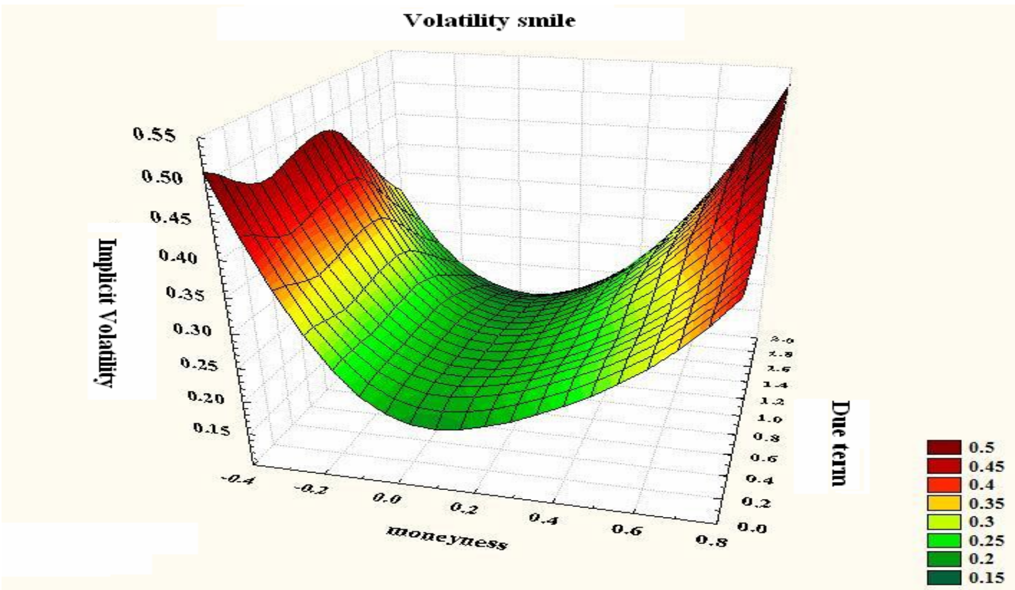
We will stop at said **n** for which the convergence criteria are observed:

$$\left|\sigma_n - \sigma_{n-1}\right| < \varepsilon \tag{8}$$

where **ε** is a fixed real positive number. Given different values which **ε** may take, Newton-Raphson algorithm may converge more rapidly, yet providing a smaller accuracy of the final result, (for a large **ε**) or may produce results close to reality, yet with the deficiency of a longer period of execution (in case of a small **ε**).

Based on this algorithm, we estimated the implicit volatilities for the Black-Scholes model. At the same time, we mention that we eliminated from analysis the “aberrant” values, namely the options with a bonus smaller than 10 m.u. and those having a due term shorter than 10 days or longer than 100 days.

It is interesting to analyze the evolution of the option support-assets implicit volatility, given its moneyness. An entire series of specialty studies mention the existence of an **implicit volatility smile**, phenomenon which manifests as a consequence of the fact that the volatilities used by operators in evaluating options differ, given the ratio of the support-assets market price to the exercise price. In order to illustrate this aspect, the way numerous specialty studies demonstrate, under the empirical (implicit) distribution of probability of the support-assets price it is more likely that the out-of-the-money options be in-the-money at the due term, than under the log-normal distribution, considered a central hypothesis in the Black-Scholes model. This means that said options will be more expensive, which leads us to the conclusion that the implicit volatility will be higher[1]. Therefore, we may ascertain that the Black-Scholes model overestimates the out-of-the-money options. In a similar way we can reach the conclusion that the Black-Scholes model underestimates the in-the-money options. At the same time, in evaluating options there must be taken into account the due term thereof. The following chart (all the diagrams and graphs presented in this article are made with the "Statistica" program) shows the volatility **smile** (the relationship between the implicit volatility and moneyness of options) combined with the duration up to the due term of options on the CAC-40 money market index:



Source: Authors' computations, on the basis of data from http://www.euronext.com/landing/indexMarket-18812-FR.html .

[1] *We remind that the bonus of an option is an increasing function with the volatility of the support-assets (the* **vega** *of an option is always positive).*

It can be noticed that the volatility picture is not flat, it shapes a smile. The implicit volatility is minimum when an at-the-money option is evaluated, and it increases as the option becomes in-the-money or out-of-the-money; yet, there must be remembered that implicit volatility used to evaluate an out-of-the-money call is much higher than the one used to evaluate an in-the-money call. For put options, the situation is reverse. That is why we deal with an asymmetrical volatility smile which, in the art, is called **smirk**[2]. As about the dependency of volatility on the option due term, there must be said that implicit volatility tends to be a maturity option increasing function when recent volatilities registered reduced values, as the operators wait for an increase in volatilities. Symmetrically, the implicit volatility is a maturity decreasing function when recent volatilities were high (as a diminishing thereof is expected).

There are several possible explanations for the volatility smile. The most plausible of all refers to the financial lever effect. When the value of a company's own capital decreases, the lever effect increases, and the volatility of the company capital increases in its turn. When a company's own capital is increased, the lever thereof decreases, and the volatility of the capital decreases. Thus, the volatility used to evaluate an out-of-the-money call is higher that that used in case of an in-the-money call.

At the same time, Mark Rubinstein explained the volatility smile by means of the crashophobia[3] concept: a volatility smirk (namely an asymmetrical smile) did not exist until after the money market crash in October 1987. It had not been a smile of implicit volatility until then. The market operators take into account the possibility of another crash to appear and will consequently evaluate the options

As a conclusion, we demonstrated that the operators take into consideration the moneyness of an option and the duration up to the due term thereof, when they calculate the volatility on account of which they evaluate the option. This is a direct consequence of the fact the Black-Scholes model cannot be applied in its original form: the prices of the financial assets do not follow log-normal distribution laws. Therefore, the volatility smile appears as a phenomenon that makes the error produced by the Black-Scholes model be systematic, as we will see in the next section of the present study.

II. Estimation of Errors Produced by the Black-Scholes Model

Once the daily implicit volatilities are estimated, we aim to see to what extent the market prices of said options stray from the theoretical prices determined by introducing implicit volatility in the Black-Scholes formula. In this regard, we calculated the evaluation errors, using the following formula:

$$\varepsilon = C_{BS}\left(\sigma_{imp.}\right) - C_{obs.} \tag{9}$$

[2] *Smart smile. In the 80's, when everybody was convinced that the market is perfectly described by the Black-Scholes model (meaning that for evaluating options a flat volatility picture was used), few operators were aware of the fact that implicit volatility shapes a smile. Obviously, they made significant profits by means of strategies of the* deep out-of-the-money *option buying type.*

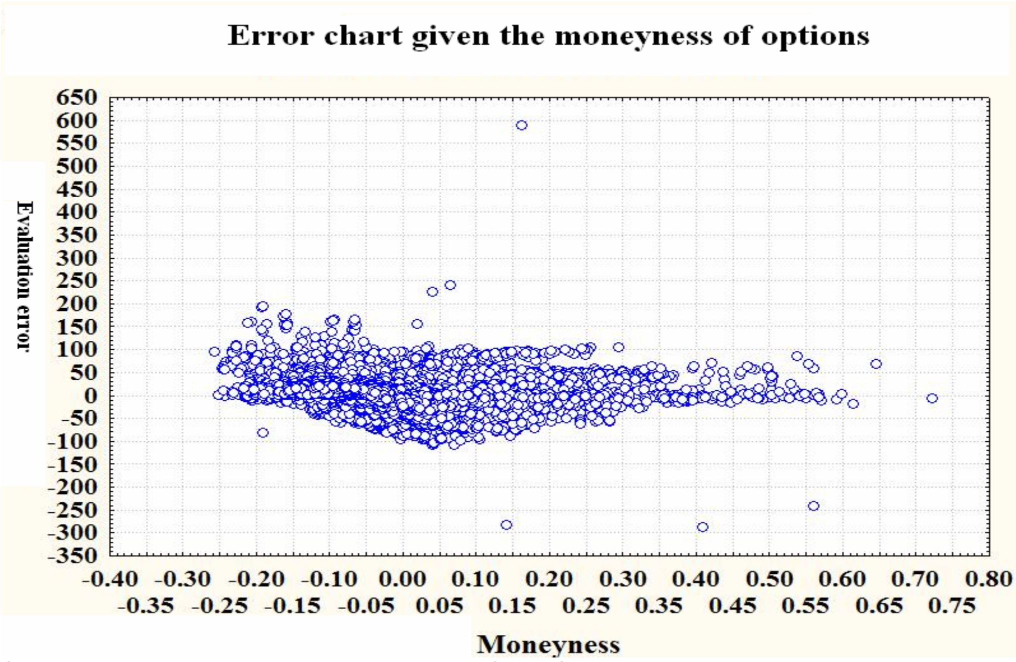
[3] ***Crashophobia.***

where: ε = evaluation error

$C_{BS}(\sigma_{imp.})$ = the theoretical Black-Scholes price of said option, calculated for the estimated implicit volatility ($\sigma_{imp.}$)

$C_{obs.}$ = the option bonus, namely the market price at which this is transacted.

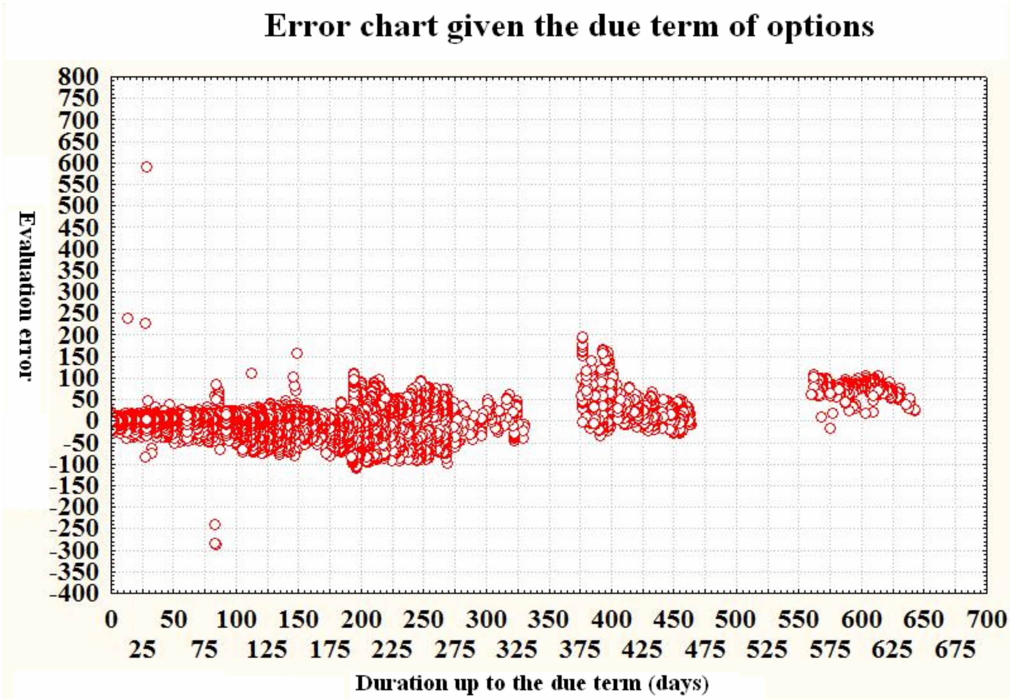
Further on, we will proceed with verifying the statement according to which the Black-Scholes model overestimates the out-of-the-money options and underestimates the in-the-money options. Below, we present the point-cloud type chart of the errors, given the call option moneyness on CAC-40 index:



Source: Authors' computations, on the basis of data from http://www.euronext.com/landing/indexMarket-18812-FR.html

Following the chart, it is clear that the Black-Scholes model massively overestimates the out-of-the-money options (for which the moneyness is negative), the errors having positive values. As the moneyness of the options increases, the evaluation errors caused by the model start to diminish. If on the at-the-money options we cannot accurately express our opinion, as regards the in-the-money ones there is a tendency to underestimate.

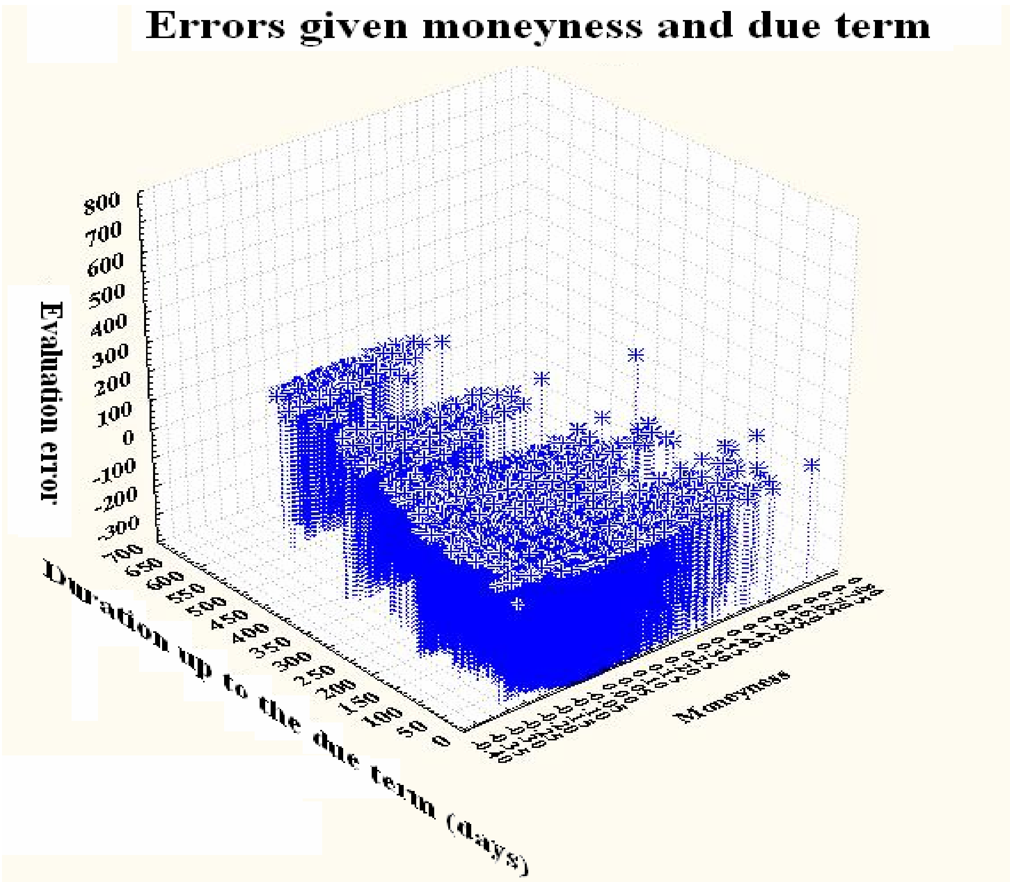
As shown in the volatility smile analysis, the time interval up to the due term of the option is an important variable, which the operators take into account when making the evaluation of options. This happens because, given the structure on terms of the implicit volatility, the Black-Scholes model will under- or overestimate options. Further on, we will present the chart of the evaluation errors, given the time duration expressed in days up to the due term of options (similar to the previous chart, wherein we disclosed the errors, given the moneyness):



Source: Authors' computations, on the basis of data from
http://www.euronext.com/landing/indexMarket-18812-FR.html

As we can notice, the short-term options (30-50 days) are mostly underestimated, as well as the medium term options (having a 50–100 days due term), although, in their case, overestimation is manifest in a quite pronounced manner. As about the long-term options, they are mostly overestimated; the options having a due term over a year are almost always overestimated. The explanations for these results must be also looked in the structure in terms of implicit volatility: it is normal that, for long or very long terms (of 1 or even 2 years), operators take into account an increased volatility when evaluating options, as they cannot accurately predict the market evolution. Therefore, the considered excess of volatility appears like a risk bonus. On a short and medium terms, the used volatility is strongly influenced by the trajectory of the implicit volatility up to the moment of evaluation (as shown before), both under- and overestimations of options being possible.

The following chart synthesizes the obtained results and represents errors in a three-dimensional manner, given moneyness and due terms.



Source: Authors' computations, on the basis of data from http://www.euronext.com/landing/indexMarket-18812-FR.html

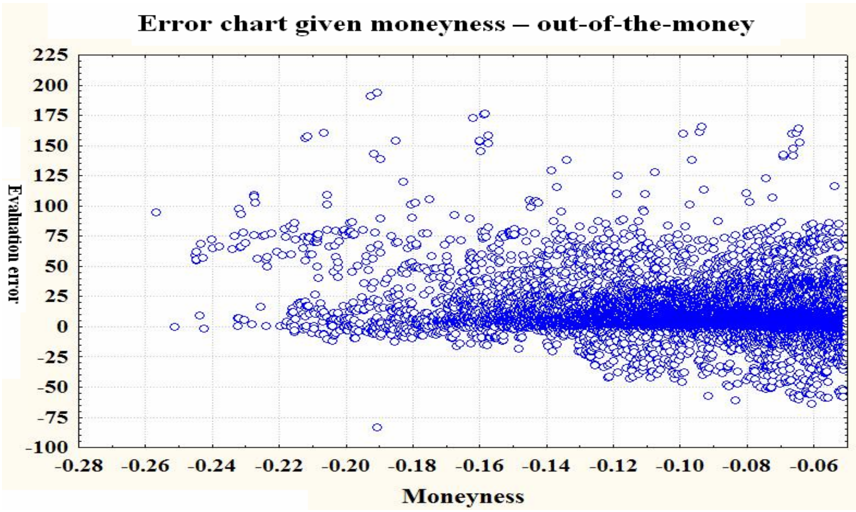
Further on we analyze in detail the performances of the Black-Scholes model. In this regard, we divided the call options on the CAC-40 money market index into three classes, namely:

- *out-of-the-money* options (with **moneyness less than -0.05**);
- *at-the-money* options (with **moneyness of between -0.05 and 0.05**);
- *in-the-money* options with **moneyness over 0.05**);

For each class of options, we will draft a chart of errors as shown by the Black-Scholes model, given moneyness and due terms.

I. Out-of-the-money Options

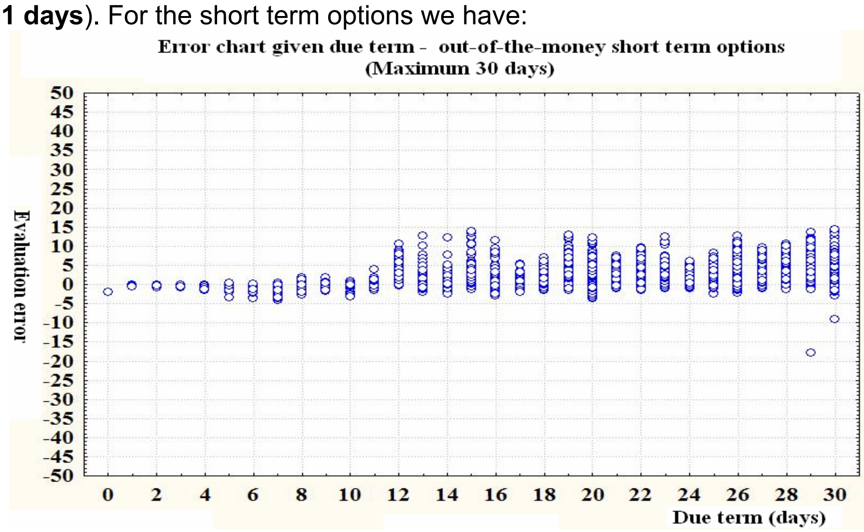
The errors chart, given moneyness, for the out-of-the-money options, is the following:



Source: Authors' computations, on the basis of data from http://www.euronext.com/landing/indexMarket-18812-FR.html

As we can notice, the Black-Scholes model overestimates most of the options having a moneyness less than -0.20 (these are *deep out-of-the-money* options), the tendency being maintained for the options with moneyness between -0.20 and -0.12. Once the moneyness of options increases, the overestimating tendency diminishes.

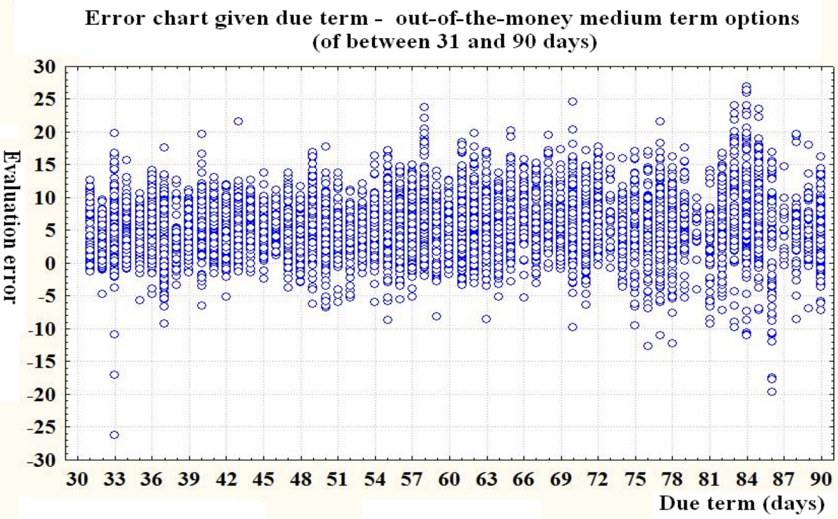
Before dealing with the error chart, given the due term of options, we will divide them into other three subgroups, namely: *out-of-the-money short-term* options, (with **due term between 0 and 30 days**), *out-of-the-money medium-term* options (with **due term between 31 and 90 days**) and *out-of-the-money long-term* options (with **due term over 91 days**). For the short term options we have:



Source: Authors' computations, on the basis of data from http://www.euronext.com/landing/indexMarket-18812-FR.html

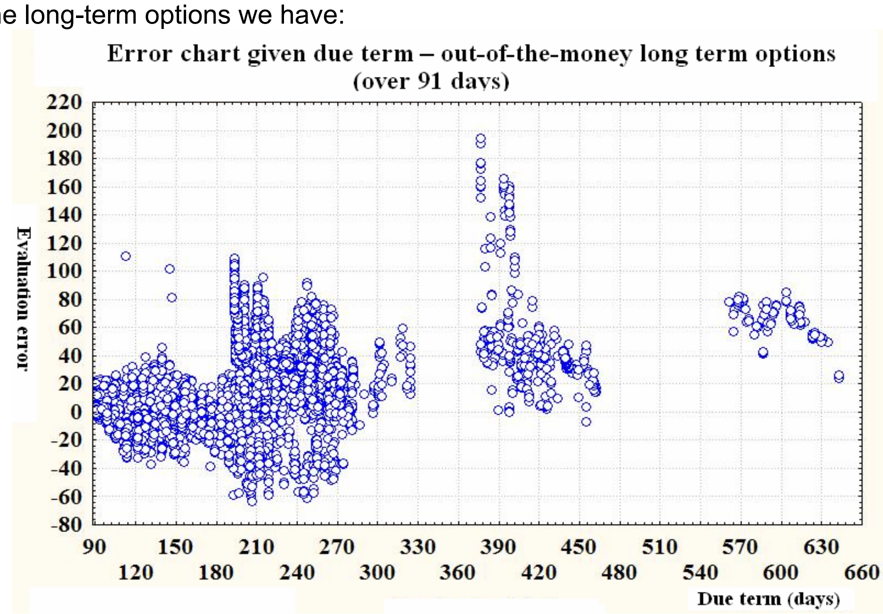
We should note that the Black-Scholes model overestimates most of the out-of-the-money options having a due term between 10 and 30 days. Meanwhile, we should note that it evaluates with quite an increased accuracy the very short-term options, (having a due term between 0 and 10 days).

The error chart for the out-of-the-money medium-term options is the following:



Source: Authors' computations, on the basis of data from http://www.euronext.com/landing/indexMarket-18812-FR.html

For the long-term options we have:

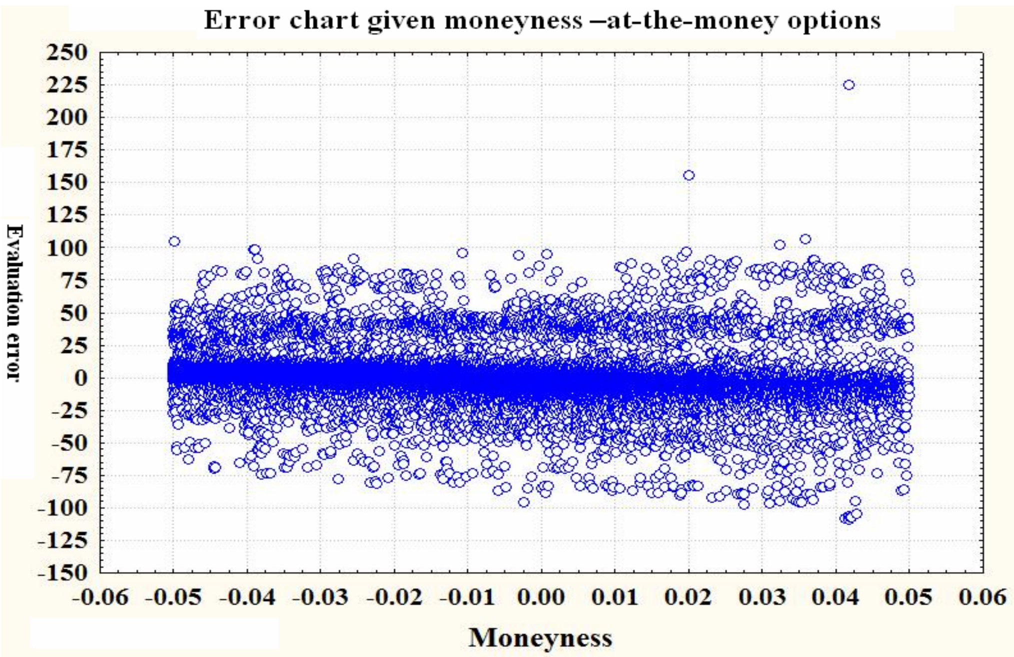


Source: Authors' computations, on the basis of data from http://www.euronext.com/landing/indexMarket-18812-FR.html

The first of the two previous charts emphasizes the fact that the Black-Scholes model overestimates a very large proportion of the out-of-the-money medium-term options. As about the long-term options, the model underestimates quite a large part of those having a due term up to one year. As fort the rest, the tendency of overestimating the out-of-the-money options is maintained.

II. At-the-money Options

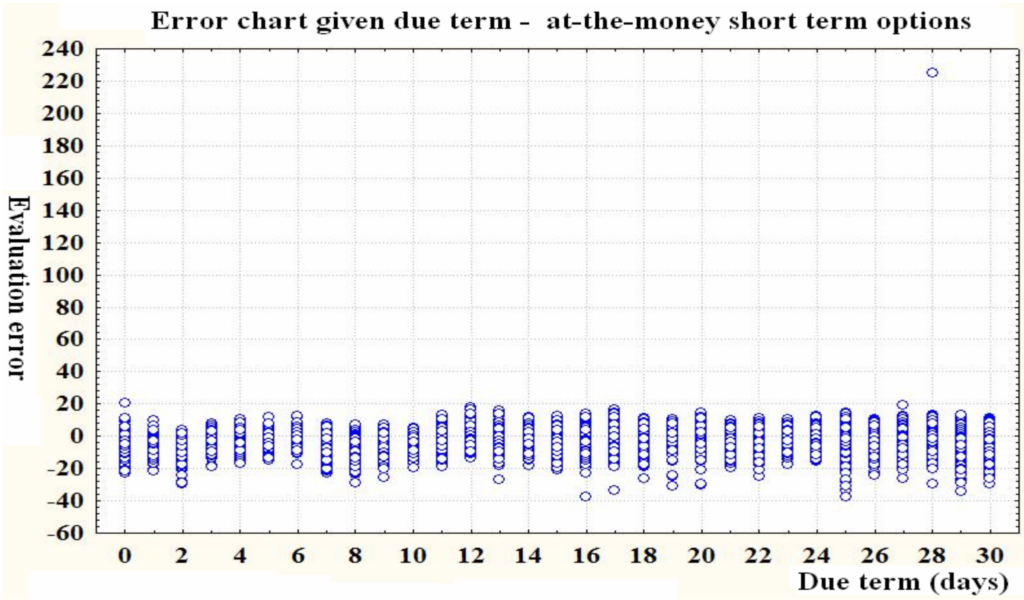
We noticed previously that we cannot precisely state which is the sense of the error given by the evaluation model, when we deal with at-the-money options. In this section, we will deal with these instruments and we will start by drafting the error chart, given moneyness:



Source: Authors' computations, on the basis of data from http://www.euronext.com/landing/indexMarket-18812-FR.html

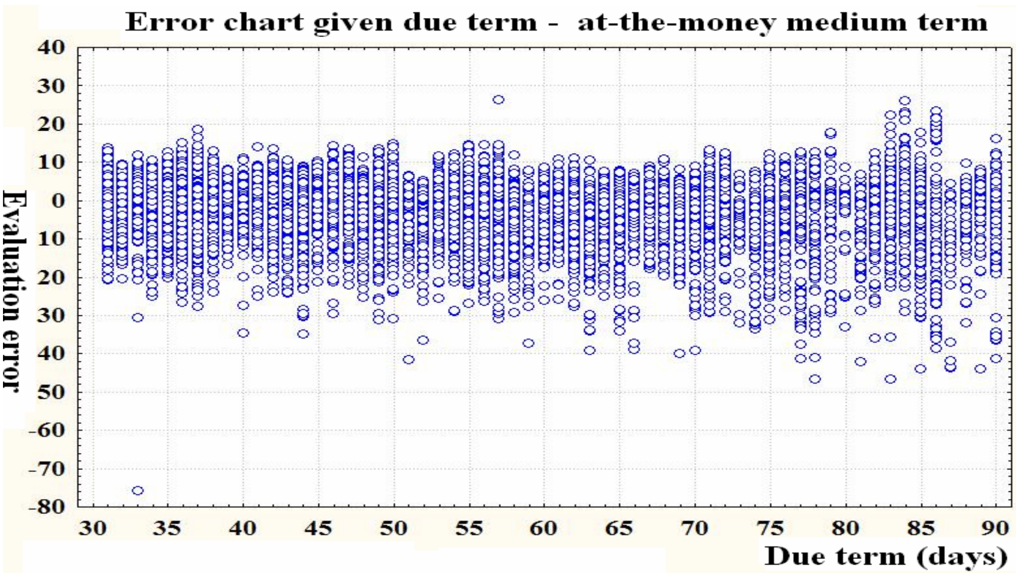
This chart confirms our initial intuition. The underestimated at-the-money options alternates with the overestimated ones, in some cases the evaluation errors being even very great. Yet it is to be mentioned that there is quite a large number of at-the-money options pertinently evaluated by the Black-Scholes model.

Further on, we will proceed as in the case of the out-of-the-money options and we will represent in a chart the errors, given the due term of options. For those on short term, we will get:



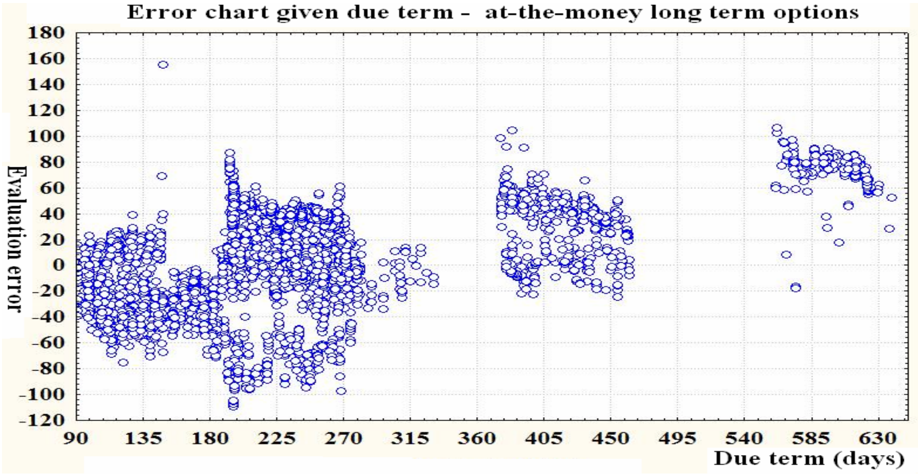
Source: Authors' computations, on the basis of data from
http://www.euronext.com/landing/indexMarket-18812-FR.html

As about the medium-term options, we get:



Source: Authors' computations, on the basis of data from
http://www.euronext.com/landing/indexMarket-18812-FR.html

At last, for the at-the-money long-term options, we get:



Source: Authors' computations, on the basis of data from
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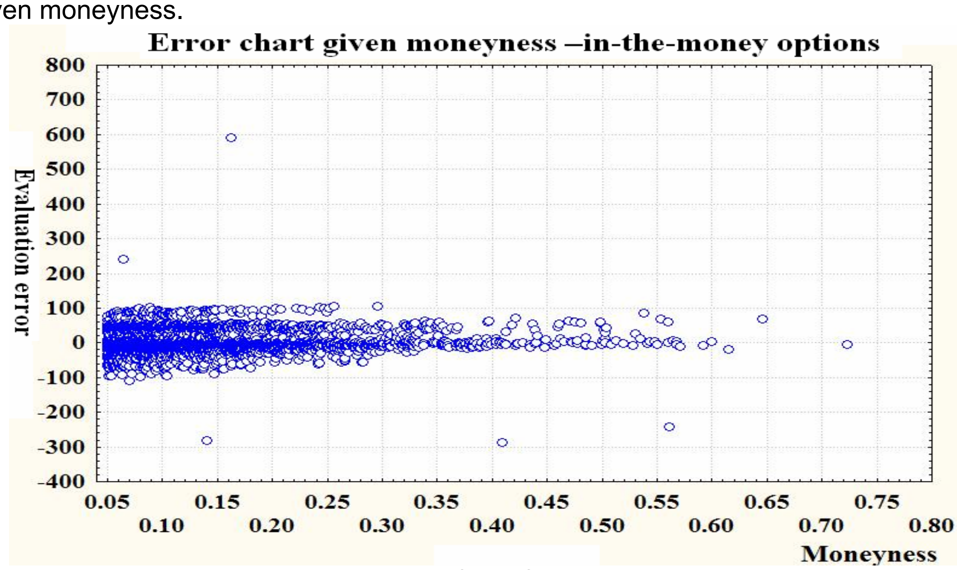
The three charts strengthen our belief that in case of the at-the-money options, we cannot accurately say in which sense the error is given by the Black-Scholes model.

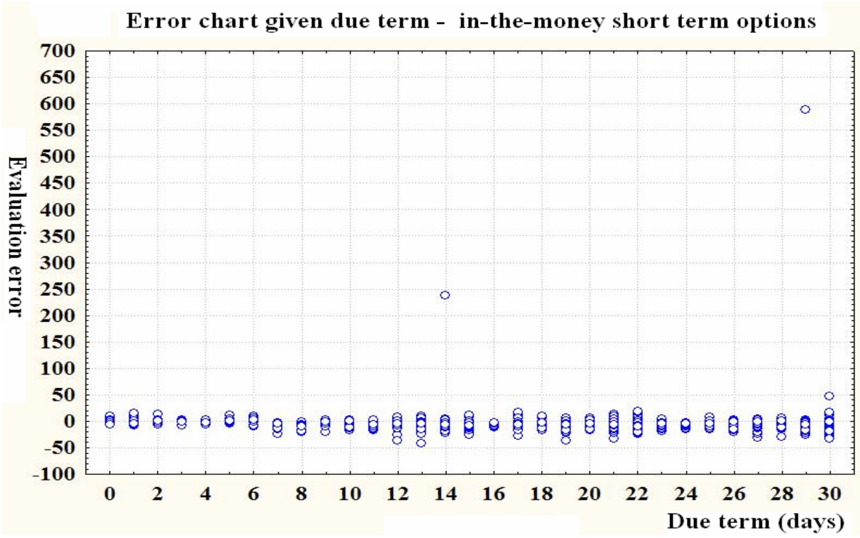
III. In-the-money Options

We noticed above that the Black-Scholes model underestimates the in-the-money options. In this section we will analyze in detail these evaluation errors, given, as before, moneyness and the due term of options. We present below the error chart given moneyness.



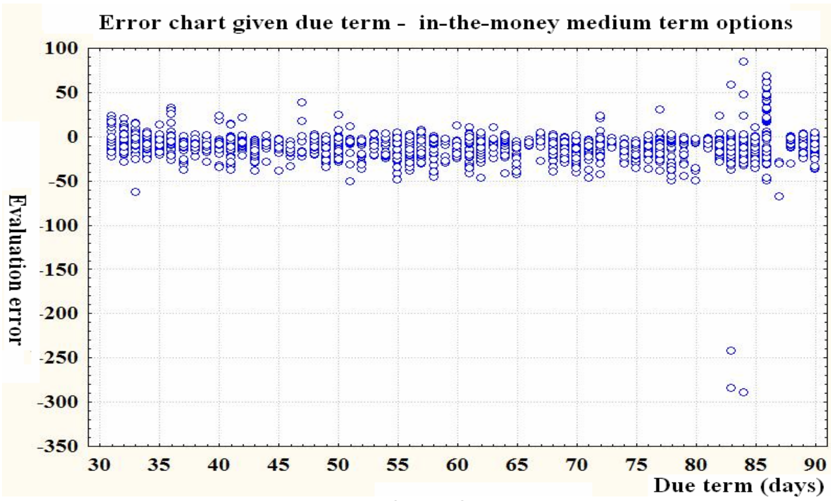
Source: Authors' computations, on the basis of data from
http://www.euronext.com/landing/indexMarket-18812-FR.html

We notice that the in-the-money options are massively underestimated when moneyness takes on values between 0.05 and 0.20. As moneyness increases, the options start to be overestimated (we remind you that, according to what we discussed in the volatility smile analysis, the implicit volatility used to evaluate a *deep in-the-money* call is very high and, consequently, the option will be overestimated). Let us further analyze the repartition of errors, given the due term of options. The Figure below presents the evaluation errors for the short-term options:



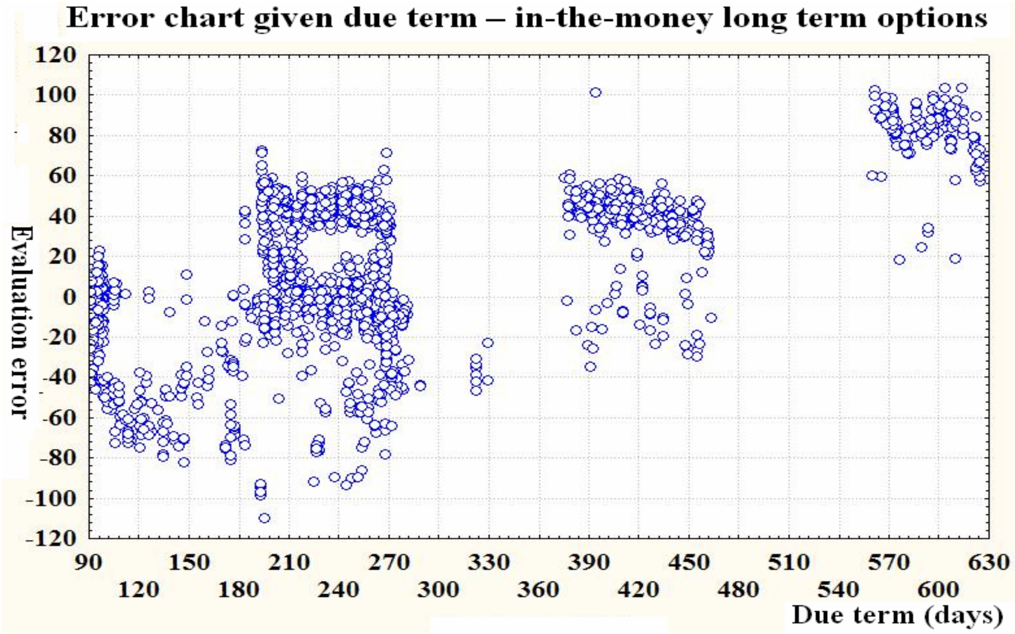
Source: Authors' computations, on the basis of data from
http://www.euronext.com/landing/indexMarket-18812-FR.html

For the medium term options, we get:



Source: Authors' computations, on the basis of data from
http://www.euronext.com/landing/indexMarket-18812-FR.html

We draw the chart for the in-the-money long-term options:



Source: Authors' computations, on the basis of data from
http://www.euronext.com/landing/indexMarket-18812-FR.html

The three charts above show that, generally, the Black-Scholes model underestimates the in-the-money options. This is obvious especially in the case of medium-term options; as about the long-term options, they are underestimated for due terms shorter than 210 days. For longer due terms, up to one year, the underestimated options alternate with the overestimated ones, while for due terms of over 500 days the overestimation of the in-the-money options is a rule. The reason was already discussed for the evaluation of very long term options, and the operators will take into account high levels of implicit volatility.

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Source of data: http://www.euronext.com/landing/indexMarket-18812-FR.html