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Resolution of SU(2) monopole singularities by oxidation

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ABSTRACT

We show how *colored* SU(2) BPS monopoles (that is: SU(2) monopoles satisfying the Bogomol'nyi equation whose Higgs field and magnetic charge vanish at infinity and which are singular at the origin) can be obtained from the BPST instanton by a singular dimensional reduction, explaining the origin of the singularity and implying that the singularity can be cured by the oxidation of the solution. We study the oxidation of other monopole solutions in this scheme.

1. Introduction: monopoles and instantons

It has been known for a long time that selfdual Yang–Mills (YM) instantons in 4-dimensional Euclidean space $\mathbb{E}^4$ and magnetic monopoles satisfying the Bogomol'nyi equation in $\mathbb{E}^3$ [1] [1] are related by dimensional reduction. In its simplest setting, this relation can be described as follows: if $\hat{A}_{\hat{\mu}}$ $(\hat{\mu}=0,1,2,3)$ [2] is the gauge potential of a selfdual YM instanton solution in $\mathbb{E}^4$ and is furthermore independent of one of the 4 Cartesian coordinates, z say, then the z-component $\hat{A}_z$ and the other three components $\hat{A}_m$ $(m=1,2,3)$ can be identified with the Higgs field $\Phi \equiv -\hat{A}_z$ and the gauge potential $A_m \equiv \hat{A}_m$ of a solution of the Yang–Mills–Higgs (YMH) system in the Prasad–Sommerfield limit satisfying the Bogomol'nyi equation:

$$\mathcal{D}_m \Phi = \tfrac{1}{2}\epsilon_{mnp} F_{np} \,. \tag{1.1}$$

The sign in the Bogomol'nyi equation depends on the orientation of the coordinates; we have taken the one corresponding to z to be x^0 and $\epsilon_{0123}=\epsilon_{123}=+1$.

The coordinate z has to be compactified for the instanton action to be finite [3]: $z \sim z+4\pi$. Thus, in practice, we are performing the dimensional reduction in $S^1\times\mathbb{E}^3$ and the z-independent solutions can be considered to be the Fourier zero modes of instanton solutions periodic in the direction z (the so-called *calorons*).

The paradigm of selfdual YM instanton in $\mathbb{E}^4$ is the BPST instanton [5], usually presented in Cartesian coordinates using the 't Hooft symbols. It belongs to a family of selfdual YM solutions depending on an arbitrary function K, harmonic on $\mathbb{E}^4$ (see *e.g.* Ref. [6] and the references therein). With K asymptotically constant and with a single point-like pole at the origin $K=1+4/(\lambda^2\rho^2)$, where $|\vec{x}_{(4)}|^2\equiv\rho^2$, the solution describes a single BPST instanton located at the origin. Replacing K by a harmonic function on $S^1\times\mathbb{E}^3$ with a single pole at the origin and asymptotically constant in $\mathbb{E}^3$, $K=1+(\sinh r/2)/[\lambda^2 r^2(\cosh r/2-\cos z/2)]$, where $r^2=|\vec{x}_{(3)}|^2$ and z is the fourth, compact, Euclidean coordinate, we get a caloron [7] whose Fourier zero mode gives, upon dimensional reduction, the spatial part of a Wu–Yang SU(2) magnetic monopole [8], which is singular at the origin.

Since the BPST instanton and caloron are regular everywhere, the singularity of the Wu–Yang solution can be understood as the result of having ignored the massive Fourier modes in the dimensional reduction, but the mere oxidation of the 3-dimensional monopole does not automatically restore them: the 4-dimensional instanton corresponding to the Fourier zero mode of the BPST caloron is singular.

The above *redox* relation was generalized by Kronheimer in Ref. [9] to a relation between selfdual Yang–Mills instanton solutions in hyper-Kähler (HK) spaces [9] and BPS monopoles in $\mathbb{E}^3$. We are going to see that Kronheimer's scheme provides an alternative reduction of the BPST instanton which relates it to the *colored* BPS monopole solution of Protogenov [10]. Colored monopoles are a rather misterious type of monopole solutions that exist for many

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[1] This is the equation satisfied by the 't Hooft–Polyakov monopole [2,3] in the Prasad–Sommerfield limit [4]. We will henceforth refer to these monopoles as BPS monopoles. Since the time direction does not play any role here, we will also refer to the spatial parts of 4-dimensional Lorentzian solutions as "3-dimensional" solutions.

[2] We dress 4-dimensional objects with a hat; hatless objects are 3-dimensional.

[3] This choice of period is unconventional but convenient for what follows.

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gauge groups [11] and are characterized by asymptotically vanishing Higgs field and magnetic charge which, nevertheless, can contribute to the Bekenstein–Hawking entropy of certain (supersymmetric) non-Abelian black holes [12,13,11].

Let us start by reviewing Kronheimer's result: consider a 4-dimensional HK space admitting a free U(1) action which shifts the adapted periodic coordinate $z \sim z + 4\pi$ by an arbitrary constant. Its metric can always be put in the form [14]

$$d\hat{s}^2 = H^{-1}(dz+\omega)^2 + H dx^m dx^m \qquad (m = 1, 2, 3)\,, \tag{1.2}$$

where the z-independent function H and 1-form ω are related by[4]

$$dH = \star d\omega\,. \tag{1.3}$$

The integrability condition of this equation implies that H is a harmonic function in $\mathbb{E}^3$ which is furthermore required to be strictly positive in order for the metric to be regular. Now, for any gauge group G, let us consider a gauge field $\hat{A}$ whose field strength $\hat{F}$ is selfdual $\hat{\star}\hat{F} = +\hat{F}$ in the above HK metric with respect to the frame and orientation

$$\hat{e}^{\,0} = H^{-1/2}(dz+\omega)\,, \qquad \hat{e}^{\,a} = H^{1/2}\delta^a{}_m dx^m\,,$$
$$\epsilon_{0123} = +1\,. \tag{1.4}$$

Then, the 3-dimensional gauge and Higgs fields A and Φ defined by

$$\Phi \equiv -H\hat{A}_z\,,$$
$$A_m \equiv \hat{A}_m - \omega_m \hat{A}_z\,, \tag{1.5}$$

satisfy the Bogomol'nyi equation in $\mathbb{E}^3$ Eq. (1.1). It is worth stressing that, had we started with an anti-selfdual YM field we would have obtained the Bogomol'nyi equation with opposite sign, which is acceptable, but also Eq. (1.3) with opposite sign, which would be a contradiction: in this setup we can only reduce YM fields which are selfdual w.r.t. the above frame and orientation.

When $H = 1$, the HK space is just $S^1 \times \mathbb{E}^3$ and one recovers the result explained at the beginning. A more interesting choice is $H = 1/r$ with $r^2 = x^m x^m$. Writing the $\mathbb{E}^3$ metric $dx^m dx^m$ as $dr^2 + r^2 d\Omega^2_{(2)}$ and then redefining $r = \rho^2/4$ the HK metric Eq. (1.2) becomes the metric of $\mathbb{E}^4$ in spherical coordinates

$$ds^2 = d\rho^2 + \rho^2 d\Omega^2_{(3)}\,, \tag{1.6}$$

where $d\Omega^2_{(3)}$ is the round metric of the 3-sphere of unit radius in Eq. (A.14). This HK space is, therefore, $\mathbb{E}^4_{-\{0\}}$ and the shifts of z act freely on it because the origin $\rho = 0$ does not belong to it.

Obviously, the standard BPST instanton is a selfdual solution in this space and, provided that the gauge field is independent of z, we can reduce it directly (avoiding the caloron step) using Kronheimer's scheme to find a monopole in $\mathbb{E}^3_{-\{0\}}$. This is what we are going to do in the next section but, before, we want to review the relation between the Euclidean action of the instanton and the monopole charge.

The gauge field strength components in the frame Eq. (1.4) are

$$\begin{cases} \hat{F}_{ab} = H^{-1}F_{ab} - H^{-2}\Phi(d\omega)_{ab}\,, \\ \hat{F}_{0a} = H^{-1}\mathcal{D}_a\Phi - H^{-2}\Phi\partial_a H\,. \end{cases} \tag{1.7}$$

Substituting them into the YM action and using repeatedly Eq. (1.3), the Bogomol'nyi equation (1.1) and Stokes' theorem we get

$$\tfrac{1}{4}\int d^4x\sqrt{|\hat{g}|}\hat{F}^2 = 4\pi \int_{V^3} \tfrac{1}{2}H^{-2}d\star dH\,\Phi^2 + 4\pi \int_{\partial V^3}\left[H^{-1}\Phi^A F^A + \tfrac{1}{2}\star dH^{-1}\Phi^2\right], \tag{1.8}$$

where V^3 is $\mathbb{E}^3$ with the singular points of H removed: this means that the first term on the r.h.s. always vanishes. The end result therefore reads

$$\tfrac{1}{4}\int d^4x\sqrt{|\hat{g}|}\hat{F}^2 = 4\pi \int_{\partial V^3}\left[H^{-1}\Phi^A F^A + \tfrac{1}{2}\star dH^{-1}\Phi^2\right], \tag{1.9}$$

and one must take into account that the boundary of V^3 includes the singularities of H as well as infinity.

For $H = 1$, $V^3 = \mathbb{E}^3$ and the r.h.s. is directly related to the monopole magnetic charge

$$p = \tfrac{1}{4\pi}\int_{S^2_\infty} \frac{\Phi^A F^A}{\sqrt{\Phi^B\Phi^B}}\,, \tag{1.10}$$

provided the Higgs field is asymptotically constant, as in the BPS 't Hooft–Polyakov monopole.

For $H = 1/r$, which is the case of interest here, $V^3 = \mathbb{E}^3_{-\{0\}}$, $\partial V^3 = \{0\} \cup S^2_\infty$, and the integral will diverge precisely for monopoles with well-defined magnetic charge at infinity and asymptotically constant Higgs fields. Thus, we can only expect convergence for colored magnetic monopoles [11]. If the selfdual YM field has a finite action, then it must lead to a colored monopole in $\mathbb{E}^3$ by Kronheimer's dimensional reduction. In the next section we are going to see that this is indeed the case for the BPST instanton.

2. Singular reduction of the BPST instanton

In order to reduce the BPST instanton *à la Kronheimer* in the HK space with $H = 1/r$, it is convenient to write it in spherical coordinates and, actually, it is easier to rederive it directly using the following ansatz for the components of the SU(2) gauge potential

$$\hat{A}^A_{\substack{L\\R}} = b_{\substack{L\\R}}(\rho)v^A_{\substack{L\\R}}\,, \qquad A = 1, 2, 3\,, \tag{2.1}$$

where the $v^A_{\substack{L\\R}}$ are the components of the SU(2) Maurer–Cartan (MC) 1-forms defined in Eqs. (A.12), satisfying Eq. (A.13), and $b_{\substack{L\\R}}(\rho)$ is a function of ρ to be determined by imposing the selfduality of the gauge field strength. To this end it is most convenient to use the frames

$$\hat{e}^{\,0}_{\substack{L\\R}} = d\rho\,, \qquad \hat{e}^{\,a}_{\substack{L\\R}} = \tfrac{1}{2}\rho\delta^a{}_A v^A_{\substack{L\\R}}\,, \tag{2.2}$$

for the metric Eq. (1.6). Using the MC 1-forms it is straightforward to compute the gauge field strength $\hat{F}^A_{\substack{L\\R}}$:

$$\hat{F}_{\substack{L\\R}}{}^A = \frac{2\dot{b}}{\rho}\delta^A{}_a\,\hat{e}_{\substack{L\\R}}{}^0 \wedge \hat{e}_{\substack{L\\R}}{}^a + \frac{2b(b\mp 1)}{\rho^2}\epsilon^A{}_{ab}\,\hat{e}_{\substack{L\\R}}{}^a \wedge \hat{e}_{\substack{L\\R}}{}^b\,. \tag{2.3}$$

Requiring $\hat{F}^A_{\substack{L\\R}}$ to be (anti-)selfdual ($\hat{F}^{A(\pm)}{}_{0a} = \pm\tfrac{1}{2}\epsilon_{abc}\hat{F}^{A(\pm)}{}_{bc}$) in these two frames we arrive at a differential equation for $b^{\pm}_{\substack{L\\R}}(\rho)$ leading to two self- and two anti-selfdual solutions describing a

[4] Unhatted objects are always defined in 3-dimensional Euclidean space $\mathbb{E}^3$.

single BPST instanton or anti-instanton, of size[5] determined by the parameter λ, at the origin:

$$\hat{\star}\hat{F}=+\hat{F}\begin{cases}\hat{A}_L^{A(+)} &= \dfrac{1}{1+\lambda^2\rho^2/4}\,v_L^A\,,\\ \hat{A}_R^{A(+)} &= -\dfrac{\lambda^2\rho^2/4}{1+\lambda^2\rho^2/4}\,v_R^A\,,\end{cases}$$
$$\hat{\star}\hat{F}=-\hat{F}\begin{cases}\hat{A}_L^{A(-)} &= +\dfrac{\lambda^2\rho^2/4}{1+\lambda^2\rho^2/4}\,v_L^A\,,\\ \hat{A}_R^{A(-)} &= -\dfrac{1}{1+\lambda^2\rho^2/4}\,v_R^A\,.\end{cases} \tag{2.4}$$

The gauge fields $\hat{A}_L^{A(\pm)}$ are gauge-equivalent to the $\hat{A}_R^{A(\pm)}$ owing to

$$U\hat{A}_L^{A(\pm)}U^{-1}+dUU^{-1}=\hat{A}_R^{A(\pm)}\,, \tag{2.5}$$

and the property Eq. (A.11). Then, we could just work with $\hat{A}_R^{A(+)}$ and $\hat{A}_L^{A(-)}$, which are regular (they vanish at $\rho=0$ while the other two are multivalued there). However, if we want to use Kronheimer's results we are forced to work with the singular ones, $\hat{A}_L^{A(+)}$ and $\hat{A}_R^{A(-)}$, because as one can see the transformation between the frame $\hat{e}^{\hat{a}}_{\substack{L\\R}}$ in Eqs. (2.2) and Kronheimer's frame $\hat{e}^{\hat{a}}$ in Eqs. (1.4) preserves the orientation for $\hat{e}^{\hat{a}}_L$ but reverses it for $\hat{e}^{\hat{a}}_R$. In other words: the regular gauge fields $\hat{A}_R^{A(+)}$ and $\hat{A}_L^{A(-)}$ are anti-selfdual in Kronheimer's frame and can therefore not be consistently reduced.

Let us, then, consider $\hat{A}_L^{A(+)}$ and $\hat{A}_R^{A(-)}$. By construction, these gauge fields are invariant under the free U(1) actions in Eqs. (A.5) and (A.4), respectively.

In other words: $\hat{A}_L^{A(+)}$ is φ-independent and $\hat{A}_R^{A(-)}$ is ψ-independent and can be dimensionally reduced along those directions because the only invariant point under these actions (the origin $\rho=0$) does not belong to our HK space. We can expect 3-dimensional monopoles which are singular there.

Using directly Eqs. (1.5), from $\hat{A}_L^{A(+)}$ we get the Yang–Mills and Higgs fields of a BPS monopole solution

$$\Phi_L^{A(+)}=\frac{1}{r(1+\lambda^2 r)}\delta^A{}_m\frac{y_L^m}{r}\,,$$
$$A_L^{A(+)}=\frac{1}{(1+\lambda^2 r)}\epsilon^A{}_{mn}d\frac{y_L^m}{r}\frac{y_L^n}{r} \tag{2.6}$$

where we have defined the Cartesian coordinates $y^m/r\equiv-\delta^m{}_A v^A_{L\,\varphi}$ [6]:

$$y_L^1\equiv r\sin\theta\cos\psi\,,\quad y_L^2\equiv r\sin\theta\sin\psi\,,\quad y_L^3\equiv r\cos\theta\,. \tag{2.7}$$

The reduction of $\hat{A}_R^{A(-)}$ gives exactly the same 3-dimensional fields upon the replacement of the Cartesian coordinates y_L^m by $y_R^m\equiv+r\delta^m{}_A v^A_{R\,\psi}$ [7]:

$$y_R^1\equiv r\sin\theta\cos\varphi\,,\quad y_R^2\equiv -r\sin\theta\sin\varphi\,,\quad y_R^3\equiv -r\cos\theta\,. \tag{2.8}$$

As predicted by the arguments based on the Euclidean action, the 3-dimensional BPS monopole obtained by this procedure is the colored monopole found by Protogenov in Ref. [10]. The Higgs field vanishes at infinity and the magnetic charge, as defined in Eq. (1.10) vanishes identically. The solution approaches the Wu–Yang monopole [8] for $r\to 0$ (which corresponds to $\lambda^2=0$) and, therefore, one can argue that the solution describes a magnetic monopole at the origin whose charge is completely screened at infinity. This interpretation is supported by the computation of the Bekenstein–Hawking entropy $S_{\rm BH}$ of non-Abelian black holes with this kind of gauge fields: there is a contribution to $S_{\rm BH}$ corresponding to a magnetic charge [12,13].

3. Oxidation of the singular Protogenov monopoles

Reversing the procedure we just carried out, we see that the singularity of the SU(2) colored BPS monopole disappears completely when it is oxidized to 4 Euclidean dimensions. Since there are other singular SU(2) BPS monopoles [10], it is natural to ask whether their singularities can also be cured by oxidizing them within this scheme.

The spherically symmetric solutions of the SU(2) Bogomol'nyi equations have the following *hedgehog* form [10]:

$$A^A=-r^2h(r)\epsilon^A{}_{mn}\frac{y^n}{r}d\left(\frac{y^m}{r}\right)\,, \tag{3.1}$$

$$\Phi^A=-rf(r)\delta^A{}_m\frac{y^m}{r}\,, \tag{3.2}$$

where the functions $f(r)$ and $h(r)$ must satisfy the differential equations

$$r\dot{h}+2h+f(1+r^2h)=0\,, \tag{3.3}$$
$$r(\dot{h}-\dot{f})-r^2h(h-f)=0\,, \tag{3.4}$$

if the above Yang–Mills and Higgs fields are to satisfy the Bogomol'nyi equation (1.1). Apart from the family of colored solutions in Eq. (2.6), there is another 2-parameter (μ and s) family of solutions given by

$$rf=-\frac{1}{r}\left[1-\mu r\coth(\mu r+s)\right]\,,$$
$$rh=\frac{1}{r}\left[\frac{\mu r}{\sinh(\mu r+s)}-1\right]\,. \tag{3.5}$$

The BPS limit of the 't Hooft–Polyakov monopole [2,3] is the $s=0$ member of this family, and the only regular one. Before oxidizing them, we can compute the action of the corresponding instanton using Eq. (1.9). The action turns out to diverge for all values of s. However, even if all hope of getting a regular instanton by oxidizing these solutions is lost, it is still worth finding the general expression of the singular instantons, since it may give us inspiration for making instanton ansätze directly in 4 dimensions. Using Kronheimer's relations, Eq. (1.5), we find

$$\hat{A}^A=-r^2f(r)v_L^A+r^2\left[f(r)-h(r)\right]u^A\,, \tag{3.6}$$

where we have defined the 1-forms

$$u^1=\cos\psi\sin\theta\cos\theta d\psi+\sin\psi d\theta\,,$$
$$u^2=\sin\psi\sin\theta\cos\theta d\psi-\cos\psi d\theta\,,$$
$$u^3=-\sin^2\theta d\psi\,. \tag{3.7}$$

These 1-forms depend only on two coordinates (ψ and θ) and they can be seen as projections of the left-invariant MC 1-forms v_L^A

$$u^A=v_L^B\left[\delta^A{}_B-\frac{y_By^A}{r^2}\right]\,. \tag{3.8}$$

[5] In the instanton literature it is customary to denote the size of the (anti-)instanton by ρ, see *e.g.* Refs. [15], but here we'll denote it by ρ_0. It is then easy to see that $\lambda=2/\rho_0$.

[6] We use the identity $v_L^A(\varphi=0)-\cos\theta\, v^A_{L\,\varphi}d\psi=\epsilon^A{}_{mn}d\frac{y_L^m}{r}\frac{y_L^n}{r}$.

[7] Now we use the identity $v_R^A(\psi=0)-\cos\theta\, v^A_{R\,\psi}d\varphi=-\epsilon^A{}_{mn}d\frac{y_R^m}{r}\frac{y_R^n}{r}$.

They satisfy differential equations identical to the ones satisfied by the left-invariant MC 1-forms v_L^A up to the 1/2 factor, *i.e.*

$$du^A = -\epsilon^A{}_{BC} u^B \wedge u^C \,, \tag{3.9}$$

which makes them well suited for a generalization of the ansatz Eq. (2.1):

$$\hat{A}^A = b(\rho) v_L^A + c(\rho) u^A \,. \tag{3.10}$$

Imposing selfduality of the corresponding field strength with the redefinition

$$b(\rho(r)) = -r^2 f(r) \,, \qquad c(\rho(r)) = -r^2 [h(r) - f(r)] \,, \tag{3.11}$$

leads to Protogenov's equations (3.3) and (3.4); the oxidation of the BPS monopoles gives all the selfdual instantons of that form.

4. Conclusions

In this paper we have shown how a misterious kind of SU(2) BPS magnetic monopoles known as *colored monopoles*, which are singular at the origin and have vanishing asymptotic charge and Higgs field, can be understood as the result of the singular dimensional reduction of the BPST instanton, which is itself globally regular. The parameter appearing in the monopole family of solutions turns out to be related to the one that measures the instantons' size.

The mechanism is analogous to the well-known mechanism curing gravitational singularities by oxidation as for example the KK-monopole [16] or in certain 4-dimensional dilatonic black holes [17], but with the twist that here the fields are non-Abelian. The mechanism that cures the singularity of the colored monopole does not, however, work for the rest of the spherically-symmetric BPS monopoles of the theory: they always have infinite action, but depending on the application this may or may not be a problem.

We have argued, based on the relation between the instanton action and the monopole magnetic charge, that this relation between regular instantons and singular, colored magnetic monopoles should be general. It has recently been shown in Ref. [11] that colored magnetic monopoles are present in the Yang–Mills–Higgs theory for all SU(N) groups and the results of that paper can be used to construct regular selfdual SU(N) instantons [18]. Possibly, the transmutation monopoles discovered in Ref. [11], which have different (non-vanishing) charges at infinity and at the origin, can be related to regular solutions by a similar mechanism.

The case studied here is just the simplest and most special of those comprised in Kronheimer's work Ref. [9], since it just involves $\mathbb{E}^4_{-\{0\}}$. One may wonder if the rest can be of any relevance in physics. It turns out that the relation between $\mathcal{N}=1$, $d=5$ and $\mathcal{N}=2$, $d=4$ super-Einstein–Yang–Mills (SEYM) theories must include the relation between selfdual instantons in HK spaces and BPS monopoles in $\mathbb{E}^3$ discovered by Kronheimer: the timelike supersymmetric solutions of $\mathcal{N}=1$, $d=5$ [19] (as it happens in the Abelian case [20]) involve a 4-dimensional Euclidean base space of HK type and the YM field strengths have a piece which is selfdual in that space. On the other hand the YM fields of the timelike supersymmetric solutions of $\mathcal{N}=2$, $d=4$ SEYM [21] are required to satisfy the Bogomol'nyi equation in $\mathbb{E}^3$ in combination with an effective *Higgs* field. These two classes of theories and their solutions are related by dimensional reduction. Explicit solutions of the latter describing non-Abelian black holes have been obtained in [22,23,12,13,11]. Some of the solutions are powered by the colored BPS monopoles that we have shown to be related to the BPST instanton. It is then natural to expect that the oxidation of the complete supergravity solutions will provide us with explicit solutions of the $\mathcal{N}=1$, $d=5$ SEYM theory[8] involving the BPST instanton. These solutions, whose form is quite intriguing, may be globally regular. The oxidation *à la Kronheimer* of solutions involving other monopoles will give potentially singular solutions, but, just as it happens with singular monopoles in $d=4$, gravity may cover the singularities with event horizons. All these new possibilities opened by the result presented in this paper are very interesting and well worth investigating. Work in this direction is already under way [24].

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Appendix A. The metrics of the round S^3 and S^2

In this appendix we will review the well-known construction of the SO(4)-invariant metric on S^3 using its identification with the SU(2) group manifold, the construction of SO(3)-invariant metric on S^2 using its identification with the SU(2)/U(1) coset space and the relation between both of them.

All matrices $U \in SU(2)$ ($U^\dagger = U^{-1}$, $\det U = +1$) can be parametrized by two complex numbers z_0, z_1

$$U \equiv \begin{pmatrix} z_0 & z_1 \\ -\bar{z}_1 & \bar{z}_0 \end{pmatrix}, \qquad |z_0|^2 + |z_1|^2 = 1 \,. \tag{A.1}$$

Therefore, the SU(2) manifold can be identified with S^3. Both are traditionally parametrized by the Euler angles $\{\theta, \varphi, \psi\}$:

$$z_0 = \cos(\theta/2)\, e^{i(\varphi+\psi)/2} \,, \qquad z_1 = \sin(\theta/2)\, e^{i(\varphi-\psi)/2} \,. \tag{A.2}$$

The main property of this parametrization is that any SU(2) rotation can be written as the product of three rotations with these angles:

$$U(\varphi, \theta, \psi) = U(\varphi, 0, 0) U(0, \theta, 0) U(0, 0, \psi) \,. \tag{A.3}$$

The Euler angles are usually assumed to take values in the intervals $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi)$, and $\psi \in [0, 4\pi)$. Other choices are possible: for instance, $\theta \in [0, \pi]$, $\varphi \in [0, 4\pi)$, and $\psi \in [0, 2\pi)$ also covers once S^3. Only the coordinate chosen to take values in $[0, 4\pi)$ should be considered periodic. There is a free U(1) action on S^3 associated to constant shifts of the periodic coordinate. For the standard choice, this action is

$$U(\varphi, \theta, \psi) \to U(\varphi, \theta, \psi) U(0, 0, 2\alpha) \,, \qquad \alpha \in [0, 2\pi) \,. \tag{A.4}$$

Being a right action, it is adequate to define the right coset space SU(2)/U(1). If we choose instead φ to be the periodic coordinate,

[8] So far, no explicit solutions of these theories have been constructed.

the U(1) action is

$$U(\varphi,\theta,\psi) \to U(2\alpha,0,0)U(\varphi,\theta,\psi)\,, \qquad \alpha \in [0,2\pi)\,. \tag{A.5}$$

Being a left action, it is adequate to define the left coset space U(1)\SU(2), which is a more unusual option.

A convenient basis of the $\mathfrak{su}(2)$ Lie algebra is provided by the anti-Hermitian matrices[9]

$$T_A = \tfrac{i}{2}\sigma^A\,, \qquad [T_A, T_B] = -\epsilon_{ABC}T_C\,. \tag{A.7}$$

In this basis

$$U(\varphi,0,0) = e^{\varphi T_3}\,, \qquad U(0,\theta,0) = e^{\theta T_2}\,,$$
$$U(0,0,\psi) = e^{\psi T_3}\,. \tag{A.8}$$

The left- (resp. right-)invariant Maurer–Cartan (MC) 1-form V_L (resp. V_R) are defined by

$$V_L \equiv -U^{-1}dU\,, \qquad V_R \equiv -dUU^{-1}\,, \tag{A.9}$$

and as a consequence of their definition they satisfy the MC equations

$$dV_{\substack{L\\R}} \mp V_{\substack{L\\R}} \wedge V_{\substack{L\\R}} = 0\,. \tag{A.10}$$

Observe that the left- and right-invariant MC 1-forms are related by the following *gauge* transformations:

$$V_R = UV_LU^{-1}\,. \tag{A.11}$$

The components of the MC 1-forms in the above basis $V_{\substack{L\\R}} \equiv v^A_{\substack{L\\R}} T_A$ are given by

$$\begin{cases} v^1_L &= \sin\psi\, d\theta - \sin\theta\cos\psi\, d\varphi\,, \\ v^2_L &= -\cos\psi\, d\theta - \sin\theta\sin\psi\, d\varphi\,, \\ v^3_L &= -(d\psi + \cos\theta\, d\varphi)\,, \end{cases}$$
$$\begin{cases} v^1_R &= -\sin\varphi\, d\theta + \sin\theta\cos\varphi\, d\psi\,, \\ v^2_R &= -\cos\varphi\, d\theta - \sin\theta\sin\varphi\, d\psi\,, \\ v^3_R &= -(d\varphi + \cos\theta\, d\psi)\,, \end{cases} \tag{A.12}$$

and the MC equations in components take the form

$$dv^A_{\substack{L\\R}} \pm \tfrac{1}{2}\epsilon_{ABC}\, v^B_{\substack{L\\R}} \wedge v^C_{\substack{L\\R}} = 0\,. \tag{A.13}$$

As their name indicates, the left- (resp. right-)invariant MC 1-forms are invariant under the left (resp. right) U(1) action in Eq. (A.5) (resp. Eq. (A.4)).

Both the left- or the right-invariant MC 1-forms can be used as Dreibeins to construct a bi-invariant (that is SU(2) × SU(2) ∼ SO(4)-invariant) metric on SU(2) (∼ S^3) with tangent space metric δ_{AB}. The result is exactly the same in both cases: normalizing the metric so as to get the volume of the 3-sphere of unit radius, we find

$$\begin{aligned} d\Omega^2_{(3)} &= \tfrac{1}{4} v^A_L v^A_L = \tfrac{1}{4} v^A_R v^A_R \\ &= \tfrac{1}{4}\Big[d\theta^2 + d\varphi^2 + d\psi^2 + 2\cos\theta\, d\varphi d\psi\Big]\,. \end{aligned} \tag{A.14}$$

It is customary to rewrite this metric so that the invariance under the chosen U(1) action is manifest. For the standard choice in which $\psi \in [0, 4\pi)$ is the periodic coordinate and there is invariance under the right action in Eq. (A.4)

$$d\Omega^2_{(3)} = \tfrac{1}{4}\Big[d\Omega^2_{(2)}(\theta,\varphi) + v^3_L v^3_L\Big]\,, \tag{A.15}$$

where $d\Omega^2_{(2)}(\theta,\varphi)$ is the standard metric of the round 2-sphere of unit radius

$$d\Omega^2_{(2)}(\theta,\varphi) = d\theta^2 + \sin^2\theta d\varphi^2 = v^1_L v^1_L + v^2_L v^2_L\,. \tag{A.16}$$

For the other choice, we just have to interchange φ and ψ and L by R in the above expressions.

[9] The σ^A are the Pauli matrices, which we take to satisfy

$$\sigma^A\sigma^B = \delta^{AB} + i\epsilon^{ABC}\sigma^C\,. \tag{A.6}$$

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