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$\mathcal{N}=2$ Einstein-Yang-Mills' static two-center solutions

Pablo Bueno[1,a], *Patrick Meessen*[2,b], *Tomás Ortín*[1,c] *and Pedro F. Ramírez*[1,d],

[1]*Instituto de Física Teórica UAM/CSIC*
C/ Nicolás Cabrera, 13–15, C.U. Cantoblanco, E-28049 Madrid, Spain

[2]*HEP Theory Group, Departamento de Física, Universidad de Oviedo*
Avda. Calvo Sotelo s/n, E-33007 Oviedo, Spain

Abstract

We construct *bona fide* one- and two-center supersymmetric solutions to $\mathcal{N}=2$, $d=4$ supergravity coupled to SU(2) non-Abelian vector multiplets. The solutions describe black holes and global monopoles alone or in equilibrium with each other and exhibit non-Abelian hairs of different kinds.

[a]E-mail: `p.bueno [at] csic.es`
[b]E-mail: `meessenpatrick [at] uniovi.es`
[c]E-mail: `Tomas.Ortin [at] csic.es`
[d]E-mail: `p.f.ramirez [at] csic.es`

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Introduction

Contrary to what one might think, multi-black hole solutions need not be related to supersymmetry or, like in the case of Kastor and Traschen's solution in Ref. [1], fake-supersymmetry. Proof of this is given by various solutions *e.g.* the ones presented in Refs. [2] and [3]. The benefit of using supersymmetry, however, is that after a few decades' worth of investigations there are workable recipes for creating supersymmetric solutions, which greatly facilitates the construction and study of multi-black hole solutions.

The construction is particularly straightforward in ungauged $\mathcal{N} = 2$, $d = 4$ supergravity coupled to vector multiplets where there are clear-cut rules for a supersymmetric multi-object solution to give rise to a well-defined multi-black hole solution [4, 5, 6, 7, 8, 9, 10, 11]: i) positive mass of the constituents, ii) the near-horizon limit has to have definite entropy, iii) the 2^{nd} law of thermodynamics must hold in the coalescence of constituents, and iv) the Denef constraints [9] must be satisfied. Depending on the charges the latter may constrain the distance between the constituents but it always implies the absence of NUT charge.

The oft forgotten case of ungauged $\mathcal{N} = 2$, $d = 4$ supergravity coupled to non-Abelian vector multiplets, which we will refer to as $\mathcal{N} = 2$ Einstein-Yang-Mills, is similar to the Abelian case in that there is a well-defined recipe for constructing supersymmetric solutions [12, 13]. However, the construction of supersymmetric solutions is greatly hindered not only by the fact that not every Abelian theory can be non-Abelianized, but doubly more so by the fact that the supersymmetric recipe requires the use of solutions of the (non-Abelian) Bogomol'nyi equation on $\mathbb{R}^3$ [15]. Our lack of knowledge of the space of all solutions to this equation is a serious limitation to the application of the supersymmetric recipe: there exists a vast literature on single monopole solutions, *i.e.* regular single-center solutions to the Bogomol'nyi equation (see *e.g.* Refs. [16]). Depending on the chosen $\mathcal{N} = 2$, $d = 4$ model, these can be used to construct globally regular supergravity solutions known as *global monopoles*. A lot less is known about the singular solutions to the Bogomol'nyi equation which are the ones which give rise to black holes with different degrees of non-Abelian hair [12, 13, 14]. Finally, even less is known about multi-center solutions to the Bogomol'nyi equation. These are the ones we need in order to to apply the supersymmetric recipe to the construction of multi-center supergravity solutions, with centers that correspond to global monopoles or black holes.

Luckily enough, some explicit solutions are known.[1] In this paper we are going to use the solutions of the SU(2) Bogomol'nyi equation found by Cherkis and Durcan [20] and Blair and Cherkis [21] (which we will generalize by adding *Protogenov hair* [14]). These solutions describe an 't Hooft-Polyakov (-Protogenov) monopole in the presence of an arbitrary number of Dirac monopoles embedded in SU(2), all having

[1]Finite-energy, multi-center solutions of the Yang-Mills or Yang-Mills-Higgs system which do not satisfy the Bogomol'nyi equation like those in Refs. [17, 18, 19] are also known.

charge opposite to the one carried by the former. These solutions can (in principle) give rise to supergravity solutions describing black holes in the presence of a global monopole. The construction of these solutions is, at the same time, our main goal and our main result.

Before we start constructing multi-black hole solutions, however, it is worth reviewing briefly some of the previous work on solutions of YM theories coupled to gravity[2]. Most of the previous work on this topic was focused on pure Einstein-Yang-Mills (EYM) theories, (the minimal non-Abelian extension of the Einstein-Maxwell theory), ignoring the possible existence of unbroken supersymmetry which is, however, one of our main concerns here.

Soon after the discovery of the 't Hooft-Polyakov monopole [23, 24] several groups found solutions to the pure EYM theory [25] whose $SU(2)$ gauge field is that of the Wu–Yang $SU(2)$ monopole [26]. The metric of all these solutions is that of the (dS or AdS) non-extremal Reissner-Nordström black hole and the singularity in the gauge field (generically expected for static YM solutions [27]) is covered by an event horizon.

This coincidence of the metrics is due to the relation between the Wu–Yang $SU(2)$ monopole and the non-Abelian embedding of the Dirac monopole Eq. (B.10): they are related by a singular gauge transformation and therefore give rise to exactly the same energy-momentum tensor as it is gauge invariant whether the gauge transformation is singular or not. For this reason, these solutions have been regarded as not truly non-Abelian, even though there are potentially measurable differences, see *e.g.* Refs. [28, 29].

Finding less trivial ("genuinely or essentially non-Abelian") solutions proved much more difficult and a *non-Abelian baldness theorem* stating that the only black-hole solutions of the EYM $SU(2)$ theory with a regular horizon and non-vanishing magnetic charge had to be non-Abelian embeddings of the Reissner–Nordström solution was proven in [30]. This theorem was subsequently generalized to prove the absence of regular monopole or dyon solutions to the EYM theory in Refs. [31, 32].

An "essentially non-Abelian" solution, globally regular [33] to EYM theory had already been found: the Bartnik-McKinnon particle [34]. The Bartnik-McKinnon particle and its black hole-type generalizations [35], are in fact families of unstable solutions indexed by a discrete parameter and evade the non-Abelian baldness theorem by being bald, *i.e.* they have no asymptotic charge. It is worth pointing out that even though these solutions are only known numerically, they have been proven to exist [36].

The classification of the possible EYM solutions for the gauge group $SU(2)$ [37] suggests that one has to add more fields to the theory in order to get "essentially non-Abelian" black-hole or gravitating monopole solutions with non-vanishing charges. Investigations of solutions to the EYM theory coupled to a Higgs field in the adjoint representation [38] in the BPS-limit, a theory that is closer to the one we are going to study than EYM, shows that a globally well-defined 't Hooft-Polyakov monopole exists and furthermore the existence of other Bartnik-McKinnon-like solutions.

[2]For more comprehensive reviews see *e.g.* Refs. [22].

As far as 4-dimensional supergravity is concerned we have the (supersymmetric) Harvey-Liu [39] and the Chamseddine-Volkov [40] regular gravitating monopole solutions to gauged $\mathcal{N} = 4$, $d = 4$ supergravity; in $\mathcal{N} = 2$, $d = 4$ theories there are analytical solutions describing global monopole solutions and non-Abelian black hole solutions with and without asymptotic magnetic charge. Needless to say, all the solutions mentioned in this little historical *exposé* describe the fields corresponding to a single object. To our knowledge, there are no known, essentially non-Abelian multi-object analytic[3] solutions and this article intends to fill this gap by constructing static solutions describing the interplay between an 't Hooft-Polyakov monopole and a Dirac monopole of opposite charge in two generic classes of gauged $\mathcal{N} = 2$, $d = 4$ models.

It is convenient to stress that in the theories we have called $\mathcal{N} = 2$, $d = 4$ SEYM the gauge group does not contain any part of the R-symmetry group. Indeed, in general (ungauged) $\mathcal{N} = 2$, $d = 4$ theories, the global symmetry group G can be written as

$$\mathrm{G} = \mathrm{G_V} \times \mathrm{G_{hyper}} \times \mathrm{SU}(2)_\mathrm{R} \times \mathrm{U}(1)_\mathrm{R}\,, \tag{0.1}$$

where $\mathrm{G_V}$ and $\mathrm{G_{hyper}}$ stand for the isometry groups of the special and quaternionic Kähler manifolds respectively. When a (necessarily non-Abelian) subgroup of $\mathrm{G_V}$ is gauged (as in $\mathcal{N} = 2$, $d = 4$ SEYM theories) the scalar potential is positive semidefinite, which allows for asymptotically De-Sitter and asymptotically flat solutions (such as the ones we construct in this paper). This is in contradistinction to theories in which a subgroup of $\mathrm{SU}(2)_\mathrm{R}$ (or the complete $\mathrm{SU}(2)_\mathrm{R}$) is gauged via Fayet-Iliopoulos terms[4] in whose case the scalar potential becomes negative definite, the solutions thus being asymptotically anti-De Sitter. Lately, an intense effort has been devoted to the construction of black-hole solutions of theories with Abelian gaugings (that is, theories in which a subgroup $\mathrm{U}(1) \in \mathrm{SU}(2)_\mathrm{R}$ has been gauged); see, for instance, Refs. [43, 44, 45, 46, 47, 48] and references therein. The case in which the full $\mathrm{SU}(2)_\mathrm{R}$ has been gauged remains as unexplored as challenging, even though the general form of the timelike supersymmetric solutions of this theory has been given in Ref. [49].

This paper is organized as follows: in Section 1 we review the theories we are going to work with ($\mathcal{N} = 2$, $d = 4$ Super-Einstein-Yang-Mills theories) and the recipe for constructing timelike supersymmetric solutions (black holes, in particular). In Section 2 we apply that recipe to construct single, static supersymmetric black-hole and monopole solutions of two particular examples of SU(2)-gauged $\mathcal{N} = 2$, $d = 4$ SEYM: the $\overline{\mathbb{CP}}^3$ model (quadratic) (2.2) and the ST[2, 4] model (cubic) (2.3.1). We use as seeds for these solutions the single-center solutions of the Bogomol'nyi equations reviewed in Section 2.1. In Section 3 we construct multi-black-hole solutions for the same models

[3]Numerical, multi-center solutions have been found previously, though. See, *e.g.* Refs. [41, 42]. Some of those solutions can be embedded in $\mathcal{N} = 1, d = 4$ supergravity. However, representing massive objects, they can never be supersymmetric in that theory. The embedding in higher-$\mathcal{N}$ supergravities is much more difficult (if possible at all). We thank J. Kunz for pointing these works to us.

[4]The overall $\mathrm{U}(1)_R$ group cannot be gauged in this way. The Abelian gaugings discussed in the literature deal with a subgroup $\mathrm{U}(1) \in \mathrm{SU}(2)_\mathrm{R}$.

using the multi-center solutions of the Bogomol'nyi equations reviewed in Section 3.1. Our conclusions are contained in Section 4. In the Appendices we review a particularly interesting single-center solution of the SU(2) Bogomol'nyi equations which appears in different guises: as a "Lorentzian meron" (Appendix A), as the Wu-Yang monopole (Appendix B) or as a solution of the Skyrme model (Appendix C). A higher-charge generalization of this solution is reviewed in Appendix D.

1 $\mathcal{N}=2$, $d=4$ SEYM and its supersymmetric black-hole solutions (SBHSs)

In this section we are going to introduce the class of theories that we have called $\mathcal{N}=2$, $d=4$ SEYM theories and we are going to review the recipe to construct all their timelike supersymmetric solutions, presented in Ref. [12]. We shall be extremely brief. The interested reader can find more details in Refs. [13, 50, 51]; our conventions are those of Refs. [12, 13, 51].

1.1 The theory

$\mathcal{N}=2$, $d=4$ SEYM theories can be seen as the simplest $\mathcal{N}=2$ supersymmetrization of the Einstein-Yang-Mills (EYM) theories. They are nothing but theories of $\mathcal{N}=2$, $d=4$ supergravity coupled to n vector multiplets in which a (necessarily non-Abelian)[5] subgroup of the isometry group of the (Special Kähler) scalar manifold has been gauged using some of the vector fields of the theory as gauge fields[6].

We will only be concerned with the bosonic sector of the theory, which consists on the metric $g_{\mu\nu}$, the vector fields $A^{\Lambda}{}_{\mu}$ ($\Lambda = 0, 1, \cdots, n$) and the complex scalars Z^i ($i = 1, \cdots, n$). The action of the bosonic sector reads

$$\begin{aligned} S[g_{\mu\nu}, A^{\Lambda}{}_{\mu}, Z^i] &= \int d^4x \sqrt{|g|}\, \Big[R + 2\mathcal{G}_{ij^*}\mathfrak{D}_{\mu}Z^i\mathfrak{D}^{\mu}Z^{*\,j^*} + 2\Im\mathfrak{m}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu}F^{\Sigma}{}_{\mu\nu} \\ &\quad -2\Re\mathfrak{e}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu}\star F^{\Sigma}{}_{\mu\nu} - V(Z,Z^*)\Big] \,. \end{aligned} \tag{1.1}$$

In this expression, $\mathcal{G}_{ij^*}$ is the Kähler metric, $\mathfrak{D}_{\mu}Z^i$ is the gauge-covariant derivative

$$\mathfrak{D}_{\mu}Z^i = \partial_{\mu}Z^i + gA^{\Lambda}{}_{\mu}k_{\Lambda}{}^{i}\,, \tag{1.2}$$

$F^{\Lambda}{}_{\mu\nu}$ is the vector field strength

$$F^{\Lambda}{}_{\mu\nu} = 2\partial_{[\mu}A^{\Lambda}{}_{\nu]} - gf_{\Sigma\Gamma}{}^{\Lambda}A^{\Sigma}{}_{\mu}A^{\Gamma}{}_{\nu}\,, \tag{1.3}$$

[5] The theory becomes identical to the ungauged one when the gauge group is Abelian.

[6] A global symmetry group can be gauged if it acts on the vector fields in the adjoint representation. Furthermore, it is required to be a symmetry of the prepotential; see *e.g.* ref. [12] for more details.

$\mathcal{N}_{\Lambda\Sigma}$ is the period matrix and, finally, $V(Z, Z^*)$ is the scalar potential

$$V(Z, Z^*) = -\tfrac{1}{4}g^2\Im\mathfrak{m}\mathcal{N}^{\Lambda\Sigma}\mathcal{P}_\Lambda\mathcal{P}_\Sigma\,. \tag{1.4}$$

Since the imaginary part of the period matrix is negative definite, the scalar potential is positive semidefinite, which leads to asymptotically-flat or -De Sitter solutions.

In the above equations, $k_\Lambda{}^i(Z)$ are the holomorphic Killing vectors of the isometries that have been gauged[7] and $\mathcal{P}_\Lambda(Z, Z^*)$ the corresponding momentum maps, which are related to the Killing vectors and to the Kähler potential $\mathcal{K}$ by

$$i\mathcal{P}_\Lambda \;=\; k_\Lambda{}^i\partial_i\mathcal{K} - \lambda_\Lambda\,, \tag{1.5}$$

$$k_{\Lambda\, i^*} \;=\; i\partial_{i^*}\mathcal{P}_\Lambda\,, \tag{1.6}$$

for some holomorphic functions $\lambda_\Lambda(Z)$. Furthermore, the holomorphic Killing vectors and the generators T_Λ of the gauge group satisfy the Lie algebras

$$[k_\Lambda, k_\Sigma] = -f_{\Lambda\Sigma}{}^\Gamma k_\Gamma\,, \qquad [T_\Lambda, T_\Sigma] = +f_{\Lambda\Sigma}{}^\Gamma T_\Gamma\,. \tag{1.7}$$

For the gauge group SU(2), which is the only one we are going to consider, we use lowercase indices[8] $a, b, c = 1, 2, 3$ and the structure constants are $f_{ab}{}^c = -\varepsilon_{abc}$, so

$$[k_a, k_b] = +\varepsilon_{abc}k_c\,, \qquad [T_a, T_b] = -\varepsilon_{abc}T_c\,. \tag{1.8}$$

We will use the fundamental representation, in which the generators are proportional to the standard Pauli matrices[9] σ^a

$$T_a \equiv +\tfrac{i}{2}\sigma^a\,, \;\;\Rightarrow\;\; \mathrm{Tr}(T_aT_b) = -\tfrac{1}{2}\delta_{ab}\,. \tag{1.10}$$

The equations of motion of the theory can be written in the following form:

[7] The employed notation associates a Killing vector to each value of the index Λ in order to avoid the introduction of yet another class of indices and the embedding tensor (See *e.g.* the reviews [52]); it is understood that not all the k_Λ need to be non-vanishing.

[8] These will be a certain subset of those represented by $\Lambda, \Sigma, \ldots$.

[9] These are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma^a\sigma^b = \delta^{ab} + i\varepsilon^{abc}\sigma^c\,. \tag{1.9}$$

$$
\begin{aligned}
G_{\mu\nu} + 2\mathcal{G}_{ij^*}[\mathfrak{D}_{(\mu}Z^i\mathfrak{D}_{\nu)}Z^{*\,j^*} - \tfrac{1}{2}g_{\mu\nu}\mathfrak{D}_\rho Z^i\mathfrak{D}^\rho Z^{*\,j^*}] & \\
+4\mathcal{M}_{MN}\mathcal{F}^M{}_\mu{}^\rho\mathcal{F}^N{}_{\nu\rho} + \tfrac{1}{2}g_{\mu\nu}V(Z,Z^*) &= 0, \qquad (1.11)\\
\mathfrak{D}^2 Z^i + \partial^i G_{\Lambda\,\mu\nu} \star F^{\Lambda\,\mu\nu} + \tfrac{1}{2}\partial^i V(Z,Z^*) &= 0, \qquad (1.12)\\
\mathfrak{D}_\nu \star G_\Lambda{}^{\nu\mu} + \tfrac{1}{4}g\left(k_{\Lambda\,i^*}\mathfrak{D}_\mu Z^{*i^*} + k^*_{\Lambda\,i}\mathfrak{D}_\mu Z^i\right) &= 0\,, \qquad (1.13)
\end{aligned}
$$

where $G_{\Lambda\,\mu\nu}$ is the dual vector field strength

$$
G_\Lambda \equiv \Re\mathfrak{e}\mathcal{N}_{\Lambda\Sigma}F^\Sigma + \Im\mathfrak{m}\mathcal{N}_{\Lambda\Sigma} \star F^\Sigma\,, \qquad (1.14)
$$

$\mathcal{F}^M{}_{\mu\nu}$ is the symplectic vector of vector field strengths

$$
\left(\mathcal{F}^M\right) \equiv \begin{pmatrix} F^\Lambda \\ G_\Lambda \end{pmatrix}, \qquad (1.15)
$$

$\mathcal{M}_{MN}$ is the symmetric $2(n+1)\times 2(n+1)$ matrix defined by

$$
(\mathcal{M}_{MN}) \equiv \begin{pmatrix} \Im\mathfrak{m}\mathcal{N}_{\Lambda\Sigma} + R_{\Lambda\Gamma}\Im\mathfrak{m}\mathcal{N}^{-1|\Gamma\Omega}R_{\Omega\Sigma} & -R_{\Lambda\Gamma}\Im\mathfrak{m}\mathcal{N}^{-1|\Gamma\Sigma} \\ & \\ -\Im\mathfrak{m}\mathcal{N}^{-1|\Lambda\Omega}R_{\Omega\Sigma} & \Im\mathfrak{m}\mathcal{N}^{-1|\Lambda\Sigma} \end{pmatrix}, \qquad (1.16)
$$

and

$$
\mathfrak{D}_\nu \star G_\Lambda{}^{\nu\mu} = \partial_\nu \star G_\Lambda{}^{\nu\mu} + gf_{\Lambda\Sigma}{}^\Gamma A^\Sigma{}_\nu \star G_\Lambda{}^{\nu\mu}\,. \qquad (1.17)
$$

Most of the literature and earlier work on non-Abelian black-hole and monopole solutions has been carried out in the context of the Einstein-Yang-Mills (EYM) and Einstein-Yang-Mills-Higgs (EYMH) theories. Before closing this introduction, it is worth discussing the relation between those and the theories we are considering here. The main differences of the latter w.r.t. the former are the complexification of the Higgs field and the presence of a non-trivial period matrix. A further difference is the possibility of having more general scalar manifolds, which is reflected in the expressions of the gauge-covariant derivatives of the scalar fields. Solutions to the $\mathcal{N}=2$, $d=4$ SEYM theory have a chance of being also solutions of the EYMH theory if they have covariantly-constant scalars with identical phases (*e.g.* all of them purely imaginary). Then, if the scalar potential vanishes on the solutions, they also have a chance of being solutions to the EYM system as well; as we are going to see, some of the solutions found in Refs. [12, 13] are also solutions of the EYM theory and have the same metric as the EYM solutions of Refs. [25, 29].

1.2 The recipe to construct SBHSs of $\mathcal{N}=2$, $d=4$ SEYM

To construct timelike supersymmetric solutions of the $\mathcal{N}=2$, $d=4$ SEYM theory, it suffices to follow this recipe [12, 13] to find the elementary building blocks of the solutions, which are the $2(n+1)$ time-independent functions $(\mathcal{I}^M) = \begin{pmatrix} \mathcal{I}^\Lambda \\ \mathcal{I}_\Lambda \end{pmatrix}$:

1. Take a solution of the Bogomol'nyi equations

$$\tilde{F}^{\Lambda}{}_{\underline{mn}} = -\tfrac{1}{\sqrt{2}}\varepsilon_{mnp}\tilde{\mathfrak{D}}_{\underline{p}}\mathcal{I}^{\Lambda}, \tag{1.18}$$

   for a gauge field $\tilde{A}^{\Lambda}{}_{\underline{m}}$ ($\underline{m}=1,2,3$ labels the 3 spatial coordinates) and a real "Higgs" field $\mathcal{I}^\Lambda$. $\tilde{\mathfrak{D}}_{\underline{p}}\mathcal{I}^\Lambda$ is the covariant derivative in the adjoint representation with gauge field $\tilde{A}^{\Lambda}{}_{\underline{m}}$. Observe that this equation has to be solved in the gauged (non-Abelian) and ungauged (Abelian) directions. The integrability condition in the Abelian directions is the familiar requirement that the $\mathcal{I}^\Lambda$ be harmonic functions on $\mathbb{R}^3$.

2. Find the functions $\mathcal{I}_\Lambda$ by solving these equations:

$$\tilde{\mathfrak{D}}_{\underline{m}}\tilde{\mathfrak{D}}_{\underline{m}}\mathcal{I}_\Lambda = \tfrac{1}{2}g^2\left[f_{\Lambda(\Sigma}{}^{\Gamma}f_{\Delta)\Gamma}{}^{\Omega}\,\mathcal{I}^\Sigma\mathcal{I}^\Delta\right]\mathcal{I}_\Omega\,. \tag{1.19}$$

   In the non-Abelian directions these equations can, in many cases, be solved by taking $\mathcal{I}_\Lambda \propto \mathcal{I}^\Lambda$, but currently we only know how to generate non-trivial solutions to them in the cases where the gauge doublet $(\tilde{A}^\Lambda, \mathcal{I}^\Lambda)$ describes a non-Abelian Wu-Yang monopole; Observe that $\mathcal{I}_\Lambda = 0$ is always a solution, but the physical fields may be singular in some models.

   In the Abelian directions, the $\mathcal{I}_\Lambda$ are just independent harmonic functions on $\mathbb{R}^3$.

3. Given the functions $\mathcal{I}^M$, we must find the 1-form on $\mathbb{R}^3$ $\omega_{\underline{m}}$ by solving the following equation:

$$\partial_{[\underline{m}}\omega_{\underline{n}]} = \varepsilon_{mnp}\mathcal{I}_M\tilde{\mathfrak{D}}_{\underline{p}}\mathcal{I}^M = \varepsilon_{mnp}\left(\mathcal{I}_\Lambda\tilde{\mathfrak{D}}_{\underline{p}}\mathcal{I}^\Lambda - \mathcal{I}^\Lambda\tilde{\mathfrak{D}}_{\underline{p}}\mathcal{I}_\Lambda\right)\,. \tag{1.20}$$

   The integrability conditions of this equation impose constraints on the integration constants of the functions $\mathcal{I}^M$ in exactly the same manner as in the ungauged case [9, 53].

   In the case of static solutions, *i.e.* when $\omega = 0$, the above equation becomes a constraint on the integration constants of the functions $\mathcal{I}^M$ that will have to be solved. Observe, however, that this constraint is independent of the specific $\mathcal{N}=2$, $d=4$ model and only depends on the choice of gauge group; possible restrictions on the solution to said constraint can come from the desired behaviour of the physical fields in the full solution.

4. To reconstruct the physical fields from the functions $\mathcal{I}^M$ we need to solve the stabilization equations, a.k.a. *Freudenthal duality equations*, which give the components of the Freudenthal dual[10] $\tilde{\mathcal{I}}^M(\mathcal{I})$ in terms of the functions $\mathcal{I}^M$ [55]; These relations completely characterize the model of $\mathcal{N}=2$, $d=4$ supergravity.

   Equivalently, the $\tilde{\mathcal{I}}$ can be derived from a homogeneous function of degree 2 $W(\mathcal{I})$ called the *Hesse potential* as [53, 56, 57]

$$\tilde{\mathcal{I}}_M = \tfrac{1}{2}\tfrac{\partial W}{\partial \mathcal{I}^M} \quad \longrightarrow \quad W(\mathcal{I}) = \tilde{\mathcal{I}}_M \mathcal{I}^M \,. \tag{1.21}$$

5. The metric takes the form

$$ds^2 = e^{2U}(dt+\omega)^2 - e^{-2U} dx^m dx^m \,, \tag{1.22}$$

   where $\omega = \omega_{\underline{m}} dx^m$ is the above spatial 1-form and the metric function e^{-2U} is given by

$$e^{-2U} = \tilde{\mathcal{I}}_M(\mathcal{I})\mathcal{I}^M = W(\mathcal{I})\,. \tag{1.23}$$

6. The scalar fields are given by

$$Z^i = \frac{\tilde{\mathcal{I}}^i + i\mathcal{I}^i}{\tilde{\mathcal{I}}^0 + i\mathcal{I}^0}\,. \tag{1.24}$$

7. The components of the vector fields are given by

$$A^{\Lambda}{}_t = -\tfrac{1}{\sqrt{2}} e^{2U} \tilde{\mathcal{I}}^{\Lambda}\,, \tag{1.25}$$

$$A^{\Lambda}{}_{\underline{m}} = \tilde{A}^{\Lambda}{}_{\underline{m}} + \omega_{\underline{m}}\, A^{\Lambda}{}_t\,. \tag{1.26}$$

After having gone through the steps of the recipe, one ends up with a supersymmetric solution to a chosen $\mathcal{N}=2$, $d=4$ EYM theory and what remains to be done is to analyze the constraints coming from imposing appropriate regularity conditions such as the absence of naked singularities.

[10] In Refs. [54, 12, 13] the components of the Freudenthal dual are denoted by $\mathcal{R}^M$.

2 Static, single-SBHSs of $\mathrm{SU}(2)$ $\mathcal{N}=2$, $d=4$ SEYM and pure EYM

Following the recipe given in Section 1.2, we are going to construct static, single-center SBHSs of SU(2) $\mathcal{N}=2$, $d=4$ SEYM. Some of the solutions will simultaneously solve the equations of motion of the EYM and EYMH theories.

The first step consists in finding a solution $\tilde{A}^{\Lambda}{}_{\underline{m}}, \mathcal{I}^{\Lambda}$ of the SU(2) Bogomol'nyi equations in $\mathbb{R}^3$ Eqs. (1.18).

2.1 Single-center solutions of the $\mathrm{SU}(2)$ Bogomol'nyi equations in $\mathbb{R}^3$

Before we search for solutions of the Bogomol'nyi equations it is worth reviewing the origin and meaning of those equations in the context of the SU(2) Yang-Mills-Higgs theory (in the Bogomol'nyi-Prasad-Sommerfield (BPS) limit in which the Higgs potential vanishes).

2.1.1 The $\mathrm{SU}(2)$ Yang-Mills-Higgs system

With the normalization in Eq. (1.10) and writing $F \equiv F^a T_a, \Phi \equiv \Phi^a T_a$, the action of the YMH theory in our conventions reads

$$S_{\mathrm{YMH}} = -2\int d^4x\, \mathrm{Tr}\left\{\tfrac{1}{2}\mathfrak{D}_{\mu}\Phi\mathfrak{D}^{\mu}\Phi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\right\}\,, \tag{2.1}$$

and the corresponding equations of motion are

$$\mathfrak{D}_{\mu}F^{\mu\nu} = g[\Phi, \mathfrak{D}^{\nu}\Phi]\,, \tag{2.2}$$

$$\mathfrak{D}^2\Phi = 0\,. \tag{2.3}$$

For static configurations $F_{t\underline{m}} = \mathfrak{D}_t\Phi = 0$, the action can be written, up to a total derivative, in the manifestly positive form

$$S_{\mathrm{YMH}} = -2\int d^4x\, \mathrm{Tr}\left\{-\tfrac{1}{4}\left(F_{\underline{mn}} \mp \varepsilon_{mnp}\mathfrak{D}_{\underline{p}}\Phi\right)\left(F_{\underline{mn}} \mp \varepsilon_{mnp}\mathfrak{D}_{\underline{p}}\Phi\right)\right\}\,, \tag{2.4}$$

which leads to the conclusion that static field configurations satisfying the first-order *Bogomol'nyi equations* [15]

$$F_{\underline{mn}} = \pm\varepsilon_{mnp}\mathfrak{D}_{\underline{p}}\Phi\,, \tag{2.5}$$

extremize the action Eq. (2.1) and are solutions of the full Yang-Mills-Higgs equations. Indeed, if we act with $\mathfrak{D}_{\underline{m}}$ on both sides of the equation and use the Ricci identity and the Bogomol'nyi equation we get the Yang-Mills equation:

$$\mathfrak{D}_{\underline{m}}F_{\underline{mn}} = \mp\varepsilon_{nmp}\mathfrak{D}_{\underline{m}}\mathfrak{D}_{\underline{p}}\Phi = \mp\tfrac{1}{2}g\varepsilon_{nmp}[F_{\underline{mp}},\Phi] = -g[\mathfrak{D}_{\underline{n}}\Phi,\Phi]\,. \tag{2.6}$$

If, instead, we act with $\varepsilon_{pmn}\mathfrak{D}_{\underline{p}}$ and use the Bianchi identity, we get the Higgs equation:

$$0 = \varepsilon_{pmn}\mathfrak{D}_{\underline{p}}F_{\underline{mn}} = \pm\mathfrak{D}_{\underline{p}}\mathfrak{D}_{\underline{p}}\Phi\,. \tag{2.7}$$

Observe that the source of the Yang-Mills field, the *Higgs current* $g[\Phi,\mathfrak{D}\Phi]$, not only vanishes when the Higgs field is covariantly constant $\mathfrak{D}\Phi = 0$ but also when Φ and $\mathfrak{D}\Phi$ are parallel in $\mathfrak{su}(2)$.

Eqs. (2.5) are identical to the ones that arise in $\mathcal{N} = 2$, $d = 4$ SEYM theory, (1.18) upon the identification of the vector fields and

$$\tfrac{1}{\sqrt{2}}\mathcal{I}^a = \mp\Phi^a\,. \tag{2.8}$$

2.1.2 The hedgehog ansatz

In order to construct static, single-center black-hole-type solutions, it is natural to look for spherically symmetric solutions of Eqs. (2.5). Substituting the *hedgehog ansatz*

$$\mp\Phi^a = \delta^a{}_m f(r)x^m\,, \qquad A^a{}_{\underline{m}} = -\varepsilon^a{}_{mn}x^n h(r) \tag{2.9}$$

in the Bogomol'nyi Eqs. (2.5) we get an equivalent system of differential equations for $f(r)$ and $h(r)$:

$$\begin{aligned} r\partial_r h + 2h - f(1+gr^2h) &= 0\,,\\ r\partial_r(h+f) - gr^2h(h+f) &= 0\,. \end{aligned} \tag{2.10}$$

After Prasad and Sommerfield [58] found the solution describing the 't Hooft-Polyakov monopole in the BPS limit, Protogenov [59] classified all spherically symmetric solutions to the SU(2) Bogomol'nyi equations: the ones that can be used to generate BH-like spacetimes are a 2-parameter family $(f_{\mu,s},h_{\mu,s})$ plus a 1-parameter family (f_λ,h_λ) given by

$$\begin{aligned} rf_{\mu,s} &= \frac{1}{gr}\left[1-\mu r\coth(\mu r+s)\right], & rh_{\mu,s} &= \frac{1}{gr}\left[\frac{\mu r}{\sinh(\mu r+s)}-1\right],\\ rf_\lambda &= \frac{1}{gr}\left[\frac{1}{1+\lambda^2 r}\right], & rh_\lambda &= -rf_\lambda\,. \end{aligned} \tag{2.11}$$

The parameter s is known in the black-hole context as the *Protogenov hair* parameter [14]. The BPS 't Hooft-Polyakov monopole [58] is the only globally regular solution of this family (which explains why it is the only one usually considered in the monopole literature[11]) and corresponds to $s = 0$. In the $s \to \infty$ limit we get

$$-rf_{\mu,\infty} = \frac{\mu}{g} - \frac{1}{gr}, \qquad rh_{\mu,\infty} = -\frac{1}{gr}, \tag{2.12}$$

which, for $\mu = 0$, coincides with the Wu-Yang monopole [26] given in Eq. (B.10), and is a solution of the pure Yang-Mills theory. This is possible because the Higgs current $g[\Phi, \mathfrak{D}\Phi]$ vanishes even though Φ is neither zero nor covariantly constant[12]. With a non-trivial Higgs field, though, we can assign a well-defined monopole charge to it: for any μ and s

$$\frac{1}{4\pi}\int_{S^2_\infty} \mathrm{Tr}(\hat{\Phi}F) = \frac{1}{g}, \qquad \hat{\Phi} \equiv \frac{\Phi}{\sqrt{|\mathrm{Tr}(\Phi^2)|}}. \tag{2.13}$$

The same field configuration can be seen as a Lorentzian meron (see Appendix A) and as a solution to the Skyrme model (see Appendix C), and, crucially, it is related to the SU(2)-embedded Dirac monopole by a singular gauge transformation (see Appendix B). Since the metric is oblivious to gauge transformations, singular or not, the Wu-Yang monopole gives rise to solutions whose metric is identical to that of Abelian case.[13] The same applies to the higher-charge generalizations of the Lorentzian meron/Wu-Yang monopole reviewed in Appendix D.

If fact, this mechanism can be used to generate Wu-Yang monopoles of higher charge from the well-known Dirac monopole solutions of charge higher than 1 embedded in SU(2), as reviewed in Appendix D. The metric cannot see the difference between the non-Abelian and the Abelian fields given in Eq. (2.12).

The 1-parameter family is singular for all values of the parameter λ, which also appears in black-hole solutions as hair. The magnetic charge measured at spatial infinity vanishes according to the above definition. However, it can be argued that these solutions do describe a magnetic monopole placed at the origin whose charge is screened: the entropy of black hole associated to this field has the same form as that of the black hole associated to the Wu-Yang monopole. Observe that, for $\lambda = 0$, the solution is identical to the Wu-Yang monopole with $\mu = 0$, Eqs. (2.12).

2.1.3 The Protogenov trick

As it turns out, many regular monopole solutions can be deformed by adding a parameter s to the argument μr, generating a family of solutions that contains the original one ($s = 0$) and, typically, a new and simpler solution in the $s \to \infty$ limit. We will refer to this procedure as *the Protogenov trick* and it can be justified as follows: let us consider, for instance, the 't Hooft-Polyakov monopole. Since the Bogomol'nyi equation is

[11] After coupling the system to gravity, the singularities of the other solutions may become "harmless" if they can be covered by regular event horizons.

[12] Actually, the only field configuration in this ansatz with a vanishing Higgs current is this one.

[13] Of course there are measurable differences between these two situations, see *e.g.* Refs. [28, 29].

polynomial, having elementary functions such as hyperbolic functions in the solution means that they must cancel amongst themselves and that only their derivatives contribute to the polynomial part of the solution. This means that one should be able to deform the dependency of the elementary functions introducing a shift s of the radial coordinate and still solve the Bogomol'nyi equations.

Of course, the cancellations necessary for having a regular solution will not work out anymore (assuming they did work for $s = 0$) and one will end up with a family of singular solutions. We will use this trick later.

2.2 Embedding in the $\mathrm{SU}(2)$-gauged $\overline{\mathbb{CP}}^3$ model

2.2.1 The $\overline{\mathbb{CP}}^3$ model

The $\overline{\mathbb{CP}}^n$ models have n vector supermultiplets and are defined by the quadratic prepotentials

$$\mathcal{F} = -\tfrac{i}{4}\eta_{\Lambda\Sigma}\mathcal{X}^{\Lambda}\mathcal{X}^{\Sigma}\,, \qquad (\eta_{\Lambda\Sigma}) = \mathrm{diag}(+-\cdots-)\,. \tag{2.14}$$

The n physical scalar fields can be defined as

$$Z^i \equiv \mathcal{X}^i/\mathcal{X}^0\,, \tag{2.15}$$

and they parametrize the symmetric space $\mathrm{U}(1,n)/(\mathrm{U}(1)\times\mathrm{U}(n))$. It is convenient to define $Z^0 \equiv 1$, $Z^{\Lambda} \equiv \mathcal{X}^{\Lambda}/\mathcal{X}^0$ and $Z_{\Lambda} \equiv \eta_{\Lambda\Sigma}Z^{\Sigma}$. In the $\mathcal{X}^0 = 1$ gauge, the Kähler potential and the Kähler metric are given by

$$\mathcal{K} = -\log\left(Z^{*\Lambda}Z_{\Lambda}\right)\,, \quad \mathcal{G}_{ij^*} = -e^{\mathcal{K}}\left(\eta_{ij^*} - e^{\mathcal{K}}Z^*_i Z_{j^*}\right)\,, \quad \Rightarrow \quad 0 \leq \sum_i |Z^i|^2 < 1\,. \tag{2.16}$$

The above metric is the standard (Bergman) metric for the $\mathrm{U}(1,n)/(\mathrm{U}(1)\times\mathrm{U}(n))$ symmetric spaces [60]. The covariantly holomorphic symplectic section $\mathcal{V}$ and the period matrix $\mathcal{N}_{\Lambda\Sigma}$ are given by

$$\mathcal{V} = e^{\mathcal{K}/2}\begin{pmatrix} Z^{\Lambda} \\ \\ -\tfrac{i}{2}Z_{\Lambda}\end{pmatrix}\,, \qquad \mathcal{N}_{\Lambda\Sigma} = \tfrac{i}{2}\left[\eta_{\Lambda\Sigma} - 2\frac{Z_{\Lambda}Z_{\Sigma}}{Z^{\Gamma}Z_{\Gamma}}\right]\,. \tag{2.17}$$

The isometry subgroup $\mathrm{SU}(1,n)$ acts linearly, in the fundamental representation, on the coordinates $\mathcal{X}^{\Lambda}$

$$\mathcal{X}'^{\Lambda} = \Lambda^{\Lambda}{}_{\Sigma}\mathcal{X}^{\Sigma}\,, \quad \text{with} \quad \Lambda^{\dagger}\eta\Lambda = \eta\,, \quad \text{and} \quad \det\Lambda = 1\,. \tag{2.18}$$

This linear action induces a non-linear action on the special coordinates:

$$Z'^{\Lambda} = \frac{\Lambda^{\Lambda}{}_{\Sigma}Z^{\Sigma}}{\Lambda^{0}{}_{\Sigma}Z^{\Sigma}}\,. \tag{2.19}$$

The Kähler potential is invariant under these transformations up to Kähler transformations $\mathcal{K}' = \mathcal{K} + f + f^*$ with

$$f(Z) = \log\left(\Lambda^0{}_\Sigma Z^\Sigma\right). \tag{2.20}$$

The $n(n+2)$ infinitesimal generators T_m of $\mathfrak{su}(1,n)$ in the fundamental representation are defined by

$$\Lambda^\Lambda{}_\Sigma \sim \delta^\Lambda{}_\Sigma + \alpha^m\, T_m{}^\Lambda{}_\Sigma\,, \quad \text{with} \quad \eta T_m^\dagger \eta = -T_m\,, \quad \text{and} \quad T_m{}^\Lambda{}_\Lambda = 0\,. \tag{2.21}$$

Substituting this definition into Eq. (2.19) we find an expression for the holomorphic Killing vectors[14].

$$Z'^\Lambda = Z^\Lambda + \alpha^m k_m{}^\Lambda(Z)\,, \qquad k_m{}^\Lambda(Z) = T_m{}^\Lambda{}_\Sigma\, Z^\Sigma - T_m{}^0{}_\Omega\, Z^\Omega Z^\Lambda\,, \tag{2.22}$$

and, from this expression, we also find explicit expressions for the holomorphic functions $\lambda_m(Z)$ and the momentum maps

$$\lambda_m = T_m{}^0{}_\Sigma Z^\Sigma\,, \qquad \mathcal{P}_m = ie^{\mathcal{K}} T_m{}^\Lambda{}_\Sigma Z^\Sigma Z^*_\Lambda = ie^{\mathcal{K}} \eta_{\Lambda\Omega} T_m{}^\Lambda{}_\Sigma Z^\Sigma Z^{*\,\Omega}\,. \tag{2.23}$$

Although the theory is invariant under the whole $\mathrm{SU}(1,n)$ group, the prepotential is invariant only under the subgroup of $\mathrm{SU}(1,n)$ with real matrices, $\mathrm{SO}(1,n)$, which is the largest group that we could eventually gauge. However, the requirements that the vectors must transform in the adjoint representation restricts the possibilities to either $\mathrm{SO}(1,2)$ or $\mathrm{SO}(3)$ (if $n \geq 2$ or $n \geq 3$, respectively); we are going to consider the latter case embedded into the minimal model admitting this gauge group, namely $\overline{\mathbb{CP}}^3$.

In this model, the adjoint indices $a, b, c, \ldots$ and the fundamental indices $i, j, k, \ldots$ take the same values $1, 2, 3$ and we will only use the latter. The infinitesimal transformations of the scalars are

$$\delta_\alpha Z^i = \alpha^j T_j{}^i{}_k Z^k\,, \quad \text{where} T_j{}^i{}_k = f_{jk}{}^i = -\epsilon_{jki}\,, \tag{2.24}$$

and the momentum maps, holomorphic Killing vectors etc. take the values

$$\mathcal{P}_i = -ie^{\mathcal{K}} \epsilon_{ijk} Z^j Z^{*\,k}\,, \qquad k_i{}^j = \epsilon_{ijk} Z^k\,, \qquad \lambda_i = 0\,. \tag{2.25}$$

This means that the gauge-covariant derivative of the scalars is just that of a complex adjoint $\mathrm{SO}(3)$ scalar

$$\mathfrak{D}_\mu Z^i = \partial_\mu Z^i - g\epsilon_{ijk} A^j{}_\mu Z^k, \tag{2.26}$$

and that the scalar potential takes the form

$$V(Z, Z^*) \;=\; -\tfrac{1}{2} g^2 e^{\mathcal{K}} \epsilon_{ijk} \epsilon_{imn} Z^j Z^{*k^*} Z^m Z^{*n^*} \;=\; \tfrac{1}{2} g^2 \left| \vec{Z} \times \vec{Z}^* \right|^2\,. \tag{2.27}$$

[14]The $k_m{}^0(Z)$ component vanishes identically, as it must, but it is convenient to keep it.

2.2.2 The solutions

To construct the solutions of this model[15] we just have to follow the recipe spelled out in Section 1.2. We will only consider static solutions (so $\omega = 0$ and $\tilde{A}^{\Lambda}{}_{\underline{m}} = A^{\Lambda}{}_{\underline{m}}$). First of all, we need a solution of the Bogomol'nyi Eqs. (1.18). These equations split into an Abelian part (the 0th component) and the non-Abelian part (the $i = 1,2,3$ components):

$$F^{0}{}_{\underline{mn}} = -\tfrac{1}{\sqrt{2}}\epsilon_{mnp}\partial_{\underline{p}}\mathcal{I}^{0}\,, \tag{2.28}$$

$$F^{i}{}_{\underline{mn}} = -\tfrac{1}{\sqrt{2}}\epsilon_{mnp}\mathfrak{D}_{\underline{p}}\mathcal{I}^{i}\,. \tag{2.29}$$

The Abelian equation is solved by

$$\mathcal{I}^{0} = A^{0} + \frac{p^{0}/\sqrt{2}}{r}\,, \tag{2.30}$$

where A^0 is an integration constant and p^0 is the normalized Abelian magnetic charge. The non-Abelian set of equations can be solved making the identification Eq. (2.8) and using Protogenov's solutions Eqs. (2.11).

The second step in the recipe (finding solutions $\mathcal{I}_{\Lambda}$ to Eqs. (1.19)) will be solved, for the sake of simplicity, by choosing another harmonic function in the Abelian direction and vanishing functions in the rest:

$$\mathcal{I}_{0} = A_{0} + \frac{q_{0}/\sqrt{2}}{r}\,, \qquad \mathcal{I}_{i} = 0\,. \tag{2.31}$$

The third point in the recipe, combined with the staticity of the solutions implies the following constraint on the integration constants:

$$A^{0}q_{0} - A_{0}p^{0} = 0\,. \tag{2.32}$$

Another constraint will arise from the normalization of the metric at infinity. The solution is completely determined and, now, we only have to write the physical fields and make, if necessary, sensible choices of the values of the physical parameters to make the solutions regular.

In order to write the physical fields we need the solutions of the Freudenthal duality equations of this model. These are given by (see, *e.g.* Ref. [61])

$$(\tilde{\mathcal{I}}^{M}) = \begin{pmatrix} \tilde{\mathcal{I}}^{\Lambda} \\ \tilde{\mathcal{I}}_{\Lambda} \end{pmatrix} = \begin{pmatrix} -2\eta^{\Lambda\Sigma}\mathcal{I}_{\Sigma} \\ \tfrac{1}{2}\eta_{\Lambda\Sigma}\mathcal{I}^{\Sigma} \end{pmatrix}, \quad \Rightarrow \quad e^{-2U} = \tfrac{1}{2}\eta_{\Lambda\Sigma}\mathcal{I}^{\Lambda}\mathcal{I}^{\Sigma} + 2\eta^{\Lambda\Sigma}\mathcal{I}_{\Lambda}\mathcal{I}_{\Sigma}\,, \tag{2.33}$$

[15] All these solutions have already been presented in Refs. [12, 14, 13]. We review them here for pedagogical reasons and also for the sake of making easier the comparison with the solutions of other models.

and the metric function and the physical scalars are given by

$$e^{-2U} = \tfrac{1}{2}(\mathcal{I}^0)^2 + 2(\mathcal{I}_0)^2 - (rf)^2, \tag{2.34}$$

$$Z^i = \frac{\sqrt{2}rf}{\mathcal{I}^0 + 2i\mathcal{I}_0}\delta^i{}_m \frac{x^m}{r}. \tag{2.35}$$

At least one of the two functions $\mathcal{I}^0, \mathcal{I}_0$ must be different from zero for the metric function to be positive. Then, there are two possible cases, depending on the vanishing of the Abelian charges p^0, q_0:

I. $p^0 = q_0 = 0$ The only regular solution is the one with $s = 0$ (the 't Hooft-Polyakov monopole). In this solution, the integration constants satisfy the normalization condition

$$\tfrac{1}{2}(A^0)^2 + 2(A_0)^2 = 1 + (\mu/g)^2\,. \tag{2.36}$$

They are also related to the asymptotic values of the scalars. These cannot be constant, in general, because the scalars transform under local $\mathrm{SU}(2)$ transformations, but they are covariantly constant and their asymptotic values are determined by a single gauge-invariant complex parameter that we call Z_∞:[16]

$$Z^i \sim Z_\infty \delta^i{}_m \frac{x^m}{r}\,, \qquad Z_\infty \equiv \frac{\mu/g}{1+(\mu/g)^2}\left(\tfrac{1}{\sqrt{2}}A^0 - \sqrt{2}iA_0\right)\,, \qquad 0 \leq |Z_\infty|^2 < 1\,. \tag{2.37}$$

These expressions lead to the following identification of the integration constant μ in terms of the physical parameters:

$$\mu^2 = \frac{|Z_\infty|^2}{1-|Z_\infty|^2}g^2\,, \tag{2.38}$$

and to the following expression for the mass of the solution

$$M_{\text{monopole}} = \sqrt{\frac{|Z_\infty|^2}{1-|Z_\infty|^2}}\frac{1}{g}\,. \tag{2.39}$$

This asymptotically flat solution has no singularities nor horizons (one finds a Minkowski spacetime in the $r \to 0$ limit, hence zero entropy and temperature). Globally-regular solutions of this kind [39, 40] are sometimes called *global monopoles*.

[16] Observe that the scalar potential of this theory, Eq. (2.27), vanishes at infinity for those solutions, which is why they are asymptotically flat.

Observe that a solution of the ungauged theory with

$$H^0 = A^0\,, \quad H_0 = A_0\,, \quad H^1 = A^1 + \frac{\sqrt{2}}{gr}\,, \tag{2.40}$$

in which the non-Abelian monopole is replaced by an Abelian monopole with the same charge, would have the same asymptotic behavior but it would mean having a naked singularity at some value of $r > 0$.

II. $p^0 q_0 \neq 0$ [17] Solving Eq. (2.32) the metric can be written in the form

$$e^{-2U} = \frac{1}{1-|Z_\infty|^2} H^2 - (rf)^2, \tag{2.41}$$

$$Z^i = \frac{2\beta}{p^0 + 2iq_0} \frac{rf}{H} \delta^i{}_m \frac{x^m}{r}\,, \tag{2.42}$$

where H is the harmonic function

$$H \equiv 1 + \frac{\beta}{r}\,, \qquad \beta^2 = (1-|Z_\infty|^2) W_{\rm RN}(\mathcal{Q})/2\,, \qquad W_{\rm RN}(\mathcal{Q}) \equiv \tfrac{1}{2}(p^0)^2 + 2(q_0)^2\,, \tag{2.43}$$

and the integration constant μ is still given by Eq. (2.38). We have expressed all the constants (except for Protogenov's hair parameter s and λ) in terms of physical constants. Observe that the isolated solution f_* has $\mu = 0$ and corresponds to $Z_\infty = 0$. These identifications allow us to compute the mass and entropy of all the possible solutions in terms of the physical parameters. We get a completely general mass formula and two formulae for the entropy, one for the $s \neq 0$ solutions and another one for the $s = 0$ and the isolated solutions (which corresponds to $Z_\infty = 0$):

$$M = \sqrt{\frac{1}{2}\frac{W_{RN}(\mathcal{Q})}{1-|Z_\infty|^2}} + M_{\rm monopole}, \tag{2.44}$$

$$S/\pi = \tfrac{1}{2} W_{\rm RN}(\mathcal{Q}) - \frac{1}{g^2}\,, \quad \text{for} \quad s \neq 0 \;\; \text{and} \;\; Z_\infty = 0, \tag{2.45}$$

$$S/\pi = \tfrac{1}{2} W_{\rm RN}(\mathcal{Q}), \quad \text{for} \quad s = 0\,, \tag{2.46}$$

[17] It is easier to work with both charges non-vanishing. The results will still be valid when we set one of them to zero.

where M_{monopole} is given by Eq. (2.39).

The entropy is moduli-independent as in the ungauged case and both the entropy and the mass are independent of the hair parameters s and λ.

Observe that the charge of the BPS 't Hooft-Polyakov monopole $s = 0$ does not contribute to the entropy which suggests that it must be associated to a pure state in the quantum theory. The non-Abelian field of the isolated solution does not contribute to the mass at infinity ($M_{\text{monopole}} = 0$ for $Z_\infty = 0$) but there is a magnetic-charge contribution to the entropy, which suggests that there really is a magnetic charge but it is screened at infinity. Further support for this interpretation comes from the near-horizon limit of the scalars, which is the covariantly-constant function of the charges

$$Z^i_{\text{h}} = \frac{1/g}{\frac{1}{2}p^0 + iq_0}\,\delta^i{}_m \frac{x^m}{r}. \tag{2.47}$$

even for the isolated case, when no magnetic charge is observed at infinity.

In the case of the 1-parameter (λ) family, neither the mass nor the entropy depend on λ.

Some of the solutions in this family can also be seen as solutions of the pure EYM theory. They are identical to those obtained in Refs. [25, 29]. As discussed at the end of Section 1.1, we need to tune the parameters of the solutions so as to get covariantly constant scalars which do not contribute to the energy-momentum tensor This is only possible for the $s \to \infty$ solutions (Wu–Yang monopoles) for which rf is a harmonic function. In that case

$$Z^i = Z\,\delta^i{}_m \frac{x^m}{r}\,, \qquad Z = \frac{1/g}{\frac{1}{2}p^0 + iq_0} = Z_\infty\,. \tag{2.48}$$

The metric is identical to that of a Reissner-Nordström black hole. These solutions were called *black hedgehogs* in Ref. [12] and *black merons* in Ref. [29] because the gauge field of the Wu–Yang monopole can also be understood as Lorentzian meron solution.

A closely related solution with non-covariantly constant scalars was obtained in a different context in Ref. [62].

2.3 Embedding in $\mathrm{SU}(2)$-gauged $\mathrm{ST}[2,n]$ models

2.3.1 The $ST[2,n]$ models

The $ST[2,n]$ models are cubic models with $n_V = n+1$ vector supermultiplets and as many complex scalars and, as all other cubic models, they can be embedded in type II String Theory compactified Calabi-Yau 3-folds and then uplifted to M-theory.

They can also be obtained from corresponding models of $N = 1$, $d = 5$ supergravity compactified on S^1.

A generic cubic model is defined by the prepotential

$$\mathcal{F} = -\frac{1}{3!}\, d_{ijk}\, \frac{\mathcal{X}^i \mathcal{X}^j \mathcal{X}^k}{\mathcal{X}^0}\,, \tag{2.49}$$

where d is completely symmetric in its indices; the $ST[2, n]$ models are characterized by d-tensors with non-vanishing components $d_{1\alpha\beta} = \eta_{\alpha\beta}$ where $(\eta_{\alpha\beta}) = \mathrm{diag}(+ - \cdots -)$ and where the indices α, β take n values between 2 and $n + 1$.

The scalar $Z^1 = \mathcal{X}^1/\mathcal{X}^0$ plays a special role and parametrizes a $\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2)$ coset space. For this and other reasons, it is called axidilaton and we will denote it by τ. The other n scalars parametrize a $\mathrm{SO}(2, n)/(\mathrm{SO}(2)\times\mathrm{SO}(n))$ coset space and will be denoted by $Z^\alpha = \mathcal{X}^\alpha/\mathcal{X}^0$ ($\alpha = 2, \cdots, n$). The Kähler metric and 1-form connection are the products of those of the two spaces.

Using this notation and using the gauge $\mathcal{X}^0 = 1$, the canonical symplectic section Ω, the Kähler potential $\mathcal{K}$ and the components of Kähler 1-form $\mathcal{Q}_i$ and of the Kähler metric $\mathcal{G}_{ij^*}$ are given by

$$\begin{aligned}
\Omega &= \begin{pmatrix} 1 \\ \tau \\ Z^\alpha \\ \frac{1}{2}\tau\eta_{\alpha\beta}Z^\alpha Z^\beta \\ -\frac{1}{2}\eta_{\alpha\beta}Z^\alpha Z^\beta \\ -\tau\eta_{\alpha\beta}Z^\beta \end{pmatrix}, &\qquad e^{-\mathcal{K}} &= 4\Im\mathrm{m}\,\tau\;\eta_{\alpha\beta}\Im\mathrm{m}\,Z^\alpha\;\Im\mathrm{m}\,Z^\beta, \\
\mathcal{Q}_\tau &= \frac{1}{4\Im\mathrm{m}\,\tau}, &\qquad \mathcal{Q}_\alpha &= \frac{\eta_{\alpha\beta}\Im\mathrm{m}\,Z^\beta}{2\eta_{\gamma\delta}\Im\mathrm{m}\,Z^\gamma\;\Im\mathrm{m}\,Z^\delta}, \\
\mathcal{G}_{\tau\tau^*} &= \frac{1}{4(\Im\mathrm{m}\,\tau)^2}, &\qquad \mathcal{G}_{\alpha\beta^*} &= \frac{\eta_{\alpha\gamma}\Im\mathrm{m}\,Z^\gamma\;\eta_{\beta\delta}\Im\mathrm{m}\,Z^\delta}{\left[\eta_{\epsilon\varphi}\Im\mathrm{m}\,Z^\epsilon\;\Im\mathrm{m}\,Z^\varphi\right]^2} - \frac{\eta_{\alpha\beta}}{2\eta_{\epsilon\varphi}\Im\mathrm{m}\,Z^\epsilon\;\Im\mathrm{m}\,Z^\varphi}\,.
\end{aligned} \tag{2.50}$$

The reality of the Kähler potential constrains the values of the scalars. The model has two branches characterized by

$$\Im\mathrm{m}\,\tau > 0\,, \qquad \eta_{\alpha\beta}\Im\mathrm{m}\,Z^\alpha\;\Im\mathrm{m}\,Z^\beta > 0\,, \tag{2.51}$$

and

$$\Im\mathrm{m}\,\tau < 0\,, \qquad \eta_{\alpha\beta}\Im\mathrm{m}\,Z^\alpha\;\Im\mathrm{m}\,Z^\beta < 0\,, \tag{2.52}$$

that will be distinguished where required by $+$ and $-$ indices, respectively.

Only the subgroup $\mathrm{SO}(1,n) \subset \mathrm{SO}(2,n)$ acts linearly (in the fundamental representation) on the special coordinates Z^α and the group $\mathrm{SO}(3)$ acts in the adjoint (for instance) on the coordinates $\alpha = 3,4,5$ if $n \geq 4$. We take $n = 4$ for simplicity and denote the $\alpha = 3,4,5$ indices by $a, b, \cdots = 1,2,3$. For the sake of simplicity we will write Z^a instead of Z^{a+2} for Z^3, Z^4, Z^5 etc. The generators and structure constants of $\mathfrak{so}(3)$ and their action on the scalars are the same as in the $\overline{\mathbb{CP}}^3$ model with obvious changes of notation:

$$(T_a)^b{}_c = f_{ac}{}^b = -\varepsilon_{acb}\,, \qquad \delta_\alpha Z^a = \alpha^b (T_b)^a{}_c Z^c = -\epsilon_{abc}\alpha^b Z^c = \alpha^b k_b{}^a(Z)\,, \tag{2.53}$$

(τ and Z^2 are inert) so the holomorphic Killing vectors and the momentum maps are

$$k_a{}^b(Z) = \epsilon_{abc} Z^c\,, \qquad \mathcal{P}_a = -\tfrac{i}{2}\frac{\epsilon_{abc} Z^b Z^{*\,c^*}}{\eta_{\alpha\beta}\Im\mathrm{m}\, Z^\alpha\; \Im\mathrm{m}\, Z^\beta}\,. \tag{2.54}$$

The scalar potential has a structure similar to that of the $\overline{\mathbb{CP}}^3$ model, but more complicated. We will not give it here since it is not needed anyway.

2.3.2 The solutions

To find solutions to this non-Abelian model we just need to follow the recipe. First, we find the functions $\mathcal{I}^\Lambda$ and the spatial components of the vector fields $A^\Lambda{}_{\underline{m}}$ by solving the Bogomol'nyi equations

$$F^\Lambda{}_{\underline{mn}} = -\tfrac{1}{\sqrt{2}}\epsilon_{mnp}\partial_{\underline{p}}\mathcal{I}^\Lambda\,, \quad I = 0,1,2, \tag{2.55}$$

$$F^{a+2}{}_{\underline{mn}} = -\tfrac{1}{\sqrt{2}}\epsilon_{mnp}\mathfrak{D}_{\underline{p}}\mathcal{I}^{a+2}\,, \quad a = 1,2,3, \tag{2.56}$$

(we will suppress the $+2$ in the non-Abelian indices in most places). The Abelian equations are solved by harmonic functions and the non-Abelian ones by making the identification Eq. (2.8) with the Higgs field and using Protogenov's solutions Eqs. (2.11), as we did in the $\overline{\mathbb{CP}}^3$ model.

Next, we have to find the functions $\mathcal{I}_\Lambda$ by solving Eqs. (1.19). In the Abelian directions $\Lambda = 0,1,2$ we can simply choose harmonic functions and in the non-Abelian ones we take $\mathcal{I}_a = 0$. This choice gives non-singular solutions, as we are going to see. We will also set some of the harmonic functions to zero for simplicity.

The Hesse potential defined in Eq. (1.21) can be found from Shmakova's solution of the stabilization (or Freudenthal duality) equations for cubic models [63]; it can be written as

$$\mathsf{W}(\mathcal{I}) = 2\sqrt{J_4(\mathcal{I})}\,, \tag{2.57}$$

with the quartic invariant $J_4(\mathcal{I})$ given by

$$J_4(\mathcal{I}) \equiv (\mathcal{I}^\alpha \mathcal{I}^\beta \eta_{\alpha\beta} + 2\mathcal{I}^0\mathcal{I}_1)(\mathcal{I}_\alpha\mathcal{I}_\beta\eta^{\alpha\beta} - 2\mathcal{I}^1\mathcal{I}_0) - (\mathcal{I}^0\mathcal{I}_0 - \mathcal{I}^1\mathcal{I}_1 + \mathcal{I}^\alpha\mathcal{I}_\alpha)^2\,. \tag{2.58}$$

This potential does not vanish for the choice $\mathcal{I}_a = 0$, as we advanced and it will remain non-singular if we set $\mathcal{I}^0 = \mathcal{I}_1 = \mathcal{I}_2 = 0$. In other words: the only non-trivial components of $\mathcal{I}^M$ are $\mathcal{I}^1, \mathcal{I}^2, \mathcal{I}^{a+2}, \mathcal{I}_0$. With this choice the metric function is given by

$$e^{-2U} = \mathsf{W}(\mathcal{I}) = 2\sqrt{-2\mathcal{I}^1\mathcal{I}_0\,\eta_{\alpha\beta}\mathcal{I}^\alpha\mathcal{I}^\beta} = 2\sqrt{-2\mathcal{I}^1\mathcal{I}_0[(\mathcal{I}^2)^2 - \mathcal{I}^a\mathcal{I}^a]}\,. \tag{2.59}$$

As instructed by the recipe in Sec. (1.2), we can calculate the $\tilde{\mathcal{I}}$ from Eq. (1.21), which for our choice of non-trivial components of $\mathcal{I}^M$ means that $\tilde{\mathcal{I}}^i = 0$ ($i = 1, \cdots, 5$); this implies that all the scalars are purely imaginary and given by

$$Z^i = i\frac{\mathcal{I}^i}{\tilde{\mathcal{I}}^0}\,, \quad \text{where} \quad \tilde{\mathcal{I}}^0 = \frac{2\mathcal{I}^1\eta_{\alpha\beta}\mathcal{I}^\alpha\mathcal{I}^\beta}{\mathsf{W}(\mathcal{I})}\,. \tag{2.60}$$

It is convenient to write all of them in terms of $\tau = Z^1$

$$Z^\alpha = \frac{\mathcal{I}^\alpha}{\mathcal{I}^1}\tau\,, \qquad \tau = i\frac{e^{-2U}}{2\eta_{\alpha\beta}\mathcal{I}^\alpha\mathcal{I}^\beta}\,. \tag{2.61}$$

In the two ($+$ and $-$) branches of the model corresponding, respectively, to the upper and lower signs $\pm\Im\mathfrak{m}\,\tau_{(\pm)} > 0$ and, since $e^{-2U} > 0$, we must choose the functions $\mathcal{I}^\alpha_{(\pm)}$ so that

$$\pm\,\eta_{\alpha\beta}\mathcal{I}^\alpha_{(\pm)}\mathcal{I}^\beta_{(\pm)} = \pm\left[(\mathcal{I}^2_{(\pm)})^2 - \mathcal{I}^a_{(\pm)}\mathcal{I}^a_{(\pm)}\right] > 0\,. \tag{2.62}$$

In order for $\mathsf{W}(\mathcal{I})$ to be real the $\mathcal{I}_{(\pm)\,0}$ and $\mathcal{I}^1_{(\pm)}$ must be chosen so as to satisfy

$$\pm\,\mathcal{I}^1_{(\pm)}\mathcal{I}_{(\pm)\,0} < 0\,. \tag{2.63}$$

(We will suppress the $\pm$ subindices in what follows, to simplify the notation, except where this may lead to confusion.)

Observe that with our choice of non-vanishing components of $\mathcal{I}^M$ the r.h.s. of Eq. (1.20) vanishes automatically, whence the staticity condition $\omega = 0$ does not impose any constraint.

According to the preceding discussions, the non-vanishing components of $\mathcal{I}^M$ will be assumed to take the form

$$\begin{aligned} \mathcal{I}^1 &= A^1 + \frac{p^1/\sqrt{2}}{r}\,, \qquad \mathcal{I}^2 = A^2 + \frac{p^2/\sqrt{2}}{r}\,, \qquad \mathcal{I}^a = \sqrt{2}\,\delta^a{}_m x^m f(r)\,, \\ \mathcal{I}_0 &= A_0 + \frac{q_0/\sqrt{2}}{r}\,, \end{aligned} \tag{2.64}$$

where $f(r)$ is $f_{\mu,s}$ or f_λ in Eqs. (2.11), p^1, p^2, q_0 are magnetic and electric charges and A^1, A^2, A_0 are integration constants to be determined in terms of the asymptotic values of the scalars and the metric. These constants must have the same sign as the corresponding charges

$$\mathrm{sign}(A^{1,2}) = \mathrm{sign}(p^{1,2})\,, \qquad \mathrm{sign}(A_0) = \mathrm{sign}(q_0)\,, \tag{2.65}$$

as the functions $\mathcal{I}^1, \mathcal{I}^2$ and $\mathcal{I}_0$ are required to have no zeroes on the interval $r \in (0, +\infty)$ in order to avoid naked singularities there. Then, the above constraint on the signs of $\mathcal{I}^1$ and $\mathcal{I}_0$ translates into the following constraints on the signs of the charges in the two branches:

$$\mathrm{sign}(p^1)\mathrm{sign}(q_0) = \mp 1\,. \tag{2.66}$$

Defining as in the $\overline{\mathbb{CP}}^3$ case the asymptotic value Z_∞ of the adjoint scalars by

$$Z^a_\infty \equiv Z_\infty\, \delta^a{}_m \frac{x^m}{r}\,, \tag{2.67}$$

and imposing the normalization of the metric at infinity it is not hard to express the integration constants μ, A^1, A^2, A_0 in terms of the moduli (the asymptotic values of the scalars $\Im\mathfrak{m}\tau_\infty$, $\Im\mathfrak{m}\, Z^2_\infty$ and $\Im\mathfrak{m}\, Z_\infty$) and the coupling constant g

$$\begin{aligned} A^1 &= \frac{\mathrm{sign}(p^1)|\Im\mathfrak{m}\tau_\infty|}{\sqrt{2}\chi_\infty}\,, \\ A^2 &= \frac{\mathrm{sign}(p^2)|\Im\mathfrak{m}\, Z^2_\infty|}{\sqrt{2}\chi_\infty}\,, \\ \mu &= \frac{g|\Im\mathfrak{m}\, Z_\infty|}{2\chi_\infty}\,, \\ A_0 &= \tfrac{1}{2\sqrt{2}}\mathrm{sign}(q_0)\chi_\infty\,, \end{aligned} \tag{2.68}$$

where we have defined the combination (real in both branches of the theory)

$$\chi_\infty \equiv \sqrt{\Im\mathfrak{m}\tau_\infty\,[(\Im\mathfrak{m}\, Z^2_\infty)^2 - (\Im\mathfrak{m}\, Z_\infty)^2]}\,. \tag{2.69}$$

The mass of the solutions in terms of the moduli and the charges is

$$M = \tfrac{1}{4}\frac{\chi_\infty}{|\Im\mathfrak{m}\tau_\infty|}|p^1| + \frac{1}{2\chi_\infty}|q_0| \pm \tfrac{1}{2}\frac{|\Im\mathfrak{m}\tau_\infty \Im\mathfrak{m}\, Z^2_\infty|}{\chi_\infty}|p^2| \pm \frac{|\Im\mathfrak{m}\tau_\infty \Im\mathfrak{m}\, Z_\infty|}{\chi_\infty}\frac{1}{g}\,. \tag{2.70}$$

In the above expressions we have used two consistency conditions:

$$\mathrm{sign}(\Im\mathfrak{m}\, Z_\infty) = \mp \mathrm{sign}(p^1)\,, \qquad \mathrm{sign}(\Im\mathfrak{m}\, Z^2_\infty) = \pm \mathrm{sign}(p^1)\mathrm{sign}(p^2)\,. \tag{2.71}$$

These expressions for the integration constants and the mass are valid both for the 2- and 1-parameter families, the latter being recovered by setting $\Im\mathfrak{m}\, Z_\infty = 0$ everywhere. The contribution of the monopole charge $1/g$ to the mass disappears because it is screened.

Observe that the positivity of the mass is not guaranteed in the $-$ branch for arbitrary values of the charges and moduli: it has to be imposed by hand.

Let us now study the behavior of the solution in the near-horizon limit $r \to 0$. For $f_{\mu,s\neq 0}$ and f_λ the metric function behaves as

$$e^{-2U} \sim \sqrt{-2p^1 q_0\left[(p^2)^2 - (2/g)^2\right]}\,\frac{1}{r^2}\,, \tag{2.72}$$

which corresponds to a regular horizon in both branches. The solutions will describe regular black holes if the charges and moduli are such that $M > 0$. Observe that in the $-$ branch it is possible to chose those such that $M = 0$ with a non-vanishing entropy.

In the $f_{\mu,s=0}$ case with $p^2 \neq 0$ the solution is only well defined in the $+$ branch because there is no $1/r$ contribution from the monopole in the $r \to 0$ limit and it is impossible to satisfy the inequality $-\eta_{\alpha\beta}\mathcal{I}^\alpha\mathcal{I}^\beta > 0$ in that limit. In this case (the $+$ branch with $p^2 \neq 0$) we have

$$e^{-2U} \sim \sqrt{-2p^1 q_0 (p^2)^2}\,\frac{1}{r^2}\,, \tag{2.73}$$

which corresponds to a regular horizon.

In the $f_{\mu,s=0}$ case with $p^2 = 0$ there are two possibilities:

1. We can set $p^1 = q_0 = 0$. Then, in the $r \to 0$ limit, e^{-2U} is the moduli-dependent constant $2\sqrt{-2A^1 A_0 (A^2)^2}$. There is neither horizon nor singularity and the solution, which is a global monopole, belongs to the $+$ branch (this also guarantees that the mass is positive).

2. We can keep both $p^1 \neq 0$ and $q_0 \neq 0$, setting $A^2 = 0$ and profit from the fact that, in this limit $\Phi^a\Phi^a$ goes to zero as r^2. The solution is only well defined in the $-$ branch. The metric function takes the constant value

$$e^{-2U} \sim \sqrt{+p^1 q_0 \frac{\mu^4}{g^2}}\,, \tag{2.74}$$

We have, as far as the metric is concerned, a global monopole solution (as long as $M > 0$), but since we need two Abelian charges switched on, namely p^1 and q_0, the scalar fields and the gauge fields are singular at $r = 0$. As before, it is possible to tune the moduli and charges so that $M = 0$.

The near-horizon limits of the scalars are, in the $f_{\mu,s\neq 0}$ and f_λ cases

$$
\begin{aligned}
\Im\mathfrak{m}\tau_{\rm h} &= \frac{\sqrt{-2p^1q_0\,[(p^2)^2-(2/g)^2]}}{2\,[(p^2)^2-(2/g)^2]}\,,\\
\Im\mathfrak{m}\,Z^2_{\rm h} &= \frac{p^2}{p^1}\Im\mathfrak{m}\tau_{\rm h}\,,\\
\Im\mathfrak{m}\,Z^a_{\rm h} &= \frac{2\Im\mathfrak{m}\tau_{\rm h}}{gp^1}\delta^a{}_m\frac{x^m}{r}\,,
\end{aligned}
\tag{2.75}
$$

and, in the $f_{\mu,s=0}$ case with $p^2 \neq 0$, we get the same results up to the contribution of the monopole which disappears (formally, $1/g = 0$).

2.4 Embedding in pure $\mathrm{SU}(2)$ EYM

The scalars can only be trivialized for the Wu-Yang monopole $s = \infty$. In that case, it is easy to construct a double-extremal black hole with constant scalars and the metric is, as usual, Reissner-Nordström's.

3 Multi-center SBHSs

To construct multi-center SBHSs we can use the same recipe as in the single-center case but we need multi-center solutions of the Bogomol'nyi equations. We start by discussing these.

3.1 Multi-center solutions of the $\mathrm{SU}(2)$ Bogomol'nyi equations on $\mathbb{R}^3$

In the Abelian case, the multicenter solutions of the Bogomol'nyi equations are associated to harmonic functions with isolated point-like singularities. They are the seed solutions of the multi-black-hole solutions of the Einstein-Maxwell theory [4, 5, 6, 7, 8, 10] and $\mathcal{N} = 2$, $d = 4$ supergravities [64, 9, 53, 11]. In the non-Abelian case, the hedgehog ansatz is clearly inappropriate and more sophisticated methods need to be used. Only a few explicit solutions are known, even though solutions describing several BPS objects in equilibrium are, on general grounds, expected to exist. For instance, there is no explicit solution describing two BPS 't Hooft-Polyakov monopoles in equilibrium (see however Ref. [65]).

Perhaps not surprisingly, the only general families of explicit solutions available involve an arbitrary number of Wu-Yang or Dirac monopoles embedded in SU(2). The simplest of these only involve Wu-Yang monopoles and formally, it can be obtained from solutions describing Dirac monopoles embedded in SU(2) via singular gauge transformations [66], generalizing the constructions reviewed in Appendices B (minimal charge) and D (higher charge). As we have explained at length in the preceding sections, the metric is completely oblivious to these gauge transformations and takes the same form as in the Abelian cases. We will not study such solutions in this section.

In Refs. [20], using the Nahm equations [67], Cherkis and Durcan found new solutions describing one or two, charge 1, Wu-Yang monopoles embedded in SU(2) in the background of a single BPS 't Hooft-Polyakov monopole.[18] We are going to use the first of them to construct multi-center solutions of the $\overline{\mathbb{CP}}^3$ and $ST[2,4]$ models of $\mathcal{N} = 2$, $d = 4$ SEYM. Let us review the Cherkis-Durcan solution first: take the BPS 't Hooft-Polyakov monopole to be located at $x^n = x^n_0$ and the Wu-Yang monopole at $x^m = x^m_1$. We define the coordinates relative to each of those centers and the relative position by

$$r^m \equiv x^m - x^m_0\,, \qquad u^m \equiv x^m - x^m_1\,, \qquad d^m \equiv u^m - r^m = x^m_0 - x^m_1\,, \tag{3.1}$$

and their norms by respectively, r, u and d. The Higgs field and gauge potential of this solution (adapted to our conventions) are given by [20]

$$\pm\,\Phi^a = \frac{1}{g}\delta^a{}_m \left\{ \left[\frac{1}{r} - \left(\mu + \frac{1}{u}\right)\frac{K}{L}\right]\frac{r^m}{r} + \frac{2r}{uL}\left(\delta^{mn} - \frac{r^m r^n}{r^2}\right) d^n \right\}\,, \tag{3.2}$$

$$\begin{aligned} A^a = {} & -\frac{1}{g}\left[\frac{1}{r} - \frac{\mu \mathrm{D} + 2d + 2u}{\mathrm{L}}\right] \frac{\varepsilon^a{}_{mn} r^m dx^n}{r} + 2\frac{\mathrm{K}}{\mathrm{L}}\frac{\varepsilon_{npq} d^n u^p dx^q}{u\mathrm{D}}\,\delta^a{}_m \frac{r^m}{r} \\ & - \frac{2r}{u\mathrm{L}}\,\delta^a{}_m \left(\delta^{mn} - \frac{r^m r^n}{r^2}\right)\varepsilon_{npq} u^p dx^q\,, \end{aligned} \tag{3.3}$$

where the functions K, L, D of u and r are defined by

$$K \equiv \left[(u+d)^2 + r^2\right]\cosh \mu r + 2r(u+d)\sinh \mu r\,, \tag{3.4}$$

$$L \equiv \left[(u+d)^2 + r^2\right]\sinh \mu r + 2r(u+d)\cosh \mu r\,, \tag{3.5}$$

$$\mathrm{D} = 2\,(ud + u^m d^m) = (d+u)^2 - r^2\,. \tag{3.6}$$

The function D is clearly zero along the direction[19] $u^m/u = -d^m/d$ signaling the

[18] In Ref. [21] Blair and Cherkis generated a solution describing an arbitrary number of charge 1 Wu-Yang monopoles in the presence of an 't Hooft-Polyakov monopole; one can easily generalize this solution to one describing an arbitrary number of charge $n(>0)$ Wu-Yang monopoles in the background of an 't Hooft-Polyakov monopole, by coalescing n charge 1 Wu-Yang monopoles. Needless to say, the Protogenov trick works as expected. For the sake of simplicity of exposition, we will not consider this more general solution in this article.

[19] This is the half of the line that joins $r = 0$ to $u = 0$ that stretches from the Dirac monopole $u = 0$ to infinity in the direction opposite to the 't Hooft-Polyakov monopole at $r = 0$

possible presence of a Dirac string in Eq. (3.3); that this is however not the case is demonstrated in Ref. [21].

In the models that we are going to study, the Higgs field enters the metric in the combination $\Phi^a\Phi^a$, which takes the value

$$\Phi^a\Phi^a = \frac{1}{g^2}\left\{\left[\frac{1}{r} - \left(\mu + \frac{1}{u}\right)\frac{K}{L}\right]^2 + \frac{4|\vec{r}\times\vec{d}|^2}{u^2L^2}\right\}. \tag{3.7}$$

To better understand this solution one will consider several limits:

1. The limit in which we take the BPS 't Hooft-Polyakov anti-monopole infinitely far away, keeping the Dirac monopole at x_1^m: in this limit $d \to \infty$, $r^m \sim -d^m$ while u remains finite. The Higgs and gauge fields take the form

$$\begin{aligned} \pm\Phi^a &\sim -\frac{1}{g}\delta^a{}_m\left(\mu + \frac{1}{u}\right)\frac{d^m}{d}, & (3.8)\\ A^a &\sim -\frac{1}{g}\left(1 + \frac{d^m}{d}\frac{u^m}{u}\right)^{-1}\varepsilon_{mnp}\frac{d^m}{d}\frac{u^m}{u}d\frac{u^p}{u}. & (3.9)\end{aligned}$$

   The gauge field should be compared with the embedding of a Dirac monopole with a string in the direction $-d^m$ into the direction $\delta^a{}_m d^m T^a$ of the gauge group, Eqs. (B.6) and (B.12) with $s^m = -d^m$.

2. The limit in which we take the Dirac monopole infinitely away, keeping the BPS 't Hooft-Polyakov anti-monopole at x_0^m: In this limit $d \to \infty$, $u^m \sim d^m$ while r remains finite. The Higgs and gauge fields become those of a single BPS 't Hooft-Polyakov anti-monopole at x_0^m.

3. In the limit in which we are infinitely far away from both monopoles ($r \to \infty$, $u \to \infty$), which remain at a finite relative distance, the Higgs and gauge fields take the form

$$\begin{aligned} \pm\Phi^a &= -\left[\frac{\mu}{g} + \mathcal{O}(|x|^{-2})\right]\delta^a{}_m\frac{x^m}{|x|}, & (3.10)\\ A^a &= -\frac{1}{g}\varepsilon^a{}_{mn}\frac{x^m dx^n}{|x|^2} + \frac{1}{2g}\delta^a{}_m\frac{x^m}{|x|}\left(\frac{\varepsilon_{npq}d^n x^p dx^q}{|x|^2}\right). & (3.11)\end{aligned}$$

   The first term in the gauge potential is identical to that of a Wu-Yang anti-monopole (compare with Eq. (A.2)). This is also the asymptotic behavior of the BPS 't Hooft-Polyakov monopole. The Higgs field is asymptotically covariantly constant and, in particular

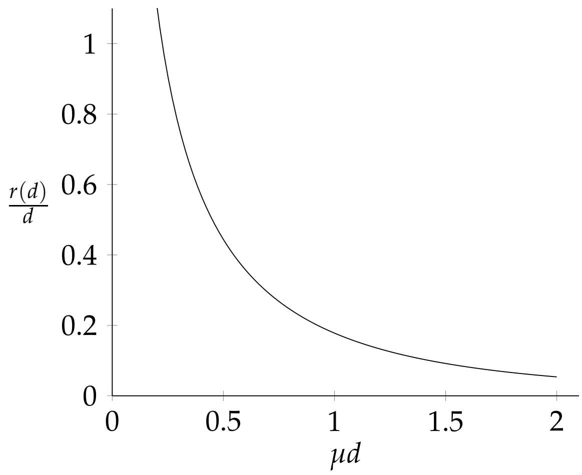

Figure 1: The zeros of the Higgs density as measured by r as a function of the dimensionless separation μd.

$$\Phi^a \Phi^a \sim \frac{\mu^2}{g^2} + \mathcal{O}(\frac{1}{|x|^2})\,. \tag{3.12}$$

4. The limit in which we approach the center of the BPS 't Hooft-Polyakov anti-monopole $r^m \to 0$, $u^m \to d^m$

$$\Phi^a \Phi^a \sim \frac{1}{4g^2 d^2 (1+\mu d)^2} + \mathcal{O}(r)\,. \tag{3.13}$$

   This limit is finite and only vanishes when the Dirac monopole is taken to infinity $d \to \infty$.

   For finite values of d, Eq. (3.7) says that $\Phi^a \Phi^a$ can only vanish along the line that stretches from $r = 0$ to $u = 0$ so $\vec{r} \times \vec{d} = 0$. Substituting $r^m = \alpha d^m$ in $\Phi^a \Phi^a$ we get a function of α and of the parameter μd. Plotting the functions of α for different values of μd we find that they have a single zero, which is also a local minimum. At this minimum the second derivative does not vanish, and therefore, there, $\Phi^a \Phi^a \sim \mathcal{O}(r^2)$, as in the single-monopole case. However, the value of this second derivative depends on the direction.

5. The limit in which we approach the singularity of the Dirac monopole $u^m \to 0$, $r^m \to -d^m$

$$\Phi^a \Phi^a \to \frac{1}{g^2} \left\{ \frac{1}{u^2} + \left(\frac{1}{d} - \mu \right) \frac{1}{u} \right\} + \mathcal{O}(1)\,. \tag{3.14}$$

3.1.1 Growing Protogenov hair

As we have argued in Sec. (2.1.3) we can add a Protogenov hair parameter s to the Cherkis & Durcan solution by simply replacing the argument μr of the hyperbolic sines and cosines in the functions K and L by the shifted on $\mu r + s$. We do not need to write explicitly the solution, but we do need to reconsider the different limits studied for the $s = 0$ case:

1. In the limit in which we take the BPS 't Hooft-Polyakov-Protogenov anti-monopole infinitely away, keeping the Dirac monopole at x_1^m the Higgs and gauge fields become, to leading order, those of the Dirac monopole with the Dirac string in the direction $-d^m$, as in the $s = 0$ case (See Eqs. (3.8) and (3.3)).
2. In the limit in which we take the Dirac monopole infinitely away, keeping the BPS 't Hooft-Polyakov-Protogenov anti-monopole at x_0^m the Higgs and gauge fields become those of a single BPS 't Hooft-Polyakov-Protogenov anti-monopole at $x^m = x_0^m$ (the first two equations (2.11)).
3. In the limit in which we are infinitely far away from both monopoles ($r \to \infty$, $u \to \infty$), which remain at a finite relative distance, the Higgs and gauge fields take the same form as in the $s = 0$ case, Eqs. (3.10-3.12).
4. The limit in which we approach the singularity of the BPS 't Hooft-Polyakov-Protogenov anti-monopole $r^m \to 0$, $u^m \to d^m$ (for $s \neq 0$)

$$\pm \Phi^a \sim \frac{1}{g}\delta^a{}_m \left[\frac{1}{r} - \left(\mu + \frac{1}{d}\right)\coth s + \mathcal{O}(r)\right]\frac{r^m}{r}, \tag{3.15}$$

$$\Rightarrow \Phi^a\Phi^a \sim \frac{1}{g^2 r^2} + \mathcal{O}\left(\frac{1}{r}\right), \tag{3.16}$$

   which is similar to the behaviour near the Dirac monopole as in Eq. (3.14) (with u replaced by r).
5. The limit in which we approach the singularity of the Dirac monopole $u^m \to 0$, $r^m \to -d^m$ we have the same behavior as in the $s = 0$ case Eq. (3.14).

The solutions with Protogenov hair have another limit, namely the one in which $s \to \infty$; this case will be studied separately.

3.1.2 The $s \to \infty$ limit solution

In this limit we get a solution that describes the same Dirac monopole together with a ($\mu \neq 0$) Wu-Yang anti-monopole:[20]

[20] One can see fairly easily that in the limiting solution one can, as far as the Bogomol'nyi equations are concerned, allow for μ to be negative; for finite values of s this is impossible.

$$\pm\,\Phi^a = \frac{1}{g}\delta^a{}_m\left[-\mu+\frac{1}{r}-\frac{1}{u}\right]\frac{r^m}{r}\,, \tag{3.17}$$

$$A^a = \frac{1}{g}\frac{\varepsilon^a{}_{mn}r^m dx^n}{r^2}+\frac{1}{g}\frac{\varepsilon_{npq}d^n u^p du^q}{u(ud+u^r d^r)}\delta^a{}_m\frac{r^m}{r}\,. \tag{3.18}$$

This solution is a particular example of a more general family describing an arbitrary number of Dirac monopoles in the background of a Wu-Yang anti-monopole. These solutions can be obtained from a solution describing only Dirac monopoles embedded in SU(2) via a singular gauge transformation that only removes the Dirac string of one of them, which becomes the Wu-Yang anti-monopole. The general family of solutions can be written in the form:

$$\Phi = \Phi_{\rm WY}+HU\,, \qquad A = A_{\rm WY}+CU\,, \tag{3.19}$$

where U is the SU(2) (and $\mathfrak{su}(2)$) matrix defined in Eq. (A.1) and where $\Phi_{\rm WY}$ and $A_{\rm WY}$ are the Higgs and Yang-Mills fields of a Wu-Yang monopole, given, respectively, by

$$\mp\,\Phi_{\rm WY} = \frac{1}{2g}\left[-\mu+\frac{1}{r}\right]U\,, \tag{3.20}$$

and by Eq. (A.2) and where H is a function and C a 1-form on $\mathbb{R}^3$. If we substitute into the Bogomol'nyi equations (2.5) and use, on the one hand, that they are satisfied by the pair $A_{\rm WY},\Phi_{\rm WY}$, and, on the other hand, that U is covariantly constant with the connection $A_{\rm WY}$ we arrive at the Dirac monopole equation

$$dC = \star_{(3)}dH\,. \tag{3.21}$$

The integrability condition of this equation is $d\star_{(3)}dH = 0$ so H is any harmonic function. We can choose it to have isolated poles at the points $x^m = x^m_i$ $i = 1,\cdots,N$

$$H = \sum_i \frac{p_i}{2u_i}\,, \qquad u^m_i \equiv x^m - x^m_i\,, \tag{3.22}$$

in which case C is the 1-form potential of N Dirac monopoles with charges p_i which can be constructed by summing over the potentials of each individual monopole:

$$C = \sum C_i\,, \qquad dC_i = \star_{(3)}d\frac{p_i}{2u_i}\,. \tag{3.23}$$

The expression for each of the C_i is of the form Eq. (B.6) where we can, in principle, choose the direction s^m_i of each Dirac string independently:

$$C_i = \frac{p_i}{2}\left(1-\frac{s^m_i}{s_i}\frac{u^m_i}{u_i}\right)^{-1}\varepsilon_{mnp}\frac{s^m_i}{s_i}\frac{u^m_i}{u_i}d\frac{u^p_i}{u_i}\,, \quad \text{(no sum over } i\text{)}. \tag{3.24}$$

This solution of the Yang-Mills-Higgs system shares two important properties with the original Wu-Yang monopole and which are related to the fact that they are related to Abelian embeddings by singular gauge transformations:

1. Both Φ and $D\Phi$ are proportional to U:

$$\Phi = \left(-\frac{\mu}{2g} + \frac{1}{2gr} + H\right) U\,, \qquad D\Phi = d\left(-\frac{\mu}{2g} + \frac{1}{2gr} + H\right) U\,, \tag{3.25}$$

   and, therefore, commute with each other, so the Higgs current vanishes and the gauge field is, by itself, a solution of the pure Yang-Mills theory.

2. The gauge field strength is also proportional to U, the coefficient being the field strength of an Abelian gauge field:

$$F(A) = d(B + C)U\,, \tag{3.26}$$

   which implies that the energy-momentum tensors are related as in the single-center case.

These solutions can be generalized even further, by allowing the the charge of the "original" Wu-Yang monopole at $r = 0$ to be n/g (that is: using the generalization of the Wu-Yang monopole due to Bais [68] which is studied in Appendix D). If we now substitute into the Bogomol'nyi equations (2.5) the ansatz

$$\Phi = \Phi_{(n)} + HU_{(n)}\,, \qquad A = A_{(n)} + CU_{(n)}\,, \tag{3.27}$$

where $U_{(n)}, A_{(n)}$ and $\Phi_{(n)}$ are given, respectively, in Eqs. (D.5),(D.6) and (D.11), H is a function and C a 1-form on $\mathbb{R}^3$, and use that they are satisfied by the pair $A_{(n)}, \Phi_{(n)}$ and that $U_{(n)}$ is covariantly constant with the connection $A_{(n)}$, we arrive again at the Dirac monopole equation (3.21).

Since all these solutions are related to Abelian embeddings, they contribute to the black-hole solutions as the Abelian solutions. We will not consider them in what follows.

3.2 Embedding in the $\mathrm{SU}(2)$-gauged $\overline{\mathbb{CP}}^3$ model

We can use the Cherkis & Durcan solution of the SU(2) Bogomol'nyi equations reviewed in the previous section as a seed solution for a multicenter solution of $\mathcal{N} = 2$, $d = 4$ SEYM, adding the same harmonic functions as in the single-center case $(\mathcal{I}^0, \mathcal{I}_0)$

or a generalization with poles at the locations of the monopoles $r = 0$[21] and $u = 0$. More explicitly, we take

$$
\begin{aligned}
\mathcal{I}^0 &= A^0 + \frac{p_r^0/\sqrt{2}}{r} + \frac{p_u^0/\sqrt{2}}{u}\,, \\
\mathcal{I}_0 &= A_0 + \frac{q_{r,0}/\sqrt{2}}{r} + \frac{q_{u,0}/\sqrt{2}}{u}\,, \\
\mathcal{I}^i &= \mp\sqrt{2}\Phi^i(r,u)\,, \\
\mathcal{I}_i &= 0\,,
\end{aligned}
\tag{3.28}
$$

where $\Phi^i(r,u)$ is the Higgs field of the Cherkis & Durcan solution. The metric and scalar fields take the form

$$
e^{-2U} = \tfrac{1}{2}(\mathcal{I}^0)^2 + 2(\mathcal{I}_0)^2 - \Phi^i\Phi^i\,, \tag{3.29}
$$

$$
Z^i = \frac{\mp\sqrt{2}\Phi^i}{\mathcal{I}^0 + 2i\mathcal{I}_0}\,. \tag{3.30}
$$

The normalization of the metric and scalars at infinity leads to the same relations between the integration constants A^0, A_0, μ and the physical constants Z_∞, g as in the single-center case, namely

$$
\tfrac{1}{\sqrt{2}}A^0 + \sqrt{2}iA_0 = \frac{Z^*_\infty}{|Z_\infty|}\frac{1}{\sqrt{1-|Z_\infty|^2}}\,, \qquad \mu = \frac{|Z_\infty|}{\sqrt{1-|Z_\infty|^2}}g\,. \tag{3.31}
$$

The integrability conditions of Eq. (1.20) are, in this case,

$$
\mathcal{I}_0\partial_{\underline{m}}\partial_{\underline{m}}\mathcal{I}^0 - \mathcal{I}^0\partial_{\underline{m}}\partial_{\underline{m}}\mathcal{I}_0 = 0\,, \tag{3.32}
$$

and lead to the following relations between the integration constants:

$$
A^0(q_{r,0}+q_{u,0}) - A_0(p_r^0+p_u^0) = 0\,, \tag{3.33}
$$

$$
J - \tfrac{1}{\sqrt{2}}d(A^0 q_{u,0} - A_0 p_u^0) = 0\,, \tag{3.34}
$$

[21]The location of the BPS 't Hooft-Polyakov anti-monopole is not completely clear: it is sometimes argued that the center of the monopole is the point at which the Higgs vanishes and the full gauge symmetry is restored. As we have discussed, that point is not $r = 0$. We could try to place the poles of the harmonic functions at that point, but, given that its location is not known analytically and the expansion of $\Phi^a\Phi^a$ around it is difficult to compute, we will not try to do that here.

where we have defined the constant

$$J \equiv p_r^0 q_{u,0} - q_{r,0} p_u^0 \,. \tag{3.35}$$

The first equation is equivalent to Eq. (2.32) for the total charges and the second equation determines the relative distance d in terms of J and $A^0 q_{u,0} - A_0 p_u^0$ provided that $J \neq 0$. When that is the case, the solution is not static and has an angular momentum J directed along the line that joins the monopoles $J^m = J d^m/d$. The corresponding 1-form ω can be constructed by the standard procedure of the Abelian case. However, since this complicates the analysis of the regularity of the solutions, we will stick to the static case and require $J = 0$.

In order to have regular solutions, the charges at each center must be chosen as in the corresponding single-center case: since there is an Abelian monopole at $u = 0$, we must switch on either p_u^0 or $q_{u,0}$ to have a regular horizon there. We can treat them both as non-vanishing with no loss of generality. Then, there are two possibilities:

I. $p_r^0 = q_{r,0} = 0$**:** Only for $s = 0$ ('t Hooft-Polyakov anti-monopole at $r = 0$) has the solution a chance of being regular at $r = 0$. Solving Eq. (3.33) the solution can be written in the form

$$e^{-2U} = \frac{1}{1 - |Z_\infty|^2} H^2 - \Phi^i \Phi^i \,, \tag{3.36}$$

$$Z^i = \frac{2\beta}{p^0 + 2iq_0} \frac{\Phi^i}{H} \,, \tag{3.37}$$

where H is the harmonic function

$$H \equiv 1 + \frac{\beta}{u} \,, \qquad \beta^2 = (1 - |Z_\infty|^2) W_{\mathrm{RN}}(\mathcal{Q}_u)/2 \,, \qquad W_{\mathrm{RN}}(\mathcal{Q}_u) \equiv \tfrac{1}{2}(p_u^0)^2 + 2(q_{u,0})^2 \,. \tag{3.38}$$

The free parameters of this solution are the charges p_u^0, $q_{u,0}$ and the single modulus $|Z_\infty|$.

Studying the $u \to 0$ limit we find a black hole with entropy

$$S_u/\pi = \tfrac{1}{2} W_{\mathrm{RN}}(\mathcal{Q}_u) - \frac{1}{g^2} \,, \tag{3.39}$$

as in the corresponding single-center case.

In the $r \to 0$ limit e^{-2U} is constant. The positivity of the constant is guaranteed if S_u is positive. The total entropy of the solution is just the entropy of the black hole at $u = 0$ and the Dirac monopole does contribute to it.

The mass of the solution, expressed in terms of the independent parameters of the solution, p^0_u, $q_{u,0}$ and $|Z_\infty|$ takes the form

$$M = M_r + M_u \,, \tag{3.40}$$

$$M_r = -M_{\text{monopole}} \,, \tag{3.41}$$

$$M_u = \sqrt{\tfrac{1}{2}\frac{W_{RN}(\mathcal{Q}_u)}{1-|Z_\infty|^2}} + M_{\text{monopole}} \,, \tag{3.42}$$

where M_{monopole} is given by Eq. (2.39). The contributions of the monopole and the 't Hooft-Polyakov monopole to the mass cancel each other.

II. p^0_r **or** $q_{r,0} \neq 0$ We can treat both charges as non-vanishing with no loss of generality. Solving Eqs. (3.33) and (3.35), we can write the solution as in Eqs. (3.36) and (3.37) where, now,

$$\begin{aligned} H &\equiv 1 + \frac{\beta_r}{r} + \frac{\beta_u}{u} \,, \qquad \beta^2_{r,u} = (1-|Z_\infty|^2) W_{\text{RN}}(\mathcal{Q}_{r,u})/2 \,, \\ W_{\text{RN}}(\mathcal{Q}_{r,u}) &\equiv \tfrac{1}{2}(p^0_{r,u})^2 + 2(q_{r,u,0})^2 \,. \end{aligned} \tag{3.43}$$

The free parameters of this solution are the charges p^0_u, $q_{u,0}$ and $|Z_\infty|$ and either p^0_r or $q_{r,0}$, since they must be proportional to those of the other center. The areas of each of the horizons are as in the single-center case. In particular, the BPS 't Hooft-Polyakov monopole ($s = 0$) does not contribute to the entropy of the $r = 0$ center. The mass is given by

$$M = M_r + M_u \,, \tag{3.44}$$

$$M_r = \sqrt{\tfrac{1}{2}\frac{W_{RN}(\mathcal{Q}_r)}{1-|Z_\infty|^2}} - M_{\text{monopole}} \,, \tag{3.45}$$

$$M_u = \sqrt{\tfrac{1}{2}\frac{W_{RN}(\mathcal{Q}_u)}{1-|Z_\infty|^2}} + M_{\text{monopole}} \,, \tag{3.46}$$

and the contributions of the monopole and anti-monopole cancel each other. In the $s \to \infty$ limit it can be easily seen that the solution is completely regular everywhere (e^{-2U} only vanishes at $r = 0$ and $u = 0$) if the Abelian charges as chosen so that the horizons are regular. This guarantees that all the terms in e^{-2U} are positive. For finite s this is more difficult to proof analytically, but, since the Higgs field has a better behavior than in the $s \to \infty$ case, it is reasonable to expect that it will also be true. We have checked numerically that this is so in several examples.

3.3 Embedding in the $\mathrm{SU}(2)$-gauged $\mathrm{ST}[2,4]$ model

The metric and scalar fields of the solution are now given by

$$e^{-2U} = 2\sqrt{-2\mathcal{I}^1\mathcal{I}_0[(\mathcal{I}^2)^2 - 2\Phi^a\Phi^a]}\,, \tag{3.47}$$

$$Z^1 \equiv \tau = i\frac{e^{-2U}}{2[(\mathcal{I}^2)^2 - 2\Phi^a\Phi^a]}\,, \qquad Z^2 = \frac{\mathcal{I}^2}{\mathcal{I}^1}\tau\,, \qquad Z^a = \frac{\sqrt{2}\Phi^a}{\mathcal{I}^1}\tau\,, \tag{3.48}$$

where Φ^a is the Higgs field of the Cherkis & Durcan solution (deformed with the Protogenov hair parameter s) and where the harmonic functions $\mathcal{I}^1$, $\mathcal{I}^2$ and $\mathcal{I}_0$ are allowed to have poles at $r = 0$ and $u = 0$:

$$\begin{aligned}
\mathcal{I}^1 &= A^1 + \frac{p^1_r/\sqrt{2}}{r} + \frac{p^1_u/\sqrt{2}}{u}\,, \qquad \mathcal{I}^2 = A^2 + \frac{p^2_r/\sqrt{2}}{r} + \frac{p^2_u/\sqrt{2}}{u}\,, \\
\mathcal{I}_0 &= A_0 + \frac{q_{r,0}/\sqrt{2}}{r} + \frac{q_{u,0}/\sqrt{2}}{u}\,.
\end{aligned} \tag{3.49}$$

As in the $\overline{\mathbb{CP}}^3$ case, the Abelian charges at each center must be chosen with the same criteria as in the corresponding single-center case. This means, in particular, that the Abelian charges at $u = 0$, $p^1_u, q_{u,0}$ must be non-vanishing. p^2_u may need to be activated, depending on the branch we are considering. At $r = 0$, for $s \neq 0$ we get exactly the same possibilities, but, for $s = 0$ there are two possibilities:

1. $p^1_r, q_{r,0}, p^2_r$ non-vanishing. We find a black hole at $r = 0$ in the $+$ branch.
2. $p^1_r = q_{r,0} = p^2_r = 0$. e^{-2U} is a complicated d-dependent constant in the $r = 0$ limit and we get a global monopole.

Here we find an important difference with the single-center case, due to the fact that $\Phi^a\Phi^a$ is a finite constant in the $r \to 0$ limit instead of going to zero as r^2: there is no solution with $p^1_r q_{r,0} \neq 0$ and $p^2_r = 0$. In order to have such a global monopole solution with $p^1 q_0 \neq 0$ and $p^2 = 0$ in equilibrium with the monopole at $u = 0$ one may try to place those charges at the point at which $\Phi^a\Phi^a = 0$, but the resulting solution may not be well defined there because the limit of the metric function depends on the direction from which we approach that point.

The entropy of the solution is the sum of the entropies of both centers (vanishing for global monopoles). As in the $\overline{\mathbb{CP}}^3$ case, the monopole at each center does contribute to the center entropy (except for global monopoles). The contributions of the monopole and anti-monopole to the mass cancel each other:

$$M = \tfrac{1}{4}\frac{\chi_\infty}{|\Im\mathfrak{m}\,\tau_\infty|}|p^1_u + p^1_r| + \frac{1}{2\chi_\infty}|q_{u,0} + q_{r,0}| \pm \tfrac{1}{2}\frac{|\Im\mathfrak{m}\,\tau_\infty \Im\mathfrak{m}\, Z^2_\infty|}{\chi_\infty}|p^2_u + p^2_r|\,. \tag{3.50}$$

4 Conclusions

In this article we have discussed the construction of supersymmetric multi-object solutions in $\mathcal{N} = 2$, $d = 4$ EYM theories, specifically in the so-called $\overline{\mathbb{CP}}^{n\geq 3}$ and ST$[2, n]$ models. These models were chosen due to their workability, the fact that they allow for a SU(2) gauging and (in the second case) for their stringy origin. Starting with a deformation of the solutions to the SU(2) Bogomol'nyi equation found by Cherkis and Durcan that adds to the 't Hooft-Polyakov monopole *Protogenov hair*, we have been able to construct *bona fide* two-center solutions. These solutions describe a Dirac monopole embedded in SU(2) in the presence of either a global monopole (the supergravity solution corresponding to the 't Hooft-Polyakov monopole) or a non-Abelian black hole (a supergravity solution with an 't Hooft-Polyakov-Protogenov monopole). In order to make the comparison with the single-object case easier, we included a detailed discussion of the embeddings of the spherically symmetric solutions to the SU(2) Bogomol'nyi equations into the two models, and expressed the whole solution in terms of charges and moduli of the physical fields.

The constructed solutions are all static. It would be very interesting to study dyonic solutions and to see how this interplays with the Denef constraint; the stumbling block in this respect is not so much the Bogomol'nyi equation as the equation (1.19); for the moment the only general solution we know of is to take $\mathcal{I}_\Lambda \sim \mathcal{I}^\Lambda$ in the gauged directions, but this automatically solves the Denef constraint. The only case for which we can find non-trivial dyonic solutions is for the multi-Wu-Yang solutions, or if you like the $s \to \infty$ limit of the deformed Cherkis and Durcan's solution; we refrain from discussing these solutions here as, due to gauge invariance, even taking into account the singular gauge transformation, the restriction coming from the Denef constraint is basically the one corresponding to the Abelian theory.

A natural question that follows from the results presented here and in Refs. [12, 14, 13] is whether we could use a charge k SU(2) monopole to construct globally regular solutions; the answer is yes: observe that the construction of globally regular solutions in Sec. (2) hinges exclusively but crucially on the fact that the used monopole solution is regular and is such that $\Phi^a\Phi^a \leq \lim_{|\vec{x}|\to\infty}\Phi^a\Phi^a$. A charge-$k$ monopole may be rather difficult to construct but the regularity is guaranteed and also the last needed ingredient is known to be satisfied: indeed, using the Bogomol'nyi equation (2.5) one can show that

$$\partial_{\underline{m}}\partial_{\underline{m}}\, \Phi^a\Phi^a \;=\; F^a{}_{\underline{mm}}F^a{}_{\underline{mm}} \;\geq\; 0\,. \tag{4.1}$$

This equation together with the Hopf maximum principle and the regularity, implies that the function $\Phi^a\Phi^a$ is bounded from above by its value on the sphere at infinity, which is exactly what one needs.

As was said in the introduction, the creation and study of non-Abelian solutions to $d = 4$ supergravity theories is in its infancy and this holds doubly so for the higher dimensional theories. One possible reason is that the structure of supersymmetric solutions to higher supergravities (see *e.g.* Refs. [69, 70]) is more entangled than the one given in the recipe in Section 1.2. For example, naively one would expect that Kronheimer's link of monopoles on $\mathbb{R}^3$ to instantons on GH-spaces, would carry over to the supersymmetric solutions as in $d = 4$ the base space is $\mathbb{R}^3$ and that in $d = 5$ must be hyper-Kähler; *i.e.* one would expect the instanton equation to show up in the recipe for cooking up 5-dimensional supersymmetric solutions. Perhaps it does, but it definitely is not obvious where and how it is making its appearance in such a clear-cut manner as in $d = 4$.

The 4- and 5-dimensional EYMH theories are, however, related by dimensional reduction/oxidation, whence the solutions to the cubic models presented in this article can be oxidized to 5-dimensions and can be studied with the hope of unraveling the structure of 5-dimensional supersymmetric solutions. Work along these lines is in progress.

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A The $\mathrm{SU}(2)$ Lorentzian meron

A Lorentzian meron is a classical solution to the pure SU(2) (Lorentzian) Yang-Mills theory such that the 1-form gauge field A defining it, is proportional to a pure-gauge configuration, which in our conventions would be $\frac{1}{g}dUU^{-1}$ where $U(x) \in \mathrm{SU}(2)$. In Ref. [29] $U(x)$ was chosen to be of the *hedgehog form*

$$U \equiv 2\frac{x^m}{r}\delta^a_m T_a\,, \qquad U^\dagger = U^{-1} = -U\,, \;\; \Rightarrow U^2 = -\mathbb{1}_{2\times 2}\,. \tag{A.1}$$

and it was shown that A solves the Yang-Mills equations if the proportionality coefficient is $1/2$, that is

$$A = \frac{1}{2g}dUU^{-1} = -\frac{1}{gr^2}\varepsilon^a{}_{mn}x^m dx^n T_a\,. \tag{A.2}$$

As we will see, this gauge field is nothing but the gauge field of the Wu-Yang SU(2) monopole given in Eq. (B.10).

Since the field strength of a pure gauge configuration vanishes, we find that $F(A)$ can be written in these two specially simple ways which we will use in Appendix C:

$$F(A) = \tfrac{1}{2}dA = g[A,A] = \star_{(3)} d\frac{1}{2gr}\,U\,, \tag{A.3}$$

Now we can write the non-Abelian field strength $F(A)$ in terms of $F(B)$, where $F(B)$ is the field strengths of the Dirac monopole of unit charge Eq. (B.1) that we will review in the next section

$$F(A) = F(B)\,U\,, \qquad F(B) = \star_{(3)} d\frac{1}{2gr}\,, \tag{A.4}$$

and the energy-momentum tensor of A in terms of that of B

$$T_{\mu\nu}(A) = -\tfrac{1}{2}\mathrm{Tr}[F_{\mu\rho}(A)F_\nu{}^\rho(A) - \tfrac{1}{4}\eta_{\mu\nu}F^2(A)] = F_{\mu\rho}(B)F_\nu{}^\rho(B) - \tfrac{1}{4}\eta_{\mu\nu}F^2(B) = T_{\mu\nu}(B)\,. \tag{A.5}$$

B The Wu-Yang $\mathrm{SU}(2)$ monopole

The Wu-Yang SU(2) monopole [26] is a solution of the SU(2) Yang-Mills theory that can be obtained from the embedding of the Dirac monopole in SU(2) via a singular gauge transformation (see, *e.g.* Ref. [71] and references therein). To fix our conventions, it is convenient to start by reviewing the Wu-Yang construction of the Dirac monopole [72].

B.1 The Dirac monopole

The U(1) field of the Dirac monopole, that we will denote by B is defined to satisfy the Dirac monopole equation[22], which can be written in several forms:

$$F(B) \equiv dB = \star_{(3)} d\frac{1}{2gr} = -\frac{1}{2g} d\Omega^2 \,, \qquad 2\partial_{[m} B_{n]} = -\frac{1}{2g}\varepsilon_{mnp}\frac{x^p}{r^3}\,, \tag{B.1}$$

where $d\Omega^2$ is the volume 2-form of the round 2-sphere of unit radius

$$d\Omega^2 = -\tfrac{1}{2}\varepsilon_{mnp}\frac{x^m}{r} d\frac{x^n}{r} \wedge d\frac{x^p}{r} = \sin\theta d\theta \wedge d\varphi\,. \tag{B.2}$$

The value of the magnetic charge has been set to g^{-1} and it is the minimal charge allowed if the unit of electric charge is g.

The above equation does not admit a global regular solution.

$$B^{(\pm)} = -\frac{1}{2g}(\cos\theta \mp 1) d\varphi\,, \tag{B.3}$$

are local solutions regular everywhere except on the negative (resp. positive) z axis (the Dirac strings). A globally regular solution can be constructed by using $B^{\pm}$ in the upper (lower) hemisphere and using the gauge transformation

$$B^{(+)} - B^{(-)} = -d\left(\frac{1}{g}\varphi\right)\,, \tag{B.4}$$

to relate them in the overlap region. If the gauge group is U(1) where the radius of the circle is the inverse coupling constant $1/g$, the gauge transformation parameter can have a periodicity $2\pi n/g$ with $n \in \mathbb{N}$. This is the well-known Abelian Wu-Yang monopole construction [72]. In our case, since the period of φ is 2π, we get $2\pi/g$, which is the smallest value allowed $p = 1/g$. The solution that describes the monopole of charge n times the minimum is n times this one $p = n/g$.

It is useful to have the expression of $B^{(\pm)}$ in Cartesian coordinates:

$$B^{(\pm)} = \frac{1}{2g}\frac{[(0,0,\mp 1) \times (x^1, x^2, x^3)] \cdot d\vec{x}}{r^2(r \pm x^3)}\,, \tag{B.5}$$

in which the singularity at $r = \mp x^3$ becomes evident. In this form, one can easily change the position of the monopole from the origin to some other point x_0^m and the position of the Dirac string from the half line that starts from the origin in the direction $-(0,0,\mp 1)$ to the half line that starts at the monopole's position x_0^m hand has the direction s^m relative to that point:

$$B^{(s)} = \frac{1}{2g}\left(1 - \frac{s^m}{s}\frac{u^m}{u}\right)^{-1}\varepsilon_{mnp}\frac{s^m}{s}\frac{u^n}{u} d\frac{u^p}{u}\,, \tag{B.6}$$

[22] This equation is just the Abelian version of the Bogomol'nyi equation.

with

$$u^m \equiv x^m - x_0^m\,, \quad u^2 \equiv u^m u^m\,, \quad s^2 \equiv s^m s^m\,. \tag{B.7}$$

B.2 From the Dirac monopole to the Wu-Yang SU(2) monopole

Let us consider the Abelian $B^{(+)}$ solution in Eq. (B.3) and let us embed it in $\mathrm{SU}(2)$ as the 3rd component of the gauge field

$$A^{(+)} \equiv 2B^{(+)}T_3\,, \qquad F(A^{(+)}) = 2F(B)T_3\,. \tag{B.8}$$

The $\mathrm{SU}(2)$ gauge transformation (which is evidently singular along the negative z axis and makes the whole Dirac string singularity, but the endpoint at the coordinate origin, disappear)

$$U^{(+)} \equiv \frac{1}{\sqrt{2(1+\frac{z}{r})}}\left[1 + \frac{z}{r} + 2\left(\frac{x}{r}T_2 - \frac{y}{r}T_1\right)\right]\,, \tag{B.9}$$

relates the gauged field $A^{(+)}$ to

$$A = \frac{1}{g}\varepsilon^a{}_{mn}dx^m\frac{x^n}{r^2}T_a\,, \qquad A^{(+)} = U^{(+)}A(U^{(+)})^{-1} + \frac{1}{g}dU^{(+)}(U^{(+)})^{-1}\,, \tag{B.10}$$

which is the gauge field of the Wu-Yang $\mathrm{SU}(2)$ monopole. As we have mentioned in the previous appendix, this is also the gauge field of the Lorentzian meron Eq. (A.2). The gauge transformation also relates T_3 to $\mathcal{U}$ in Eq. (A.1) and the Abelian vector

$$U^{(+)}\mathcal{U}(U^{(+)})^{-1} = 2T_3\,. \tag{B.11}$$

The fact that the Lorentzian meron is the Wu-Yang monopole, which is related by a gauge transformation to the Dirac monopole makes the relation Eq. (A.5) trivial.

This construction can be generalized to more general positions of the Dirac string: if we consider embedding of the Dirac monopole solution $B^{(s)}$ in Eq. (B.6) into $\mathrm{SU}(2)$

$$A^{(s)} \equiv -2B^{(s)}\frac{s^m}{s}\delta_m{}^a T_a\,, \tag{B.12}$$

it is easy to see that the gauge transformation

$$U^{(s)} \equiv \frac{1}{\sqrt{2\left(1-\frac{s^m}{s}\frac{u^m}{u}\right)}}\left[1 - \frac{s^m}{s}\frac{u^m}{u} - 2\varepsilon_{mn}{}^a\frac{s^m}{s}\frac{u^n}{u}T_a\right]\,, \tag{B.13}$$

relates it to the same Wu-Yang monopole field Eq. (B.10)

$$A^{(s)} = U^{(s)}A(U^{(s)})^{-1} + \frac{1}{g}dU^{(s)}(U^{(s)})^{-1}\,. \tag{B.14}$$

C The SU(2) Skyrme model

In this appendix we are going to show that the Lorentzian meron (Wu-Yang monopole) is also associated to a solution of the equations of motion of the SU(2) Skyrme model [73] written in the form [74]

$$S_{\text{Skyrme}} = -\tfrac{1}{2}\int d^4x \left\{\tfrac{1}{2}R_\mu R^\mu + \frac{\lambda}{16}S_{\mu\nu}S^{\mu\nu}\right\}\,, \tag{C.1}$$

where

$$R_\mu \equiv V^{-1}\partial_\mu V\,, \qquad S_{\mu\nu} \equiv [R_\mu, R_\nu]\,, \qquad V(x) \in \mathrm{SU}(2)\,. \tag{C.2}$$

The equations of motion are

$$\partial_\mu R^\mu + \frac{\lambda}{4}\partial_\mu [R_\nu, F^{\mu\nu}] = 0\,. \tag{C.3}$$

If we take $V = U^{-1}$ (U given by Eq. (A.1)), then we can write $R = 2gA$ where A is Lorentzian meron's gauge field Eq. (A.2) and

$$\begin{aligned}
\partial_\mu R^{i\,\mu} &= -2g\partial_m A^i{}_m = 0\,,\\
\partial_\mu [R_\nu, F^{\mu\nu}]^i &\sim \partial_m\left(\frac{A^i{}_m}{r^2}\right) = 0\,.
\end{aligned} \tag{C.4}$$

D Higher-charge Lorentzian merons and Wu-Yang monopoles

The construction of a Lorentzian meron can be generalized by using a generalization of the unit outward-pointing vector x^m/r denoted by ξ^m and defined by [68]

$$(\xi^m) \equiv \frac{1}{r}\left(\frac{\mathfrak{Im}(x^2+ix^1)^n}{\rho^{n-1}}, \frac{\mathfrak{Re}(x^2+ix^1)^n}{\rho^{n-1}}, x^3\right)\,, \qquad \rho^2 \equiv (x^1)^2 + (x^2)^2\,, \tag{D.1}$$

or, in spherical coordinates,

$$(\xi^m) \equiv (\sin\theta \sin n\varphi, \sin\theta\cos n\varphi, \cos\theta)\,, \tag{D.2}$$

and which reduces to x^m/r for $n = 1$. The essential properties of ξ^m are

$$d\xi^m \wedge d\xi^n = -n\varepsilon_{mnp}\xi^p d\Omega^2\,, \tag{D.3}$$

$$-\tfrac{1}{2}\varepsilon_{mnp}\xi^m d\xi^n \wedge d\xi^p = n d\Omega^2 = \star_{(3)} d\frac{n}{r}\,, \tag{D.4}$$

The generalization of the meron solution is constructed in terms of the generalization SU(2) matrix in Eq. (A.1)

$$U_{(n)} \equiv 2\xi^m \delta^a_m T_a\,, \qquad U^\dagger_{(n)} = U^{-1}_{(n)} = -U_{(n)}\,, \tag{D.5}$$

and takes the form

$$A \equiv \frac{1}{2g} dU_{(n)} U^{-1}_{(n)}\,. \tag{D.6}$$

The field strength is given by

$$F(A_{(n)}) = \tfrac{1}{2} dA = g[A, A] = \star_{(3)} d\frac{n}{2gr}\, U_{(n)}\,, \tag{D.7}$$

and can be related to that of a Dirac monopole of charge $p = n/g$

$$F(B_{(n)}) = \star_{(3)} d\frac{n}{2gr}\,, \quad F(A_{(n)}) = F(B_{(n)})\, U_{(n)}\,, \tag{D.8}$$

which is given by the expressions studied at the beginning. The energy-momentum tensor of A is also equal to that of the Abelian monopole of charge n/g B. These fields can also be related to the embedding of the charge n/g Dirac monopole into SU(2) with a generalization of the gauge transformation Eq. (B.13)

$$U^{(s)}_{(n)} \equiv \frac{1}{\sqrt{2\left(1 - \frac{s^m}{s}\xi^m\right)}} \left[1 - \frac{s^m}{s}\xi^m - 2\varepsilon_{mn}{}^a \frac{s^m}{s}\xi^n T_a\right]\,, \tag{D.9}$$

relates it to the meron gauge field:

$$U^{(s)}_{(n)} U_{(n)} (U^{(s)}_{(n)})^{-1} = -2\frac{s^m}{s}\delta_m{}^a T_a\,, \qquad U^{(s)}_{(n)} A_{(n)} (U^{(s)}_{(n)})^{-1} + \frac{1}{g} dU^{(s)}_{(n)} (U^{(s)}_{(n)})^{-1} = nB^{(s)}_{(n)} 2\frac{s^m}{s}\delta_m{}^a T_a\,. \tag{D.10}$$

To check that this gauge field solves the Yang-Mills equations of motion we first stress that, with the above connection, $U_{(n)}$ is a covariantly-constant adjoint field. Then, auxiliary the adjoint Higgs field

$$\Phi_{(n)} \equiv \left(-\frac{\mu}{2g} + \frac{n}{2gr}\right) U_{(n)}\,, \tag{D.11}$$

satisfies

$$D\Phi_{(n)} = d\frac{n}{2gr}\, U_{(n)}\,, \tag{D.12}$$

and the pair $A_{(n)}, \Phi_{(n)}$ satisfies the Bogomol'nyi equations (2.5) and, as a consequence the equations of motion of the Yang-Mills-Higgs system. The last equation implies that $\Phi_{(n)}$ and $D\Phi_{(n)}$ commute so the Higgs current vanishes and $A_{(n)}$ also solves the sourceless Yang-Mills equations.

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