IRREDUCIBLE REPRESENTATIONS OF THE EXCEPTIONAL CHENG-KAC SUPERALGEBRA

Consuelo Martínez*
Departamento de Matemáticas, Universidad de Oviedo,
C/ Calvo Sotelo, s/n, 33007 Oviedo SPAIN

Efim Zelmanov†
Department of Mathematics, University of California at San Diego
9500 Gilman Drive, La Jolla, CA 92093-0112 USA

To the memory of our dear friend Hyo Chul Myung

Abstract

We classify all conformal irreducible modules of finite type over the Cheng Kac superalgebra CK_6.

1 Introduction

The study of Lie conformal superalgebras and their representations was initiated by V. Kac ([K2]) in view of their connections to the free fields realizations in conformal field theory. A complete classification of simple Lie conformal superalgebras of finite type was achieved in [FK]. The list consists of current Lie superalgebras, $Cur(\mathcal{G})$, where $\mathcal{G}$ is a simple finite dimensional

*Partially supported by MTM 2010-67884-C04-01
†Partially supported the NSF

Lie superalgebra; four series of Lie conformal superalgebras of Cartan type and the exceptional Lie conformal superalgebra CK_6.

For classification of representations of finite type of current Lie superalgebras and Lie superalgebras of Cartan type see [BKLR], [BKL1], [BKL2], [CK1].

In this paper we classify all conformal irreducible modules of finite type over the superalgebra CK_6. We use this classification and the results of [MZ4] to classify conformal irreducible Jordan bimodules of finite type over the Jordan superalgebra $JCK(6)$.

For a different approach to this classification see [BKL2].

2 Basic Definitions

Let A be an arbitrary (not necessarily associative) algebra over $\mathbf{C}$. By a formal distribution

$$a(z) = \sum_{i \in \mathbf{Z}} a(i) z^{-i-1} \in A[[z]]$$

we mean a power series over A, which is infinite in both directions.

Two formal distributions $a(z)$, $b(z)$ are said to be mutually local if there exists an integer $N = N(a,b) \geq 0$ such that $a(z)b(w)(z-w)^N = b(w)a(z)(z-w)^N = 0$.

We will consider a countable family of operations:

$$a(z) \circ_n b(z) = Res_w a(w) b(z) (w-z)^n, \ n \geq 0, \ n \in \mathbf{Z}.$$

Here Res_w means the coefficient at w^{-1}.

If $a(z)$, $b(z)$ are mutually local then only finitely many products $a \circ_n b$ may be different from zero.

Definition 2.1 *A vector space $C \subseteq A[[z^{-1}, z]]$ is called a conformal algebra of formal distributions if $\partial C \subseteq C$, $\partial = \frac{d}{dz}$, $C \circ_n C \subseteq C$ for an arbitrary $n \geq 0$ and every two elements from C are mutually local.*

By Dong Lemma (see [K2]) if A is an associative or Lie algebra then for an arbitrary collection C of pairwise mutually local distributions the closure of C with respect to the action of ∂ and to all operations $\circ_n$, $n \geq 0$, is a conformal algebra of formal distributions.

Examples 2.1 *(1) Let $\mathcal{G}$ be an arbitrary algebra and let $A = \mathcal{G}[t, t^{-1}]$ be the algebra of Laurent polynomials over $\mathcal{G}$. For an arbitrary element $a \in \mathcal{G}$ let $\tilde{a} = \sum_{i \in \mathbf{Z}} (at^i) z^{-i-1} \in A[[z^{-1}, z]]$.*

Any two formal distributions $\tilde{a}$, $\tilde{b}$ are mutually local.

(2) Let $\mathcal{V}ir$ =$Der\mathbf{C}[\mathbf{t^{-1}}, \mathbf{t}]$ be the (centerless) Virasoro algebra. The formal distribution

$$L = \sum_{i \in \mathbf{Z}} t^{i+1} \frac{d}{dt} z^{-i-2} \in \mathcal{V}ir[[z^{-1}, z]]$$

is mutually local with itself.

(3) Let $W =< t^{-1}, t, \frac{d}{dt} >$ be the (associative) Weyl algebra of differential operators on $C[t^{-1}, t]$. Let $J_k = \sum_{i \in Z} t^i (\frac{d}{dt})^k z^{-i-1}$, $k \geq 0$. Any two formal distributions J_k, J_l are mutually local.

In all three cases (1), (2) and (3) we can talk about the conformal algebras $Cur(\mathcal{G})$, $\mathcal{V}ir$, W respectively, generated by them.

Now we are ready to introduce an abstract definition of a conformal algebra.

Let C be a module over a polynomial algebra $\mathbf{C}[\partial]$, which is equipped with countably many binary bilinear operations $C \circ_n C \to C$, $n \geq 0$.

Definition 2.2 *We say that $(C, \partial, \circ_n)$ is an abstract conformal algebra if for arbitrary elements $a, b \in C$ arbitrary $n \geq 0$, we have:*

1) $\partial(a \circ_n b) = \partial a \circ_n b + a \circ_n \partial b$,

2) $\partial a \circ_n b = -na \circ_{n-1} b$; for $n = 0$ the condition turns into $\partial a \circ_0 b = 0$.

3) (Locality) There exists an integer $N = N(a, b) \geq 0$ such that for an arbitrary $n \geq N$ we have $a \circ_n b = 0$.

Every conformal algebra of formal distributions is an abstract conformal algebra. The converse is also true: every conformal algebra can be realized as an algebra of formal distributions over some algebra of coefficients. Moreover, among these algebras of coefficients there is a universal one $Coeff(C)$.

Definition 2.3 *We say that a conformal algebra C is a Lie (resp. associative, Jordan) algebra iff $Coeff(C)$ is a Lie (associative, Jordan) algebra.*

Now let C be a Lie conformal algebra and let M be another $\mathbf{C}[\partial]$-module. Suppose that we have a family of bilinear maps $C \circ_n M \subseteq M$, $n \geq 0$.

Definition 2.4 *We say that M is a conformal C-module if the null split extension $C + M$ is a Lie conformal algebra.*

As above, M can be realized as a space of formal distributions over $Coeff(M)$, where $Coeff(M)$ is a universal (with this property) Lie module over $Coeff(C)$.

Important Remark If there is a natural (and standard) way to arrange elements of a (super)algebra L in formal distributions then we will talk about L and modules over L even if we have in mind their conformal counterparts.

3 The Cheng-Kac Superalgebra

The exceptional conformal superalgebra CK_6 was introduced in [CK2] and in [GLS]. In [MZ1] we constructed, for an arbitrary associative commutative superalgebra R with an even derivation $d : R \to R$, a superalgebra $CK(R, d)$ so that $CK_6 \simeq CK(\mathbf{C}[\mathbf{t^{-1}}, \mathbf{t}], \frac{\mathbf{d}}{\mathbf{dt}})$.

Lets recall the construction of $CK(R, d)$ from [MZ1].

Consider the associative Weyl algebra $W = \sum_{i\geq 0} Rd^i$, where the variable d does not commute with a coefficient $a \in R$, but $da = ad + d(a)$. We will realize the $CK(R, d)$ as a superalgebra of 8×8 matrices over W.

The simple finite dimensional Lie superalgebra $P(n-1)$ is the superalgebra of $2n \times 2n$ matrices of the type $\begin{pmatrix} a & k \\ h & -a^t \end{pmatrix}$, where a, h, k are $n \times n$-matrices over $\mathbf{C}$, $tr(a) = 0$, $k^t = -k$, $h^t = h$. The superalgebras $P(n)$, $n \neq 3$, are centrally closed. However, $P(3)$ has a nontrivial central cover $\hat{P(3)}$. Its existence comes from the fact that the Lie algebra $K_4(\mathbf{C})$ of skew-symmetric 4×4 matrices is a direct sum of two ideals $K_4(\mathbf{C}) = sl_2(\mathbf{C}) \oplus sl_2(\mathbf{C})$. For an arbitrary element $k \in K_4(\mathbf{C})$ we consider its decomposition $k = k' + k''$ and let $\varphi(k) = k' - k''$. The universal central cover $\hat{P(3)}$ of $P(3)$ can be realized as a superalgebra of 8×8-matrices over the polynomial algebra $\mathbf{C}[d]$ of the type

$$\begin{pmatrix} a & k \\ \varphi(k)d + h & -a^t \end{pmatrix} + \alpha d I_8,$$

where a, k, h are 4×4 matrices over $\mathbf{C}$, $tr(a) = 0$, $k = -k^t$, $h = h^t$, $\alpha \in \mathbf{C}$ and I_8 is the identity matrix.

The superalgebra $CK(R, d)$ is a subsuperalgebra of 8×8 matrices over W generated by $\hat{P(3)}$ and by all matrices $\begin{pmatrix} e_{ij}(a) & 0 \\ 0 & -e_{ji}(a) \end{pmatrix}$ where $a \in R$, $1 \leq i \neq j \leq 4$.

The Cartan subalgebra H of $CK(R, d)$ consists of diagonal matrices

$$H = \{h = diag(a_1, \dots, a_4, -a_1, \dots, -a_4),\ a_i \in \mathbf{C},\ \sum_{i=1}^{4} a_i = 0\},$$

the even and the odd roots of the $CK(R, d)$ with respect to the action of H are:

$$\Delta_{\bar{0}} = \{w_i - w_j \,|\, 1 \leq i \neq j \leq 4\},$$

$$\Delta_{\bar{1}} = \{w_i + w_j,\ 1 \leq i \neq j \leq 4,\ -w_i - w_j,\ 1 \leq i, j \leq 4\}.$$

Notice that $w_i(a) = a_i$, $1 \leq i \leq 4$.

Thus, the superalgebra $CK(R, d)$ is graded by the abelian group

$$\sum_{i=1}^{4} \mathbf{Z} w_i / \mathbf{Z}(w_1 + w_2 + w_3 + w_4),$$

$CK(R, d) = \sum_{\alpha \in \Delta \cup \{0\}} CK(R, d)_\alpha$.

Let us fix the notation for the following weight elements:

$$e_{w_i - w_j} = \begin{pmatrix} e_{ij} & 0 \\ 0 & -e_{ji} \end{pmatrix},\ e_{w_i - w_j}(a) = \begin{pmatrix} e_{ij}(a) & 0 \\ 0 & -e_{ji}(a) \end{pmatrix},$$

$$h_{w_i - w_j}(a) = \begin{pmatrix} e_{ii}(a) - e_{jj}(a) & 0 \\ 0 & e_{jj}(a) - e_{ii}(a) \end{pmatrix},\ q_{-w_i - w_j} = \begin{pmatrix} 0 & 0 \\ e_{ij} + e_{ji} & 0 \end{pmatrix},$$

$$q_{-w_i - w_j}(a) = \begin{pmatrix} 0 & 0 \\ e_{ij}(a) + e_{ji}(a) & 0 \end{pmatrix},\ q_{w_i + w_j} = \begin{pmatrix} 0 & e_{ij} - e_{ji} \\ \varphi(e_{ij} - e_{ji})d & 0 \end{pmatrix},$$

$a \in R$.

In [MZ3] it was shown that $CK(R,d)_{w_i-w_j}=e_{w_i-w_j}(R)$, $1\le i\ne j\le 4$; $CK(R,d)_{-2w_i}=q_{-2w_i}(R)$; $CK(R,d)_{w_i+w_j}=[q_{w_i+w_k},e_{w_j-w_k}(R)]+q_{-w_k-w_l}(R)$, where $\{i,j,k,l\}=\{1,2,3,4\}$.

For an arbitrary element $a\in R$ consider the element

$$[[e_{w_4-w_1}(a),q_{w_3+w_1}],q_{w_2+w_1}]=$$

$$\begin{pmatrix} e_{11}(da)+e_{22}(ad)+e_{33}(ad)+e_{44}(ad) & 0 \\ 0 & e_{11}(ad)+e_{22}(da)+e_{33}(da)+e_{44}(da) \end{pmatrix}$$

$$=I_8(ad)-\begin{pmatrix} e_{11}(a') & 0 \\ 0 & -e_{11}(a')+I_4(a') \end{pmatrix},\ a'=[a,d]=d(a)$$

We will denote the element on the right hand side as $Vir(a)$. The mapping $ad\to Vir(a)$ from $Rd\to \mathcal{V}ir(R)$ is and isomorphism of Lie algebras.

It was shown in [MZ3] that $CK(R,d)_0=H\otimes R+\mathcal{V}ir(R)$.

Consider the functional

$$f:\sum_{i=1}^{4}\mathbf{Z}w_i/\mathbf{Z}(\sum_{i=1}^{4}w_i)\to\mathbf{Z}$$

given by $f(w_1)=5,\ f(w_2)=-3,\ f(w_3)=2,\ f(w_4)=-4.$

Notice that $f(\pm w_i\pm w_j)\ne 0$, unless $\pm w_i\pm w_j=0$.

From now on we will denote $L=CK_6=CK(\mathbf{C}[t^{-1},t],\frac{d}{dt})$.

Note that $L_0=H\otimes\mathbf{C}[t^{-1},t]\rtimes\mathcal{V}ir(R)\le Cur(sl_4)\rtimes\mathcal{V}ir(R)\le L$.

The algebra L has a triangular decomposition $L=L_-+L_0+L_+$, $L_-=\sum_{f(\alpha)<0}L_\alpha$, $L_+=\sum_{f(\alpha)>0}L_\alpha$.

Let M be a conformal module of finite type over the Lie conformal algebra CK_6. Then the subalgebra $sl_4\subseteq L$ acts on M and the action of sl_4 commutes with the action of the polynomial algebra $\mathbf{C}[\partial]$. Hence M decomposes into a finite direct sum of eigenspaces with respect to the action of H,

$$M=\sum_{\gamma\in H^*}M_\gamma.$$

If M is irreducible, then there exists a unique highest weight $\lambda \in H^*$ such that $M_\lambda \neq (0)$ and $L_+ \circ_n M_\lambda = (0)$ for all $n \geq 0$; M_λ is an irreducible conformal module over L_0.

We have mentioned above that $L_0 \subset Cur(sl_4) + \mathcal{V}ir \subset L$.

Let M' be the $Cur(sl_4) + \mathcal{V}ir$-module generated by M_λ. Let M'' be the largest submodule of M' such that $M'' \cap M_\lambda = 0$. Then M'/M'' is an irreducible $Cur(sl_4)+\mathcal{V}ir$-module and $(M'/M'')_\lambda = M_\lambda$. Let $V = Coeff(M)$ be an L-module.

From the description of irreducible modules of finite type over $Cur(sl_4) \bowtie \mathcal{V}ir(R)$ (see [CK1]) it follows that the module V_λ can be identified with $\mathbf{C}[t^{-1}, t]$, say $V_\lambda = \overline{\mathbf{C}[t^{-1}, t]}$. For arbitrary elements $a, b \in \mathbf{C}[t^{-1}, t]$, $h \in H$ we have $(h \otimes a)\bar{b} =< \lambda, h > \overline{ab}$. Moreover, there exist scalars $\alpha, \beta \in \mathbf{C}$ such that for arbitrary $a, b \in \mathbf{C}[t^{-1}, t]$ we have

$$Vir(a)\bar{b} = \overline{-ab' + \beta a'b + \alpha ab}.$$

Denote this L_0-module as $V(\lambda, \beta, \alpha)$. It is well known that, for an arbitrary $\lambda \in H^*$, given an irreducible L_0-module W such that the elements $h \in H$ act on W as scalar multiplications $< \lambda, h >$, there exists a unique L-module with the highest weight λ under the action of H, whose λ-space is isomorphic to W as L_0-module. If we consider the irreducible L_0-module $V(\lambda, \beta, \alpha)$, then the corresponding irreducible L-module will be denoted as $Irr(\lambda, \beta, \alpha)$.

It follows from the above that every irreducible conformal module over CK_6 is isomorphic to $Irr(\lambda, \beta, \alpha)$ for some $\lambda \in H^*$, $\beta, \alpha \in \mathbf{C}$. This gives rise to the question:

For which parameters $\lambda \in H^*$, $\beta, \alpha \in \mathbf{C}$, the irreducible conformal module $Irr(\lambda, \Delta, \alpha)$ is of finite type?

Let λ be an integral dominant weight, that is, $< \lambda, w_1 - w_3 >$, $< \lambda, w_3 - w_2 >$, $< \lambda, w_2 - w_4 >$ all lie in $\mathbf{Z}_{\geq 0}$.

Theorem 3.1 *The conformal module $Irr(\lambda, \beta, \alpha)$ is of finite type if and only if*

(1)$< \lambda, h_{w_1 - w_3} >\geq 2$; $\beta, \alpha \in \mathbf{C}$, or

(2)$< \lambda, h_{w_1 - w_3} >= 1$; $< \lambda, h_{w_2 - w_3} >= 0$, $\beta = -1$, $\alpha \in \mathbf{C}$.

These modules exhaust all conformal irreducible CK_6-modules of finite type.

Since $V(\lambda, \beta, \alpha)$ are known to be conformal modules of finite type over L_0 (see [CK1]), we can easily conclude that $Irr(\lambda, \beta, \alpha)$ is of finite type if and only if it has finitely many weights with respect to the action of H. At this point we can forget about conformal modules and address the question:

For which $\lambda \in H^*$, $\beta, \alpha \in \mathbf{C}$, the L-module $Irr(\lambda, \beta, \alpha)$ has finitely many weights?

Lemma 3.1 *Let $\alpha = w_i - w_j$ or $-w_i - w_j$. For an element $a \in R$, let $X_\alpha(a) = e_{w_i - w_j}(a)$ or $q_{-w_i - w_j}(a)$ defined as above. Suppose that $\alpha < 0$ and for any decomposition $-\alpha = \alpha_1 + \cdots + \alpha_r$ into a sum of positive roots, for any elements $x_i \in L_{\alpha_i}$, $1 \leq i \leq r$,there exist an element $h \in H$ and an element $b \in R$ such that $[x_1, [x_2, \ldots [x_r, X_\alpha(a)] \cdots] = h \otimes ab$ for an arbitrary $a \in R$. Then for an arbitrary element $v_\lambda \in V_\lambda$ we have $X_\alpha(a) v_\lambda = X_\alpha(1)(av)_\lambda$.*

Proof: It is sufficient to show that

$$U(L_+)(X_\alpha(a)v_\lambda - X_\alpha(1)(av)_\lambda) \cap V_\lambda = (0).$$

Otherwise there exists a decomposition $-\alpha = \alpha_1 + \cdots + \alpha_r$, $\alpha_i > 0$ and elements $x_i \in L_{\alpha_i}$ such that $x_1 \cdots x_r (X_\alpha(a) v_\alpha - X_\alpha(1)(av)_\lambda) \neq 0$.

But

$$x_1 \cdots x_r X_\alpha(a) v_\lambda = [x_1, [x_2, \ldots, [x_r, X_\alpha(a)] \ldots] v_\lambda = (h \otimes ab) v_\lambda =$$

$$h(abv)_\lambda = [x_1, [x_2, \ldots, [x_r, X_\alpha(1)] \ldots](av)_\lambda = x_1 \ldots x_r X_\alpha(1)(av)_\lambda,$$

a contradiction. The lemma is proved.

Lemma 3.2 *The negative roots $w_2 - w_3$, $w_4 - w_3$, $-w_1 - w_2$, $-w_1 - w_3$, $-w_1 - w_4$, $w_4 - w_2$ satisfy the assumptions of Lemma 3.1*

Proof: We list all possible decompositions. The roots $w_3 - w_2$, $w_1 + w_4$ and $w_2 - w_4$ do not have nontrivial decompositions. Then, $w_1 + w_2 = (w_1 + w_4) + (w_2 - w_4)$, $w_1 + w_3 = (w_1 + w_2) + (w_3 - w_2) = (w_1 + w_4) + (w_2 - w_4) + (w_3 - w_2) = (w_3 - w_4) + (w_4 + w_1)$; $w_3 - w_4 = (w_2 - w_4) + (w_3 - w_2)$.

The condition of Lemma 3.1 is checked by a straightforward computation in the superalgebra L. The lemma is proved.

Lemma 3.3 *For arbitrary elements $a, b \in R$ we have*

$$[q_{w_1+w_4}, e_{w_2-w_4}(a)][q_{w_1+w_4}, e_{w_3-w_4}(b)]q_{-w_1-w_2}q_{-w_1-w_3}v_\lambda = \gamma(abv)_\lambda,$$

where $\gamma =< \lambda, h_{w_1-w_2} > (1- < \lambda, h_{w_1-w_3} >)$.

Proof: We have $[q_{w_1+w_4}, e_{w_3-w_4}(b)]q_{-w_1-w_2}q_{-w_1-w_3}v_\lambda = (I) - (II)$, where

$$(I) = q_{w_1+w_4}e_{w_3-w_4}(b)q_{-w_1-w_2}q_{-w_1-w_3}v_\lambda =$$

$$q_{w_1+w_4}q_{-w_1-w_2}e_{w_3-w_4}(b)q_{-w_1-w_3}v_\lambda =$$

$$q_{w_1+w_4}q_{-w_1-w_2}[e_{w_3-w_4}(b), q_{-w_1-w_3}]v_\lambda + q_{w_1+w_4}q_{-w_1-w_2}q_{-w_1-w_3}e_{w_3-w_4}(b)v_\lambda =$$

$$-q_{w_1+w_4}q_{-w_1-w_2}q_{-w_1-w_4}(b)v_\lambda = -q_{w_1+w_4}q_{-w_1-w_2}q_{-w_1-w_4}(bv)_\lambda,$$

by Lemma 3.1.

$$(II) = e_{w_3-w_4}(b)q_{w_1+w_4}q_{-w_1-w_2}q_{-w_1-w_3}v_\lambda =$$

$$-e_{w_3-w_4}(b)e_{w_4-w_2}q_{-w_1-w_3}v_\lambda - e_{w_3-w_4}(b)q_{-w_1-w_2}q_{w_1+w_4}q_{-w_1-w_3}v_\lambda =$$

$$-e_{w_3-w_4}(b)q_{-w_1-w_3}e_{w_4-w_2}v_\lambda - q_{-w_1-w_2}e_{w_3-w_4}(b)q_{w_1+w_4}q_{-w_1-w_3}v_\lambda.$$

Now

$$e_{w_3-w_4}(b)q_{-w_1-w_3}e_{w_4-w_2}v_\lambda =$$

$$[e_{w_3-w_4}(b), q_{-w_1-w_3}]e_{w_4-w_2}v_\lambda + q_{-w_1-w_3}e_{w_3-w_4}(b)e_{w_4-w_2}v_\lambda.$$

The second summand is 0 since $f(w_3-w_4) = 6$ whereas $f(w_4-w_2) = -1$. The first summand is

$$-q_{-w_1-w_4}(b)e_{w_4-w_2}v_\lambda =$$

$$-[q_{-w_1-w_4}e_{w_4-w_2}]v_\lambda - e_{w_4-w_2}q_{-w_1-w_4}(b)v_\lambda =$$

$$-q_{-w_1-w_2}(b)v_\lambda - e_{w_4-w_2}q_{-w_1-w_4}(b)v_\lambda =$$

$$-q_{-w_1-w_2}(bv)_\lambda - e_{w_4-w_2}q_{-w_1-w_4}(bv_\lambda),$$

by Lemma 3.2.

As for the other summand of (II),

$$q_{-w_1-w_2}e_{w_3-w_4}(b)q_{w_1+w_4}q_{-w_1-w_3}v_\lambda =$$

$$q_{-w_1-w_2}[e_{w_3-w_4}(b),[q_{w_1+w_4},q_{-w_1-w_3}]]v_\lambda =$$
$$q_{-w_1-w_2}h_{w_3-w_4}(b)v_\lambda = q_{-w_1-w_2}h_{w_3-w_4}(bv)_\lambda.$$

We proved that

$$[q_{w_1+w_4},e_{w_3-w_4}(b)]q_{-w_1-w_2}q_{-w_1-w_3}v_\lambda = P(bv)_\lambda,$$

where P is an operator that does not involve b. Choosing $b=1$ we get

$$P = ad([q_{w_1+w_4},e_{w_3-w_4}])ad(q_{-w_1-w_2})ad(q_{-w_1-w_3}) =$$
$$ad(q_{w_3+w_1})ad(q_{-w_1-w_2})ad(q_{-w_1-w_3}).$$

Now we have to consider the element

$$[q_{w_1+w_4},e_{w_2-w_4}(a)]q_{w_3+w_1}q_{-w_1-w_2}q_{-w_1-w_3}(bv)_\lambda.$$

Remark that $[[q_{w_1+w_4},e_{w_2-w_4}(a)],q_{w_3+w_1}]\in e_{w_1-w_4}(R)$ and

$$e_{w_1-w_4}(R)q_{-w_1-w_2}q_{-w_1-w_3}(bv)_\lambda = [[e_{w_1-w_4}(R),q_{-w_1-w_2}],q_{-w_1-w_4}](bv)_\lambda = (0).$$

Hence our expression becomes

$$-q_{w_1+w_3}[q_{w_1+w_4},e_{w_2-w_4}(a)]q_{-w_1-w_2}q_{-w_1-w_3}(bv)_\lambda.$$

Denote $X=[q_{w_1+w_4},e_{w_2-w_4}(a)]$, $Y=q_{-w_1-w_2}$, $Z=q_{-w_1-w_3}$. Then $XYZ=[[X,Y],Z]-Y[X,Z]+YZX+Z[X,Y]$, $[X,Y]=h_{w_2-w_1}(a)$, $[X,Z]=e_{w_2-w_3}(a)$, $[[X,Y],Z]=q_{-w_1-w_3}(a)$. By Lemma 2,

$$h_{w_2-w_1}(a)(bv)_\lambda = h_{w_2-w_1}(abv)_\lambda,\ e_{w_2-w_3}(a)(bv)_\lambda = e_{w_2-w_3}(abv)_\lambda$$

and

$$q_{-w_1-w_3}(a)(bv)_\lambda = q_{-w_2-w_3}(abv)_\lambda.$$

As we did above, we can conclude that

$$[q_{w_1+w_4},e_{w_2-w_4}(a)][q_{w_1+w_4},e_{w_3-w_4}(b)]q_{-w_1-w_2}q_{-w_1-w_3}v_\lambda = \tilde{P}(abv)_\lambda,$$

where $\tilde{P}$ is an operator that does not involve a or b. Choosing $a=b=1$ we get

$$\tilde{P} = ad(q_{w_2+w_1})ad(q_{w_3+w_1})ad(q_{-w_1-w_2})ad(q_{-w_1-w_3}).$$

Now

$$\tilde{P}(abv)_\lambda = q_{w_2+w_1}[q_{w_3+w_1}, q_{-w_1-w_2}]q_{-w_1-w_3}(abv)_\lambda -$$
$$q_{w_2+w_1}q_{-w_1-w_2}q_{w_3+w_1}q_{-w_1-w_3}(abv)_\lambda =$$
$$[q_{w_2+w_1}, [[q_{w_3+w_1}, q_{-w_1-w_2}], q_{-w_1-w_3}]](abv)_\lambda -$$
$$[q_{w_2+w_1}, q_{-w_1-w_2}], [q_{w_3+w_1}, q_{-w_1-w_3}]](abv)_\lambda =$$
$$-h_{w_2-w_1}(abv)_\lambda - h_{w_2-w_1}h_{w_3-w_1}(abv)_\lambda = \gamma(abv)_\lambda,$$

$\gamma =< \lambda, h_{w_1-w_2} > (1- < \lambda, h_{w_1-w_3} >)$. The lemma is proved.

Remark. In what follows $[x_1, \ldots, x_n]$ denotes the left-normed commutator $[...[x_1, x_2], x_3], \ldots, x_n]$.

Lemma 3.4 $[[[e_{w_4-w_1}(a), [q_{w_1+w_4}, e_{w_3-w_4}(c)]], [q_{w_1+w_4}, e_{w_2-w_4}(b)]] = h_{w_1-w_4}(ab'c) - Vir(abc)$.

Proof: Since $[e_{w_4-w_1}(a), q_{w_1+w_4}] = 0$ the left hand side is equal to

$$[e_{w_4-w_1}(a), e_{w_3-w_4}(c), q_{w_1+w_4}, e_{w_2-w_4}(b), q_{w_1+w_4}] =$$

$$-[e_{w_3-w_1}(ac), q_{w_1+w_4}, e_{w_2-w_4}(b), q_{w_1+w_4}].$$

Furthermore, $e_{w_2-w_4}(b) = -[q_{w_1+w_2}, q_{-w_1-w_4}(b)]$. Substituting this expression we get:

$$LHS = [e_{w_3-w_1}(ac), q_{w_1+w_4}, q_{w_1+w_2}, q_{-w_1-w_4}(b), q_{w_1+w_4}] +$$
$$[e_{w_3-w_1}(ac), q_{w_1+w_4}, q_{-w_1-w_4}(b), q_{w_1+w_2}, q_{w_1+w_4}] = (I) + (II).$$

Recall that we use the notation $[e_{w_4-w_1}(a), q_{w_3+w_1}, q_{w_2+w_1}] = Vir(a)$. Now,

$$[e_{w_3-w_1}(ac), q_{w_1+w_4}, q_{w_1+w_2}] = -[e_{w_4-w_1}(ac), e_{w_3-w_4}, q_{w_1+w_4}, q_{w_1+w_2}] =$$

$$-[e_{w_4-w_1}(ac), [e_{w_3-w_4}, q_{w_1+w_4}], q_{w_1+w_2}]$$

since $[e_{w_4-w_1}(ac), q_{w_1+w_4}] \subseteq L_{2w_4} = (0)$.

Using that $[e_{w_3-w_4}, q_{w_1+w_4}] = -q_{w_3+w_1}$, our expression becomes

$$[e_{w_4-w_1}(ac), q_{w_3+w_1}, q_{w_2+w_1}] = Vir(ac).$$

Hence,

$$(I) = [e_{w_3-w_1}(ac), q_{w_1+w_4}, q_{w_1+w_2}, [q_{-w_1-w_4}(b), q_{w_1+w_4}]] =$$

$$-[Vir(ac), h_{w_1-w_4}(b)] = -h_{w_1-w_4}(ab'c).$$

On the other hand,

$$(II) = [e_{w_3-w_1}(ac), q_{w_1+w_4}, q_{-w_1-w_4}(b), q_{w_1+w_2}, q_{w_1+w_4}] =$$

$$[e_{w_3-w_1}(ac), [q_{w_1+w_4}, q_{-w_1-w_4}(b)], q_{w_1+w_2}, q_{w_1+w_4}] =$$

$$[e_{w_3-w_1}(ac), h_{w_1-w_4}(b), q_{w_1+w_2}, q_{w_1+w_4}] = [e_{w_3-w_1}(abc), q_{w_1+w_2}, q_{w_1+w_4}] =$$

$$-[e_{w_3-w_1}(abc), q_{w_1+w_4}, q_{w_1+w_2}] = -Vir(abc),$$

as we have seen above. This proves the lemma.

Lemma 3.4 implies that

$$[q_{w_1+w_4}, e_{w_2-w_4}(b)][q_{w_1+w_4}, e_{w_3-w_4}(c)]e_{w_4-w_1}(a)v_\lambda =$$

$$-[e_{w_4-w_1}(a), [q_{w_1+w_4}, e_{w_3-w_4}(c)], [q_{w_1+w_4}, e_{w_2-w_4}(b)]]v_\lambda =$$

$$(-h_{w_1-w_4}(ab'c) - Vir(abc))v_\lambda =$$

$$- < \lambda, h_{w_1-w_4} > (abcv')_\lambda + ((\mu_0 abc + \mu_1 a'bc + \mu_2 ab'c + \mu_3 abc')v)_\lambda,$$

here v is viewed as an element from $R = F[t^{-1}, t]$; $\mu_0,\ \mu_1,\ \mu_2,\ \mu_3$ are scalars from F.

Choosing $a = 1$ we get:

$$[q_{w_1+w_4}, e_{w_2-w_4}(b)][q_{w_1+w_4}, e_{w_3-w_4}(c)]e_{w_4-w_1}v_\lambda =$$

$$((\mu_0 bc + \mu_2 b'c + \mu_3 bc')v)_\lambda - < \lambda, h_{w_1-w_4} > (bcv')_\lambda.$$

Hence, $[q_{w_1+w_4}, e_{w_2-w_4}(b)][q_{w_1+w_4}, e_{w_3-w_4}(c)]e_{w_4-w_1}(av)_\lambda =$

$$((\mu_0 abc + \mu_2 ab'c + \mu_3 abc')v)_\lambda - < \lambda, h_{w_1-w_4} > ((abcv')_\lambda + (a'bcv)_\lambda).$$

This implies

$$[q_{w_1+w_4}, e_{w_2-w_4}(b)][q_{w_1+w_4}, e_{w_3-w_4}(c)](e_{w_4-w_1}(a)(v)_\lambda - e_{w_4-w_1}(av)_\lambda =$$

$$(\mu_1 + < \lambda, h_{w_1-w_4} >)(a'bcv)_\lambda. \qquad (*)$$

4 The case $<\lambda, h_{w_1-w_3}> \geq 2$

In this section we will prove that if λ is an integral dominant functional and $<\lambda, h_{w_1-w_3}>\geq 2$, then for arbitrary $\beta, \alpha \in F$ the irreducible module $V(\lambda, \beta, \alpha)$ has only finitely many weights with respect to the action of H.

Let $\gamma =<\lambda, h_{w_1-w_2}> (1-<\lambda, h_{w_1-w_3}>)$ (see Lemma 3.3). Since $w_1 - w_2 = (w_1 - w_3) + (w_3 - w_2)$ and the root $w_3 - w_2$ is positive, we conclude that $<\lambda, h_{w_1-w_2}>\geq 2$ and therefore $\gamma \neq 0$. Let $\xi = \frac{\mu_1+<\lambda, h_{w_1-w_4}>}{\gamma}$.

Lemma 4.1 $e_{w_4-w_1}(a)v_\lambda - e_{w_4-w_1}(av)_\lambda - \xi q_{-w_1-w_2}q_{-w_1-w_3}(a'v)_\lambda = 0.$

Proof. Denote the left hand side of the above equality as w. In order to prove that $w = 0$ we need only to check that $L_+w \cap V_\lambda = (0)$. From the equality (*) and Lemma 3.3 it follows that

$$[q_{w_1+w_4}, e_{w_2-w_4}(b)][q_{w_1+w_4}, e_{w_3-w_4}(c)]w = 0.$$

Consider the element $q_{-w_3-w_4}(b)w$. We have $q_{-w_3-w_4}(b)e_{w_4-w_1}(a)v_\lambda = q_{-w_1-w_3}(ab)v_\lambda = q_{-w_1-w_3}(abv)_\lambda$ by Lemma 3.2. The last expression is equal also to $q_{-w_3-w_4}(b)e_{w_3-w_1}(av)_\lambda$. Furthermore,

$$q_{-w_3-w_4}(b)q_{-w_1-w_2}q_{-w_1-w_3}v_\lambda = q_{-w_1-w_2}q_{-w_1-w_3}q_{-w_3-w_4}(b)v_\lambda = 0,$$

since $-w_3 - w_4$ is positive.

We have shown that $q_{-w_3-w_4}(R)w = (0)$.
Similarly $q_{-w_2-w_4}(R)w = (0)$.

Let us show that $e_{w_1-w_4}(R)w = (0)$. Indeed, $f(w_1 - w_4) = 9$, $f(-w_1 - w_2) = -2$, $f(-w_1 - w_3) = -7$. Hence,

$$e_{w_1-w_4}(b)q_{-w_1-w_2}q_{-w_1-w_3}v_\lambda = [e_{w_1-w_4}(b), q_{-w_1-w_2}, q_{-w_1-w_3}]v_\lambda = 0.$$

Now

$$e_{w_1-w_4}(b)w = e_{w_1-w_4}(b)e_{w_4-w_1}(a)v_\lambda - e_{w_1-w_4}(b)e_{w_4-w_1}(av)_\lambda =$$

$$h_{w_1-w_4}(ab)v_\lambda - h_{w_1-w_4}(b)(av)_\lambda = 0.$$

Since $[L_{w_1+w_2}, L_{w_1+w_3}] \subseteq e_{w_1-w_4}(R)$ it follows that for arbitrary elements $x \in L_{w_1+w_2}$, $y \in L_{w_1+w_3}$, $xyw = -yxw$.

We have $L_{w_1+w_2}=[q_{w_1+w_4},e_{w_2-w_4}(R)]+q_{-w_3-w_4}(R)$, $L_{w_1+w_3}=[q_{w_1+w_4},e_{w_3-w_4}(R)]+q_{-w_2-w_4}(R)$.

From what we proved above, it follows that $L_{w_1+w_2}L_{w_1+w_3}w=(0)$. Together with $e_{w_1-w_4}(R)w=(0)$ it implies that $U(L_+)w\cap V_\lambda=(0)$ and therefore $w=0$. Lemma is proved.

Lemma 4.2 $e_{w_3-w_1}(a)v_\lambda=e_{w_3-w_1}(av)_\lambda-\xi q_{-w_1-w_2}q_{-w_1-w_4}(a'v)_\lambda$.

Proof. By Lemma 2.5,

$$e_{w_3-w_4}e_{w_4-w_1}(a)v_\lambda=e_{w_3-w_4}e_{w_4-w_1}(av)_\lambda+\xi e_{w_3-w_4}q_{-w_1-w_2}q_{-w_1-w_3}(a'v)_\lambda.$$

Since w_3-w_4 is positive, it implies

$$[e_{w_3-w_4},e_{w_4-w_1}(a)]v_\lambda=[e_{w_3-w_4},e_{w_4-w_1}](av)_\lambda+$$
$$\xi q_{-w_1-w_2}[e_{w_3-w_4},q_{-w_1-w_3}](a'v)_\lambda=e_{w_3-w_1}(av)_\lambda-\xi q_{-w_1-w_2}q_{-w_1-w_4}(a'v)_\lambda.$$

This implies the result and proves the lemma.

From now on in this section, unless otherwise stated, we will assume that $<\lambda,h_{w_1-w_3}>=2$. Our first aim is to show that $e^3_{w_3-w_1}V_\lambda=(0)$.

Lemma 4.3 $e_{w_1-w_3}(a)e^3_{w_3-w_1}v_\lambda=6\,\xi\,e_{w_3-w_1}q_{-w_1-w_2}q_{-w_1-w_4}(a'v)$.

Proof. Taking into account that

$$[e_{w_1-w_3}(a),e_{w_3-w_1},e_{w_3-w_1},e_{w_3-w_1}]=e_{w_1-w_3}(a)e^3_{w_3-w_1}-$$
$$3e_{w_3-w_1}e_{w_1-w_3}(a)e^2_{w_3-w_1}+3e^2_{w_3-w_1}e_{w_1-w_3}(a)e_{w_3-w_1}-e^3_{w_3-w_1}e_{w_1-w_3}(a)=0$$

and

$$e_{w_1-w_3}(a)e^2_{w_3-w_1}=[e_{w_1-w_3}(a),e_{w_3-w_1},e_{w_3-w_1}]+$$
$$2e_{w_3-w_1}e_{w_1-w_3}(a)e_{w_3-w_1}-e^2_{w_3-w_1}e_{w_1-w_3}(a)=$$
$$-2e_{w_3-w_1}(a)+2e_{w_3-w_1}e_{w_1-w_3}(a)e_{w_3-w_1}-e^2_{w_3-w_1}e_{w_1-w_3}(a),$$

we get

$$e_{w_1-w_3}(a)e^3_{w_3-w_1}v_\lambda=3e_{w_3-w_1}(-2e_{w_3-w_1}(a)+2e_{w_3-w_1}e_{w_1-w_3}(a)e_{w_3-w_1})v_\lambda-$$
$$3e^2_{w_3-w_1}e_{w_1-w_3}(a)e_{w_3-w_1}v_\lambda=-6e_{w_3-w_1}e_{w_3-w_1}(a)v_\lambda+3e^2_{w_3-w_1}h_{w_1-w_3}(a)v_\lambda=$$
$$-6e_{w_3-w_1}(e_{w_3-w_1}(av)_\lambda-\xi q_{-w_1-w_2}q_{-w_1-w_4}(a'v)_\lambda+6e^2_{w_3-w_1}(av)_\lambda=$$
$$6\,\xi e_{w_3-w_1}q_{-w_1-w_2}q_{-w_1-w_4}(a'v)_\lambda.$$

The lemma is proved.

Lemma 4.4 $e_{w_1-w_3}(a)e_{w_1-w_3}(b)e^3_{w_3-w_1}v_\lambda = 0.$

Proof. By Lemma 4.3, the left hand side is equal to

$$6\,\xi e_{w_1-w_3}(a)e_{w_3-w_1}q_{-w_1-w_2}q_{-w_1-w_4}(b'v)_\lambda.$$

We notice that

$$e_{w_1-w_3}(a)q_{-w_1-w_2}q_{-w_1-w_4}(b'v)_\lambda = [e_{w_1-w_3}(a), q_{-w_1-w_2}, q_{-w_1-w_4}](b'v)_\lambda = 0.$$

Hence, $$e_{w_1-w_3}(a)e_{w_3-w_1}q_{-w_1-w_2}q_{-w_1-w_4}(b'v)_\lambda =$$

$$[e_{w_1-w_3}(a), e_{w_3-w_1}]q_{-w_1-w_2}q_{-w_1-w_4}(b'v)_\lambda = h_{w_1-w_3}(a)q_{-w_1-w_2}q_{-w_1-w_4}(b'v)_\lambda.$$

From Lemma 3.2 it follows that the last expression is equal to

$$h_{w_1-w_3}(1)q_{-w_1-w_2}q_{-w_1-w_4}(ab'v)_\lambda =$$

$$< \lambda + w_3 - w_1, h_{w_1-w_3}(1) > q_{-w_1-w_2}q_{-w_1-w_4}(ab'v)_\lambda = 0.$$

The lemma is proved.

Lemma 4.5 $[q_{w_1+w_3}, e_{w_4-w_3}(a)]e_{w_3-w_1}q_{-w_1-w_2}q_{-w_1-w_4}v_\lambda =$
$q_{w_4+w_1}e_{w_3-w_1}q_{-w_1-w_2}q_{-w_1-w_4}(av)_\lambda.$

Proof. We have $[q_{w_1+w_3}, e_{w_4-w_3}(a)] = q_{w_1+w_3}e_{w_4-w_3}(a) - e_{w_4-w_3}(a)q_{w_1+w_3}$. Now, since the total weight of the expression $q_{w_1+w_3}e_{w_3-w_1}q_{-w_1-w_2}q_{-w_1-w_4}$ is $3w_3 - w_1$ that is positive, we only need to consider the expression

$$q_{w_1+w_3}e_{w_4-w_3}(a)e_{w_3-w_1}q_{-w_1-w_2}q_{-w_1-w_4}v_\lambda = (I) + (II)$$

where $(I) = q_{w_1+w_3}e_{w_4-w_1}(a)q_{-w_1-w_2}q_{-w_1-w_4}v_\lambda$, and
$(II) = q_{w_1+w_3}e_{w_3-w_1}e_{w_4-w_3}(a)q_{-w_1-w_2}q_{-w_1-w_4}v_\lambda$.

Let us consider these expressions separately.

$$(I) = q_{w_1+w_3}q_{-w_1-w_2}e_{w_4-w_1}(a)q_{-w_1-w_4}v_\lambda =$$

$$\underbrace{q_{w_1+w_3}q_{-w_1-w_2}q_{-w_1-w_4}e_{w_4-w_1}(a)v_\lambda}_{I.1} + \underbrace{q_{w_1+w_3}q_{-w_1-w_2}[e_{w_4-w_1}(a), q_{-w_1-w_4}]v_\lambda}_{I.2}.$$

$$(I.1) = q_{w_1+w_3}q_{-w_1-w_2}q_{-w_1-w_4}e_{w_4-w_1}(a)v_\lambda =$$

$$q_{w_1+w_3}q_{-w_1-w_2}q_{-w_1-w_4}(e_{w_4-w_1}(av)_\lambda + \xi q_{-w_1-w_2}q_{-w_1-w_3}(a'v)_\lambda) =$$
$$q_{w_1+w_3}q_{-w_1-w_2}q_{-w_1-w_4}e_{w_4-w_1}(av)_\lambda,$$

because $q_{-w_1-w_2}q_{-w_1-w_4}q_{-w_1-w_2} = 0$;

$$(I.2) = -q_{w_1+w_3}q_{-w_1-w_2}q_{-2w_1}(a)v_\lambda =$$
$$q_{-w_1-w_2}q_{w_1+w_3}q_{-2w_1}(a)v_\lambda - [q_{w_1+w_3}, q_{-w_1-w_2}]q_{-2w_1}(a)v_\lambda = (I.2.1) + (I.2.2);$$

$$(I.2.1) = q_{-w_1-w_2}[q_{w_1+w_3}, q_{-2w_1}(a)]v_\lambda = -2q_{-w_1-w_2}e_{w_3-w_1}(a)v_\lambda =$$
$$-2q_{-w_1-w_2}(e_{w_3-w_1}(av)_\lambda - \xi q_{-w_1-w_2}q_{-w_1-w_4}(a'v)_\lambda) = -2q_{-w_1-w_2}e_{w_3-w_1}(av)_\lambda;$$

$$(I.2.2) = e_{w_3-w_2}q_{-2w_1}(a)v_\lambda = q_{-2w_1}(a)e_{w_3-w_2}v_\lambda = 0$$

since $w_3 - w_2$ is positive.

$$(II) = q_{w_1+w_3}e_{w_3-w_1}e_{w_4-w_3}(a)q_{-w_1-w_2}q_{-w_1-w_4}v_\lambda =$$
$$q_{w_1+w_3}e_{w_3-w_1}q_{-w_1-w_2}e_{w_4-w_3}(a)q_{-w_1-w_4}v_\lambda =$$
$$\underbrace{q_{w_1+w_3}e_{w_3-w_1}q_{-w_1-w_2}q_{-w_1-w_4}e_{w_4-w_3}(a)v_\lambda}_{II.1} +$$
$$\underbrace{q_{w_1+w_3}e_{w_3-w_1}q_{-w_1-w_2}[e_{w_4-w_3}(a), q_{-w_1-w_4}]v_\lambda}_{II.2}.$$

But

$$(II.1) = q_{w_1+w_3}e_{w_3-w_1}q_{-w_1-w_2}q_{-w_1-w_4}e_{w_4-w_3}(a)v_\lambda =$$
$$q_{w_1+w_3}e_{w_3-w_1}q_{-w_1-w_2}q_{-w_1-w_4}e_{w_4-w_3}(av)_\lambda$$

by Lemma 3.2;

$$(II.2) = -q_{w_1+w_3}e_{w_3-w_1}q_{-w_1-w_2}q_{-w_1-w_3}(a)v_\lambda =$$
$$-q_{w_1+w_3}e_{w_3-w_1}q_{-w_1-w_2}q_{-w_1-w_3}(av)_\lambda$$

again by Lemma 3.2.

To summarize, we have proved that

$$[q_{w_1+w_3}, e_{w_4-w_3}(a)]e_{w_3-w_1}q_{-w_1-w_3}q_{-w_1-w_4}v_\lambda = P(av)_\lambda,$$

where P is an operator that does not involve a. Choosing $a = 1$, we get $P = ad(q_{w_4+w_1})ad(e_{w_3-w_1})ad(q_{-w_1-w_2})ad(q_{-w_1-w_4})$. The lemma is proved.

Lemma 4.6 *(i)* $[q_{w_1+w_3}, e_{w_4-w_3}(a)]e^3_{w_3-w_1}v_\lambda =$

$$-q_{w_1+w_4}e^3_{w_3-w_1}(av)_\lambda - 3\,\xi e^2_{w_3-w_1}q_{-w_1-w_2}(a'v)_\lambda.$$

(ii) $[q_{w_1+w_3}, e_{w_4-w_3}(a)][q_{w_1+w_3}, e_{w_4-w_3}(b)]e^3_{w_3-w_1}v_\lambda =$

$$3\,\xi q_{w_1+w_4}e^2_{w_3-w_1}q_{-w_1-w_2}((ab'-a'b)v)_\lambda.$$

(iii) $[q_{w_1+w_3}, e_{w_4-w_3}(a)][q_{w_1+w_3}, e_{w_4-w_3}(b)][q_{w_1+w_3}, e_{w_4-w_3}(c)]e^3_{w_3-w_1}v_\lambda = 0.$

Proof. (i) The element $q_{w_1+w_3}$ commutes with $e_{w_3-w_1}$ and w_1+w_3 is positive. Hence, $q_{w_1+w_3}e^3_{w_3-w_1}v_\lambda = 0$. Furthermore, $[e_{w_4-w_3}(a), e_{w_3-w_1}, e_{w_3-w_1}] = 0$. Hence,

$$e_{w_4-w_3}(a)e^3_{w_3-w_1}v_\lambda = 3\,e^2_{w_3-w_1}e_{w_4-w_3}(a)e_{w_3-w_1}v_\lambda - 2\,e^3_{w_3-w_1}e_{w_4-w_3}(a)v_\lambda =$$

$$3\,e^2_{w_3-w_1}e_{w_4-w_1}(a)v_\lambda + e^3_{w_3-w_1}e_{w_4-w_3}(a)v_\lambda =$$

$$3\,e^2_{w_3-w_1}(e_{w_4-w_1}(av)_\lambda + \xi q_{-w_1-w_2}q_{-w_1-w_3}(a'v)_\lambda) + e^3_{w_3-w_1}e_{w_4-w_3}(av)_\lambda.$$

We have proved that $[q_{w_1+w_3}, e_{w_4-w_3}(a)]e^3_{w_3-w_1}v_\lambda =$

$$[q_{w_1+w_3}, e_{w_4-w_3}]e^3_{w_3-w_1}(av)_\lambda + 3\,\xi q_{w_1+w_3}e^2_{w_3-w_1}q_{-w_1-w_2}q_{-w_1-w_3}(a'v)_\lambda =$$

$$-q_{w_1+w_4}e^3_{w_3-w_1}(av)_\lambda + 3\,\xi e^2_{w_3-w_1}q_{w_1+w_3}q_{-w_1-w_2}q_{-w_1-w_3}(a'v)_\lambda =$$

$$-q_{w_1+w_4}e^3_{w_3-w_1}(av)_\lambda - 3\,\xi e^2_{w_3-w_1}e_{w_3-w_2}q_{-w_1-w_3}(a'v)_\lambda -$$

$$3\,\xi e^2_{w_3-w_1}q_{-w_1-w_2}q_{w_1+w_3}q_{-w_1-w_3}(a'v)_\lambda =$$

$$-q_{w_1+w_4}e^3_{w_3-w_1}(av)_\lambda + 3\,\xi e^2_{w_3-w_1}q_{-w_1-w_2}(a'v)_\lambda - 3\,\xi e^2_{w_3-w_1}q_{-w_1-w_2}h_{w_1-w_3}(a'v)_\lambda =$$

$$-q_{w_1+w_4}e^3_{w_3-w_1}(av)_\lambda - 3\,\xi e^2_{w_3-w_1}q_{-w_1-w_2}(a'v)_\lambda.$$

The assertion (i) is proved.

(ii) Let us apply (i) to $[q_{w_1+w_3}, e_{w_4-w_3}(b)]e^3_{w_3-w_1}v_\lambda$ and consider both summands of the right hand side of (i) separately.

We have, $[q_{w_1+w_3}, e_{w_4-w_3}(a)]q_{w_1+w_4}e^3_{w_3-w_1}(bv)_\lambda =$

$$-q_{w_1+w_4}[q_{w_1+w_3}, e_{w_4-w_3}(a)]e^3_{w_3-w_1}(bv)_\lambda =$$

$$q_{w_1+w_4}(q_{w_1+w_4}e^3_{w_3-w_1}(abv)_\lambda + 3\,\xi e^2_{w_3-w_1}q_{-w_1-w_2}(a'bv)_\lambda) =$$

$$3\,\xi q_{w_1+w_4} e^2_{w_3-w_1} q_{-w_1-w_2}(a'bv)_\lambda).$$

Acting on the second summand, we get

$$[q_{w_1+w_3}, e_{w_4-w_3}(a)] e^2_{w_3-w_1} q_{-w_1-w_2}(b'v)_\lambda =$$
$$-e^2_{w_3-w_1}[q_{w_1+w_3}, e_{w_4-w_3}(a)] q_{-w_1-w_2}(b'v)_\lambda) +$$
$$2e_{w_3-w_1}[q_{w_1+w_3}, e_{w_4-w_3}(a)] e_{w_3-w_1} q_{-w_1-w_2}(b'v)_\lambda =$$
$$e^2_{w_3-w_1}[q_{w_1+w_3}, e_{w_4-w_3}(a)] q_{-w_1-w_2}(b'v)_\lambda +$$
$$2e_{w_3-w_1}[q_{w_1+w_3}, e_{w_4-w_3}(a), e_{w_3-w_1}] q_{-w_1-w_2}(b'v)_\lambda =$$
$$e^2_{w_3-w_1}[q_{w_1+w_3}, e_{w_4-w_3}(a), q_{-w_1-w_2}](b'v)_\lambda +$$
$$2e_{w_3-w_1}[q_{w_1+w_3}, e_{w_4-w_1}(a)] q_{-w_1-w_2}(b'v)_\lambda =$$
$$e^2_{w_3-w_1} e_{w_4-w_2}(a)(b'v)_\lambda \;+\; 2e_{w_3-w_1}[q_{w_1+w_3}, e_{w_4-w_1}(a)] q_{-w_1-w_2}(b'v)_\lambda.$$

The first summand of this sum is equal to $e^2_{w_3-w_1} e_{w_4-w_2}(ab'v)_\lambda$ by Lemma 3.2. As for the second summand,

$$e_{w_3-w_1}[q_{w_1+w_3}, e_{w_4-w_1}(a)] q_{-w_1-w_2}(b'v)_\lambda =$$
$$e_{w_3-w_1} q_{w_1+w_3} e_{w_4-w_1}(a) q_{-w_1-w_2}(b'v)_\lambda =$$
$$e_{w_3-w_1} q_{w_1+w_3} q_{-w_1-w_2} e_{w_4-w_1}(a)(b'v)_\lambda =$$
$$e_{w_3-w_1} q_{w_1+w_3} q_{-w_1-w_2} e_{w_4-w_1}(ab'v)_\lambda +$$
$$\xi e_{w_3-w_1} q_{w_1+w_3} \underbrace{q_{-w_1-w_2} q_{-w_1-w_2}}_{0} q_{-w_1-w_3}(a'b'v)_\lambda =$$
$$e_{w_3-w_1} q_{w_1+w_3} q_{-w_1-w_2} e_{w_4-w_1}(ab'v)_\lambda.$$

We have shown that

$$[q_{w_1+w_3}, e_{w_4-w_3}(a)] e^2_{w_3-w_1} q_{-w_1-w_2}(b'v)_\lambda = P(ab'v)_\lambda,$$

where P is an operator which does not involve a or b. Choosing $a = 1$, $b = t$, we get $P = -ad(q_{w_1+w_4}) ad(e_{w_3-w_1})^2 ad(q_{-w_1-w_2})$ which finishes the proof of (ii).

(iii) By using (ii) we need only to show that

$$[q_{w_1+w_3}, e_{w_4-w_3}(a)] q_{w_1+w_4} e^2_{w_3-w_1} q_{-w_1-w_2} v_\lambda = 0.$$

Since $[q_{w_1+w_3}, e_{w_4-w_3}(a)]$ and $q_{w_1+w_4}$ commute, the expression above is $-q_{w_1+w_4}[q_{w_1+w_3}, e_{w_4-w_3}(a)]e^2_{w_3-w_1}q_{-w_1-w_2}v_\lambda$.

We proved above that

$$[q_{w_1+w_3}, e_{w_4-w_3}(a)]e^2_{w_3-w_1}q_{-w_1-w_2}v_\lambda = -q_{w_1+w_4}e^2_{w_3-w_1}q_{-w_1-w_2}(av)_\lambda.$$

Now multiplying this expression on the left by $q_{w_1+w_4}$ we get 0. This concludes the proof of the lemma.

Lemma 4.7 $[q_{w_1+w_3}, e_{w_2-w_3}(R)]^3 e^3_{w_3-w_1}v_\lambda = (0)$.

Proof. Apply $ad(e_{w_2-w_4})^3$ to the equality $[q_{w_1+w_3}, e_{w_4-w_3}(R)]^3 e^3_{w_3-w_1}v_\lambda = (0)$ of Lemma 4.6(iii).

Since $[e_{w_2-w_4}, q_{w_1+w_3}] = [e_{w_2-w_4}, e_{w_3-w_1}] = [e_{w_4-w_3}(R), e_{w_2-w_4}, e_{w_2-w_4}] = (0)$, we will get

$$[q_{w_1+w_3}, [e_{w_2-w_4}, e_{w_4-w_3}(R)]]^3 e^3_{w_3-w_1}v_\lambda = (0),$$

completing the proof of the lemma.

Lemma 4.8 $e^3_{w_3-w_1}v_\lambda = 0$.

Proof. If $e^3_{w_3-w_1}v_\lambda \neq 0$, then there exist positive roots $\alpha_1, \ldots, \alpha_s$ such that $(0) \neq L_{\alpha_1} \cdots L_{\alpha_s} e^3_{w_3-w_1}v_\lambda \subseteq V_\lambda$. Let s be the minimal number with this property. Since we can move each L_{α_i} to the right modulo shorter products, we can assume that for each i, $1 \leq i \leq s$, $\alpha_i + w_3 - w_1$ is a root or 0 and $f(\alpha_i) \leq f(w_1 - w_3) = 3$. Among all positive roots, only $w_1 - w_3$, $w_1 + w_2$, $w_1 + w_4$ have this properties. Suppose that

$$(0) \neq L^i_{w_1+w_2} L^j_{w_1+w_4} e_{w_1-w_3}(R)^k e^3_{w_3-w_1}v_\lambda \subseteq V_\lambda.$$

Then $i(w_1 + w_2) + j(w_1 + w_4) + (3 - k)(w_3 - w_1) = m(w_1 + w_2 + w_3 + w_4)$; $0 \leq i, j, k \in \mathbf{Z}$, $m \in \mathbf{Z}$.

This implies $i = j = m$, $k = 3 - m \geq 0$. Hence, we have 3 options:

1) $k = 2$ or 3. This contradicts Lemma 4.4.

2) $k = 1$. By Lemma 4.3

$$L^2_{w_1+w_2} L^2_{w_1+w_4} e_{w_1-w_3}(R) e^3_{w_3-w_1}v_\lambda \subseteq L^2_{w_1+w_2} L^2_{w_1+w_4} e_{w_3-w_1}q_{-w_1-w_2}q_{-w_1-w_4}v_\lambda.$$

The factors in $L^2_{w_1+w_2}L^2_{w_1+w_4}$ on the right hand side anticommute, because of the minimality of s and the fact that $[L_{w_1+w_2}, L_{w_1+w_4}] \subseteq e_{w_1-w_3}(R)$, which leads to the case 2).

Suppose that at least one of the two $L_{w_1+w_4}$ factors lies in $q_{-w_2-w_3}(R)$. Then

$$\underbrace{q_{-w_2-w_3}(a)e_{w_3-w_1}}\, q_{-w_1-w_2}q_{-w_1-w_4}v_\lambda =$$

$$e_{w_3-w_1}q_{-w_2-w_3}(a)q_{-w_1-w_2}q_{-w_1-w_4}v_\lambda + q_{-w_2-w_1}(a)q_{-w_1-w_2}q_{-w_1-w_4}v_\lambda.$$

The first summand is 0 because $-w_2 - w_3$ is positive. The second summand is equal to

$$q_{-w_1-w_2}q_{-w_1-w_4}q_{-w_2-w_1}(a)v_\lambda = q_{-w_1-w_2}q_{-w_1-w_4}q_{-w_2-w_1}(av)_\lambda$$

by Lemma 3.2. Now it remains to notice that $q_{-w_1-w_2}q_{-w_1-w_4}q_{-w_1-w_2} = 0$.

Thus, we can assume that both factors from $L_{w_1+w_4}$ are $[q_{w_1+w_3}, e_{w_4-w_3}(a_i)]$, $i = 1, 2$.

By Lemma 4.5 we have

$$[q_{w_1+w_3}, e_{w_4-w_3}(a_1)][q_{w_1+w_3}, e_{w_4-w_3}(a_2)]e_{w_3-w_1}q_{-w_1-w_2}q_{-w_1-w_4}v_\lambda =$$

$$[q_{w_1+w_3}, e_{w_4-w_3}(a_1)]q_{w_4+w_1}e_{w_3-w_1}q_{-w_1-w_2}q_{-w_1-w_4}(a_2v)_\lambda.$$

The element $[q_{w_1+w_3}, e_{w_4-w_3}(a_1)]$ anticommutes with $q_{w_4+w_1}$. Hence again by Lemma 4.5

$$[q_{w_1+w_3}, e_{w_4-w_3}(a_1)]q_{w_4+w_1}e_{w_3-w_1}q_{-w_1-w_2}q_{-w_1-w_4}(a_2v)_\lambda =$$

$$-q_{w_4+w_1}q_{w_4+w_1}e_{w_3-w_1}q_{-w_1-w_2}q_{-w_1-w_4}(a_1a_2v)_\lambda = 0.$$

3) $k = 0$. We have to examine $L^3_{w_1+w_2}L^3_{w_1+w_4}e^3_{w_3-w_1}v_\lambda$. As above, we conclude that factors from $L_{w_1+w_2}$ and $L_{w_1+w_4}$ anticommute module the previous cases ($k \geq 1$).

From $q_{-w_3-w_4}(R)^2e^3_{w_3-w_1}v_\lambda = (0)$, it follows that no more than one factor from $L_{w_1+w_4}$ lies in $q_{-w_2-w_3}(R)$.

On the other hand, Lemma 4.6 (iii) implies that exactly one factor from $L_{w_1+w_4}$ lies in $q_{-w_2-w_3}(R)$. Similarly, $q_{-w_3-w_4}(R)^2e^3_{w_3-w_1}v_\lambda = (0)$ and Lemma 4.7 imply that exactly one factor from $L_{w_1+w_2}$ lies in $q_{-w_3-w_4}(R)$.

Now we need to show that

$[q_{w_1+w_3}, e_{w_2-w_3}(a_1)][q_{w_1+w_3}, e_{w_2-w_3}(a_2)][q_{w_1+w_3}, e_{w_4-w_3}(b_1)][q_{w_1+w_3}, e_{w_4-w_3}(b_2)]$

$q_{-w_3-w_4}(c_1)q_{-w_2-w_3}(c_2)e^3_{w_3-w_1}v_\lambda = 0.$

First notice that $q_{-w_2-w_3}(c_2)e^3_{w_3-w_1}v_\lambda =$

$$3e^2_{w_3-w_1}[q_{-w_2-w_3}(c_2), e_{w_3-w_1}]v_\lambda =$$

$$3e^2_{w_3-w_1}q_{-w_1-w_2}(c_2)v_\lambda = 3e^2_{w_3-w_1}q_{-w_1-w_2}(c_2v)_\lambda$$

by Lemma 3.2. Hence, without loss of generality, we can assume that $c_2 = 1$ and similarly, $c_1 = 1$. Moreover,

$$q_{-w_3-w_4}q_{-w_2-w_3}e^3_{w_3-w_1}v_\lambda =$$

$$6e_{w_3-w_1}[q_{-w_3-w_4}, e_{w_3-w_1}][q_{-w_2-w_3}, e_{w_3-w_1}]v_\lambda = 6e_{w_3-w_1}q_{-w_1-w_4}q_{-w_1-w_2}v_\lambda.$$

Now,

$$[q_{w_1+w_3}, e_{w_4-w_3}(b_1)][q_{w_1+w_3}, e_{w_4-w_3}(b_2)]e_{w_3-w_1}q_{-w_1-w_4}q_{-w_1-w_2}v_\lambda = 0$$

follows from Lemma 4.5. This concludes the proof of the lemma.

Lemma 4.9 *1)* $(e_{w_4-w_3})^{<\lambda, h_{w_3-w_4}>+1}v_\lambda = 0,$

2) $(e_{w_4-w_2})^{<\lambda, h_{w_2-w_4}>+1}v_\lambda = 0.$

Proof. The only positive roots α such that $\alpha + w_4 - w_3$ is a root and $f(\alpha) \leq f(w_3 - w_4) = 6$ are $w_3 - w_4$, $w_1 + w_2$, $w_3 - w_2$, $w_2 - w_4$. Suppose that

$$(0) \neq L^i_{w_1+w_2}L^j_{w_3-w_2}L^k_{w_2-w_4}L^l_{w_3-w_4}e^q_{w_4-w_3}v_\lambda \subseteq V_\lambda, \;\; q =< \lambda, h_{w_3-w_4} > +1.$$

Then $i(w_1+w_2)+j(w_3-w_2)+k(w_2-w_4)+(q-l)(w_4-w_3) = m(w_1+w_2+w_3+w_4)$, where $i, j, k, l, m \in \mathbf{Z}$.

This implies $i = m, i-j+k = m, j-(q-l) = m, -k+(q-l) = m$. Hence $i = m = 0$, $j = k = q-l \geq 0$.

Now we have to examine the expression $L^j_{w_3-w_2}L^j_{w_2-w_4}L^l_{w_3-w_4}e^q_{w_4-w_3}v_\lambda$, where $l = q - j$.

Suppose that $l \geq 1$. There exist rational numbers $\mu,\ \nu$ such that for an arbitrary element $a \in R$,

$$e_{w_3-w_4}(a)e^q_{w_4-w_3}v_\lambda = \mu e^{q-2}_{w_4-w_3}e_{w_4-w_3}(a)v_\lambda + \nu e^{q-1}_{w_4-w_3}h_{w_3-w_4}v_\lambda =$$

$$\mu e^{q-2}_{w_4-w_3}e_{w_4-w_3}(av)_\lambda + \nu e^{q-1}_{w_4-w_3} < \lambda, h_{w_3-w_4} > (av)_\lambda$$

by Lemma 3.2.

This implies that

$$e_{w_3-w_4}(a)e^q_{w_4-w_3}v_\lambda = e_{w_3-w_4}e^q_{w_4-w_3}(av)_\lambda = 0$$

since $q =< \lambda, h_{w_3-w_4} > +1$.

Now let $l = 0, j = q$. As above

$$e_{w_2-w_4}(a)e^q_{w_4-w_3}v_\lambda = qe^{q-1}_{w_4-w_3}e_{w_2-w_3}(a)v_\lambda = qe^{q-1}_{w_4-w_3}e_{w_2-w_3}(av)_\lambda.$$

This implies that

$$L^q_{w_3-w_2}L^q_{w_2-w_4}e^q_{w_4-w_3}V_\lambda = L^q_{w_3-w_2}e^q_{w_2-w_4}e^q_{w_4-w_3}V_\lambda$$

and, similarly, this expression is equal to $e^q_{w_3-w_2}e^q_{w_2-w_4}e^q_{w_4-w_3}V_\lambda$. We have shown above that

$$e^q_{w_2-w_4}e^q_{w_4-w_3}v_\lambda = q!e^q_{w_2-w_3}v_\lambda.$$

Now $e^q_{w_3-w_2}e^q_{w_2-w_3}v_\lambda = 0$ because $q =< \lambda, h_{w_3-w_4} > +1 \geq< \lambda, h_{w_3-w_2} > +1$. This proves 1). Let us prove now assertion 2). The only positive roots α such that $\alpha + w_4 - w_2$ is a root and $f(\alpha) \leq f(w_2 - w_4) = 1$ are $w_2 - w_4$ and $w_1 + w_4$. If

$$(0) \neq L^i_{w_1+w_4}L^j_{w_2-w_4}e^p_{w_4-w_2}v_\lambda \subseteq V_\lambda,\ p =< \lambda, h_{w_2-w_4} > +1,$$

then $i(w_1 + w_4) + (p - j)(w_4 - w_2) = m(w_1 + w_2 + w_3 + w_4)$, which implies $i = m = 0,\ j = p$. Arguing as above, we see that $L^p_{w_2-w_4}e^p_{w_4-w_2}V_\lambda = e^p_{w_2-w_4}e^p_{w_4-w_2}V_\lambda = 0$. This completes the proof of the lemma.

Lemma 4.10 *Let $M \subseteq L$ a subspace such that $M^n v_\lambda = (0)$, where $v_\lambda \in V_\lambda$. Let $1 \leq i \neq j \leq 4$ and $e^m_{w_i-w_j}v_\lambda = 0$. Suppose further that $[e_{w_i-w_j}, M, M] = (0)$. Then $[M, e_{w_i-w_j}]^{m+n}v_\lambda = (0)$.*

Proof. From $[M, e_{w_i-w_j}, e_{w_i-w_j}] = (0)$ it follows that

$$[M, e_{w_i-w_j}]^{m+n} = [\underbrace{M \cdots M}_{m+n}, \underbrace{e_{w_i-w_j}, \ldots, e_{w_i-w_j}}_{m+n}],$$

where products on the left hand side and in $M \cdots M$ are taken in the associative algebra $End_F(V)$. Hence,

$$[M, e_{w_i-w_j}]^{m+n} v_\lambda \subseteq \sum_{s+r=m+n} e^s_{w_i-w_j} \underbrace{M \cdots M}\, e^r_{w_i-w_j} v_\lambda.$$

In each nonzero summand on the right hand side $r \leq m-1$.

From $[\underbrace{M, [M, [M, \ldots [M}_{r+1}, e^r_{w_i-w_j}]] \cdots] = (0)$ it follows that

$$\underbrace{M \cdots M}_{m+n} e^r_{w_i-w_j} \subseteq \sum_{p+q=m+n, p<m} M^p e^r_{w_i-w_j} M^q$$

which implies that $q \geq n$ and therefore $M^q v_\lambda = (0)$.

Lemma 4.11 *There exists $m \geq 1$ such that $e^m_{w_i-w_j} V_\lambda = (0)$ for any $1 \leq i \neq j \leq m$.*

Proof. By Lemmas 4.8 and 4.9 the elements $e_{\pm(w_1-w_3)}$, $e_{\pm(w_3-w_4)}$, $e_{\pm(w_2-w_4)}$ act nilpotently on V_λ. Now it remains to notice that those elements generate $sl(4)$ and to use Lemma 4.10.

Lemma 4.12 *For an arbitrary root α the subspace L_α acts nilpotently on V_λ.*

Proof. Let $\alpha = w_i - w_j$, $1 \leq i \neq j \leq 4$, $w_i - w_j$ negative. We have shown that $e^m_{w_i-w_j} V_\lambda = (0)$. Now, $L_{w_i-w_j} = [e_{w_j-w_i}(R), e_{w_i-w_j}, e_{w_i-w_j}]$ and $[e_{w_j-w_i}(R), e_{w_i-w_j}, e_{w_i-w_j}, e_{w_i-w_j}] = (0)$. This implies that

$$L^m_{w_i-w_j} = [L^m_{w_j-w_i}, \underbrace{e_{w_i-w_j}, \ldots, e_{w_i-w_j}}_{2m}] \subseteq \sum_{p+q=2m} e^p_{w_i-w_j} L^m_{w_j-w_i} e^q_{w_i-w_j}.$$

If $q \geq m$ then $e^q_{w_i-w_j} V_\lambda = (0)$. If $q \leq m-1$, then $f(m(w_j - w_i) + q(w_i - w_j)) > 0$ and again $L^m_{w_j-w_i} e^q_{w_i-w_j} V_\lambda = (0)$. We have shown that $L^m_{w_j-w_i} V_\lambda = (0)$.

Let α be an odd root such that L_α acts on V_α nilpotently, α is not of the form $-2w_k$. Then for arbitrary $1 \le i \ne j \le 4$ the subspace $[L_\alpha, e_{w_i-w_j}]$ acts on V_λ nilpotently. Indeed, since $\alpha \ne -2w_i$, we have $[L_\alpha, e_{w_i-w_j}, e_{w_i-w_j}] = [e_{w_i-e_j}, L_\alpha, L_\alpha] = (0)$. Now the claim follows from Lemma 4.10.

Consider a root space $L_{w_i+w_j}$, $1 \le i \ne j \le 4$. If one of i, j is equal to 1, then $w_i + w_j > 0$. Let $i \ne 1$, $j \ne 1$. Then $[L_{w_i+w_j}, e_{w_i-w_1}] = (0)$, but $< w_i + w_j, h_{w_i-w_1} > \ne 0$. Hence, $L_{w_i+w_j} = [[e_{w_i-w_1}, e_{w_1-w_i}], L_{w_i+w_j}] \subseteq [e_{w_i-w_1}, L_{w_1+w_j}]$.

From what we proved above it follows that $L_{w_i+w_j}$ acts on V_λ nilpotently.

Next, $L_{-2w_i} = [[e_{w_j-w_i}, e_{w_i-w_j}], L_{-2w_i}] \subseteq [e_{w_j-w_i}, L_{-w_i-w_j}]$, which implies that L_{-2w_i} acts on V_λ nilpotently. This completes the proof of the lemma.

5 Tensor product of modules $V(\lambda, \beta, \alpha)$

In this section we will discuss a realization of modules $V(\beta, \alpha)$ and define a tensor product in this class.

Let R be an arbitrary commutative F-algebra with a derivation $d : R \to R$. Recall that the Weyl algebra W is $W = \sum_{i=0}^{\infty} Rd^i$, $da = ad + d(a)$. For an arbitrary scalar $\beta \in F$ consider the vector space $W_\beta(R, d) = \{a_0 d^\beta + a_1 d^{\beta-1} + a_2 d^{\beta-2} + \cdots, a_i \in R\}$, the (infinite) sums are understood formally, $\tilde{W}(R, d) = \sum_{\beta \in F} W_\beta(R, d)$. The rule $d^\gamma a = \sum_{i=0}^{\infty} \binom{\gamma}{i} d^i(a) d^{\gamma-i}$, where $d^i(a)$ is the i-th derivative of the element a, makes $\tilde{W}(R, d)$ an associative algebra, $W \subseteq \tilde{W}(R, d)$. Moreover, for each $\beta \in F$ we have $[Rd, W_\beta(R, d)] \subseteq W_\beta(R, d)$. Hence $W_\beta(R, d)$ is a module over the Virasoro algebra Rd.

Now consider the associative commutative algebra $\tilde{R} = R + Rv$, $v^2 = 0$. Extend the derivation d via $d(v) = -\alpha v$, $\alpha \in F$. Then the subspace $W_\beta(R, v, d) = \sum_{i=0}^{\infty} Rvd^{\beta-i} \subset W_\beta(\tilde{R}, d)$ is an Rd-submodule of $W_\beta(\tilde{R}, d)$.

The following proposition is streightforward.

Proposition 5.1 $W_\beta(R, v, d)/W_{\beta-1}(R, v, d) \simeq V(\beta, \alpha)$.

The tensor product $V(\beta_1, \alpha_1) \otimes_F V(\beta_2, \alpha_2)$ can be identified with $W_{\beta_1+\beta_2}(R, v_1 v_2, d)/W_{\beta_1+\beta_2-1}(R, v_1 v_2, d)$, where $R = F[t_1^{-1}, t_1, t_2^{-1} t_2]$, $d = -d/dt_1 - d/dt_2$.

Since $d(t_1 - t_2) = 0$ it follows that $(t_1 - t_2)(V(\beta_1, \alpha_1) \otimes V(\beta_2, \alpha_2))$ is a submodule of $V(\beta_1, \alpha_1) \otimes V(\beta_2, \alpha_2)$.

Clearly, $V(\beta_1, \alpha_1) \otimes V(\beta_2, \alpha_2)/(t_1 - t_2) \simeq V(\beta_1 + \beta_2, \alpha_1 + \alpha_2)$.

Proposition 5.2 *If $V(\lambda_i, \beta_i, \alpha_i)$, $i = 1, 2$ are conformal modules of finite type, then so is $V(\lambda_1 + \lambda_2, \beta_1 + \beta_2, \alpha_1 + \alpha_2)$.*

Proof. The L-modules $V(\lambda_i, \beta_i, \alpha_i)$ have finitely many weight spaces with respect to the Cartan subalgebra H of L. The tensor product $V = V(\lambda_1, \beta_1, \alpha_1) \otimes V(\lambda_2, \beta_2, \alpha_2)$ also has finitely many weight spaces. The subspace of V of weight $\lambda_1 + \lambda_2$ can be identified with $V(\beta_1, \alpha_1) \otimes V(\beta_2, \alpha_2)$.Let M be the submodule of V generated by $(t_1 - t_2)(V(\beta_1, \alpha_1) \otimes V(\beta_2, \alpha_2))$. Then $(V/M)_{\lambda_1+\lambda_2} \simeq V(\beta_1 + \beta_2, \alpha_1 + \alpha_2)$. The L-module $V(\lambda_1 + \lambda_2, \beta_1 + \beta_2, \alpha_1 + \alpha_2)$ is a homomorphic image of the submodule of V/M generated by $(V/M)_{\lambda_1+\lambda_2}$. Hence $V(\lambda_1 + \lambda_2, \beta_1 + \beta_2, \alpha_1 + \alpha_2)$ has finitely many weight spaces with respect to H. This concludes the proof of the proposition.

Consider a copy of the algebra of Laurent polynomials $\overline{F[t, t^{-1}]}$ and make it a W-module via $a\bar{b} = \overline{ab}$, $d\bar{b} = -\bar{b}'$, $a, b \in F[t^{-1}, t]$. Then the space of 8-columns $\overline{F[t, t^{-1}]}^8$ becomes a left module over $M_8(W)$, hence a $CK(6)$-module. It is easy to see that this $CK(6)$-module is irreducible.

If we define the form $(w_i/w_j) = \delta_{ij}$ on $\sum_{i=1}^4 Fw_i$ and view functionals on H as elements of $\sum_{i=1}^4 Fw_i$, then the highest weight of the module $\overline{F[t, t^{-1}]}^8$ is w_1, $(h_{w_i - w_j} \otimes a)(\bar{b}, 0, \ldots, 0)^T = (w_i - w_j/w_1)(\bar{b}, 0, \ldots, 0)^T$. Moreover $Vir(a)(\bar{b}, 0, \ldots, 0)^T = (\overline{-ab' - a'b}, 0, \ldots, 0)^T$. Hence $\overline{F[t, t^{-1}]}^8 \simeq V(w_1, -1, 0)$.

Proposition 5.3 *If λ is an integral dominant functional and $< \lambda, h_{w_1 - w_3} > \geq 2$, then for arbitrary $\beta, \alpha \in F$ the irreducible module $V(\lambda, \beta, \alpha)$ has only finitely many weights with respect to the action of H.*

Proof. Let $< \lambda, h_{w_1 - w_3} >= k \geq 2$. By Proposition 5.2 the module $V' = V(\lambda - (k-2)w_1, \beta + (k-2), \alpha)$ has finitely many H-weights. Tensoring V' with $\overline{F[t, t^{-1}]}^8 \simeq V(w_1, -1, 0)$ $k - 2$ times and using Proposition 5.2 we get the result.

6 The case $< \lambda, h_{w_1-w_3} >= 1$

The aim of this section is to prove the following

Proposition 6.1 *Let λ be an integral dominant weight, such that $< \lambda, h_{w_1-w_3} >= 1$. Then $V(\lambda, \beta, \alpha)$ has finitely many weights with respect to H if and only if $< \lambda, h_{w_3-w_2} >= 0$ and $\beta = -1$.*

Suppose at first that λ is an integral dominant weight such that $< \lambda, h_{w_1-w_3} >= 1$ and $V(\lambda, \beta, \alpha)$ has finitely many H-weights.

Lemma 6.1 *For arbitrary elements $a \in R$, $v_\lambda \in V_\lambda$ we have $e_{w_3-w_1}(a)v_\lambda = e_{w_3-w_1}(av)_\lambda$.*

Proof. Since $V(\lambda, \beta, \alpha)$ is a finite sum of eigenspaces with respect to the H it follows that the element $e_{w_3-w_1}$ acts on V_λ nilpotently. The standard argument shows that $e^2_{w_3-w_1} V_\lambda = (0)$. Now for an arbitrary $a \in R$ we have

$$0 = e_{w_1-w_3}(a)e^2_{w_3-w_1} v_\lambda =$$
$$[e_{w_1-w_3}(a), e_{w_3-w_1}, e_{w_3-w_1}]v_\lambda + 2e_{w_3-w_1} e_{w_1-w_3}(a)e_{w_3-w_1} v_\lambda =$$
$$-2e_{w_1-w_3}(a)v_\lambda + 2e_{w_3-w_1} h_{w_1-w_3}(a)v_\lambda = -2(e_{w_3-w_1}(a)v_\lambda - e_{w_1-w_3}(av)_\lambda).$$

This concludes the proof of the lemma.

Lemma 6.2 $\beta = -1$.

Proof.

$$[q_{w_1+w_3}, e_{w_2-w_3}(b)][q_{w_1+w_3}, e_{w_4-w_3}(c)]e_{w_3-w_1}(a)v_\lambda =$$
$$[[q_{w_1+w_3}, e_{w_2-w_3}(b)], [[q_{w_1+w_3}, e_{w_4-w_3}(c)], e_{w_3-w_1}(a)]]v_\lambda =$$
$$(-h_{w_1-w_3}(ab'c) + Vir(abc))v_\lambda$$

as in Lemma 3.4. This element is equal to $(-ab'cv - abcv' + \beta(abc)'v + \alpha abcv)_\lambda$.

On the other hand, by Lemma 3.1 we have

$$[q_{w_1+w_3}, e_{w_2-w_3}(b)][q_{w_1+w_3}, e_{w_4-w_3}(c)]e_{w_3-w_1}(a)v_\lambda =$$
$$[q_{w_1+w_3}, e_{w_2-w_3}(b)][q_{w_1+w_3}, e_{w_4-w_3}(c)]e_{w_3-w_1}(av)_\lambda =$$
$$(-h_{w_1-w_3}(b'c) + Vir(bc))(av)_\lambda = (-ab'cv - (bc)(av)' + \beta(bc)'(av) + \alpha abcv)_\lambda.$$

Comparing these two expressions we see that $\beta a'bcv = -bca'v$, so $\beta = -1$. This finishes the proof of the lemma.

Lemma 6.3 $<\lambda, h_{w_3-w_2}>=0$.

Proof. We have $q_{w_1+w_2}q_{w_1+w_4}q_{-w_3-w_4}q_{-w_2-w_3}e^2_{w_3-w_1}v_\lambda=0$. Now,

$$q_{-w_2-w_3}e^2_{w_3-w_1}v_\lambda=2e_{w_3-w_1}[q_{-w_2-w_3},e_{w_3-w_1}]v_\lambda=2e_{w_3-w_1}q_{-w_1-w_2}v_\lambda.$$

Hence,

$$0=q_{w_1+w_2}q_{w_1+w_4}q_{-w_3-w_4}e_{w_3-w_1}q_{-w_1-w_2}v_\lambda=$$
$$q_{w_1+w_4}q_{-w_3-w_4}q_{w_1+w_2}e_{w_3-w_1}q_{-w_1-w_2}v_\lambda=$$
$$q_{w_1+w_4}q_{-w_3-w_4}[q_{w_1+w_2},e_{w_3-w_1}]q_{-w_1-w_2}v_\lambda+$$
$$q_{w_1+w_4}q_{-w_3-w_4}e_{w_3-w_1}[q_{w_1+w_2},q_{-w_1-w_2}]v_\lambda=$$
$$q_{w_1+w_4}q_{-w_3-w_4}q_{w_2+w_3}q_{-w_1-w_2}v_\lambda+<\lambda,h_{w_1-w_2}>q_{w_1+w_4}q_{-w_3-w_4}e_{w_3-w_1}v_\lambda=$$
$$q_{w_1+w_4}[q_{-w_3-w_4},q_{w_2+w_3}]q_{-w_1-w_2}v_\lambda-q_{w_1+w_4}q_{w_2+w_3}q_{-w_3-w_4}q_{-w_1-w_2}v_\lambda+$$
$$<\lambda,h_{w_1-w_2}>[q_{w_1+w_4},[q_{-w_3-w_4},e_{w_3-w_1}]]v_\lambda=$$
$$-q_{w_1+w_4}q_{-w_1-w_4}v_\lambda+<\lambda,h_{w_1-w_2}><\lambda,h_{w_1-w_4}>v_\lambda=$$
$$<\lambda,h_{w_1-w_4}>(-1+<\lambda,h_{w_1-w_2}>)v_\lambda.$$

Since $<\lambda,h_{w_1-w_4}>\geq<\lambda,h_{w_1-w_3}>=1$ it follows that $<\lambda,h_{w_1-w_2}>=1$ and therefore $<\lambda,h_{w_3-w_2}>=0$. This concludes the proof of the lemma.

Now we will assume that λ is an integral dominant weight such that $<\lambda,h_{w_1-w_3}>=1$, $<\lambda,h_{w_3-w_2}>=0$. Let $\beta=-1$. We will prove that $V(\lambda,\beta,\alpha)$ is a finite sum of eigenspaces with respect to H.

Lemma 6.4 *Under the assumptions above,* $e_{w_3-w_1}(a)v_\lambda=e_{w_3-w_1}(av)_\lambda$ *for arbitrary* $a\in R$, $v_\lambda\in V_\lambda$.

Proof. The computations of Lemma 6.3 show that for $<\lambda,h_{w_1-w_3}>=1$, $\beta=-1$ we have

$$[q_{w_1+w_3},e_{w_2-w_3}(R)][q_{w_1+w_3},e_{w_4-w_3}(R)](e_{w_3-w_1}(a)v_\lambda-e_{w_3-w_1}(av))_\lambda=0.$$

Also, $q_{-w_3-w_4}(R)(e_{w_3-w_1}(a)v_\lambda-e_{w_3-w_1}(av)_\lambda)=q_{-w_2-w_3}(R)(e_{w_3-w_1}(a)v_\lambda-e_{w_3-w_1}(av)_\lambda)=(0)$ by Lemma 3.2 . This implies that $U(L_+)(e_{w_3-w_1}(a)v_\lambda=e_{w_3-w_1}(av)_\lambda$. Lemma is proved.

Lemma 6.5 $e_{w_4-w_1}(a)v_\lambda=e_{w_4-w_1}(av)_\lambda$ *for an arbitrary* $a\in R$.

Proof. Denote $w = e_{w_4-w_1}(a)v_\lambda - e_{w_4-w_1}(av)_\lambda$. Clearly, $e_{w_1-w_4}(R)w = (0)$. Since $f(w_3 - w_4) > 0$, it follows that $e_{w_3-w_4}(b)w = e_{w_3-w_1}(ab)v_\lambda - e_{w_3-w_1}(b)(av)_\lambda = 0$ by Lemma 6.4.

From $[q_{w_1+w_4}, e_{w_4-w_1}(R)] = (0)$ we conclude that $q_{w_1+w_4}w = 0$. Hence, $[q_{w_1+w_4}, e_{w_3-w_4}(R)]w = (0)$. Also, $q_{-w_2-w_4}(R)w = (0)$ by Lemma 3.2 applied to the root $-w_1 - w_2$. We proved that $L_{w_1+w_3}w = (0)$.

If $w \neq 0$ then $U(L_+)w \cap V_\lambda \neq (0)$.

It means that there exist positive roots $\alpha_1, \ldots, \alpha_s$ such that $\alpha_1 + \cdots + \alpha_s + w_4 - w_1 \in \mathbf{Z}(w_1 + w_2 + w_3 + w_4)$ and, moreover, $\alpha_i + w_4 - w_1$ is a negative root or 0 for any i. If α_i is an even root and $\alpha_i + w_4 - w_1$ is one of the roots of Lemma 3.2 or 0 then $e_{\alpha_i}(b)w = 0$ again by Lemma 3.2. By Lemma 6.4 α_i is not supposed to be $w_3 - w_4$ as well. This rules out all even roots except $w_2 - w_4$.

Of odd roots, we have to examine $w_1 + w_2$ and $w_1 + w_3$, but the latter one has been ruled out above. Hence, $i(w_2 - w_4) + j(w_1 + w_2) + (w_4 - w_3) = k(w_1 + w_2 + w_3 + w_4)$; $i, j, k \in \mathbf{Z}$; $i, j \geq 0$. This equation does not have a solution. Hence $w = 0$. This finishes the proof of the lemma.

Lemma 6.6 $e^2_{w_3-w_1}v_\lambda = 0$.

Proof. For an arbitrary element $a \in R$ we have $e_{w_1-w_3}(a)e^2_{w_3-w_1}v_\lambda = [e_{w_1-w_3}(a), e_{w_3-w_1}, e_{w_3-w_1}]v_\lambda + 2e_{w_3-w_1}h_{w_1-w_3}(a)v_\lambda = -2e_{w_3-w_1}(a)v_\lambda + 2e_{w_3-w_1}h_{w_1-w_3}(a)v_\lambda = 0$ by Lemma 6.5.

Now, as in Section 4 we see that

$$U(L_+)e^2_{w_3-w_1}v_\lambda \cap V_\lambda = L^2_{w_1+w_2}L^2_{w_1+w_4}e^2_{w_3-w_1}v_\lambda.$$

We have $L_{w_1+w_4} = [q_{w_1+w_3}, e_{w_4-w_3}(R)] + q_{-w_2-w_3}(R)$.

Claim 1: $[q_{w_1+w_3}, e_{w_4-w_3}(a)]e^2_{w_3-w_1}v_\lambda = [q_{w_1+w_3}, e_{w_4-w_3}]e^2_{w_3-w_1}(av)_\lambda$ for an arbitrary $a \in R$.

Indeed, $[L_{w_1+w_4}, e_{w_3-w_1}, e_{w_3-w_1}] = (0)$ implies

$$[q_{w_1+w_3}, e_{w_4-w_3}(a)]e^2_{w_3-w_1}v_\lambda = 2e_{w_3-w_1}[[q_{w_1+w_3}, e_{w_4-w_3}(a)], e_{w_3-w_1}]v_\lambda =$$

$$2e_{w_3-w_1}q_{w_1+w_3}e_{w_4-w_1}(a)v_\lambda = 2e_{w_3-w_1}q_{w_1+w_3}e_{w_4-w_1}(av)_\lambda$$

by Lemma 6.5. This proves the claim.

Claim 2: $q_{-w_2-w_3}(a)e^2_{w_3-w_1}v_\lambda = q_{-w_2-w_3}e^2_{w_3-w_1}(av)_\lambda$.

Indeed, $q_{-w_2-w_3}(a)e^2_{w_2-w_1}v_\lambda = 2e_{w_3-w_1}[q_{-w_2-w_3}(a), e_{w_3-w_1}]v_\lambda = 2e_{w_3-w_1}q_{-w_1-w_2}(a)v_\lambda = 2e_{w_3-w_1}q_{-w_1-w_2}(av)_\lambda$ by Lemma 3.2. This proves the claim.

These claims and the similar assertions for $L_{w_1+w_2}v_\lambda$ show that

$$L^2_{w_1+w_2}L^2_{w_1+w_4}e^2_{w_3-w_1}V_\lambda =$$

$$(Fq_{w_1+w_2} + Fq_{-w_3-w_4})^2(Fq_{w_1+w_4} + Fq_{-w_2-w_3})^2e^2_{w_3-w_1}V_\lambda =$$

$$q_{w_1+w_2}q_{w_1+w_4}q_{-w_3-w_4}q_{-w_2-w_3}V_\lambda.$$

The computations of Lemma 6.3 show that, under the assumption $< \lambda, h_{w_2-w_3} >= 0$, this expression is equal to 0. This finishes the proof of the lemma.

Lemma 6.7 $e_{w_2-w_3}v_\lambda = 0$.

Proof. If $e_{w_2-w_3}v_\lambda \neq 0$ then there exist positive roots $\alpha_1, \ldots, \alpha_s$ such that $\alpha_1 + \cdots + \alpha_s = w_3 - w_2$, for each α_1 the sum $\alpha_i + w_2 - w_3$ is a negative root or 0 and $L_{\alpha_1} \cdots L_{\alpha_s} e_{w_2-w_3}v_\lambda \neq (0)$.

The only positive roots with the properties above are $w_3 - w_2$ and $w_1 + w_4$. But

$$e_{w_3-w_2}(a)e_{w_2-w_3}v_\lambda = h_{w_3-w_2}(a)v_\lambda =< \lambda, w_3 - w_2 > (av)_\lambda = 0.$$

Hence, all α_i have to be equal to $w_1 + w_4$, $L^s_{w_1+w_4}e_{w_2-w_3}v_\lambda \neq (0)$. But $[L_{w_1+w_4}, [L_{w_1+w_4}, e_{w_2-w_3}]] = (0)$, which leads to a contradiction and finishes the proof of the lemma.

Lemma 6.8 $e_{w_4-w_3}(a)v_\lambda = e_{w_4-w_3}(av)_\lambda$.

We have $e_{w_4-w_3}(a) = [e_{w_4-w_1}(a), e_{w_1-w_3}]$. Hence

$$e_{w_4-w_3}(a) = -e_{w_1-w_3}e_{w_4-w_1}(a)v_\lambda =$$

$$-e_{w_1-w_3}e_{w_4-w_1}(av)_\lambda = [e_{w_4-w_1}, e_{w_1-w_3}](av)_\lambda = e_{w_4-w_3}(av)_\lambda$$

by Lemma 6.5.

Lemma 6.9 $e^{<\lambda,w_3-w_4>+1}_{w_4-w_3}v_\lambda = 0$.

Proof. As in the proof of Lemma 6.7, if the assertion is not true, then there exist positive roots $\alpha_1, \dots, \alpha_s$, such that $\alpha_1 + \cdots + \alpha_s + (< \lambda, w_3 - w_4 > +1)(w_4 - w_3) \in \mathbf{Z}(w_1 + \cdots + w_4)$, $\alpha_i + w_4 - w_3$ is a negative root or 0. In fact, 0 is also excluded, because

$$e_{w_3-w_4}(a)e_{w_4-w_3}^{<\lambda,w_3-w_4>+1}v_\lambda = e_{w_3-w_4}e_{w_4-w_3}^{<\lambda,w_3-w_4>+1}(av)_\lambda = 0$$

by Lemma 6.8.

The only such positive roots are: $w_3 - w_2$, $w_2 - w_4$, $w_1 + w_2$, $-2w_4$.

Hence, there exist $i, j, k, l \in \mathbf{Z}_{\geq 0}$, $p \in \mathbf{Z}$, such that $i(w_3 - w_2) + j(w_2 - w_4) + k(w_1 + w_2) - 2lw_4 + m(w_4 - w_3) = p(w_1 + w_2 + w_3 + w_4)$, where $m =< \lambda, w_3 - w_4 > +1$. It means, that $k = p$, $-i + j + k = p$, $i - m = p$, $-j - 2l = p$. The first two equalities imply that $p = k \in \mathbf{Z}_{\geq 0}$, $i = j$. Now, adding the last two equalities we get $-m - 2l = 2p$, where the left hand side is negative, whereas the right hand side is positive. This concludes the proof of the lemma.

The element $e_{\pm(w_1-w_3)}$, $e_{\pm(w_2-w_3)}$, $e_{\pm(w_4-w_3)}$ generate $sl(4)$ and act on V_λ nilpotently. Arguing as in the proof of Lemmas 4.11 and 4.12 we get

Lemma 6.10 *(1) There exists $m \geq 1$ such that $e_{w_i-w_j}^m V_\lambda = (0)$ for any $1 \leq i \neq j \leq 4$.*
(2) There exist $m \geq 1$ such that $L_\alpha^m V_\lambda = (0)$ for an arbitrary root α.

This implies that for an integral dominant weight λ such that $1 = < \lambda, h_{w_1-w_3} >$, $0 =< \lambda, h_{w_3-w_2} >$, the module $V(\lambda, -1, \alpha)$ is a finite sum of weights spaces with respect to the action of H.

7 The case $< \lambda, h_{w_1-w_3} >= 0$

We have

$$[[e_{w_4-w_1}(a), q_{w_3+w_1}], q_{w_2+w_1}] = -[[e_{w_3-w_1}(a), q_{w_1+w_4}], q_{w_2+w_1}] = Vir(a).$$

If $< \lambda, h_{w_1-w_3} >= 0$ and nevertheless $V(\lambda, \beta, \alpha)$ is of finite type, then $e_{w_3-w_1}V_\lambda = (0)$. This implies that

$$e_{w_3-w_1}(a)V_\lambda = -\frac{1}{2}[e_{w_1-w_3}(a), e_{w_3-w_1}, e_{w_3-w_1}]V_\lambda = (0).$$

Hence

$$q_{w_2+w_1}q_{w_1+w_4}e_{w_3-w_1}(a)V_\lambda = [q_{w_2+w_1}, [q_{w_1+w_4}, e_{w_3-w_1}(a)]]V_\lambda = Vir(a)V_\lambda = (0).$$

Since $H \subseteq [H \otimes R, Vir(R)]$ it follows that $HV_\lambda = (0)$, $\lambda = 0$. Then V is a 1-dimensional module with zero multiplication, which is not viewed as irreducible.

This concludes the proof of Theorem 3.1.

8 Jordan bimodules

Let V be a Jordan bimodule over a unital Jordan (super)algebra J. Then V can be represented as a direct sum $V = V_0 \oplus V_{1/2} \oplus V_1$, where $JV_0 = (0)$, $V_{1/2}$ is a one-sided Jordan bimodule (see [MZ2]), V_1 is a unital Jordan bimodule.

In [MZ2] it was shown that the universal associative enveloping algebra of $JCK(R,d)$ is $M_4(W(R,d)$. It means that one-sided Jordan $JCK(R,d)$-bimodules are left modules over $M_4(W(R,d))$ or, equivalently, 4-tuples U^4, where U is a left module over $W(R,d)$.

Now suppose that $R = F[t^{-1},t]$, $d = \frac{d}{dt}$, and the one-sided Jordan bimodule over $JCK(6)$ is conformal.Then the left module M over $W(R,d)$ is a unital conformal module. Such modules correspond to left unital $\mathbf{C}[d]$-modules. Irreducible $\mathbf{C}[d]$-modules are one-dimensional and parametrized by scalars $\alpha \in F$. Indecomposable conformal modules of finite type correspond to Jordan blocks.

Let's be more precise. let N be a left $\mathbf{C}[d]$-module. Then $N[t^{-1},t] = \{\sum n_i t^i, n_i \in N, i \in \mathbf{Z}\}$ is a conformal left $W(F[t^{-1},t], \frac{d}{dt})$-module. It is of finite type if and only if $dim_{\mathbf{C}} N < \infty$. The space of 4-tuples $N[t^{-1},t]^4$ is a left associative conformal module over $M_4(W)$, hence a one-sided Jordan conformal module over $JCK(6)$.

Proposition 8.1 *(1) Every one-sided Jordan conformal bimodule over $JCK(6)$ is of the type $N[t^{-1},t]^4$;*

(2) The module $N[t^{-1},t]^4$ is irreducible if and only if N is one-dimensional, $N = \mathbf{C}v$, $vd = \alpha v$, $\alpha \in F$;

(3) $N[t^{-1},t]^4$ is an indecomposable module of finite type if and only if N is a Jordan block.

Now let V be a unital irreducible conformal Jordan bimodule of finite type over $J = JCK(6)$. In [MZ4], [Z] it was shown that the Tits-Kantor-Koecher construction $K(V)$ is a Lie conformal module of finite type over the TKK-algebra $K(J) = CK(6)$. In [MZ4] we proved that: (1) the reduced module $\bar{K}(V)$ is irreducible over $K(J)$ and uniquely determines the J-bimodule V; (2) the action of the Cartan subalgebra H on $\bar{K}(V)$ is diagonalizable, all weights of $\bar{K}(V)$ belong to the set $\{\pm w_i \pm w_j, 1 \leq i, j \leq 4\}$. Let λ be the highest weight of $\bar{K}(V)$. Since the Weyl group of sl_4 is the permutation group P_4 and $f(\lambda) \geq f(\sigma(\lambda))$ for all $\sigma \in P_4$, the only possibilities for λ are : $2w_1$, $w_1 - w_4$, $w_1 + w_3$, $-2w_4$. The last two cases are ruled out by Theorem 3.1. The modules $V(2w_1, \beta, \alpha)$, $\beta, \alpha \in F$; and $V(w_1 - w_4, -1, \alpha)$, $\alpha \in F$ indeed have Jordan structures.

Proposition 8.2 *Unital irreducible conformal Jordan $JCK(6)$-bimodules of finite type form two parametric families, which correspond to $V(2w_1, \beta, \alpha)$, $\beta, \alpha \in F$ and $V(w_1 - w_4, -1, \alpha)$, $\alpha \in F$.*

Proof. We need to show that $V(2w_1, \beta, \alpha)$, $V(w_1 - w_4, -1, \alpha)$, $\alpha, \beta \in F$ are reduced Tits-Kantor-Koecher modules of the form $\bar{K}(V)$ for some unital Jordan bimodules V over $J = JCK(6)$.

As in section 5 consider the associative commutative algebra $\tilde{R} = R + Rv$, $R = \mathbf{C}[t^{-1}, t]$, $v^2 = 0$ with the derivation d, $d(t) = -1$, $d(v) = \alpha v$. Consider the algebra $\tilde{W}(\tilde{R}, d) = \sum_{\beta \in F} W_\beta(\tilde{R}, d)$, $W_\beta(R, v, d) = \sum_{i=0}^{\infty} Rvd^{\beta - i}$. The Lie superalgebra $L = CK(6)$ is embedded into $M_8(W)$, hence into $M_8(\tilde{W})$.

Consider the subspace $(W_\beta(R, v, d))_{1,5}$ of matrices having $W_\beta(R, v, d)$ at the intersection of the 1st row and 5th column and 0 elsewhere. It is easy to see that $[L_+, (W_\beta(R, v, d))_{1,5}] = (0)$ and for an arbitrary element $u \in (W_\beta(R, v, d))_{1,5}$, arbitrary $1 \leq i \neq j \leq 4$, we have $[h_{w_i - w_j}, u] = (2w_1 / w_i - w_j)u$. Let $U_\beta = U(L)(W_\beta(R, v, d))_{1,5}$ be the L-submodule generated by $(W_\beta(R, v, d))_{1,5}$ in $M_8(\tilde{W})$. Clearly, $2w_1$ is the highest weight of this submodule and $(U_\beta)_{2w_1} = (W_\beta(R, v, d))_{1,5}$.

The L-submodule $U_{\beta-1}$ of U_β is generated by $(W_{\beta-1}(R, v, d))_{1,5}$, $U_\beta / U_{\beta-1} = U(L)(W_\beta(R, v, d)/W_{\beta-1}(R, v, d))$, hence $U_\beta / U_{\beta-1} = U(L)(U_\beta / U_{\beta-1})_{2w_1}$, and $(U_\beta / U_{\beta-1})_{2w_1} \simeq V(\beta, \alpha)$.

Hence $V(2w_1, \beta, \alpha)$ is a homomorphic image of the module $U_\beta / U_{\beta-1}$. Then all weights of $V(2w_1, \beta, \alpha)$ belong to the set $\{\pm w_i \pm w_j . 1 \leq i, j \leq 4\}$, which implies that $V(2w_1, \beta, \alpha) \simeq K(\bar{V})$ for some irreducible unital Jordan J-bimodule V.

Now let's turn to bimodules $V(2w_1, \beta, \alpha)$. Consider the Cheng-Kac superalgebra $CK(\tilde{R}, d)$ and the subspace $e_{w_1-w_4}(Rv)$. This subspace generates the L-submodule which is isomorphic to $V(w_1 - w_4, -1, \alpha)$. This concludes the proof of the proposition.

References

[BKL1] C. Boyallian, V. G. Kac and J. I. Liberati, Irreducible modules over finite simple Lie conformal superalgebras of type K , *J. Math. Phys.* **51**,063507, (2010).

[BKL2] C. Boyallian, V. G. Kac and J. I. Liberati, Classification of finite irreducible modules over the Lie conformal superalgebra CK_6, Preprint.

[BKLR] C. Boyallian, V. G. Kac, J. I. Liberati and A. Rudakov, Representations of simple finite Lie conformal superalgebras of type W and S, *J. Math. Phys.* **47**, 043513, (2006).

[CK1] S. Cheng and V.G. Kac, Conformal modules, *Asian J. Math* **1**, 181-193, (1997). Erratum: 2 ,153-156, (1998).

[CK2] S. Cheng and V.G. Kac, A new N = 6 superconformal algebra, *Comm. Math. Phys.* **186** no. 1, 219-231, (1997).

[FK] D. Fattori and V.G. Kac, Classivication of finite simple Lie conformal superalgebras, *J. of Algebra* **258**, no.1, 23-59,(2002).

[GLS] P. Grozman, D. Leites and I. Shchepochkina, Lie superalgebras of string theories, *Acta Math. Vietnam* **26** no. 1, 27-63, (2001).

[K1] V.G. Kac, Classification of simple Z-graded Lie superalgebras and simple Jordan superalgebras, Comm. in Algebra 5 (13), 1375-1400, (1977).

[K2] V.G. Kac, Vertex algebras for beginners, University Lecture Series 10, American Mathematical Society, Providence, RI 1996, Second Edition 1998.

[MZ1] C. Martínez and E. Zelmanov, Simple finite-dimensional Jordan Superalgebras in prime characteristic , *Journal of Algebra* **236** no.2, 575-629, (2001).

[MZ2] C. Martínez and E. Zelmanov, Specializations of Simple Jordan Superalgebras, *Canad. Math. Bull.* **45** (4), 653-671, (2002).

[MZ3] C. Martínez and E. Zelmanov, Lie superalgebras graded by P(n) and Q(n), *Proc. Natl. Acad. Sci. USA* **100** no.14, 8130-8137, (2003).

[MZ4]C. Martínez and E. Zelmanov, Representation Theory of Jordan Superalgebras I, *Trans. Amer. Math. Soc* **362** no. 2, 815-846, (2010).

[Z] E. Zelmanov, On the structure of conformal algebras, *Contemporary Math.* **264**, 139-153 , (2000).