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A Double-Sided Multiunit Combinatorial Auction for Substitutes: Theory and Algorithms.

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Abstract

Combinatorial exchanges have existed for a long time in securities markets. In these auctions buyers and sellers can place orders on combinations, or bundles of different securities. These orders are conjunctive: they are matched only if the full bundle is available. On business-to-business (B2B) exchanges, buyers have the choice to receive the same product with different attributes; for instance the same product can be produced by different sellers. A buyer indicates his preference by submitting a disjunctive order, where he specifies how much of the product he wants, and how much he values each attribute. Only the goods with the best attributes and prices will be matched. This article considers a doubled-sided multi-unit combinatorial auction for substitutes, that is, a uniform price auction where buyers and sellers place both types of orders, conjunctive and disjunctive. We prove the existence of a linear price which is both competitive and surplus-maximizing when goods are perfectly divisible, and nearly so otherwise. We describe an algorithm to clear the market, which is particularly efficient when the number of traders is large.

Keywords: Combinatorial auction, economic equilibrium

JEL Classification: C62, C63, D44

1. Introduction

To quote de Vries and Vohra (2003) "because of complementarities or substitution effects between different assets, bidders have preferences not just for particular items, but for sets of items, sometimes called bundles" or packages, or combinations. Combinatorial auctions are becoming more and more popular with the Internet revolution, because, as noted by Klein (1997), one effect of the Internet is the ability to conduct complex auctions.

This paper focuses on the description of a (uniform price) "double-sided, multiunit, combinatorial, with substitutes" auction, or DMCS auction [1]. So far, the literature has surveyed almost exclusively the single-sided combinatorial auction, that is, auctions with one seller, or auctions with only one buyer (the procurement auction). Pinker et al (2003) in their survey are "not aware of any true double online auctions beyond the financial and commodity markets", with the exception of www.chemconnect.com for the chemical industry. Pekec and Rothkopf (2003) point out the paucity of "documentation and public information on details of combinatorial auction design and implementations in the E-business arena". The author, who worked as a consultant for both the Eurex securities exchange and the Covisint automotive B2B exchange had the privilege to compare the design issues facing each industry from within. It is well-known that the main criticisms of procurement auctions is that they are focussed on "squeezing the suppliers" (Rothkopf and Harstad 1994). However, as Milgrom (2000) notes "if the mechanism is designed to extract all the entire surplus from the sellers, it will be difficult to attract sellers to the auction site". But why are securities exchanges double-sided (and even sometimes combinatorial, like Eurex, the largest options exchange in the world, or briefly, the Scandinavian and Osaka exchanges) whereas B2B exchanges are still mostly one-sided? Besides monopolistic behavior, the answer is to be found is the greater complexity of the goods offered on a B2B exchange, in particular the importance of different attributes in the different (substitute) items. We believe that the lack of development of double-sided B2B exchanges is partly due to a lack of accepted best practices, whereas the relative lack of recent academic interest came partly from the ignorance of the business issues of B2B exchanges (notable exceptions are Beil and Wein (2003) on MS auctions, and Ba et al (2001) on DC auctions for public goods).

Any order in our DMCS auction consists of "OR of XOR of AND bids": a combination order is a conjunctive, or "AND" bid on complement items; the "OR" operator implements the (possible) multi-unit feature of any order, which can be partially matched, whereas the disjunctive, or XOR parts allows to choose across substitutes. The choice between butter and cannons can be expressed as "butter XOR (2 wheels AND 1 gun) ", expressing the fact that a cannon is made of 3 parts: 2 wheels and 1 gun. Pekec and Rothkopf (2003) observe that such a bidding language is fully expressive. We believe that any bill of materials can be implemented in such a fashion, and that this language is a good compromise between flexibility and practicability (see Wohl 1997 for a discussion). In a manufacturing environment, it is also quite natural to express the "utility function" of the goods needed in terms of limit orders: the reservation or *limit price*(s), for instance can be equated to the shadow prices, which most production planning systems calculate anyway.

There are two main originalities in our design. The first one is the "commoditization" of

[1] We use D for double-sided, M for multi-unit, C for combinatorial, and S for substitutes. When one of these attributes is not present, we drop the corresponding letter. For instance, an MC auction is a one-sided, multiunit, combinatorial, without substitutes auction. A "simple auction" is neither D, nor M, nor C, nor S. See section 3 for examples of our terminology.

identical goods, which entails the law of one price: the price offered for the first wheel has to be identical to the price of the second wheel (we call such prices *uniform* or *nondiscriminatory*); this is the reason why we call our auction "multiunit", as opposed to most other designs, which are mostly variants of the DC, or CS auctions. As has been observed at Eurex, it is much easier to implement a continuous-time combinatorial exchange (which, in terms of pricing algorithm resembles a DC auction), than the "opening-of-the day", or batch clearing system (i.e., a DMC auction). Second is the fact that, due to commoditization, combination vectors contain all possible integer values, and not only $\{1, -1, 0\}$; in our example, a cannon needs 2 wheels.

We are not at liberty of saying how much this model resembles existing designs of improvements to an existing B2B exchange, but the limitations of the design - most of them coming from hard theoretical bounds - have been widely discussed with practitioners. They are: lack of built-in incentive compatibility, absence of competitive prices when indivisibilities are significant, and the necessity of a tie-breaking mechanism. We briefly address each of these issues.

For one-sided auctions, Rothkopf and Harstad (1994) already pointed out the "limitations of traditional game-theoretic approaches to auction design". Pekec and Rothkopf (2003) mention that, for combinatorial auctions, "any serious strategic analysis [..] of optimal bidding strategies is impossible without mastery of the determination of auction winners"; in particular the Vickrey-Clarke-Groves, (Vickrey 1961, Clarke 1971, Groves 1973) scheme is "impractical", among other reasons because among others it is based on a simple model of beliefs, and that it does not prevent shill bidding. The difficulties of implementing incentive compatibility are compounded in the more general setting of double-sided auctions. Such a mechanism exists for DC auctions for public goods, see Ba et al (2001). Gul and Stachetti (2000) establish that the Vickrey mechanism cannot be implemented in an English auction when buyers are strategic and values are interdependent. In a DC auction, a Vickrey-type mechanism may not be budget-balanced (Parkes et al 2001), i.e., the Vickrey payments may absorb more than the surplus. Pekec and Rothkopf (2003) point out to the greater "cooperative" (as opposed to collusive) behavior of bidding in combinatorial auctions, thereby mitigating the incentive compatibility effect. To conclude, we do not claim that it is either possible or impossible in our DMCS auction to build a mechanism to ensure incentive compatibility, but think that the winner determination problem should be addressed first. An incentive compatible mechanism should anyway, like in McAfee (1992) for DM auctions, first calculate uniform prices (provided they exist) and then offer side-payments.

When goods are indivisible, competitive prices may not exist (see e.g., Gul and Stachetti 1999, Bikchandani and Ostroy 1997). Dierker (1971) and Broome (1972) investigated the "convexifying" effect of a large number of traders, which explains why "various approximation algorithms are likely to produce solutions which are not far from optimal" (Pekec and Rothkopf 2003). Satterthwaite and Williams (1993) prove that the indivisible equilibrium converges to the perfectly divisible equilibrium when the number of traders increases in a DM auction. In particular, the allocation obtained by our algorithm is (approximately) efficient, and linear prices (i.e., prices such that the price of the sum is the sum of the prices) obtain in a DM auction. This robustness effect was also observed by Isaac and James (1998). Parkes (2001) gives a systematic survey on the effect of indivisibilities in combinatorial auctions. Pekec and Rothkopf again (2003) note that linear (or additive) prices are the most popular approximation method. Interestingly, another positive effect of a large number of traders, besides making prices competitive, is to make uniform price double auctions incentive compatible (Wilson 1985).

Pekec and Rothkopf (2003), note that ties in an auction are undesirable. In a DMCS auction, ties across substitutes turn out to be as much as an issue as ties across participants: we resolve this issue by requiring every participant to define a strict preference ordering on all items contained in an order. Technically, we implement this constraint by restricting clearing prices to be discrete, as mentioned in Milgrom (2003) p. 317.

Our main contributions are to show that in a DMCS auction: (i) linear (nondiscriminatory) competitive prices exist (ii) under mild conditions, any such price maximizes surplus when goods are perfectly divisible (iii) under additional conditions, at least one linear competitive price is surplus-maximizing when goods are indivisible (iv) there is an efficient algorithm to calculate a linear competitive price[2]. In the indivisible case, surplus-maximizing (nondiscriminatory) prices may not be competitive, i.e., some orders remain unfulfilled although their limit price is strictly better than the clearing price; the function of prices is therefore mostly informative. A high auction volume reduces the number of unmatched orders at clearing, by the convexifying effect mentioned earlier, rendering prices competitive. Besides their "algorithmic" importance, linear prices have a crucial informative role in the economy, because auctions are usually a repetitive business, and participate in the price discovery process.

The paper is structured as follows. In section 2, we specify the DMCS auction and provide theoretical results for the perfectly divisible good case. We also show how our DMCS auction can generalize Wohl's (1997) price-contingent mechanism. In section 3, we discuss the indivisible case, and relate our terminology and our results to the existing literature. In section 4, we describe an algorithm to clear a DMCS auction and provide numerical results. All figures, as well as one table and one proof, are included in the online companion paper.

2. Divisible Goods

Notation The superscripts ^{B} and ^{S} apply to buyer, respectively seller-related functions, such as demand/supply, allocation, ... Because of space constraints, we fully define only the buyer side in the text. The seller side can be defined symmetrically. The superscripts ^{N} and ^{C} are mnemonics for primitive (normal) goods and, respectively, combinations. We sometimes concatenate the variables that use these superscripts, e.g. $f = \begin{bmatrix} f^B & f^S \end{bmatrix}$, and $x = \begin{bmatrix} x^N & x^C \end{bmatrix}$. Double superscripts are separated by commas. $\lceil x \rceil$ is the smallest integer larger than or equal to x, whereas $[x]^+$ is the maximum of x and 0. The vector e^i is the i-th unit vector, and e is the vector of ones. The indicator $1\{A\}$ is equal to one if A is true, and zero otherwise. $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ are the sets of positive integers, positive and negative integers, and real numbers. Other sets are denoted by a calligraphic letter, e.g., $\mathcal{B}$. Some of the notation used in sections 3 and 4 is defined in the proofs of the theorems of section 2, which are in the appendix.

Combination Trading The exchange trades m primitive commodities (or goods) and $n-m$ combinations of commodities. Potentially, all combinations of commodities can be traded, that is $n \leq 2^m - 1$. By definition, when $z_j \in \mathbb{R}$ units of commodity j are allocated to a buyer, he physically receives $A(i,j)z_j$ of each primitive commodity i, with $A(i,j) \in \mathbb{Z}$. In the butter vs cannon example given earlier, the *clearing matrix* would be:

[2]Most of these results are an extension to DMCS auctions of the results on DMC auctions contained in Schellhorn (1997).

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad (1)$$

where the 1st, 2nd, 3rd and 4th commodities are butter, wheel, rifle, and cannon respectively. The coefficients $A(i, j)$ can be positive or negative. For instance, in the interest-rate swap market, the 1st (2nd) primitive commodity is a fixed (floating) bond, and a swap is equivalent to one fixed minus one floating bond. The clearing matrix is then:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

The structure of the clearing matrix is always $A = \begin{bmatrix} I & A^C \end{bmatrix}$.

Order Specification Because of space constraints, we refrain from defining preferences in terms of utility and consumption set. We just mention that utility is transferable. In our DMCS auction, buyers and sellers submit what we call a *joint limit order*, that is, a limit order over possibly many substitute goods. When a buyer desires only one good, his limit order is identical to a traditional limit order, or *simple limit order*. In a (joint limit) order, each buyer (seller) specifies a bid (an ask), a list of preferences over substitute goods (in case of price ties), and the quantity of each good desired.

We require prices (bids and asks) to be integer [3]. This is not very restrictive, since the numeraire can be expressed in any unit ($, cent, thousandth of a cent,...). We will see that a surplus-maximizing equilibrium allocates only the preferred good of each order in (i) a full match (i.e., received quantity=limit quantity) when the preferred good's marginal utility is strictly positive (ii) a partial match when the latter is zero (iii) nothing, when the latter is negative. Observe that such an allocation will be *competitive*: the case when not all the limit quantity of an order's preferred good can be allocated at equilibrium (i.e., one wheel of a cannon) can correspond only to the case where the preferred good's marginal utility is zero, therefore, it does not make sense to allocate the second-best good (butter), since its marginal utility is negative.

We note that participants who want to reduce the risk of non-execution have a tendency to submit many small orders with different limit prices; this has the effect of reducing the probability of having tie(s) at clearing. Since the profit realized from a sum of limit orders is the sum of each limit orders profit, and since our equilibrium concept (surplus maximization) is also additive, we may as well suppose that orders are anonymous. For notational ease also, we suppose there is only one order per bid. Thus each buy (sell) order comprises:

- a limit price b (respectively s) $\in \mathbb{R}^n$
- a bid $y^B(b)$ (respectively $y^B(s)$) $\in \mathbb{N}^n$
- a limit quantity $q^B(b)$ (respectively $q^S(s)$) $\in \mathbb{R}^n$, with components positive, and not all equal to zero.

[3] The advantage of this requirement is that demand and supply functions will turn out to be single-valued with our tie-breaking mechanism if clearing prices are also integer.

Marginal profit is equal to limit price minus clearing price. The limit price is different from the bid only in order to guarantee the rank order of preferences in case of ties. That is, all the quantities $y_i^B(b) - b_i$ are different, positive, and less than one. By convention, the limit price is a price per unit of good exchanged, and not for the full lot. When good k is not a possible substitute (for the buyer), the corresponding bid and limit price are set to $-\infty$. For simplicity, we also use the term buyer (seller) for b (s). The set of all buyers (sellers) is denoted $\mathcal{B}$ (respectively $\mathcal{S}$).

Commodity i will be allocated to buyer b only if the clearing price $p \in \mathcal{P}_i^B(b)$, where:

$$\mathcal{P}_i^B(b) = \{p | p_i \leq y_i^B(b), p_i \leq p_j - y_j(b) + y_i(b) - 1\{y_i^B(b) - b_i > y_j^B(b) - b_j\}, j \neq i\} \qquad (2)$$

The indicator function in (2) is equal to zero if i is the preferred good at p, and zero otherwise. In 2 dimensions, the intersection of $\mathcal{P}_i^B(p)$ with the square containing all bids is a rectangular trapeze, with the single non rectangular vertex located at $y(b)$, where the two faces intersect at an angle of 135^0. Figure 1 illustrates this definition, in a case where good 2 is preferred to good 1. Note that, by construction $\mathcal{P}_1^B(b)$ does not overlap with $\mathcal{P}_2^B(b)$.

Aggregate Demand We can then define the aggregate demand function $F^B : \mathbb{R}^n \to \mathbb{R}^n$ by:

$$F_i^B(p) = \sum_{b \in \mathcal{B}} q_i^B(b) 1\{p \in \mathcal{P}_i^B(b)\}$$

For reasons that will become apparent later, it is convenient to introduce an operator $p^{i,+} : \mathbb{R}^n \to \mathbb{R}^n$ which purpose is to "locate" the "next" point of decrease of demand in the direction i, starting from p, keeping all other directions fixed (see figure 3). If $j \neq i$, we set then $p_j^{+,i}(p) = p_j$, otherwise:

$$p_i^{B,+,i}(p) = \left\{ \begin{array}{l} p_i + 1 \\ \sup\{\pi_i | F_i^B(p + (\pi_i - p_i)e^i) = F_i^B(p) \end{array} \right\} \text{if} \left\{ \begin{array}{ll} F_i^B(p) > F_i^B(p + \varepsilon e^i) & \forall \varepsilon > 0 \\ \text{else} & \end{array} \right. \qquad (3)$$

Note that $p_i^{i,+}(p) = \infty$ is possible. We can then define the incremental demand ΔF^B by:

$$\Delta F_i^B(p) = F_i^B(p) - F_i^B(p^{i,+}(p))$$

It is the demand of the "marginal buyers at p", that is those who would not want commodity i if its price marginally increased. We define similarly aggregate supply F^B, previous sell limit price in the i direction $p^{i,-}$ and incremental supply ΔF^S.

Allocation We define an incremental buy allocation function $\Delta f^B : \mathbb{Z}^n \to \mathbb{R}^n$. The value $\Delta f_i^B(p)$ is equal to the allocated quantity at equilibrium of good i of all the "marginal buyers" at p, that is of all buyers b with $p \in \mathcal{P}_i^{B,bd}(b)$, where $\mathcal{P}_i^{B,bd}(b)$ is the boundary of $\mathcal{P}_i^B(b)$. The buy allocation function $f^B : \mathbb{Z}^n \to \mathbb{R}^n$ is defined indirectly (but unequivocally) by:

$$\begin{array}{rcll} \Delta f_i^B(p) & = & f_i^B(p) - f_i^B(p^{i,+}(p)) & \\ f_i^B(p) & = & 0 & \text{if} \ \ p_i = \infty \end{array}$$

Surplus Given a clearing price p and an allocation f, surplus is the sum of the incremental allocations at each price, times the marginal utility (profit) per trade, that is:

$$S = \sum_{i=1}^{n} \sum_{y \in \mathcal{P}_i^{B,bd} \cap l^{+,i}(p)} \Delta f_i^B(y) \alpha_i^B(y,p) + \sum_{i=1}^{n} \sum_{y \in \mathcal{P}_i^{S,bd} \cap l^{-,i}(p)} \Delta f_i^B(y) \alpha_i^B(y,p) \tag{4}$$

where

- $l^{+,i}$ ($l^{-,i}$) is the half line starting at p in the direction e^i ($-e^i$)
- $\mathcal{P}_i^{B,bd} = \bigcup_{b \in \mathcal{B}} \mathcal{P}_i^{B,bd}(b)$
- $\alpha_i^B(y,p)$ is the average profit realized by the "marginal buyers" at y.

For definiteness, we can suppose a proportional allocation between all the marginal buyers in order to determine α. What really matters in this definition however is that the average profit is always strictly positive if $y_i > p_i$. In the special (but common) case where there is only one marginal buyer b at price y, then:

$$\alpha_i^B(y,p) = b_i - p_i \tag{5}$$

We give an example of the construction of surplus in figure 2.

Clearing The exchange maximizes surplus by choosing a clearing price $p = \begin{bmatrix} p^N & p^C \end{bmatrix} \in \mathbb{Z}^n$ and an allocation function f that solves the following problem:

$$\max_{p, f \geq 0} S \tag{6}$$

$$\Delta f_i^B(y) \leq \Delta F_i^B(y) \qquad \forall y \in \mathcal{P}_i^{B,bd} \tag{7}$$
$$\Delta f_i^S(y) \leq \Delta F_i^S(y) \qquad \forall y \in \mathcal{P}_i^{S,bd} \tag{8}$$
$$\Delta f_i^B(y)(y_i - p_i) \geq 0 \qquad \forall y \in \mathcal{P}_i^{B,bd} \tag{9}$$
$$\Delta f_i^S(y)(p_i - y_i) \geq 0 \qquad \forall y \in \mathcal{P}_i^{S,bd} \tag{10}$$
$$A(f^B(p) - f^S(p)) = 0 \tag{11}$$

Constraints (9),(10) implement incentive rationality. An equilibrium (p, f) is *surplus-maximizing* if (p, f) solves (6)-(11) and *competitive* if:

$$F_i^B(p) \geq f_i^B(p) \geq F_i^B(p^{B,+,i}(p)) \tag{12}$$
$$F_i^S(p) \geq f_i^S(p) \geq F_i^S(p^{S,-,i}(p)) \tag{13}$$

An equilibrium [4] is *welfare-maximizing* if it maximizes the sum of the values of the utility functions of the participants. (For a full definition, please refer to appendix 1.) The only

[4] To lighten notation, we use the term "equilibrium" even if the variables that compose a welfare-maximizing equilibrium are different than the ones in a surplus-maximizing equilibrium. The mapping between them is explained in the proof of theorem 2.1.

difference with a surplus-maximizing equilibrium is that (i) prices may be discriminatory in a welfare-maximizing equilibrium (ii) prices are not constrained to be integer. It is well-known that, in simple D auctions (where $A = I$, and all orders are simple), the three equilibrium concepts coincide, mutatis mutandis. We now investigate the relationships between these concepts in DMCS auctions. Because of the tie-breaking rule, a surplus-maximizing equilibrium may not be welfare-maximizing, but the difference between the optimal welfare and the optimal surplus is small. As mentioned earlier, when there are many orders, utilities become almost strictly concave and ties disappear, therefore the gap between the optimal welfare and surplus also disappears.

PROPOSITION 2.1. *Every surplus-maximizing equilibrium (p, f) is competitive.*

DEFINITION: An equilibrium (p, f) is *linear* if $p = A^T p^N$.

THEOREM 2.1. *If a surplus-maximizing equilibrium (p, f) is such that*

$$f(p) > 0 \tag{14}$$

then it is linear.

Condition (14) is mild. It ensures that, at equilibrium, all goods are traded. Whereas surplus-maximizing equilibria obviously exist, theorem 2.1. does not guarantee the existence of a linear competitive equilibrium, to which we now turn. Note that the proof of the theorem is constructive and, as we shall see in section 4, yields an efficient algorithm to calculate a linear competitive equilibrium.

THEOREM 2.2. *There exists a linear competitive equilibrium (p, f) such that $p \in \mathbb{R}^n$.*

DEFINITION: A bid y is *combination-multiple* if it can be expressed as the following linear combination of integers:

$$y_j = \sum_{i=1}^{m} A(i,j) n_i \qquad n_i \in \mathbb{Z}$$

If a linear equilibrium exists, it makes sense for traders to place combination-multiple bids and asks. We also remind the reader that the unit of numeraire can be very small, so that combination-multiple prices can be made dense in $\mathbb{R}^n$, the space of prices expressed in \$.

COROLLARY. *If bids and asks are combination-multiples, there exists a linear competitive equilibrium (p', f) such that $p' \in \mathbb{Z}^n$.*

PROOF: Let (p, f) be a linear competitive equilibrium, with $p \in \mathbb{R}^n$. We define:

$$p'_j = \sum_{i=1}^{m} A(i,j)[p_i] \tag{15}$$

Since bids and asks are combination-multiples, (p', f) is linear.

DEFINITION: We say that goods are *one-to-one buy substitutes* if for all $b \in \mathcal{B}, q_i^B(b) \in \{0, q^B\}$.

Goods are one-to-one buy substitutes if the buyer wants the same quantity of each substitute. For a baker, milk and cream (expressed in ounces) are not one-to-one substitutes, but a carton of 10 pints of milk is a one-to-one substitute of a carton of 1 pint of cream. This one-to-one substitution requirement is in practice not very restrictive: a car manufacturer needs 4 wheels per car; she has no real need for 4 aluminum wheels XOR 5 steel wheels.

THEOREM 2.3. *If goods are one-to-one buy substitutes and sellers submit only simple limit orders, the set of linear competitive prices is connected.*

PROOF: In the companion paper.

In theorem 2.3, prices belong to $\mathbb{Z}^n$. By "connected" we mean, that the price of every primitive good must be either (i) unique or (ii) located at the vertex of a unit cube that contains at least one other price linear competitive price as a vertex. In terms of algorithms, this theorem greatly facilitates the search of all equilibria. We suspect that this result could be used to show that all linear competitive equilibria yield the same surplus. We also suspect that theorem 2.3 would hold true if sellers were allowed to submit joint limit orders. In practice though, we experienced that few sellers need to trade substitutes.

Application: Price-Contingent Orders Wohl (1997) discusses an exchange mechanism where traders can submit market orders and limit orders contingent on the price of an index. One advantage of our DMCS is to generalize this mechanism to orders contingent on the price of several indices. Suppose $p_{1,...,}p_m$ are the prices of the normal securities, whereas $p_{m+1}, ..., p_n$ are the prices of $n-m$ indices. We formally add to our tradeable security list securities $n+1, ..., n+m$ which correspond to "negative indices": buying security p_{n+j} $(j \geq 1)$ is equivalent to selling index $n+1-j$. A linear equilibrium will result then in:

$$p_{n+j} = -p_{n+1-j} \tag{16}$$

A trader wishing to buy a number q of security $k \leq m$ at price β_k only if the prices of indices $m+1, .., n$ are (respectively) less than $\beta_{m+1}, .., \beta_n$ will therefore do it only if the prices of negative indices $n+1, ..., n+m$ are (respectively) more than $-\beta_{m+1}, .., -\beta_n$. In all other cases, he will buy nothing. The order specification in our framework will therefore be:

$$q_j^B(b) = \left\{ \begin{array}{c} q \\ 0 \end{array} \right\} \quad \text{if} \left\{ \begin{array}{l} j = k \\ \text{else} \end{array} \right. \tag{17}$$

$$b_j = \left\{ \begin{array}{c} \beta_k \\ -\beta_{j-2i} \\ -\infty \end{array} \right\} \text{if} \left\{ \begin{array}{l} j = k \\ j \in \{m+1, .., n\} \\ \text{else} \end{array} \right. \tag{18}$$

Theorem 2.2 therefore proves that an equilibrium exists that respects the buyer's order, even when there is no trading activity on the combinations. Besides, all the cumbersome data-handling contained in (17) to (18) can be abstracted from the user in a software program.

3. Indivisible Goods

Many researchers have worked on the question of existence of a nonempty core or of an equilibrium of "market games", where indivisible goods are traded. To illustrate our terminology, the horse market of Bohm-Bawerk is a D auction, whereas the house market is a DS auction. Both examples are described in Shapley and Shubik (1972). The counterexample in the same article, where each buyer wants 3 houses if they are all on the same row is a DCS auction. The FCC auction analyzed by Rothkopf et al (1998) is a C auction.

When utility is transferable, competitive equilibria exist in the following cases: D auction with 2 goods (Quinzii 1984 and Gale 1984), DM auction with 2 goods (Henry 1970), DM auction with single-unit bids on one side (Kaneko et al 1986), DS auction when goods are gross substitutes (Kelso and Crawford 1982, Gul and Stachetti 1999), DMS auctions with polymatroidal cost functions (Danilov, Koshevoi and Lang 2003), DC auctions with supermodular preferences[5] (Bikchandani and Mamer 1997). Examples where equilibria may not exist are: general DMS auction (Henry 1970) and general DCS auction (Bikchandani and Ostroy 1997). The latter result removes all hope to find an equilibrium price for a DMCS auction, even if we relax the law of one price (as is done in one-sided auctions) or the integrality of prices. We say that an equilibrium (p, f) is *subcompetitive* if only (7) to (11) hold. The following statement is then trivial:

PROPOSITION 3.1. *Every surplus-maximizing equilibrium (p, f) is subcompetitive.*

A weaker form of theorem 2.1 exists in the indivisible case, where (14) is replaced by a stronger condition, the maximal synthetic trade condition, which is defined in the appendix. Roughly speaking, it requires that, at an equilibrium, a buy order for a combination cannot be matched only with a sell order of the same combination.

THEOREM 3.1. *Suppose all bids and asks are combination-multiples. If a surplus-maximizing equilibrium (p, f) satisfies the maximal synthetic trade condition, then it is linear.*

The proof of theorem 3.1 is identical to the proof of theorem 2.1, because, by assumption, f is indivisible, and the corresponding welfare-maximizing problem has no duality gap. We note that the indivisibility requirement potentially reduces the number of synthetic trades and therefore the likelihood that a linear competitive equilibrium be surplus-maximizing.

4. Algorithms

The proof of theorem 2.2 is constructive, and can be applied to determine a linear competitive equilibrium in the perfectly divisible case. As described in Schellhorn for a DMC auction (1997), we use the Eaves-Saigal algorithm (Eaves 1972) to compute the fixed point of the correspondence $\mathcal{F}$, which is defined in the proof of theorem 2.1. Convergence is achieved in a finite number of steps. The computational differences between determining the equilibrium of DMC and a DMCS auction are negligible. We therefore report without proof the results on algorithm complexity contained in our previous article. The average number of arithmetic operations of the fixed point algorithm is:

[5] Or, equivalently, if the same equilibrium exists in the perfectly divisible case.

$$T_{FP} = O(m^5 + mn^3) \tag{19}$$

where we use the common approximation that the average number of iterations of the primal simplex method is proportional to the number of rows (see e.g. Dantzig 1963). The benchmark against which the fixed point algorithm needs to be compared is the welfare-maximizing[6] linear program (28),(29),(32). For simplicity, we assume that the number of buyers t is equal to the number of sellers, and that for all buyers and sellers, all goods are substitutes. The maximum number of normal trades x^N is equal to nt^2, and the maximum number of synthetic trades x^C is $2(n-m)t!/n!$ if $t \geq n$. Using Stirling's formula, the average number of arithmetic operations of the linear program is:

$$T_{PL} = O(nt^2 + 2(n-m)\frac{t^t}{n^n}e^{n-t}\sqrt{\frac{t}{n}}) \tag{20}$$

Taking $t = 200, m = 10, n = 100$, we obtain for instance $T_{FP} \approx 10^{12}$ and $T_{PL} \approx 10^{218}$. Table 1 in the companion paper shows some experimental comparisons of the 2 algorithms in the case $m = 1$, $n = 2$ for a DMC. With only 30 traders, our fixed point algorithm is already 2 orders of magnitude faster.

In the indivisible case, a natural algorithm would be to first look for the perfectly divisible solution, and then search in the neighborhood of the solution for a linear indivisible allocation (p, f), which gives us a lower bound for the optimal surplus. In general, the knowledge of a good lower bound accelerates the calculation of the solution of a mathematical program. Rothkopf et al (1998) observe that a C auction is equivalent to the set-packing problem on a hypergraph, which is NP-hard; it is therefore critical to design good heuristics for the DMCS auction.

Finally, it may not be optimal for the exchange to clear the whole market in batch: the indivisibility requirement may severely hamper the benefits of combinatorial trading, by reducing the quantity of synthetic trades. One can then envision a sequential auction, where in the first round, integer allocations are traded and removed from the order books. In the second round, only the non-integer allocations that would have been traded if the goods had been divisible are sent back, using non-combinatorial orders. This would then realize a compromise between the risk of non-execution and the risk of owning non-complementary items.

5. Conclusion

We showed in this article that linear competitive prices exist, and can be used to clear a DMCS auction. When goods are perfectly divisible it is difficult to argue against this equilibrium concept, especially if an incentive-compatible mechanism is superposed to the clearing mechanism. This equilibrium is not only surplus-maximizing, non-discriminatory (and therefore not illegal) but, if participants bid their true reservation value, is the best way to reveal the good's intrinsic value. If the clearing price is not linear, arbitrageurs will exchange goods after the auction clears for positive riskless profit. Finally, these types of prices have a clear algorithmic advantage over other approaches: because of the aggregation of several orders into generalized demand and supply functions, our algorithm is particularly efficient when the number of participants is large.

[6] For a DMC both algorithms yield the same welfare.

Finally, we note that non-discriminatory linear prices exist in DMC auctions for more general types of orders (Schellhorn 1995), i.e., orders corresponding to general concave utility functions. The same fact should hold true for DMCS auctions.

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Appendix

Proof of Proposition 2.1

Since an allocation must be positive, we can express the constraints (7)(8) in terms of allocation/demand/supply, instead of incremental allocation/demand/supply, that is, in (7)(8), get rid of the symbol Δ. Fixing p, the necessary optimality conditions corresponding to the variable $f_i^B(p^{B,+,i}(p))$ are:

$$\lambda_i^B(p^{B,+,i}(p)) - \nu_i^B(p^{B,+,i}(p))(p_i^{B,+,i}(p) - p_i) = \alpha_i^B(p^{B,+,i}(p), p) - \alpha_i^B(p, p) \tag{21}$$

where $\lambda_i^B(p^{+,i}(p)) \geq 0$ is the multiplier of equation (7), and $\nu_i^B(p^{+,i}(p)) \geq 0$ the multiplier of (9). Since $p_i^{B,+,i}(p) > p_i$ and $\alpha_i^B(p,p) = 0$, we have $\lambda_i^B(p^{B,+,i}(p)) > 0$, and, by complementary slackness:

$$f_i^B(p^{B,+,i}(p)) = F_i^B(p^{B,+,i}(p)) \tag{22}$$

Proof of Theorem 2.1

Given a surplus-maximizing allocation f, we can construct another set of orders (a "modified economy"), in such a way that (i) all limit orders are simple (ii) all orders are matched at the optimum (iii) the optimal equilibrium and surplus of the modified and the original economy coincide (iv) we group buyers (and sellers) with the same bids (asks). As a consequence of the fact that all orders are simple, (5) holds for all y with $y_i = b_i$. We then write $u^B(i,b)$ instead of $\alpha_i^B(y,p)$ for the marginal utility (or profit) of such a buyer b, for clarity. Marginal utilities of buyers and sellers are then:

$$\begin{aligned} u^B(i,b) &= b_i - p_i && (23) \\ u^S(i,s) &= p_i - s_i && (24) \end{aligned}$$

As a consequence of the full matching of every order, the marginal allocation $\Delta f_i^B(y)$ is equal to $q_i^B(b)$ when $y_i = b_i$. The optimal surplus in (4) becomes:

$$S = \sum_{i=1}^{n} \sum_{b \in \mathcal{B}} q_i^B u^B(i,b) + \sum_{i=1}^{n} \sum_{s \in S} q_i^S u^S(i,s) \tag{25}$$

We take a detour to define a notion (the notion of a minimal set of synthetics) that will be useful in the proof of theorem 3.1. Two types of trades emanate from the clearing process: normal trades and synthetic trades. In a *normal trade* $x^N(i,b,s)$ seller s gives a quantity $x^N(i,b,s)$ of good i to buyer b. In a *synthetic trade* $x^C(w,d)$, seller d_k^S gives a quantity $w_k^S x^C(w,d)$ of good k to buyer d_k^B. Each *trader group* $d^B(d^S) \in \mathbb{R}^{n^2}$ is a vector of buyers (sellers), with the following restrictions: (i) the component k of vector d_k^B is buyer $b \in \{b' \in \mathcal{B} | b'_k \neq -\infty\} \cup \{n^B\}$ where $\{n^B\}$ is the "null-buyer" singleton, that is, a limit price vector with all components equal to $-\infty$ (ii) two components of d_l^B and d_k^B are identical if and only if $d_k^B = n^B$. By definition a *synthetic* vector $w = \begin{bmatrix} w^B & w^S \end{bmatrix}$, where $0 \leq w^B, w^S \in \mathbb{R}^n$, corresponds to a self-sufficient exchange of goods, with, in each good, either one buyer or one seller, i.e.:

$$\begin{aligned} \sum_{j=1}^{n} w_j^B A(i,j) &= \sum_{j=1}^{n} w_j^S A(i,j) \quad \forall i && (26) \\ w^B w^S &= 0 && (27) \end{aligned}$$

Condition (27) is not necessary, but it facilitates intuition. We thus define a *minimal set of synthetics* $\mathcal{W} = \{w^1, ..., w^{2n-2m}\}$, which is an ordered set of synthetic vectors such that:

- there exists a collection of positive scalars $x^C(w^k,.)$ with $\sum_{k=1}^{n-m} x^C(w^k,.)(w^{k,B} - w^{k,S})$ spanning the null set of A
- the second half of the set consists of the same synthetic trade as the first half, but "permuting buyers and sellers", i.e., for $k \leq n-m$, and $\bar{k} = k + n - m$ then $w^{k,B} = w^{\bar{k},S}$ and $w^{k,S} = w^{\bar{k},B}$.

In other terms, $\{w^{k,B} - w^{k,S}\}$ for $k = 1..n-m$ is a basis of the null set of A. We call $\mathcal{D}$, the set of dealer groups, and $\mathcal{D}_{i,b}^B$ the set of all dealer groups d^B such that $d_i^B = b$. Normal and synthetic trades satisfy then, for any minimal set of synthetics:

$$\begin{aligned} \sum_{s \in \mathcal{S}} x^N(i,b,s) + \sum_{d \in \mathcal{D}_{i,b}^B} \sum_{k=1}^{2n-2m} w_i^{k,B} x^C(w^k,d) &= q_i^B(b) \qquad \forall (i,b) && (28) \\ \sum_{b \in \mathcal{B}} x^N(i,b,s) + \sum_{d \in \mathcal{D}_{i,s}^B} \sum_{k=1}^{2n-2m} w_i^{k,S} x^C(w^k,d) &= q_i^S(s) \qquad \forall (i,s) && (29) \end{aligned}$$

We define the marginal utility of each trade:

$$\begin{aligned} \gamma^N(i,b,s) &= b_i - s_i && (30) \\ \gamma^C(w,d) &= \sum_{i=1}^{n} w_i^B d_i^B(i) - w_i^S d_i^S(i) && (31) \end{aligned}$$

Welfare is

$$W = \sum_{i=1}^{n} \sum_{b \in \mathcal{B}} \sum_{s \in \mathcal{S}} \gamma^N(i,b,s) x^N(i,b,s) + \sum_{d = \mathcal{D}} \sum_{k=1}^{2n-2m} \gamma^N(w^k,d) x^C(w^k,d) \qquad (32)$$

An equilibrium x is *welfare-maximizing* for a DMC auction if it maximizes (32) under the constraints (28) to (31). With these definitions, it is easy to see that the market clearing condition (11) and the limit quantity constraints (7),(8) hold if (28), (29) hold. Since any more complicated synthetic trade can be decomposed in these basic trades, then (11) holds only if (28), (29) hold for some x. Incentive rationality (9),(10) requires:

$$\begin{aligned} u^B(i,b) + u^S(i,s) &\geq \gamma^N(i,b,s) && (33)\\ \sum_{i=1}^{n} w_i^B u^B(i,b) + w_i^S u^S(i,s) &\geq \gamma^C(w,d) && (34) \end{aligned}$$

Since the welfare-maximizing problem is less constrained than the surplus-maximizing problem, optimal welfare is higher than or equal to optimal surplus in (4). By duality, optimal welfare W is less than of equal to the surplus S in (25), over the domain defined by (33)(34) and positivity of u. The only possibility, namely $W = S$, is that $u \geq 0$ minimizes (25) under (33)(34). Necessary and sufficient conditions for (p, f) to be surplus-maximizing are then (28) to (31), (33)(34), positivity of u and:

$$\begin{aligned} x^N(i,b,s)(u^B(i,b) + u^S(i,s) - \gamma^N(i,b,s)) &= 0 && (35)\\ x^C(w,d)(\sum_{i=1}^{n} w_i^B u^B(i,d_i^B) + w_i^S u^S(i,d_i^S) - \gamma^C(w,d)) &= 0 && (36) \end{aligned}$$

Inserting (31), (23), and (24) in (36) results in:

$$x^C(w,d)(p^t(w^{k,B} - w^{k,S})) = 0 \tag{37}$$

A sufficient condition for p to be linear is that, for all $w^k \in \mathcal{W}$:

$$p^t(w^{k,B} - w^{k,S}) = 0 \tag{38}$$

Let $\bar{k} = k + n - m$. Clearly, if for each $k = 1..n - m$ there exists $d \in \mathcal{D}$ such that either $x^C(w^k, d)$ or $x^C(w^{\bar{k}}, d)$ is strictly positive, then p is linear. We now verify that under the stronger condition (14), which is much easier to verify computationally, p must also be linear. We define a particular minimal set of synthetics $\mathcal{W}^N$ followingly. For all $k = 1..n$ and $\bar{k} = k + n - m$ then w^k ($w^{\bar{k}}$) is the synthetic vector consisting of buying (selling) one unit of combination $k + m$, against only primitive goods, i.e.:

$$\begin{aligned} w_i^{B,k} &= \left\{ \begin{array}{c} [-A(i,k+m)]^+ \\ 1 \\ 0 \end{array} \right\} \text{ if } \left\{ \begin{array}{l} i \leq m \\ i = k+m \\ \text{else} \end{array} \right. && (39)\\ w_i^{S,k} &= \left\{ \begin{array}{c} [A(i,k+m)]^+ \\ 0 \end{array} \right\} \text{ if } \left\{ \begin{array}{l} i \leq m \\ \text{else} \end{array} \right. && (40) \end{aligned}$$

Let us fix k and $\bar{k} = k+n-m$. Suppose that for all $d \in \mathcal{D}$ and any optimal x (corresponding to the minimal set of synthetics $\mathcal{W}^N$) we have:

$$x^C(w^k, d) = x^C(w^{\bar{k}}, d) = 0 \tag{41}$$

Therefore $x^N(i,b,s)$ is equal to $q^B(i,b,s)$ (which is strictly positive by 14)for all b and s. As a consequence the optimal basis (in terms of linear programming) does not include either $x^C(w^k,d)$ or $x^C(w^{\bar{k}},d)$. Let us identify all goods i such that $w_i^{B,k} > 0$. By (14) there exists for each of these goods a buyer who is fully matched. We pick for each good i the buyer with the lowest limit price, which we call $b^*(i)$. We perform the symmetric operation for all sellers of primitive goods i such that $w_i^{S,k} > 0$. We now decrease (increase) the limit prices of the economy for all these "star" buyers (sellers), to obtain new limit prices $\bar{b}^*(i)$ and $\bar{s}^*(i)$ equal to the clearing price, i.e.:

$$
\begin{aligned}
b^*(i) &\rightarrow \bar{b}^*(i) = p_i && (42)\\
s^*(i) &\rightarrow \bar{s}^*(i) = p_i && (43)
\end{aligned}
$$

It is known from sensitivity analysis in linear programming that, provided the solution remains feasible, the current basis remains optimal. We call d^* the dealer group such that $d_i^{B,*} = \bar{b}^*(i)$ and $d_i^{S,*} = \bar{s}^*(i)$. The reduced cost of $\gamma^N(w^k,d^*)$ is then:

$$
\gamma^C(w^k,d^*) + \sum_{i=1}^{n} w_i^{B,k} u^B(i,d_i^{B,*}) + \sum_{i=1}^{n} w_i^{S,k} u^S(i,d_i^{S,*}) \geq 0 \qquad (44)
$$

But, by construction, $u^B(i,d_i^{B,*}) = u^S(i,d_i^{S,*}) = 0$, therefore:

$$
\begin{aligned}
0 \leq \gamma^C(w^k,d^*) &= \sum_{i=1}^{m}[-A(i,k+m)]^+ d_i^{B,*}(i) - \sum_{i=1}^{m}[A(i,k+m)]^+ d_i^{S,*}(i) + d_{k+m}^{B,*} && (45)\\
&= \sum_{i=1}^{m}[-A(i,k+m)]^+ p_i - \sum_{i=1}^{m}[A(i,k+m)]^+ p_i + p_{k+m} && (46)\\
&= -\sum_{i=1}^{m} A(i,k+m) p_i + p_{k+m} && (47)
\end{aligned}
$$

Therefore:

$$
p_{k+m} \geq \sum_{i=1}^{m} A(i,k+m) p_i \qquad (48)
$$

We can do the same operation for the synthetic vector $w^{\bar{k}}$, and we conclude that;

$$
p_{k+m} \leq \sum_{i=1}^{m} A(i,k+m) p_i \qquad (49)
$$

Therefore the price of combination $k+m$ is linear, and the same reasoning can be applied to all other combinations.

Proof of Theorem 2.2

The competitive demand/supply correspondence (which maps prices to the competitive allocation respecting (12)(13)) is unfortunately not upper hemi-continuous. We therefore construct two continuous functions F' and D' in such a way that their convex combination is a subset of the range of the competitive demand/supply correspondence. Let $0 \leq \delta < \frac{1}{2}$. We define two continuous bid functions:

$$q_i^{B,\sup} = \left\{ \begin{array}{l} q_i^B(b) \\ q_i^B(b)\frac{\pi_i - p_i}{\delta} \\ 0 \end{array} \right\} \text{if} \tag{50}$$

$$\left\{ \begin{array}{l} p + \delta e^i \in \mathcal{P}_i^B(b) \\ p \in \mathcal{P}_i(b), p + \delta e^i \notin \mathcal{P}_i^B(b), \pi_j = p_j (i \neq j), \pi \in \mathcal{P}_i^{B,bd}(b) \\ \text{else} \end{array} \right\} \tag{51}$$

$$q_i^{B,\inf} = \left\{ \begin{array}{l} q_i^B(b) \\ q_i^B(b)\frac{\pi_i - p_i - \delta}{\delta} \\ 0 \end{array} \right\} \text{if} \tag{52}$$

$$\left\{ \begin{array}{l} p + 2\delta e^i \in \mathcal{P}_i^B(b) \\ p + \delta e^i \in \mathcal{P}_i^B(b), p + 2\delta e^i \notin \mathcal{P}_i^B(b), \pi_j = p_j (i \neq j), \pi \in \mathcal{P}_i^{B,bd}(b) \\ \text{else} \end{array} \right\} \tag{53}$$

We set:

$$F^{B'}(p) = \sum_{b \in \mathcal{B}} q^{B,\sup}(b,p)$$
$$D^{B'}(p) = \sum_{b \in \mathcal{B}} q^{B,\inf}(b,p) - q^{B,\sup}(b,p)$$

which are obviously continuous (see figure 3 for an illustration of the 1-good case). Therefore,

$$F_i^B(p) \geq F_i^{B'}(p) \geq F_i^{B'}(p) + D_i^{B\prime}(p) \geq F_i^B(p_i^{B,+,i}(p))$$

This shows that the convex combination $f = F^{'}(p) + \tau D'(p)$, with $0 \leq \tau \leq e$ is competitive. Let $\mathcal{X} \subset \mathbb{R}^m$ be a box such that $A^T\mathcal{X}$ contains all the bids and asks. We now define as in Schellhorn (1997) a correspondence $\mathcal{F} : \mathcal{X} \to 2^{\mathcal{X}}$. Its value is the set of all prices $\{\bar{p}\}$ such that there exist $(\bar{f}, \bar{\tau})$ where $(\bar{p}, \bar{f}, \bar{\tau})$ is a solution of the following linear program:

$$\max_{p', f', \tau'} ef'$$

$$\begin{aligned}
f^{B'} + A^T(p'-p) &\le F^{B'}(p^TA) + D^{B'}(p^TA)\tau^{B'} && (54)\\
f^{S'} - A^T(p'-p) &\le F^{B'}(p^TA) + D^{S'}(p^TA)\tau^{S'} && (55)\\
A(f^{B'} - f^{S'}) &= 0 && (56)\\
-f' &\le 0 && (57)\\
ef' &\le \sum_{i=1}^{n} \max_{\pi} F_i^B(\pi) + \sum_{i=1}^{n} \max_{\pi} F_i^S(\pi) + 1 && (58)\\
0 &\le \tau' \le e && (59)
\end{aligned}$$

Observe that the constraint correspondence γ that maps p to the domain determined by (54) to (59) is compact-valued. Constraint (58) was introduced just to ensure compactness of the latter, even if this constraint is never tight at the optimum. Clearly each constraint in (54) to (59) is both upper and lower hemi-continuous, since F' and D' are continuous. Therefore, by theorem 3 p. 120 in Berge (1959), γ is also continuous. By Berge's maximum theorem, $\mathcal{F}$ is closed and upper hemi-continuous. Clearly, $(f', p', \tau) = (0, p, 0)$ is always feasible for (54) to (59), so that $\mathcal{F}(p)$ is non-empty. Since $\mathcal{F}(p)$ is a section of the solution-set of a convex program, it is convex and compact-valued. Kakutani's fixed point theorem asserts then the existence of a fixed point of $\mathcal{F}$. Let $\lambda^B \ge 0$, $\lambda^S \ge 0$, μ, $\nu \ge 0$ be the multipliers of relations (54) to (57). The following system is a subset of the optimality conditions at a fixed point:

$$\begin{bmatrix} I & 0 & A^T \\ 0 & I & -A^T \\ A & -A & 0 \end{bmatrix} \begin{bmatrix} \lambda^B \\ \lambda^S \\ \mu \end{bmatrix} = \begin{bmatrix} e^B + \nu^B \\ e^S + \nu^S \\ 0 \end{bmatrix} \tag{60}$$

Since the matrix in (60) has full rank, the only solution is $\lambda = e + \nu \ge 0$. By complementary slackness, this imposes:

$$\bar{f} = F(p^TA) + \tau D'(p^TA)$$

Therefore $(\bar{p}, \bar{f})$ is competitive.

The Maximal Synthetic Trade Condition

DEFINITION: We say that a surplus-maximizing equilibrium (p, f) *satisfies the maximal synthetic trade condition* if there exists a minimal (ordered) set of synthetics $\mathcal{W} = \{w^1, .., w^{2n-2m}\}$and a welfare-optimal allocation x of the corresponding modified economy such that, for all $k = 1..n-m$

(i) either $x^C(w^k, d) > 0$, for some $d \in \mathcal{D}$

(ii) or $x^C(w^{\bar{k}}, d) > 0$, with $\bar{k} = k + n - m$ for some $d \in \mathcal{D}$

The definitions of $\mathcal{D}$, of a minimal set of synthetics, and of the modified economy corresponding to an equilibrium are in the proof of theorem 2.1 above.

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