Pr cting Tail-related Risk M ures: The Consequences of ing GARCH Filters for N GARCH Data

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Predicting tail-related risk measures: The consequences of using GARCH filters for non GARCH data

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Abstract

We investigate the consequences for value-at-risk and expected shortfall purposes of using a GARCH filter on various mis-specified processes. We show that careful investigation of the adequacy of the GARCH filter is necessary since under mis-specifications a GARCH filter appears to do more harm than good. Using an unconditional non filtered tail estimate appears to perform satisfactorily for dependent data with a degree of dependency corresponding to actual market conditions.

Keywords: Extreme value theory, Value at Risk (VaR), Expected Shortfall, GARCH, Markov Switching, Jump Diffusion, Backtesting.

JEL classification: G12, C32.

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1 Introduction

Extreme value theory has become a precious tool to assess the likelihood of rare but large events in stock markets. In the finance literature, such estimations have become very popular.[1] In this strand of literature the estimations are typically performed under the assumption that the return generating process is i.i.d.. Actual returns do not obey this relation however, returns' variability clusters. As pointed out by Mandelbrot (1963), large events tend to be followed by other large events. Such phenomena are typically modeled as ARCH or GARCH processes, see Engel (1982) and Bollerslev (1986). There exists in the statistics literature elements on how to deal with certain types of dependency, especially to correct standard errors, see Leadbetter, Lindgren and Rootzén (1983), and Hsing (1991b). There also exist links between the ARCH literature and extreme value theory. For instance, de Haan et al. (1989) establish the extremal index for a simple ARCH model. They hint at how to obtain the extremal index of the general GARCH model, and an actual derivation thereof may be found in Stariça and Mikosch (2000). Further bridges between the two literatures may be found in Quintos, Fan and Phillips (2001). In these contributions it is shown how to test for the stability of the estimates of the tail index as well as how to adjust the standard errors under ARCH or GARCH specifications.

As an alternative to adjusting standard errors, within a VaR context, McNeil and Frey (2000) propose an interesting technique consisting in first filtering the data, then applying extreme value techniques to the tails of the

[1]Without dressing a complete list, such estimations are discussed in Danielsson, de Haan, Peng and de Vries (1998), Danielsson and de Vries (1997), de Haan et al. (1989), Hols and de Vries (1991), Huisman, Koedijk, Kool, and Palm (2001), Longin (1986), Jondeau and Rockinger (2003). At the textbook level, one may mention Embrechts, Klüppelberg, and Mikosch (1997) as well as Reiss and Thomas (1997).

innovations while bootstrapping the central part of the distribution. From there on, it is possible to obtain realistically behaved returns using simulation techniques that may be useful for VaR purposes. Our contribution is inspired by this work in that we investigate the consequences of following such a methodology when the GARCH process is mis-specified. To do so we consider various return generating processes such as a GARCH (1,1), a switching-regime model inspired by the work of Hamilton (1994), as well as a stochastic volatility model such as described by Pan (2000). Last, we consider a pure jump model as is often assumed in the finance literature.

Our findings may be summarized as follows. We find that filtering the data introduces a downward bias of the tail thickness for GARCH (1,1), switching-regime data, and stochastic volatility models. The bias tends to decrease as the threshold increases. These findings suggest that the GARCH filter 'grabs' for such specifications too much of the tails. On the other hand, when the data is actually generated by a pure jump process, then the GARCH filter induces an upward bias of the tail thickness. We show that for data obtained for market-type parameters, the conventional extreme value theory estimate leads to crash predictions with the smallest bias.

These findings suggest that data should only be filtered after careful verification that the GARCH filter is truly adequate for a given set of data. In case the true data generating process is not of the GARCH type, filtering may induce a bias.

The structure of this paper is as follows. In the next section we very briefly recall the working of the GARCH model mainly to introduce notation, and explain the extreme value method used to describe the tail behavior. In section 3 we describe the possible non GARCH specifications used as hypothetical true data generating processes in the simulations. In section 4 we show how expected shortfall is affected in this setting. Section 5 concludes.

2 Methods

Consider $(X_t, t \in \mathbb{Z})$ a strictly stationary time series representing daily observation of the negative log-return computed for a financial asset price. The dynamics of X_t are assumed to be

$$X_t = \mu_t + \sigma_t Z_t,$$

when the innovations Z_t are a strict white noise process, independent and identically distributed, with zero mean, unit variance and marginal distribution function $F_Z(z)$. The possibly time varying parameters μ_t and σ_t are measurable with respect to I_{t-1}, the information available up to time $t-1$. Let $F_{X_t}(x)$ denote the marginal distribution of X_t and for a horizon $h \in \mathbb{N}$, let $F_{X_{t+1}+\ldots+X_{t+h}|I_t}(x)$ denote the predictive distribution of the return over the next h days, given the knowledge of returns up to and including day t.

We are interested in estimating unconditional and conditional quantiles in the tails of the negative log-return distribution. We remind that for $0 < q < 1$, the qth unconditional quantile is a quantile of the marginal distribution denoted by

$$x_q = \inf\{x \in \mathbb{R} : F_X(x) \geq q\},$$

and a conditional quantile is a quantile of the predictive distribution for the return over the next h days denoted by

$$x_q^t(h) = \inf\{x \in \mathbb{R} : F_{X_{t+1}+\ldots+X_{t+h}|I_t}(x) \geq q\}.$$

We also consider the expected shortfall (ES), known to be a measure of risk for the tail of a distribution.[2] The ES is a coherent measure of risk in the sense of Artzner, Delbaen, Elsner, and Heath (2000). The unconditional expected shortfall is defined to be

$$S_q = E[X|X > x_q],$$

[2]The expected shortfall is sometimes called a conditional value at risk or CVaR.

and the conditional expected shortfall is written as

$$S_q^t(h) = E[\sum_{j=1}^{h} X_{t+j} | \sum_{j=1}^{h} X_{t+j} > x_q^t(h), I_t].$$

In this paper we restrict ourselves to the $h = 1$ step predictive distribution. Thus, we denote the quantiles respectively by x_q^t and S_q^t. Since

$$F_{X_{t+1}|I_t}(x) = P\{\mu_{t+1} + \sigma_{t+1} Z_{t+1} \leq x | I_t\},$$

trivially it holds that

$$F_{X_{t+1}|I_t}(x) = F_Z((x - \mu_{t+1})/\sigma_{t+1}).$$

As a consequence, the quantile and expected shortfall become

$$\begin{aligned} x_q^t &= \mu_{t+1} + \sigma_{t+1} z_q, \\ S_q^t &= \mu_{t+1} + \sigma_{t+1} E[Z | Z > z_q], \end{aligned}$$

where z_q is the upper qth quantile of the marginal distribution of Z_t, which by assumption does not depend on t.

2.1 Estimating μ_{t+1} and σ_{t+1}

We estimate the conditional mean as an AR(1) process, i.e. $\mu_t = \phi X_{t-1}$. In empirical work, the description of data with the GARCH(1,1) model is a popular way of modelling volatility. We follow this road and model the volatility of the mean-adjusted series, $\varepsilon_t = X_t - \mu_t$, by

$$\sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \beta \sigma_{t-1}^2,$$

where $\alpha_0 > 0$, $\alpha_1 > 0$, and $\beta > 0$.

This model is fitted using the Pseudo Maximum Likelihood (PML) method of Gourieroux, Monfort, and Trognon (1984). If we consider a GARCH(1,1) model with normal innovations, the likelihood is maximized to obtain the parameter estimates $\hat{\theta} = (\hat{\phi}, \hat{\alpha}_0, \hat{\alpha}_1, \hat{\beta})$. It has been shown that the PML method yields a consistent and asymptotically normal estimator. Another approach consists in assuming that the innovations have a leptokurtic distribution such as a Student's t distribution, scaled to have variance 1, see Bollerslev and Wooldgridge (1984). Note that the additional parameter, v representing the degrees of freedom of the Student t, can be estimated, along with the other parameters by PML. Furthermore, for our estimation and simulation experiment, since the focus is on the tail rather than on the central part of the distribution, the choice of the innovations' distribution is not a key issue.

In order to make predictions, we fix a constant memory n so that at the end of day t, the data consist of the last n negative log returns $(x_{t-n+1}, ..., x_{t-1}, x_t)$. Estimates of the conditional mean and standard deviation series $(\hat{\mu}_{t-n+1}, ..., \hat{\mu}_t)$ and $(\hat{\sigma}_{t-n+1}, ..., \hat{\sigma}_t)$ can be calculated from the equations above, after substitution of some sensible starting values. Residuals are calculated both to check the adequacy of the GARCH modelling and as an input for the second stage of the method. The estimates of the conditional mean and variance for day $t+1$, are calculated as

$$\begin{aligned} \hat{\mu}_{t+1} &= \hat{\phi} x_t, \\ \hat{\sigma}^2_{t+1} &= \hat{\alpha}_0 + \hat{\alpha}_1 \hat{\varepsilon}^2_t + \hat{\beta} \hat{\sigma}^2_t, \\ \text{where } \hat{\varepsilon}_t &= x_t - \hat{\mu}_t. \end{aligned}$$

To validate our GARCH filter, we perform a simulation of a GARCH (1,1), with parameters $\hat{\phi} = 0.05$, $\hat{\alpha}_0 = 0$, $\hat{\alpha}_1 = 0.037, \hat{\beta} = 0.95$, and Gaussian innovations. Besides recovering the correct parameter values, as the upper part of Figure 1 shows, the autocorrelation function of the raw simulated returns displays no autocorrelation but some dependency of the absolute value of returns. The lower part of Figure 1 applies the same tests to the residuals of the GARCH (1,1). Now, both residuals and their absolute values display no longer correlation.

2.2 Estimating z_q using EVT

We fix a high threshold u and we assume that excess residuals over this threshold have a generalized Pareto distribution (GPD) with tail index ξ,

$$G_{\xi,\beta}(y) = \begin{cases} 1-(1+\xi y/\beta)^{-1/\xi} & \text{if } \xi \neq 0, \\ 1-\exp(y/\beta) & \text{if } \xi = 0, \end{cases}$$

where $\beta > 0$, and the support is $y \geq 0$ when $\xi \geq 0$ and $0 \leq y \leq -\beta/\xi$ when $\xi < 0$. The choice of this distribution is motivated by a limit result in EVT. Consider a general distribution function F and the corresponding excess distribution above the threshold u defined by

$$F_u(y) = P\{X-u \leq y | X > u\} = \frac{F(y+u)-F(u)}{1-F(u)},$$

for $0 \leq y < x_0 - u$, where x_0 is the right endpoint of F. It is possible to find, for a large class of distributions F, a positive measurable function $\beta(u)$ such that

$$\lim_{u \to x_0} \sup_{0 \leq y < x_0 - u} |F_u(y) - G_{\xi,\beta(u)}(y)| = 0.$$

This result was shown by Balkema and de Haan (1974) and Pickands (1975). This result holds for most continuous distributions used in statistics. According to the value of the parameter ξ, the GPD approximation may be subdivided into three groups. The heavy tailed distributions corresponds to the case $\xi > 0$, such as the Pareto, Student's t, Cauchy, and Fréchet distributions. The tails of this heavy tailed distributions, decay like power functions. The case $\xi = 0$ corresponds to distributions like the normal, exponential, gamma, and lognormal, whose tails decay exponentially. Finally, distributions with $\xi < 0$ are short-tailed with a finite right endpoint, such as the uniform and beta distributions.

In our case we assume that the tail of the underlying distribution begins at the threshold u. We assume that the excesses over the threshold are i.i.d. with an exact GPD distribution. The parameters ξ and β are estimated by maximum likelihood. Smith (1985) has shown that maximum likelihood estimates $\hat{\xi}$ and $\hat{\beta}$ of the GPD parameters ξ and β are consistent and asymptotically normal as $N \to \infty$, provided $\xi > -1/2$. The following equality holds for points $x > u$ in the tail of F

$$1 - F(x) = (1 - F(u))(1 - F_u(x - u)).$$

We assume that in our sample of n points, the number of exceedances above threshold u is N. If we estimate the first term in the right hand side of this equation, using the random proportion of the data in the tail, i.e. N/n, and if we estimate the second term by approximating the excess distribution with a GPD fitted by maximum likelihood, we get the tail estimator

$$\hat{F}(x) = 1 - \frac{N}{n}\left(1 + \hat{\xi}\frac{x-u}{\hat{\beta}}\right)^{-1/\hat{\xi}}$$

for $x > u$. Let $z_{(1)} \geq z_{(2)} \geq z_{(3)} \geq ... \geq z_{(n)}$ represent the ordered residuals. If we fix the number of data in the tail to be $N = k$, this give us a random threshold at the (k+1)th order statistic. The GPD with parameters ξ and β is fitted to the data $(z_{(1)} - z_{(k+1)}, ..., z_{(k)} - z_{(k+1)})$, the excess amounts over the threshold for all residuals exceeding the threshold. The form of the tail estimator for $F_Z(Z)$ is then

$$\hat{F}_Z(Z) = 1 - \frac{k}{n}\left(1 + \hat{\xi}\frac{z - z_{(k+1)}}{\hat{\beta}}\right)^{-1/\hat{\xi}}.$$

For $q > 1 - k/n$ we can invert this tail formula to get

$$\hat{Z}_q = z_{(k+1)} + \frac{\hat{\beta}}{\hat{\xi}}\left(\left(\frac{1-q}{k/n}\right)^{-\hat{\xi}} - 1\right).$$

We also use the GPD tail estimator to estimate the right tail of the negative return distribution $F_X(x)$ by applying it directly to the raw return data $x_{t-n+1}, ..., x_t$. In this way, we calculate the unconditional EVT quantile estimate $\hat{x}_q$. According to McNeil and Frey (2000), it should be noted that the assumption of independent excesses over threshold is much less satisfactory for the raw return data. In fact, the procedure gives much more unstable results when applied to non-i.i.d. data. For more details, see Embrechts et al. (1997).

3 Various non-GARCH data generating processes

The method presented above shows satisfactory empirical results when applied to return series. However, if a series is mis-specified with respect to the dynamics that is initially assumed, one is not certain that the method is still consistent. To answer this problem, we assume two different behavior for financial asset return series, which will constitute our non-GARCH, specifications. We then apply the method following McNeil and Frey (2000) to these processes, and check wether or not the method remains valid. We select a model that dynamically incorporates both stochastic volatility and jumps, and another one with a switching regime volatility. These two dynamics are assumed to be potentially true DGPs for actual asset return series. Presently, we turn to describing how we perform the various simulations.

3.1 Jump diffusion models

Following Pan (1997), we present a model for asset returns that incorporates jumps. At each point of time, the occurrence of a jump is dictated by Bernoulli trials whereas the jump-size is assumed to be normally distributed. Under a discrete-time setting, let $\varepsilon = \{\varepsilon_t : t = 1, 2, ...\}$ be a sequence of i.i.d. random variables with standard normal distribution, $J = \{J_t : t = 1, 2, ...\}$ be Bernoulli trials with success probability p, and $Z = \{Z_t : t = 1, 2, ...\}$ be a sequence of i.i.d. random variables normally distributed with mean μ_Z and variance σ_Z^2. We assume that the various sets $\{\varepsilon\}$, $\{J\}$, and $\{Z\}$ are mutually independent. As a first DGP, we model the return process as follows

$$\begin{aligned} X_t &= \mu + \sigma_t \varepsilon_t + Z_t J_t, \\ \sigma_t^2 &= c J_{t-1} + a_0 + a_1 (X_{t-1} - \mu)^2 + a_2 \sigma_{t-1}^2 \end{aligned}$$

where $\mu, a_0, a_1, a_2 \in \mathbb{R}$. In that model, the time-t jump arrival is dictated by J_t, while the jump size is modelled by Z_t. At time t, the marginal movement in returns is modelled by ε_t with a stochastic volatility σ_t. Both J and ε contribute to the dynamics of σ_t. In this paper we simulate the general model above and a restriction thereof where volatility is held constant, hence, where only the mean is allowed to jump. Despite the fact that in Pan (1997), the fit of the restricted model is far less satisfactory than the general model with stochastic volatility, we consider that this model could possess the appearance of a certain financial returns series and, therefore, can represent the dynamic of an asset return.

3.2 Switching regime volatility model

Switching regime models present a further alternative that proved useful in modelling financial time series. See for instance Duecker (1997), Gray (1996), Hamilton (1989), Hamilton and Liu (1996), as well as Hamilton and Susmel (1994). See also van Norden and Schaller (1997) and Timmermann (2000) who have proposed a Markov switching regime volatility model assuming that returns are a mixture of normal distributions. This means that returns are drawn from a normal distribution where the mean and the variance can take different values depending on the state of the Markov chain. Following van Norden and Schaller (1997), such a model may be written as

$$X_t = \mu + [\sigma_1 S_t + \sigma_0(1 - S_t)]\varepsilon_t,$$

The innovations ε_t are independent and identically distributed normal innovations with mean 0 and variance 1. The state variable, S_t, is a Markov chain taking the values 0 and 1 and with transition probabilities $p = [p_{00}, p_{01}, p_{10}, p_{11}]$ such that

$$Pr[S_t = 1|S_{t-1} = 1] = p_{11}, \quad Pr[S_t = 0|S_{t-1} = 1] = p_{01},$$
$$Pr[S_t = 1|S_{t-1} = 0] = p_{10}, \quad Pr[S_t = 0|S_{t-1} = 0] = p_{00},$$

where $p_{11} + p_{01} = 1$ and $p_{10} + p_{00} = 1$. Such a model may be easily estimated with the EM algorithm, see Kitagawa (1987) or Hamilton (1989), or via PML as in van Norden and Schaller (1997).

4 Implementation and empirical results

Presently, we wish to discuss the way we simulate the various series. The samples we use for our simulations involve $n = 1000$ observations, this would correspond to somewhat less than four years of daily data. Concerning the number of observations, k, that should belong to the tail, according to McNeil and Frey (2000), the GPD-based quantile estimator is stable in terms of mean squared error for a choice of k, with k taking a value of approximately 80. For this reason, in the applications a value of 100 seems reasonable for k. This means that the 90th percentile of the estimation distribution is estimated by historical simulation, but that higher percentiles are estimated using the GPD tail estimator. On each day $t \in T$ we fit a new AR(1)-GARCH(1,1) model and determine a new GPD tail estimate.

To access the quality of the VaR prediction capability, we compare x_q^t with x_{t+1} for $q \in \{0.95, 0.99, 0.995\}$. A violation is said to occur whenever $x_{t+1} > x_q^t$.

As mentioned in the previous sections, to test the method, we simulate four processes. A GARCH(1,1) as benchmark data generating process that serves as a reference for the empirical results. Our alternative specifications consist in a switching volatility regime model, a stochastic volatility process with jumps, and a pure jump diffusion. Parameters for both the stochastic volatility with jumps process and the pure jump process have been estimated in Pan (1997) on daily returns of the $SP500$ composite index using data from 1986 to 1997.[3] Concerning the switching regime volatility model, parameters have been estimated in van Norden and Schaller (1997) based on CRSP value-monthly returns over the period January 1927 to December 1989.[4]

Insert Figures 1,2,3, and 4 somewhere here

We have plotted realizations of our various processes and the corresponding conditional EVT quantile estimate $\hat{x}_{0.95}^t$. Figure 2 shows clearly that the conditional EVT estimate responds quickly to increases in volatility, but tends to overestimate the series in periods of lower volatility. This is specially evident for the switching regime volatility model graph. We develop a binomial test of the success of this quantile estimation method based on the number of violations. Assuming the dynamics initially introduced and described in

[3]The parameter values used for the stochastic volatility with jumps model are the following: $\mu = 0$ (in Pan (1997) μ was set to 0.1842), $\mu_Z = -0.0183$, $\sigma_Z^2 = 0.0024$, $a_0 = 1.1273 * 10^{-4}$, $a_1 = 0.0363$, $a_2 = 0.9494$, $c = 0.0275$, and $p = 0.0124$. For the pure jumps process the parameters are: $\mu = 0$ (in Pan (1997) $\mu = 0.1938$),), $\mu_Z = -0.0043$, $\sigma_Z^2 = 0.007$, $a_0 = 0.0113$.

[4]The parameters used following van Norden and Schaller (1997) are: $\mu = 0.0071$, $\sigma_0 = 0.0392$, $\sigma_1 = 0.1180$, and the transition probabilities are $p_{00} = 0.991$, and $p_{11} = 0.9452$.

section 1, the indicator for a violation at time $t \in T$ is Bernoulli

$$I_t := \mathbf{1}_{\{X_{t+1} > x_q^t\}} = \mathbf{1}_{\{Z_{t+1} > z_q\}} \sim Be(1-q).$$

For t, $s \in T$ and $t \neq s$, since Z_{t+1} and Z_{s+1} are independent, I_s and I_t are independent. Therefore, the total number of violations is binomially distributed under the model,

$$\sum\nolimits_{t \in T} I_t \sim B(card(T), 1-q).$$

Under the null hypothesis that a method correctly estimates the conditional quantiles, the empirical version of the statistic $\sum_{t \in T} \mathbf{1}_{\{X_{t+1} > x_q^t\}}$ is from the binomial distribution $B(card(T), 1-q)$. We perform a two-sided binomial test of the null hypothesis against the alternative that the method has a systematic estimation error and gives too few or too many violations. A p-value less than or equal to 0.05 will be interpreted as evidence against the null hypothesis. The corresponding binomial probabilities are given in Table 2 alongside the numbers of violations for each method and each process simulation. Table 2 shows that on no occasion the approach fails. Then, following these empirical results we note that the approach does not fail for any misspecification with stochastic volatility, as well as for the constant volatility case.

However, we clearly note that filtering the data introduces a downward bias for GARCH (1,1) ,switching-regime data, and stochastic volatility with jumps model. The bias tends to decrease as the threshold increases. This observation suggest that the GARCH filter snatch too much of the tail for such specifications. Concerning the data generated by a pure jump process, the GARCH filter induces an upward bias of the tail thickness. The second important observation from these empirical results, is that in 11 out of 12 the unconditional EVT quantile estimate has the smallest bias and therefore is closest to the mark.

Insert Table 1 somewhere here

5 Expected shortfall

The concept of Value-at-Risk,VaR, is a quantile-based risk measure. Depending on the assumptions, VaR condenses all of the risk in a portfolio into a single number that describes the magnitude of the likely losses on the portfolio. It has undesirable properties such as lack of sub-additivity, i.e., VaR of a portfolio with two instruments may be greater than the sum of individual VaRs of these two instruments, and total absence of information on the size of the loss exceeding the VaR. The expected shortfall, ES, is an alternative risk measure to the quantile-based risk-measures such as VaR, which overcomes the deficiencies of the latter, see Artzner et al. (2000). The ES provides information of the average size of a potential loss given that a loss bigger than VaR has occurred.

5.1 Estimation

The conditional one-step expected shortfall is given by

$$S_q^t = \mu_{t+1} + \sigma_{t+1} E[Z|Z > z_q]$$

where μ_{t+1} and σ_{t+1} have been already estimated in the previous section. Thus, we need to estimate $E[Z|Z > z_q]$. For a random variable W with an exact GPD distribution with parameters $\xi < 1$ and β, we know that

$$E[W|W > w] = (w + \beta)/(1 - \xi),$$

where $w\xi + \beta > 0$. By noting that for $z_q > u$ we can write

$$Z - z_q|Z > z_q = (Z - u) - (z_q - u)|(Z - u) > (z_q - u),$$

and then that

$$Z - z_q | Z > z_q \sim G_{\xi, \beta + \xi(z_q - u)}.$$

We note that excesses over the higher threshold z_q also have a GPD distribution with the same shape parameter ξ but a different scaling parameter. By using the equations introduced above, we get

$$E[Z|Z > z_q] = z_q \left[\frac{1}{1-\xi} + \frac{\beta - \xi u}{(1-\xi) z_q} \right].$$

These GPD-based estimates gives us the conditional expected shortfall estimate

$$\hat{S}_q^t = \hat{\mu}_{t+1} + \hat{\sigma}_{t+1} \hat{z}_q \left[\frac{1}{1-\hat{\xi}} + \frac{\hat{\beta} - \hat{\xi} \hat{z}_q}{(1-\hat{\xi}) \hat{z}_q} \right].$$

5.2 Backtesting

To backtest the method, we are interested in the size of the discrepancy between X_{t+1} and S_q^t in the event of a quantile violation, i.e., $x_{t+1} > \hat{x}_q^t$. We define residuals as the random variables

$$E_{t+1} = \frac{X_{t+1} - S_q^t}{\sigma_{t+1}} = Z_{t+1} - E[Z|Z > z_q].$$

Under the model specification, these residuals are i.i.d., and, conditional on $\{X_{t+1} > x_q^t\}$ or equivalently $\{Z_{t+1} > z_q\}$, they have an expected value of zero. We construct empirical estimates of these residuals on days when violations occurs denoted by

$$\{e_{t+1} : t \in T, x_{t+1} > \hat{x}_q^t\}, \text{ where } e_{t+1} = \frac{x_{t+1} - \hat{S}_q^t}{\hat{\sigma}_{t+1}}.$$

Under the null hypothesis that we correctly estimate the dynamics of the process and $E[Z|Z > z_q]$, these residuals should behave like an i.i.d. sample with mean zero. To test the hypothesis of mean zero we use a bootstrap test that makes no assumption about the underlying distribution of the residuals.

We apply a one-sided test against the alternative hypothesis that the residuals have mean greater than zero or, equivalently, that conditional expected shortfall is systematically underestimated. For more details about this test, report to Efron and Tibshirani (1993). Following McNeil and Frey (2000) the residuals derived under an assumption of normality always fail the test with p-value, and in opposition, the GPD-based residuals are much more plausibly mean zero. In Table 2 we show p-values for the test applied to the GPD residuals for the GARCH(1,1) simulation, and for the three processes supposed to describe our alternative dynamics.

Insert Table 2 somewhere here

We note that p-values are always greater than 0.05, thus we conclude that on no occasion does the null hypothesis of the model being correct get rejected.

6 Conclusion

The true temporal dependency of financial returns is a complex issue. As a way to improve relevant measures for risk management, one could consider a two step procedure: First, filtering the returns through a more or less complex GARCH model, and, second, estimating the tail parameters using the assumption of i.i.d data. The actual measures for risk management can then be obtained following the two steps. In this paper we investigate the consequences of using GARCH filtered returns when the GARCH process is mis-specified. We assume as mis-specified GARCH series a simulated stochastic volatility process with jumps, a pure jump process, and a switching regime volatility model. Our findings may be summarized as follows. Filtering the data introduces a bias of the tail thickness for our various models. The bias tends to decrease as the threshold increases, suggesting that the GARCH filter absorbes too much of the tail of such specifications. The second important observation from these simulation results is that for data corresponding to market type parameters, the unconditional EVT quantile estimate has the smallest bias. These results suggest that great care should be exercised before applying EVT techniques to GARCH filtered processes.

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Captions

Table 1: Theoretically expected number of violations (exceedence of various thresholds) and actual number of violations obtained using the approach, a GARCH-model with normally distributed innovations, a GARCH-model with Student t-innovations, and quantile estimates obtained from unconditional EVT, for our four simulations. p-values for a binomial test are given in brackets.

Table 2: p-values for a one-sided bootstrap test of the hypothesis that the exceedence residuals in the GPD case have mean zero against the alternative that the mean is greater than zero.

Figure 1: A GARCH(1,1) simulation and the corresponding conditional EVT quantile estimate $\hat{x}^t_{0.95}$ represented with a continuous line.

Figure 2: Simulated data obtained with a Markov switching-regime volatility model and the corresponding conditional EVT quantile estimate $\hat{x}^t_{0.95}$ represented with a continuous line.

Figure 3: Simulated data obtained with a Stochastic volatility with jumps model and the conditional EVT quantile estimate $\hat{x}^t_{0.95}$ represented with a continuous line.

Figure 5: Simulated data obtained with a pure jump process with constant volatility and the conditional EVT quantile estimate $\hat{x}^t_{0.95}$ represented with a continuous line.

Process	GARCH (1,1)	switching regime volatility	Stochastic-volatility with jumps	Pure Jumps
0.95 Quantile				
Expected	50	50	50	50
Conditional EVT	42 (0.24)	46 (0.61)	41 (0.19)	57 (0.31)
Conditional normal	45 (0.51)	47 (0.71)	42 (0.24)	57 (0.31)
Conditional t	43 (0.34)	49 (0.94)	43 (0.34)	57(0.31)
Unconditional EVT	50 (1)	50 (1)	48 (0.82)	48 (0.82)
0.99 Quantile				
Expected	10	10	10	10
Conditional EVT	6 (0.21)	8 (0.63)	8 (0.63)	14 (0.26)
Conditional normal	6 (0.21)	10 (1)	6 (0.21)	12 (0.52)
Conditional t	9 (0.87)	10 (1)	9 (0.87)	12 (0.52)
Unconditional EVT	11 (0.74)	8 (0.63)	11 (0.74)	13 (0.33)
0.995 Quantile				
Expected	5	5	5	5
Conditional EVT	4 (0.82)	4 (0.82)	4 (0.82)	8 (0.26)
Conditional normal	4 (0.55)	4 (0.82)	4 (0.82)	7 (0.36)
Conditional t	3 (0.50)	4 (0.82)	4 (0.82)	7 (0.36)
Unconditional EVT	6 (0.64)	4 (0.82)	5 (1)	6 (0.64)

q	0.95	0.99	0.995
GARCH (1,1)	0.50	0.54	0.57
switching regime volatility	0.49	0.48	0.55
Stochastic-volatility with jumps	0.47	0.49	0.51
Pure Jumps	0.49	0.52	0.54

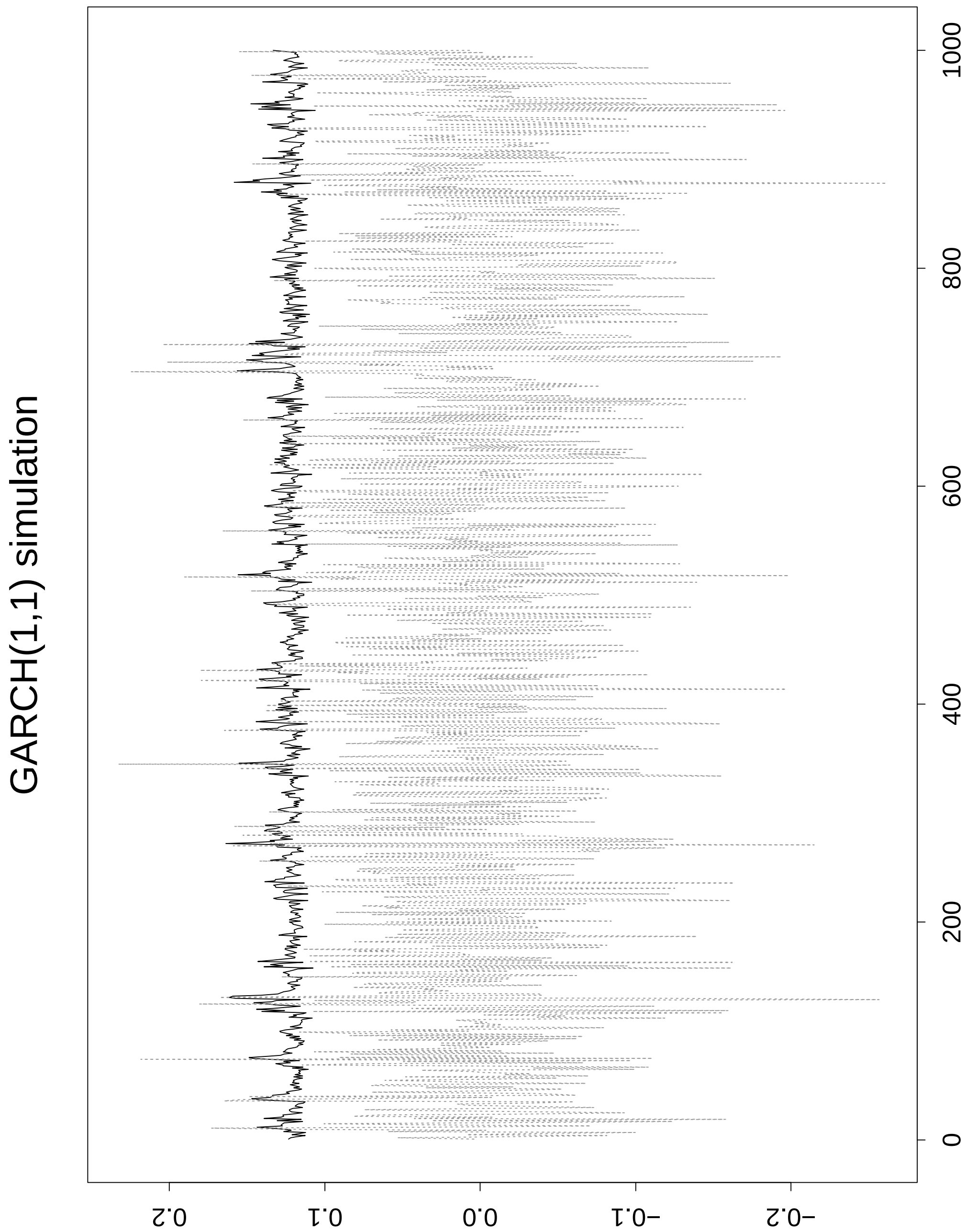

Figure 1:

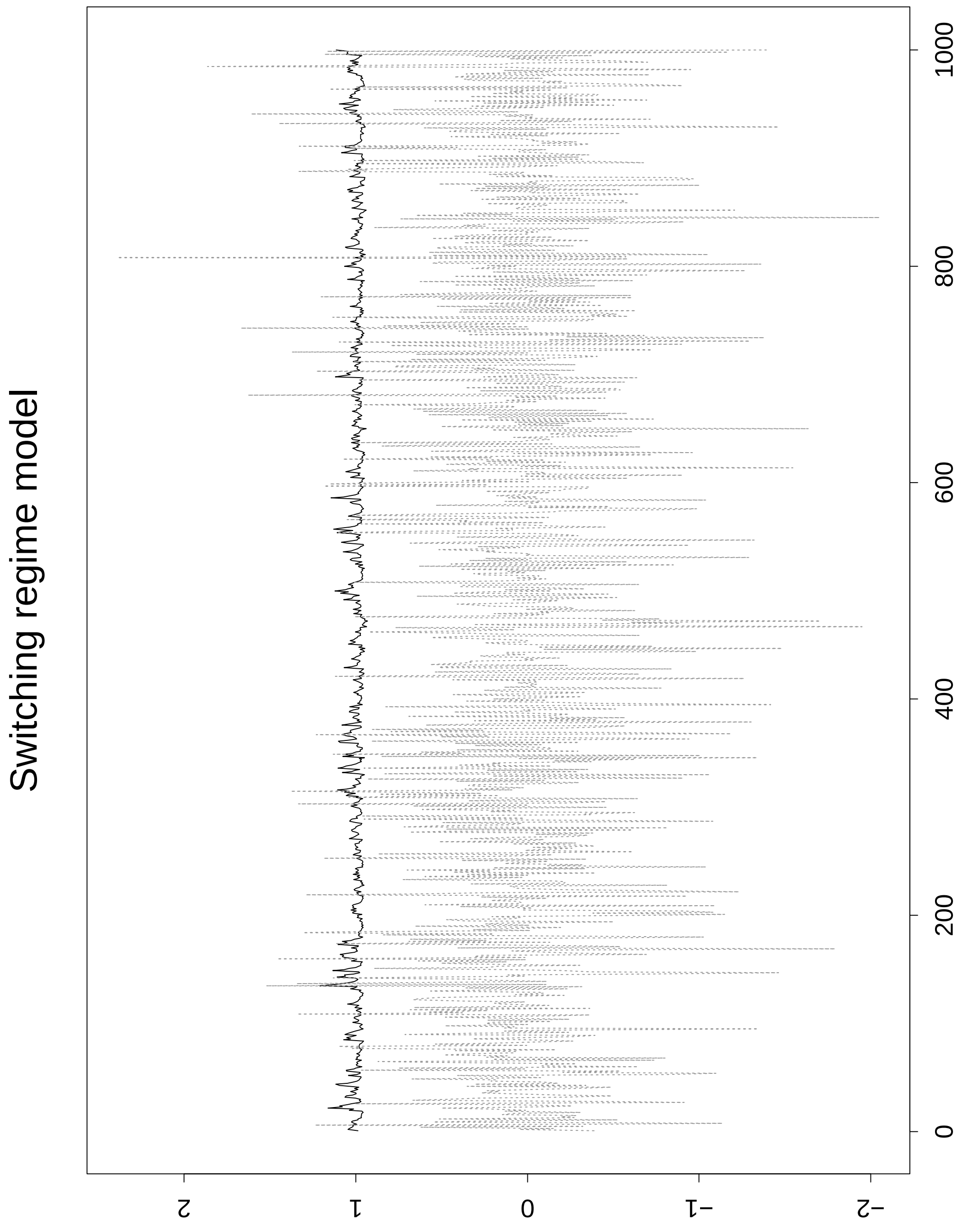

Figure 2:

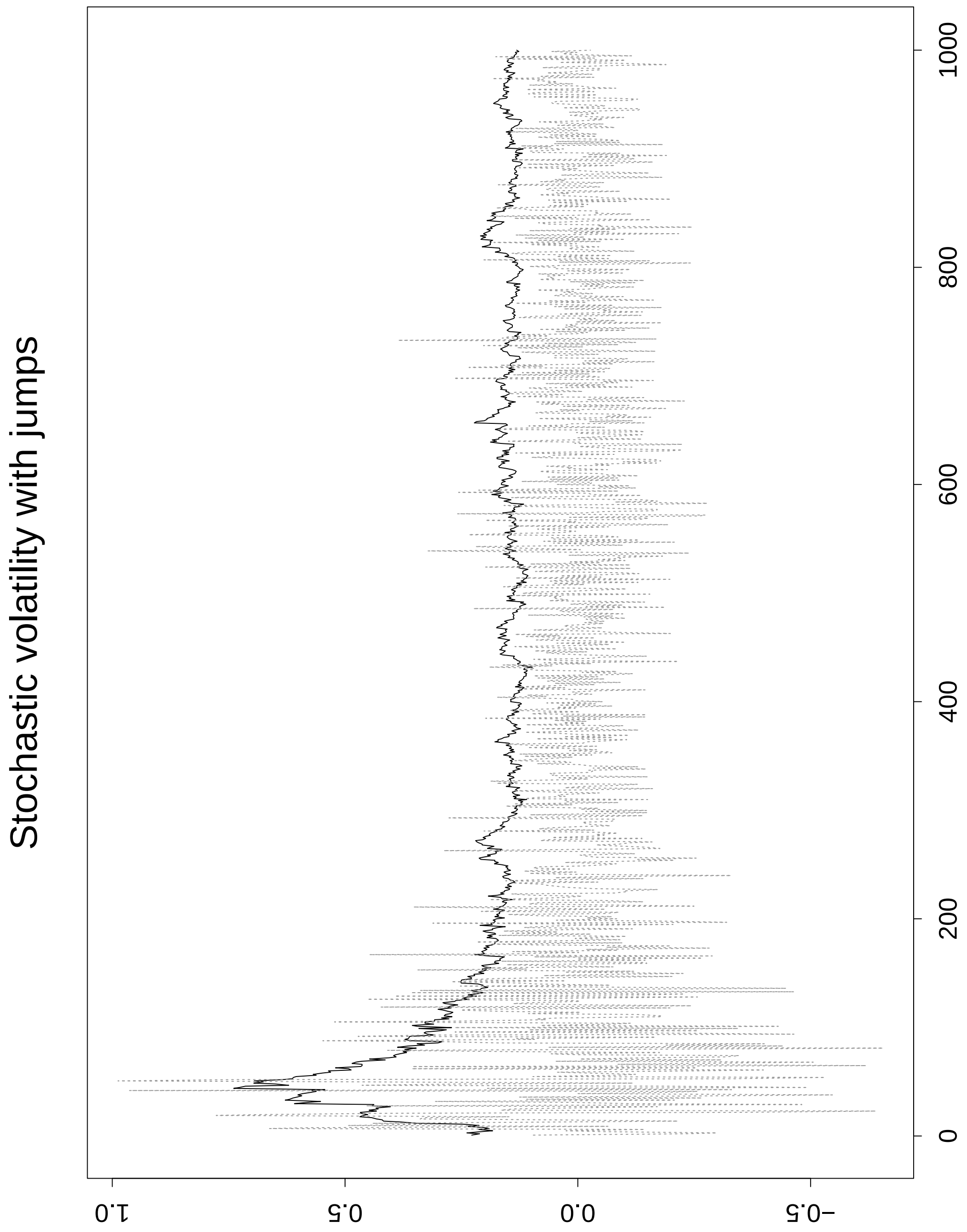
Stochastic volatility with jumps
0
200
400
600
800
1000
1.0
0.5
0.0
−0.5

Figure 3:

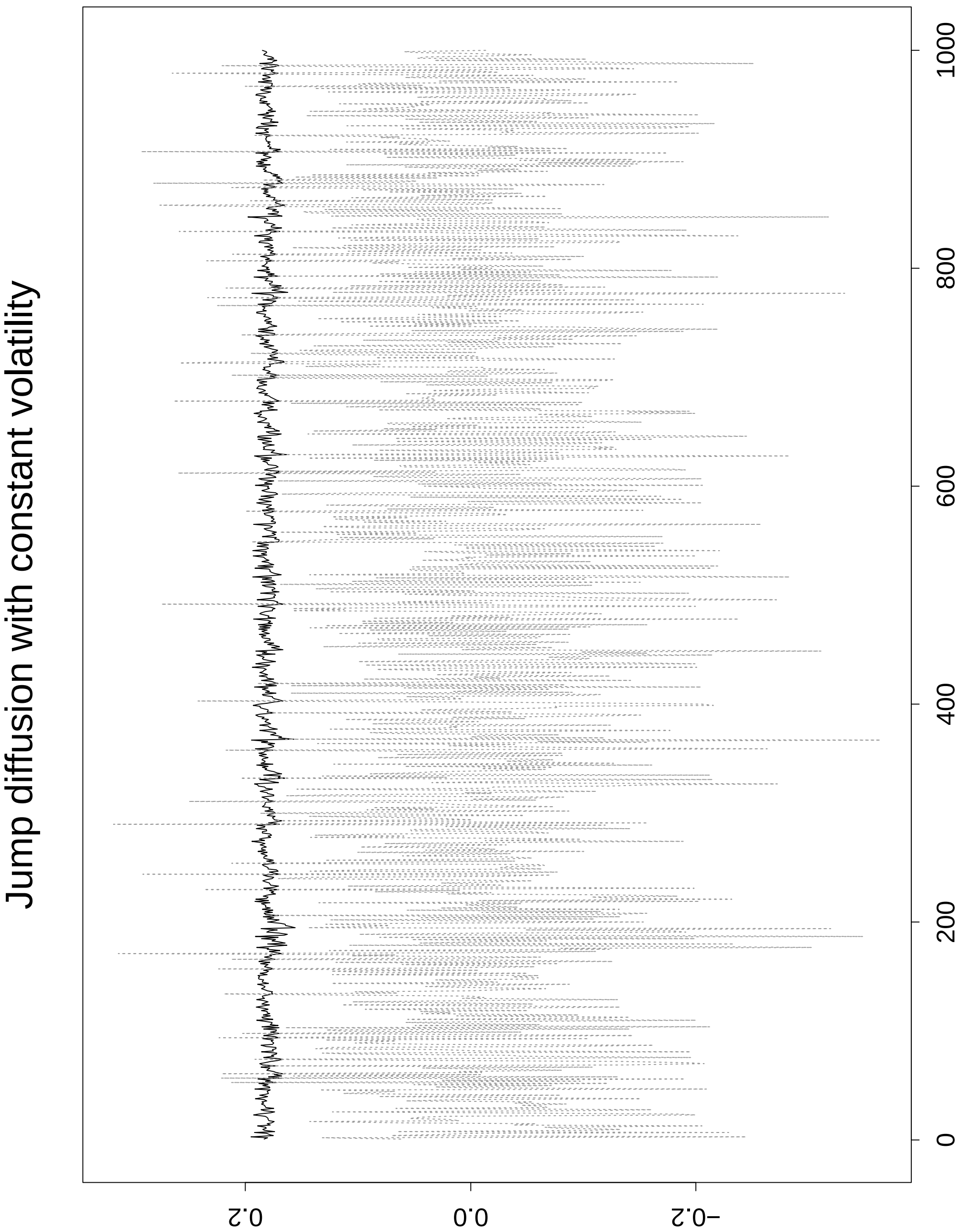
Jump diffusion with constant volatility
0
200
400
600
800
1000
0.2
0.0
−0.2

Figure 4:

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