Optimal Changes of Gaussian Measures, with Applications to Finance

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Abstract. We derive optimality conditions and calculate approximate solutions to the problem of determining the optimal speed of mean reversion to be applied to a Gaussian state variable. The optimality criterion is the minimization of the variance of the Radon-Nikodym derivative of the measure "with mean-reversion" with respect to the measure "without mean-reversion" under constraints. Our results have two main applications. First, we show that we can increase the speed of performing resimulation and sensitivity analysis in a Monte Carlo simulation. Second, we show that there is some phase delay between the optimal speed of mean-reversion and volatility. Incorporating this effect into preference modelling could contribute to solve the equity premium puzzle in finance.

Keywords: Equity premium puzzle; Monte Carlo simulation, change of measure

JEL Classification: C65, G12

1. Introduction

We examine the problem of characterizing a "target" Gaussian probability measure by doing an optimal change of measure from an "initial" (completely characterized) Gaussian probability measure. This problem has several applications in finance, in both complete and incomplete markets. When markets are complete, the traditional approach to calibrate option pricing models is to estimate a "physical", or "real" measure by time-series analysis (see for instance Chan et al. (1992) in the case of interest rate models), and then calibrate the market price of risk in such a way that (discounted) traded instruments are martingales in the risk-neutral measure; in this context, the initial measure is the physical measure, and the target measure is the risk-neutral measure. This paper also describes as an application the converse problem, where the initial measure is the risk-neutral measure, and the target measure is the physical measure; this problem has been less discussed in the literature, but is nevertheless quite important in practice. In incomplete markets, there are not enough traded instruments to fully determine the target (risk-neutral) measure, therefore the target measure is often defined by changing optimally the initial (physical) measure or, more generally, by selecting an "optimal" measure from a class of probability measures (see e.g. Rouge and El Karoui 2000).

We restrict our attention to probability measures where the state variables follow Gaussian diffusion namely, Ornstein-Uhlenbeck processes or arithmetic Brownian motions, because their analytical tractability makes them quite important in practice. The target measure is the solution of an optimization problem, whereby the variance of the Radon-Nykodim derivative of the target measure with respect to the initial measure is minimized subject to constraints on some characteristics of the target measure. We will in particular examine two different constraints: a constraint on the variance of the state variables at a terminal time, and a constraint on the (time-)average variance of the state variables. In
our application section, we describe a case where the state variable in the target measure should have a lower variance than in the initial measure; as a result, the overall speed of mean-reversion should be higher in the target measure than in the initial measure; our methodology suggests the optimal shape of the speed of mean-reversion curve. The objective chosen, the minimization of the variance of the Radon-Nykodim derivative, is important in several different contexts. We will cite only two reasons, one in the area of simulation, and one in financial economics.

First, in Monte Carlo simulation, it is often useful to resimulate the problem using a different (target) probability measure; when the integrand is expensive to compute, we advocate, rather than performing two independent simulations, to use the so-called change of measure (CM) resimulation scheme, whereby we calculate (scenario by scenario) the integrand only once in the initial measure and then multiply it by the Radon-Nikodym derivative of the target measure w.r.t the initial measure to obtain the expected value of the integrand in the target measure.

Secondly, it is a well-known fact in financial economics that traditional utility functions can hardly account to explain the abnormally high level of asset returns observed on the market ("the "equity premium puzzle", Mehra and Prescott 1985). In particular, the degree of risk-aversion would have to be much higher than plausible. In our approach, the pricing kernel (i.e., the Radon-Nikodym derivative) is determined in such a way that the difference between the physical (initial) measure and risk-neutral (target) measure is minimized; from reverse-engineering the pricing kernel into a utility function, we think our work can be applied to refine the characterization of utility functions, with more realistic risk-aversion.

Our main contributions are the following. We first show that the variance of the Radon-Nikodym derivative is the exponential of the time-integral of the solution of an ordinary differential equation, thereby making our optimal control problem deterministic. We then explore the properties of this optimal control problem subject to (i) a constraint on the average variance of the state variable in the target measure and (ii) a constraint on the terminal variance.

In the second part of our paper, we describe the aforementioned application of our results to a Monte Carlo (re)simulation problem in finance, the pricing of interest rate options under the BGM model. In that problem the initial measure is the rolling forward measure, and the target measure the physical measure. It is well-known that the calibration of the BGM model to caps and swaptions results in an implausibly high dispersion of interest rates, while information is lacking to determine the variance of interest rates at all times in the physical measure. The constraint on the (terminal) variance of interest rates in the physical measure and the minimization of the Radon-Nikodym derivative complete the determination of the optimal drift of interest rates. The analytical result on the optimal variance of the Radon-Nikodym is then useful to determine when it is advantageous to use our resimulation scheme; we also show experimental results. So far we described our scheme when there is only one initial measure and one target measure, and the resimulation applies to the target measure. We note that our resimulation scheme can apply to $n$ resimulations in $n$ target measures (for instance for sensitivity analysis).

We document the following example to fix orders of magnitude. Without loss of generality the first target measure is the most distant from the initial measure (and the $n$-th one the closest), our scheme reduces the processing time by 25% when $n = 1$, by 50% when $n = 2$, and by 85% when $n = 10$.

In the last part of the paper, we do a qualitative analysis of the cyclicality of the optimal
speed of mean-reversion with respect to the volatility. We show that the derivative of the speed of mean-reversion is approximately proportional to the derivative of the volatility, but with a phase-shift of 90 degrees. This means that the speed of mean-reversion is a quarter of a period late with respect to the volatility. This has a direct impact on the shape of the optimal utility function, in terms of minimization of the risk-aversion.

2. Model and Results

Notation: The complete filtered probability space \((\Omega, \mathcal{F}, P^I)\) supports a Brownian motion \(W^I\). We use the superscript \(^I\) and \(^T\) to refer to the probability measure, expectation operator and Brownian motion in the initial/terminal measure. When not shown otherwise, the expectation operator is taken at time zero.

The dynamics of the variable \(y\) of interest are:

\[
g(t) = x^*(t) + \alpha(t) \\
dx^*(t) = \sigma(t) dW^I(t) \\
x^*(0) = x^*_0
\]

where \(\alpha\) and \(\sigma\) are deterministic functions of time. We note that the average/terminal variance of \(y\) are the same as the ones of the scaled variable \(x = x^* - x^*_0\), which we choose as our state variable. The terminal measure \(P^T\) supports one Brownian motion \(W^T\), with:

\[
\begin{align*}
    dx(t) &= \sigma(t) dW^T(t) \\
x(0) &= 0 \\
dW^T(t) &= dW^I(t) + \frac{a(t)x(t)}{\sigma(t)} dt
\end{align*}
\]

Once the speed of mean-reversion \(a(t)\) is specified, \(P^T\) becomes fully specified. We now proceed to determine an expression for the second moment of the value at the horizon time \(T\) of the Radon-Nikodym derivative:

\[
g(T) \equiv \frac{dP^T}{dP^I}|_T
\]

Lemma:

\[
E^I[g^2(T)] = \exp\left[\int_0^T \sigma^2(t)f(t)dt\right]
\]

where:

\[
\begin{align*}
    \frac{df}{dt} &= -\frac{a^2(t)}{\sigma^2(t)} + 4a(t)f(t) - 2\sigma^2(t)f^2(t) \\
f(T) &= 0
\end{align*}
\]

As can be seen from the proof of the lemma, this expression is reminiscent of the formula for the value of a discount bond in the Cox, Ingersoll, and Ross model (see e.g.,
Duffie 1996). As Levendorskii (2004) points out, there is no “truly analytical” formula in that case, unless $a$ and $\sigma$ are constant. Various expansions exist for the solution of this problem though (see, e.g., Grasselli and Hurd 2003).

Although both time-average variance and terminal variance constraints can be incorporated into the same optimal control problem, it is simpler for analysis to consider both problems and optimality conditions separately. This also enables, in our result section, to decompose the effect of each constraint on the shape of the optimal control.

2.1. Average Variance Constraint (AVC) Problem. The AVC problem consists of selecting $a$ so that, for a fixed constant $A$ (the average variance):

$$\min_a E^T[y^2(T)]$$

$$E^T[\int_0^T x^2(t)dt] \leq A$$

$$dx = -a(t)x(t)dt + \sigma(t)dW^T(t)$$

$$x(0) = 0$$

THEOREM 1 A necessary condition for $a$ to solve the AVC problem is to set:

$$a = \sigma^2\left[2f - \frac{2v}{y}\right]$$

where, for some value of $v(T)$ and $\lambda \geq 0$, the functions $f, z, y, v$ solve the following boundary value problem:

$$\int_0^T v(t)dt \leq A$$

$$\frac{df}{dt} = -\frac{a^2}{\sigma} + 4af - 2\sigma^2 f^2$$

$$f(T) = 0$$

$$\frac{dv}{dt} = -2av + \sigma^2$$

$$v(0) = 0$$

$$\frac{dy}{dt} = \sigma^2 + 4[y\sigma^2f - a]$$

$$y(T) = 0$$

$$\frac{dz}{dt} = \lambda + 2az$$

$$z(T) = 0$$

Proof: This is a direct application of the maximum principle and lemma 1; $\lambda$ is the Lagrange multiplier of the constraint (22).

The advantage of this theorem is to replace a complicated stochastic control problem by a system of ordinary differential equations. A solution method for this type of problem is, for each value of the parameter $\lambda$ and $v(T)$ to solve the system (9) to (13) until a solution is found that respects (8) and $v(0) = 0$. Our experience shows that, for all
standard values of the parameters $\sigma(t)$ and $A$ used in finance problems\(^1\), this system either (i) is chaotic or (ii) does not respect (9) and $z(0) = 0$. We had then to resort to approximating $\text{Var}^I[g^2(T)]$ by $\text{Var}^\text{app}I[g^2(T)]$, where:

\[
\text{Var}^\text{app}I[g^2(T)] = \exp \left[ \int_0^T t \frac{a^2(t)}{\sigma^2(t)} dt \right] - 1
\]

For small values of $\sigma$ ($\sigma \leq 40\%$) and substantial average variance requirement (i.e., high $a$), $\text{Var}^\text{app}I[g^2(T)]$ is an upper bound of $\text{Var}^I[g^2(T)]$, as the next proposition shows.

**PROPOSITION II**, for $t \in [0, T]$

\[
a(t) \geq \frac{\sigma^2(t) f(t)}{2}
\]

then:

\[
\text{Var}^\text{app}I[g^2(T)] \geq \text{Var}^I[g^2(T)]
\]

**Proof.** Let $x = \frac{a^2}{\sigma^2}$, $y(t) = f(T - t)$, $z(t) = h(T - t)$ and:

\[
\frac{dy}{dt} = F(x, y) = -2ay + 2\sigma^2y^2 + x
\]

\[
\frac{dz}{dt} = G(x, z) = x
\]

Since $a(t) \geq \frac{f(0)\sigma^2(t)}{2}$, then:

\[
F(x, y) \leq G(x, z)
\]

Since $y(0) = z(0) (= 0)$, the Sturm comparison theorem (see Birkhoff and Rota 1978) says that $y(t) \leq z(t)$, and $f(t) \leq h(t)$.

Instead of solving the ACV problem, we turned to solving the approximated ACV problem, which is the same as ACV, but for the goal (21), which is replaced by the minimization of $\text{Var}^\text{app}I[g^2(T)]$, for which the optimality conditions are much simpler, i.e., to find

\[
a = -\frac{z(t)v(t)}{t}
\]

so that, for some value of $\lambda \geq 0$ and $v_T$, then $z, v$ solve (11) and (13) and $v(0) = 0$. That new system turned out to be non-chaotic and always have a solution for reasonable values of $A$. Indeed, in all cases, the value of $\text{Var}^\text{app}I[g^2(T)]$ was reasonably close to the value of $\text{Var}^I[g^2(T)]$.

---

\(^1\)We tried a very large combination of values such that $0 \leq \sigma(t) \leq 0.5$, $1 \leq T \leq 10$ and $\frac{1}{2} \int_0^T \sigma^2(t) dt \leq M \leq \frac{1}{2} \int_0^T \sigma^2(t) dt$. The collection of our test problems is available from the author upon request.
Results. We report in figures 1 to 3 the tradeoff between \( A \) and \( \text{Var}^f[g^2(T)] \) for our suboptimal control (20). To show that our approximation is quite good, we show on the figure the corresponding value of \( \text{Var}^f_{\text{app}}[g^2(T)] \). We also compare our suboptimal control to the most naive control, that is, a constant speed of mean reversion such that (22) is met. In all cases our suboptimal control beats the constant speed of mean reversion control. In figure 1 and 2 and the horizon was \( T = 1 \) year, and in figure 3, \( T = 3 \) years. We observed in all these results that the suboptimal control \( a(t) \) followed a downward trend.

\[
\begin{align*}
\text{sigma}(t) &= 0.2 \\
\text{Var}[g(1)]_{\text{approx}} &\quad \text{Var}[g(1)]_{\text{subopt}} & \quad \text{Var}[g(1)]_{\text{const speed}}
\end{align*}
\]

% Average Variance

Figure 1: Tradeoff variance of \( g \) and average variance.

2.2. Terminal Variance Constraint (TVC) Problem. The TVC problem consists of selecting \( a \) so that, for a fixed constants \( M \) (the terminal variance):

\[
\begin{align*}
\min_a E\left[g^2(T)\right] \\
E^T[x^2(T)dt] &\leq M \\
dx &\quad = -2a(t)x(t)dt + \sigma(t)dW^T(t) \\
x(0) &= 0 \\
\end{align*}
\]

**THEOREM** A necessary condition for \( a \) to solve the TVC problem is to set:

\[
a = 2\sigma^2\left[f - \frac{z}{y}\right]
\]
\[ \sigma(t) = 0.2(1 + 0.2\cos \frac{t}{4}) \]

Figure 2: Tradeoff variance of \( g \) and average variance.

\[ \sigma(t) = 0.2 \]

Figure 3: Tradeoff variance of \( g \) and average variance.
where, for some value of $z_T$ the functions $f, z, y, v$ solve the following boundary value problem:

$$\int_0^T v(t) dt \leq A$$  \hspace{1cm} (25)

$$\frac{df}{dt} = -\frac{a^2}{\sigma} + 4af - 2f^2\sigma^2 \hspace{1cm} f(T) = 0$$  \hspace{1cm} (26)

$$\frac{dv}{dt} = -4av + \sigma^2 \hspace{1cm} v(T) = M$$  \hspace{1cm} (27)

$$\frac{dy}{dt} = \sigma^2 + [4a - 4f]y \hspace{1cm} y(T) = 0$$  \hspace{1cm} (28)

$$\frac{dz}{dt} = 4az \hspace{1cm} z(T) = z_T$$  \hspace{1cm} (29)

Solving the TVC problem is easier than solving the AVC problem because only one parameter $z_T$ needs to be varied. However, we observed the same chaotic behaviour as in the previous section, and resorted instead to the same approximation as above, i.e., minimizing (14), to obtain the same optimal control (20). The last theoretical result of this section gives some intuition on the relationship between (an upper bound on) $E^I[g^2(T)]$ and $M$.

**PROPOSITION** When the volatility is constant, an upper bound for $E^I[g^2(T)]$ as a function of the terminal variance $M$ is:

$$E^I[g^2(T)] = \sqrt{\exp\left(\frac{\sigma_T^2}{4M}\right)\cos\left(-\frac{\sigma_T^2}{4M}\right)}$$  \hspace{1cm} (30)

**Proof:** Using for instance Pruefer substitution to solve (1) for constant values of $a$ and $\sigma$, one finds (check again):

$$E^I[g^2(T)] = \sqrt{\exp\left(2aT\right)\cos\left(-2aT\right) + \sin\left(-2aT\right)}$$  \hspace{1cm} (31)

which is increasing in $a$. On the other hand,

$$v(T) = \frac{\sigma^2}{4a}(1 - \exp(-4aT)) \leq \frac{\sigma^2}{4a}$$  \hspace{1cm} (32)

The terminal variance constraint is met if we set:

$$\frac{\sigma^2}{4a} = M$$  \hspace{1cm} (33)

Inserting (33) in (31) yields the result.
Results. We report in figures 4 to 7 the tradeoff between $M$ and $\text{Var}[g^2(T)]$ for our suboptimal control (20). We also compare our suboptimal control to the most naive control, that is, a constant speed of mean reversion such that (22) is met. In all cases our suboptimal control beats the constant speed of mean reversion control. In figure 4 and 5 the horizon was $T = 1$ year, and in figure 3 the horizon was $T = 3$ years. We observed in all these results that the suboptimal control $a(t)$ followed an upward trend. For the terminal variance ratio and horizons chosen, the terminal variance turns out to be not significantly lower than the maximum variance across the path.

\[ \text{sigma}(t)=0.2 \]

Figure 4: Tradeoff variance of $g$ and average variance.

3. Application to Simulating the BGM Model.

The BGM/J Libor model is currently one of the most widely used models for the pricing of interest rate options. In a one-factor BGM model, forward rates $F_i$ for a loan between period $T_i$ and $T_{i+1} = T_i + \tau$ (with $i = 1..m$) follow the system of SDE:

\[
\frac{dF_i}{F_i} = -\sigma_i \sum_{k=i+1}^m \frac{\sigma_k F_k \tau}{1 + F_k \tau} dt + \sigma_i dW^F \\
F_i(0) = F_{i,0}
\] (34) (35)

where $W^F$ is Brownian motion in the measure used for pricing, namely the rolling forward measure. The drift term in (34) is in practice very small, so that forward rates
\[ \sigma(t) = 0.2 \times (1 + 0.2 \times \cos(t / 4)) \]

Figure 5: Tradeoff variance of \( g \) and average variance.

\[ \text{sigma(t)}=0.2 \]

Figure 6: Tradeoff variance of \( g \) and average variance.
are approximately lognormal. Normality (of the logarithm of the forward rate) is a key advantage for a successful and intuitive calibration to caps and swaptions, as explained for instance in Rebonato(1999)(2002). Likewise, a joint normal distribution in the true measure of the logarithm of forward rates is much easier to interpret than any other more sophisticated and statistically more correct distribution.

Although the academic literature favours to first infer the real measure and then adjust it with a market price of risk (see Heath, Jarrow, and Morton 1992) to obtain the rolling forward measure, information often flows the other way round in practice. In many bank departments the key requirement is to do a correct pricing. To this effect, the rolling forward measure is calibrated first to the prices of caps and swaptions and/or historical correlations. The initial measure is then the rolling forward measure. The target measure is then either the physical measure (to calculate Value-at-Risk or Earnings-at-Risk), or some measure derived from either the rolling forward or the physical measure, to perform sensitivity analysis (e.g., what happens to the Value-at-Risk the average volatility changes by 1%, 2%, 5%).

We provide hereafter a comparison between the traditional resimulation scheme and the CM resimulation scheme. To keep the scheme as general as possible, we added the possibility to calculate a different measure of interest in the initial and the target measure, namely $E^T[Z]$ and $E^T[h(Z)]$. In our application to portfolio pricing, the integrand $Z$ is the sum of the discounted cash flows, which is a functional of the forward interest rates $F$, and, $E^T[h(Z)]$ is, for instance, the probability that $Z$ is less than $z$, so that $h(Z) = 1[Z \leq z]$. Of course, we could choose $h$ to be the identity function (to conduct sensitivity analysis on market prices).

Algorithm Traditional Resimulation
1. Calculate integrand $Z(\omega)$ for each scenario $\omega = 1..\Omega_{\text{trad}}$ in the initial measure

Figure 7: Tradeoff variance of $g$ and average variance.
2. Calculate the estimator in the initial measure $V_{\text{trad}}^I = \frac{1}{\Omega_{\text{trad}}} \sum_{\omega=1}^{\Omega_{\text{trad}}} Z(\omega)$

3. Calculate integrand $Z(\omega')$ for each scenario $\omega' = 1..\Omega_{\text{trad}}$ in the target measure

4. Calculate the estimator in the target measure $V_{\text{trad}}^T = \frac{1}{\Omega_{\text{CM}}} \sum_{\omega=1}^{\Omega_{\text{CM}}} h(Z(\omega'))$

**Algorithm CM Resimulation**

I. Calculate integrand $Z(\omega)$ for each scenario $\omega = 1..\Omega_{\text{CM}}$ in the initial measure

II. Calculate the estimator in the initial measure $V_{\text{CM}}^I = \frac{1}{\Omega_{\text{CM}}} \sum_{\omega=1}^{\Omega_{\text{CM}}} Z(\omega)$

III. Calculate the Radon-Nikodym derivative $g(\omega)$ for each scenario $\omega = 1..\Omega_{\text{CM}}$

IV. Calculate the estimator in the target measure $V_{\text{CM}}^T = \frac{1}{\Omega_{\text{CM}}} \sum_{\omega=1}^{\Omega_{\text{CM}}} g(\omega)h(Z(\omega))$

The CM Resimulation algorithm should be applied only when step I is more time-consuming than step III. We reported in an earlier paper (Schellhorn and Kidani 2000) that, for large portfolios of mortgage-backed securities, calculating the (discounted) cash flows can take more than 100 more times than simulating the state variable and its Radon-Nikodym derivative $g$.

However, the estimator $V_{\text{CM}}^T$ can have higher variance than the estimator $V_{\text{trad}}^T$, ceteris paribus, i.e. with the same number of scenarios $\Omega_{\text{CM}} = \Omega_{\text{trad}}$. Indeed, letting (for expositional clarity) $h$ be the identity matrix, $E$ and $\text{Var}$ the expectation and variance operators in the initial measure and $\rho_n$ the correlation between $g^2$ and $Z^{4n}$, we use H"{o}lder’s inequality to obtain:

$$E[(V_{\text{CM}}^T)^2] \leq E[g^2] - \frac{1}{\Omega_{\text{CM}}} \sum_{\omega=1}^{\Omega_{\text{CM}}} (\text{Var}[g^2] Z^{4n})^\frac{1}{4}$$

(36)

$$\leq E[g^2] - \frac{1}{\Omega_{\text{CM}}} \sum_{\omega=1}^{\Omega_{\text{CM}}} (\text{Var}[g^2] E[Z^{4n}])^\frac{1}{4}$$

(37)

$$\leq E[g^2] - \frac{1}{\Omega_{\text{CM}}} \sum_{\omega=1}^{\Omega_{\text{CM}}} (\text{Var}[g^2] Z^{4n})^\frac{1}{4}$$

(38)

For finite variance of $g^2$ the last term goes to one when $n$ goes to infinity. We can then write, for large $n$

$$E[(V_{\text{CM}}^T)^2] \leq E[g^2] E[(V_{\text{trad}}^T)^2]$$

(40)

To maintain the same accuracy in $V_{\text{CM}}^T$ and $V_{\text{trad}}^T$, one needs then in general $\Omega_{\text{CM}} > \Omega_{\text{trad}}$, thereby weakening the computational advantage of the CM scheme.

If there is a degree of freedom in selecting $g$, then (40) shows that it is optimal for simulation efficiency to minimize $E[g^2]$, thereby providing a first application of our methodology. We note that proposition 2 gives us worst case bounds on when to use the CM resimulation as a function on the bound on the terminal variance.

Calibrating the BGM/Libor model to caps and swaptions results in a dispersion of rates forecast, in the rolling forward measure, that is much higher than the plausible dispersion.
of real rates in the US, because of the high skewness of the lognormal distribution. This is one of the reasons why alternate models like Hull and White (1993), where rates are Gaussian when they are large and lognormal when they are small, were designed to prevent a too rapid increase in risk-neutral rates; while widely used, for instance at Bank of America in the 1990s (Williams 1999) this model is less practical to calibrate to caps and swaptions than the BGM/Libor. To summarize, we advocate to calibrate the rolling forward measure first, and then to derive the true measure by a mean reversion adjustment, such that rates will still be lognormal in the true measure, but with a smaller dispersion. Depending on how the “smaller dispersion” constraint is specified, the optimal speed of mean reversion solves either problem (TVC), or problem (AVC), where the state variable

\[ g(t) = \log(F_i(t)) - \log(F_{i,0}) \]

for some well-chosen forward rate \( F_i \).

**Results.** Our own experience showed that a good fit is obtained to cap prices in the US and UK when volatility takes the form:

\[ \sigma(t) = \sigma_0(1 - 0.8\exp(-2t) - mt) \]  

This corresponds to the stylized cap curve observed in Rebonato (2004 p. 232): an initial very steep portion, a plateau area, followed by a rapid decline. We arbitrarily decided to set the variance in the target measure equal to 60% of the variance in the initial measure. In figure 10, we report the variance of \( g(T) \) for \( \sigma_0 = 0.35, T = 15 \) for different declines in the long volatilities:

\[ m(k) = 0.036 + 0.004k \]  

Figure 9 shows that in all cases we obtain lower variances for our suboptimal speed of mean reversion, compared to a constant speed of mean reversion, but the effect is more pronounced for steeper declines in the volatility curve. For \( k = 10 \), the CM resimulation scheme with suboptimal speed of mean reversion needs 50% more scenarios than the traditional resimulation scheme, whereas with constant speed it needs 93% more, rendering it completely ineffective. We observed some interesting qualitative features in teh shape of the suboptimal speed of mean-reversion, namely, some delay compared to the volatility curve; this phenomenon is elaborated on in more details in the next section, which also gives some economic foundations to our selection of the objective function to minimize.

### 3.1. Application to Financial Economics.

In financial economics, equilibrium is often analyzed in terms of a representative investor maximizing her hyperbolic absolute risk aversion (HARA) utility of consumption. However, this approach leads to the equity premium puzzle: a constant degree of risk aversion cannot justify the high equity premium, i.e., the spread between the expected return on equities and the risk-free rate observed on the markets. For a coherent model to hold, the degree of risk aversion should not exceed 10, whereas, when observing the markets, it should be much superior to 20.

In these models (see e.g. Jin and Glasserman 2001 for a lucid summary on fixed income securities) the pricing kernel is equal to the discounted value of the marginal utility consumption of the representative investor. The pricing kernel is itself the discounted value of the Radon-Nikodym derivative of the risk-neutral measure w.r.t. the physical measure. In the example from the previous section, the initial measure was the rolling forward measure, whereas the target measure was the physical measure. The state variable is then
The optimal marginal utility of consumption is therefore equal to the Radon-Nikodym derivative of the risk-neutral measure w.r.t the physical measure:

\[
\frac{\partial U}{\partial c}(c^*,t) = \exp\left[ \int_0^t \frac{a(t)}{\sigma(t)} x(t) dt - \frac{1}{2} \sigma(t)^2 \int_0^t a(s) ds \right] \quad (45)
\]

Let \( \kappa \) be the expected value of the product of the state variable by the marginal utility of consumption. The time-derivative of the optimal marginal utility of consumption is then:

\[
E^{PH}\left[ \frac{\partial^2 U}{\partial c \partial t}(c^*,t) \right] = \frac{a(t)}{\sigma(t)} \kappa(t) \quad (46)
\]

Economic data shows that \( \kappa(t) \) varies very little. The qualitative analysis from the last section provides then some guidance on the cyclicity of the derivative of the marginal utility of consumption as a function of volatility.

Combining equations (20)(11)(13), we see that:

\[
\frac{da}{dt} = \frac{1}{t} \left[ a(t) - \lambda \exp\left( -2 \int_0^T a(s) ds \| \sigma^2(s) \| \right) - 2a^2(t) \right] \quad (47)
\]

(note that the control does not diverge at \( t = 0 \) since the bracket term in (47) is zero).

Taking \( \lambda \) high enough, we see that:
\[
\frac{da}{dt} = \lambda \frac{exp(-2\int_0^T a(s)ds)}{t}\sigma^2(t) + O(\varepsilon)
\]  \hspace{1cm} (48)

If the volatility follows

\[
\sigma(t) = \sigma_0 + \frac{\varepsilon}{2\sigma_0} \cos(\omega t)
\]  \hspace{1cm} (49)

We see that:

\[
\sigma^2(t) = \sigma_0^2 + \varepsilon \cos(\omega t) + O(\varepsilon^2)
\]  \hspace{1cm} (50)

For \( T - \tau = O(\varepsilon) \), we try the approximation:

\[
a(t) = a_0 + K\varepsilon \cos(\omega t - \varphi) + O(\varepsilon^2)
\]  \hspace{1cm} (51)

Inserting (50) and (51) in (48), we see that \( a \) should follow the variation of \( \sigma \) with a phase shift \( \varphi \) of 90 degrees.

**Results.** Figure 9 shows the Fast Fourier Transform (FFT) of \( a(t) \) when \( \sigma_0 = 0.2 \), \( \lambda = 10 \) and \( T = 10 \). Indeed there is a strong peak at \( \omega \). We report in table 1 and phase in figure 10 of the strongest peak of the spectrum of \( \frac{da}{dt} \), as obtained by the FFT, as a function of \( \frac{1}{\tau} \) (the amplitude of the sinusoidal part of volatility) and \( f\omega \) (the frequency of the latter). The results in these tables are proxies of \( K\varepsilon \) and \( \varphi \), respectively. Of course, if \( a \) were a linear functional of \( \sigma \) the amplitude of \( \frac{da}{dt} \) would vary linearly with \( \varepsilon \). Figure 10 shows that this, in this range of parameters, the linear approximation is quite satisfactory. Table 1 shows some variation in the phase (of up to 30 degrees), which is normal, because the Fourier analysis included the whole series, from \( \tau = 0 \) to \( T \).

<table>
<thead>
<tr>
<th>( \frac{1}{\tau} )</th>
<th>0.15</th>
<th>0.2</th>
<th>0.25</th>
<th>0.3</th>
<th>0.35</th>
<th>0.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>25</td>
<td>4</td>
<td>9</td>
<td>5</td>
<td>-2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>14</td>
<td>33</td>
<td>32</td>
<td>27</td>
<td>18</td>
<td>23</td>
</tr>
<tr>
<td>1.5</td>
<td>26</td>
<td>4</td>
<td>20</td>
<td>11</td>
<td>13</td>
<td>13</td>
</tr>
<tr>
<td>2</td>
<td>27</td>
<td>33</td>
<td>32</td>
<td>27</td>
<td>30</td>
<td>31</td>
</tr>
</tbody>
</table>

Table 1: Phase (in degrees) of the strongest peak of \( \frac{da}{dt} \).

4. Conclusion

5. Appendix: Proof of Lemma

We introduce the measure \( P^S \) where \( W^S \) is Brownian motion with:

\[
dW^S(t) = dW^T(t) + \frac{a(t)r(t)}{\sigma(t)}dt
\]  \hspace{1cm} (52)

By Girsanov theorem:
Figure 9: Spectrum of $\frac{d\sigma}{dt}$ for $\omega = 2\pi$, $\sigma_0 = 0.2$, $\varepsilon = 1$

Figure 10: Amplitude of $\frac{d\sigma}{dt}$ as a function of frequency and amplitude of $\sigma(t)$
\[ E^r \{ g^2(T) \} = E^S \left[ \frac{dP^T}{dP^r} g^2(T) \right] \]

\[ = E^S \left[ \exp \left( \int_0^T \frac{\alpha_t x_t}{\sigma_t} dW_t^S - 2 \int_0^T \frac{\alpha_t x_t}{\sigma_t}^2 dt \right) \right] \]

\[ \exp \left[ -2 \int_0^T \frac{\alpha_t x_t}{\sigma_t} dW_t^I - \int_0^T \left( \frac{\alpha_t x_t}{\sigma_t} \right)^2 dt \right] \]

\[ = E^S \left[ \exp \left( \int_0^T \left[ \frac{\alpha_t x_t}{\sigma_t} \right]^2 dt \right) \right] \]

We can rewrite (1) followingly:

\[ dx = -2a(t)x(t)dt + \sigma(t)dW^S(t) \] (57)

We came up with 2 equivalent approaches to evaluate (56) one based on the Feynman-Kac formula, the other one utilizing discrete approximations. We present only the Feynman-Kac proof, because of its elegance. Since the Feynman-Kac theorem is usually expressed "backward", that is, with a terminal condition rather than an initial condition, we recap here informally all the main steps of the approach. Let us set:

\[ l = \left( \frac{\alpha}{\sigma} \right)^2 \]

\[ y = lx^2 \]

Applying Itô’s lemma:

\[ dy = \left[ \left( \frac{dy}{dt} - 4a \right)y + \sigma^2 \right]dt + 2\sigma \sqrt{l} \sqrt{y} dW^S \] (60)

We use the relationship:

\[ \exp \left( \int_t^T y(s) ds \right) - \int_{y=a}^T y(u) \exp \left( \int_u^T y(s) ds \right) du = 1 \] (61)

Therefore:

\[ E_t[\exp(\int_t^T y(s) ds)] = 1 + E_t[\int_{y=a}^T y(u) \exp(\int_u^T y(s) ds) du] \]

\[ = 1 + E_t[\int_{y=a}^T y(u) E_u[\exp(\int_{s=y}^T y(s) ds) du]] \] (62)

Letting:

\[ h(y(t), t) = E_t[\exp(\int_t^T y(s) ds)] \]

Relation (63) becomes:
By definition,

\[ h(y(t), t) = E_t[h(y(T), T)] + \int_{t}^{T} y(u) h(y(u), u) \, du \]  

(65)

We try the functional form

\[ h(y, t) = \exp \left( ct(T - t) + bt(T - t)y \right) \]  

(67)

Applying Ito's lemma to (66), and equating the integrands in (65)(66) yields (with \( \tau = T - t \)):

\[
\left[ \left( \frac{\partial}{\partial t} - 4at \right) y + \sigma^2 \frac{\partial^2}{\partial t^2} y \right] + \frac{dc}{d\tau} - \frac{db}{d\tau} y + y = 0
\]

(68)

Separating constant and \( y \) terms yields, with \( c(0) = b(0) = 0 \):

\[
\frac{dc}{d\tau} \bigg|_{\tau=t} = \sigma^2(t) l(t) \dot{b}(T - t)
\]

(69)

\[
\frac{db}{d\tau} \bigg|_{\tau=t} = \left( \frac{l(t)}{l(T)} - 4a(T) \dot{b}(T - t) \right) + 2\sigma^2(t) l(t) \dot{b}^2(T - t) + 1
\]

(70)

If \( t = 0 \), \( y(0) = 0 \), we have then:

\[
E_t^I[y^2(T)] = \exp[c(T)]
\]

\[
= \exp \left( \int_{\tau=0}^{T} \sigma^2(T - \tau) l(T - \tau) b(\tau) \, d\tau \right)
\]

(72)

Let \( \mathcal{f}(\tau) = l(T - \tau) b(\tau) \). Then:

\[
c(T) = \int_{\tau=0}^{T} \sigma^2(T - \tau) \mathcal{f}(\tau) \, d\tau
\]

(73)

\[
\frac{d\mathcal{f}}{d\tau} = -4a(T - \tau) \mathcal{f}(\tau) + 2\sigma^2(T - \tau) \mathcal{f}^2(\tau) + \frac{a^2(T - \tau)}{\sigma^2(T - \tau)}
\]

(74)

\[
\mathcal{f}(0) = 0
\]

(75)

Setting \( f(t) = \mathcal{f}(T - t) \), the lemma obtains.
6. References


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