A Heuristic Approach to Portfolio Optimization

Manfred Gilli and Evis Këllezi*

First draft: May 2000
This draft: October 2000

Comments are welcome

*M. Gilli: Department of Econometrics, 40 Bd du Pont d'Arve, University of Geneva, 1211 Geneva 4, Switzerland, email Manfred.Gilli@metri.unige.ch. E. Këllezi: Department of Econometrics and FAME, University of Geneva, 40 Bd du Pont d'Arve, 1211 Geneva 4, Switzerland, email Evis.Kellezi@metri.unige.ch. This paper benefited significantly from conversations with Peter Winker; we would like to thank him for valuable suggestions and comments. Financial support from the Swiss National Science Foundation (project 12–5248.97) is gratefully acknowledged.

Abstract

Constraints on downside risk, measured by shortfall probability, expected shortfall, semi-variance etc., lead to optimal asset allocations which differ from the mean-variance optimum. The resulting optimization problem can become quite complex as it exhibits multiple local extrema and discontinuities, in particular if we also introduce constraints restricting the trading variables to integers, constraints on the holding size of assets or on the maximum number of different assets in the portfolio. In such situations classical optimization methods fail to work efficiently and heuristic optimization techniques can be the only way out. The paper shows how a particular optimization heuristic, called threshold accepting, can be successfully used to solve complex portfolio choice problems.

JEL codes: G11, C61, C63.

Keywords: Portfolio Optimization, Downside Risk Measures, Heuristic Optimization, Threshold Accepting.

Executive Summary

Mean-variance optimization is certainly the most popular approach to portfolio choice. In this framework, the investor is faced with a trade-off between the profitability of his portfolio — characterized by the expected return — and the risk, measured by the variance of the portfolio returns. Notwithstanding its popularity, this approach has been subject to a lot of criticism. Alternative approaches attempt to conform the fundamental assumptions to reality by dismissing the normality hypothesis in order to account for the fat-tailedness and the asymmetry of the asset returns.

Consequently, other measures of risk, such as Value at Risk (VaR), expected shortfall, mean absolute deviation, semi-variance and so on are employed which leads to problems that can not always be reduced to standard linear or quadratic programs. The resulting optimization problem often becomes quite complex as it exhibits multiple local extrema and discontinuities, in particular if we introduce constraints restricting the trading variables to integers, limits in the proportions held in a given asset, constraints on the maximum number of different assets in the portfolio, class constraints, etc.

In this paper, we illustrate how a heuristic optimization algorithm, called threshold accepting, can be successfully applied to solve realistic non-convex portfolio optimization problems arising in situations where we have to deal with downside risk and the constraints described above.

The working of the threshold accepting algorithm is first illustrated to solve a standard mean-variance optimization problem for which the solution is also computed with the quadratic programming algorithm which is used as a benchmark and thus provides some insight into the quality of the threshold accepting heuristic. Second the threshold accepting algorithm is used to solve a non-convex optimization problem where we maximize the future return for a given shortfall probability, i.e. we restrict the probability that the future portfolio value falls below a given VaR level.

From our results we conclude that the threshold accepting algorithm opens new perspectives in the practice of portfolio management as it allows to deal easily with all sort of constraints of practical importance, it provides useful approximations of the optimal solutions, it appears to be computationally efficient and is relatively easy to implement. We also observed that the algorithm is robust to changes in problem characteristics.

1 Introduction

The fundamental goal of an investor is to optimally allocate his investments among different assets. The pioneering work of (Markowitz, 1952) introduced mean-variance optimization as a quantitative tool which carries out this allocation by considering the trade-off between risk (measured by the variance of the future asset returns) and return. Assuming the normality of the returns and quadratic investor's preferences allow the simplification of the problem in a relatively easy to solve quadratic program.

Notwithstanding its popularity, this approach has also been subject to a lot of criticism. Alternative approaches attempt to conform the fundamental assumptions to reality by dismissing the normality hypothesis in order to account for the fat-tailedness and the asymmetry of the asset returns. Consequently, other measures of risk, such as Value at Risk (VaR), expected shortfall, mean absolute deviation, semi-variance and so on are used, leading to problems that cannot always be reduced to standard linear or quadratic programs. The resulting optimization problem often becomes quite complex as it exhibits multiple local extrema and discontinuities, in particular if we introduce constraints restricting the trading variables to integers, constraints on the holding size of assets, on the maximum number of different assets in the portfolio, etc.

In such situations, classical optimization methods do not work efficiently and heuristic optimization techniques can be the only way out. They are relatively easy to implement and computationally attractive. The use of heuristic optimization techniques to portfolio selection has already been suggested by (Mansini and Speranza, 1999), (Chang *et al.*, 2000) and (Speranza, 1996). This paper builds on work by (Dueck and Winker, 1992) who first applied a heuristic optimization technique, called Threshold Accepting, to portfolio choice problems. We show how this technique can be successfully employed to solve complex portfolio choice problems where risk is characterized by Value at Risk and Expected Shortfall.

In Section 2, we outline the different frameworks for portfolio choice as well as the most frequently used risk measures. Section 3 gives a general representation of the threshold accepting heuristic we use. The performance and efficiency of the algorithm is discussed in Section 4 by, first, comparing it with the quadratic programming solutions in the mean-variance framework and, second, applying the algorithm to problems minimizing the portfolio expected shortfall or VaR conditional to some return constraints. Section 5 concludes.

2 Approaches to the portfolio choice problem

2.1 The mean-variance approach

Mean-variance optimization is certainly the most popular approach to portfolio choice. In this framework, the investor is faced with a trade-off between the profitability of his portfolio, characterized by the expected return, and the risk, measured by the variance of the portfolio returns. The first two moments of the portfolio future return are sufficient to define a complete ordering of the investors preferences. This strong result is due to the simplistic hypothesis that the investors' preferences are quadratic and the returns are normally distributed.

Denoting by x_i, $i = 1, \ldots, n_A$, the amount invested in asset i out of an initial capital v^0 and by r_i, $i = 1, \ldots, n_A$, the log-returns for each asset over the planning period, then the expected return on the portfolio defined by the vector $x = (x_1, x_2, \ldots, x_{n_A})'$ is given as

$$\mu(x) = \frac{1}{v^0} \sum_{i=1}^{n_A} E(r_i)\, x_i = \frac{1}{v^0}\, x' E(r)\,.$$

The variance of the portfolio return is

$$\sigma^2(x) = x'\, Q\, x\,,$$

where Q is the matrix of variances and covariances of the vector of returns r.

Thus the mean-variance efficient portfolios, defined as having the highest expected return for a given variance and the minimum variance for a given expected return, are obtained by solving the following quadratic program

$$\begin{array}{rcll} & \min\limits_x & \frac{1}{2}\, x'\, Q\, x & \\ \sum_j x_j r_j & \geq & \rho\, v^0 & \\ \sum_j x_j & = & v^0 & \\ x_j^\ell \leq x_j \leq x_j^u & & & j = 1, \ldots, n_A\,. \end{array} \tag{1}$$

for different values of ρ, where ρ is the required return on the portfolio. The vectors x_j^ℓ, x_j^u, $j = 1, \ldots, n_A$ represent constraints on the minimum and maximum holding size of the individual assets.

The implementation of the Markowitz model with n_A assets requires n_A estimates of expected returns, n_A estimates of variances and $n_A(n_A - 1)/2$ correlation coefficients.

Several efficient algorithms exist to compute the mean-variance portfolios. Early successful parametric quadratic programming methods include the critical-line algorithm and the simplex method.

2.2 Scenario generation

An alternative approach to the above optimization setting is the scenario analysis where uncertainty about future returns is modeled through a set of possible realizations, called scenarios. Scenarios of future outcomes can be generated relying on a model, past returns or experts' opinions.

A simple approach is to use empirical distributions computed from past returns as equiprobable scenarios. Observations of returns over n_S overlapping periods of length Δt are considered as the n_S possible outcomes (or scenarios) of the future returns and a probability of $1/n_S$ is assigned to each of them.

Assume that we have T historical prices p^h, $h = 1, \ldots, T$ of the assets under consideration. For each point in time, we can compute the realized return vector over the previous period of length Δt, which will further be considered as one of the n_S scenarios for the future returns on the assets. Thus, for example, a scenario r_j^s for the return on asset j is obtained as

$$r_j^s = \log(p_j^{t+\Delta t}/p_j^t)\,. \tag{2}$$

For each asset, we obtain as many scenarios as there are overlapping periods of length Δt, i.e. n_S. In this setting, problem (1) becomes

$$\begin{aligned}
&\min_x \; \frac{1}{n_S}\sum_{s=1}^{n_S}\left(\sum_{j=1}^{n_A} r_j^s x_j - \frac{1}{n_S}\sum_{j=1}^{n_A}\sum_{s=1}^{n_S} r_j^s x_j\right)^2 \\
&\frac{1}{n_S}\sum_{j=1}^{n_A}\sum_{s=1}^{n_S} r_j^s x_j \;\geq\; \rho\, v^0 \\
&\textstyle\sum_j x_j \;=\; v^0 \\
&x_j^\ell \leq x_j \leq x_j^u \qquad j = 1, \ldots, n_A\,.
\end{aligned} \tag{3}$$

2.3 Mean downside-risk framework

If we denote by v the future portfolio value, i.e. the value of the portfolio by the end of the planning period, then the probability

$$P(v < \mathtt{VaR}) \tag{4}$$

that the portfolio value falls below the $\mathtt{VaR}$ level, is called the *shortfall probability*. The conditional mean value of the portfolio given that the portfolio value has fallen below $\mathtt{VaR}$, called the *expected shortfall*, is defined as

$$E(v \mid v < \mathtt{VaR})\,. \tag{5}$$

Other risk measures used in practice are the *mean absolute deviation*

$$E(|v - Ev|)$$

and the *semi-variance*

$$E((v - Ev)^2 \mid v < Ev)$$

where we consider only the negative deviations from the mean.

Maximizing the expected value of the portfolio for a certain level of risk characterized by one of the measures defined above leads to alternative ways of describing the investor's problem (e.g. (Leibowitz and Kogelman, 1991), (Lucas and Klaassen, 1998) and (Palmquist, Uryasev and Krokhmal, 1999)). Earlier related work had suggested a safety-first approach (see e.g. (Arzac and Bawa, 1977) and (Roy, 1952)).

For example, if the risk profile of the investor is determined in terms of VaR, a mean-VaR efficient portfolio would be the solution of the following optimization problem:

$$\begin{array}{rcll} \max\limits_{x} & Ev & & \\ P(v < \mathtt{VaR}) & \leq & \beta & \\ \sum_j x_j & = & v^0 & \\ x_j^{\ell} \leq x_j \leq x_j^u & & & j = 1, \ldots, n_A\,. \end{array} \tag{6}$$

In other words, such an investor is trying to maximize the future value of his portfolio, which requires the probability that the future value of his portfolio falls below VaR not to be greater than β.

If the uncertainty in the future asset returns is handled via scenario generation, the above optimization can be further explicited as follows:

$$
\begin{array}{rcll}
\min\limits_{x} & -\dfrac{1}{n_S}\displaystyle\sum_{s=1}^{n_S} v^s & & \\
\#\{s \mid v^s < \texttt{VaR}\} & \leq & \beta\, n_S & \\
\sum_j x_j & = & v^0 & \\
x_j^\ell \leq x_j \leq x_j^u & & & j = 1, \ldots, n_A\,.
\end{array}
\tag{7}
$$

Furthermore, it would be realistic to consider an investor who cares not only for the shortfall probability, but also for the extent to which his portfolio value can fall below the VaR level. In this case, the investor's risk profile is defined via a constraint on the expected shortfall tolerated ν if the portfolio value falls below VaR. Then the mean-expected shortfall efficient portfolios are solutions of the following program for different values of ν:

$$
\begin{array}{rcll}
\max\limits_{x} & Ev & & \\
E[v \mid v < \texttt{VaR}] & \geq & \nu & \\
\sum_j x_j & = & v^0 & \\
x_j^\ell \leq x_j \leq x_j^u & & & j = 1, \ldots, n_A\,.
\end{array}
\tag{8}
$$

Again if the future returns are generated by scenarios, the optimization problem becomes:

$$
\begin{array}{rcll}
\min\limits_{x} & -\dfrac{1}{n_S}\displaystyle\sum_{s=1}^{n_S} v^s & & \\
\frac{1}{\#\{s|v^s<\texttt{VaR}\}} \displaystyle\sum_{s|v^s<\texttt{VaR}} v^s & \geq & \nu & \\
\#\{s \mid v^s < \texttt{VaR}\} & \leq & \beta\, n_S & \\
\sum_j x_j & = & v^0 & \\
x_j^\ell \leq x_j \leq x_j^u & & & j = 1, \ldots, n_A\,.
\end{array}
\tag{9}
$$

3 The threshold accepting optimization heuristic

Heuristic approaches prove useful in situations where the classical optimization methods fail to work efficiently. Heuristic optimization techniques like simulated annealing (Kirkpatrick *et al.*, 1983) and genetic algorithms (Holland, 1975) are used with increasing success in a variety of disciplines. The reason for their success is that they are relatively easy to implement and that the cost of computing power is no longer a matter of concern.

Threshold accepting (TA) was introduced by (Dueck and Scheuer, 1990) as a deterministic analog to simulated annealing. It is a refined local search procedure which escapes local minima by accepting solutions which are not worse by more than a given threshold. The algorithm is deterministic in the sense that we fix a number of iterations and explore the neighborhood with a fixed number of steps during each iteration. The threshold is decreased successively and reaches the value of zero in the last round.

The threshold accepting algorithm has the advantage of an easy parameterization, it is robust to changes in problem characteristics and works well for many problem instances. An extensive introduction to threshold accepting is given in (Winker, 2000).

Let us formalize our optimization problem as $f : \mathcal{X} \to \mathbb{R}$ where $\mathcal{X}$ is a discrete set and where we may have more then one optimal solution defined by the set

$$\mathcal{X}_{\text{min}} = \{x \in \mathcal{X} \mid f(x) = f_{\text{opt}}\} \tag{10}$$

with

$$f_{\text{opt}} = \min_{x \in \mathcal{X}} f(x)\,. \tag{11}$$

The threshold accepting heuristic described in algorithm 1 will, after completion, provide us with a solution $x \in \mathcal{X}_{\text{min}}$ or a solution close to an element in $\mathcal{X}_{\text{min}}$. The complexity of the algorithm is $\mathcal{O}(niter \times steps)$.

Algorithm 1 Pseudo-code for the threshold accepting algorithm.

Initialize $niter$ and $steps$
Initialize sequence of thresholds th_r, $r = 1, 2, \ldots, niter$
Generate starting point $x^0 \in \mathcal{X}$
for $r = 1$ to $niter$ **do**
    for $i = 1$ to $steps$ **do**
        Generate $x^1 \in \mathcal{N}_{x^0}$ (neighbor of x^0)
        if $f(x^1) < f(x^0) + th_r$ **then**
            $x^0 = x^1$
        end if
    end for
end for

The parameters of the algorithm are the number of iterations $niter$, the number of steps per iteration $steps$ and the sequence of thresholds th. In practice, we start with the definition of the objective function, which can be a non-trivial task if f comprises several dimensions. Second, we construct a mapping $\mathcal{N} : \mathcal{X} \rightarrow 2^{\mathcal{X}}$ which defines for each $x \in \mathcal{X}$ a neighborhood $\mathcal{N}(x) \subset \mathcal{X}$. Third, we define the sequence of thresholds by exploring the neighborhood of randomly selected elements $x \in \mathcal{X}$.

These different steps of the implementation and parameterization of the algorithm will be illustrated with the application presented in the following section.

4 Application

The working of the TA algorithm is first illustrated to solve a standard mean-variance optimization problem for which the solution is also computed with the quadratic programming algorithm which will be used as a benchmark. Second we apply the TA algorithm to a non-convex optimization problem with integer variables and a variety of constraints such as holding and trading size.

4.1 Mean-variance optimization

In the following application we consider an investment opportunity set of ten assets from the Swiss Market Index (SMI) and cash. The annual mean return r and the matrix of variances and covariances Q are based on the closing prices of the last 90 trading days before June 30, 1999.

The mean-variance optimization problem has already been defined in (1). The following is a reformulation of the problem where the initial capital v^0 has been normalized to one:

$$\begin{array}{rcll} \min\limits_{\omega} & \frac{1}{2}\,\omega'\,Q\,\omega & & \\ \omega' r & \geq & \rho & \\ \iota'\omega & = & 1 & \\ \omega_j^l \leq \omega_j \leq \omega_j^u & & & j = 1, \ldots, n_A + 1\,. \end{array}$$

The composition of the portfolio is defined by the shares $\omega_i = x_i / v^0$ and ω_{n_A+1} is the proportion of cash in the portfolio. The risk-free return of cash is r_{n_A+1}.

Definition of objective function

The variance can now be minimized by exploring with the threshold accepting algorithm 1 the elements in the set $\mathcal{X}$ which satisfy the constraints. However, a better way is to accept solutions which violate the return constraint in the search process. This can be done by minimizing the following objective function

$$F(\omega) = V(\omega) + p\,(\rho - R(\omega))$$

where p is a penalty function defined as

$$p = \begin{cases} \frac{V_{\max} - V_{\min}}{\rho - \overline{R}} & \text{if} \quad \rho > R(\omega) \\ 0 & \text{otherwise.} \end{cases}$$

$V(\omega)$ and $R(\omega)$ denote respectively the variance and the return of a portfolio defined by ω. The values for $V_{\max}$, $V_{\min}$ and $\overline{R}$ which define the scaling constant $(V_{\max} - V_{\min})/(\rho - \overline{R})$ are estimated from 1000 randomly drawn portfolios.

Definition of neighborhood

To generate a point x^1 in the neighborhood $\mathcal{N}_{x^0}$ of a given point x^0 we draw with a probability $1/(n_A + 1)$ two assets i and j out of all n_A assets and cash. The amount of i and j in the portfolio is ω_i, respectively ω_j. We then sell a fraction q of asset i, i.e. $q\,\omega_i$ and buy for the corresponding amount asset j. After this move the amount of i and j in the portfolio is $(1-q)\omega_i$, respectively $\omega_j + q\,\omega_i$. The fraction q is a fixed parameter.

Algorithm 2 Definition of neighborhood.

```
Select two assets i and j with probability 1/(n_A + 1)
t = q ω_i
if (ω_i − t) ≥ ω_i^l then
    ω_i = ω_i − t
else
    t = ω_i
    ω_i = 0
end if
if (ω_j + t) ≤ ω_j^u then
    ω_j = ω_j + t
else
    ω_{nA+1} = ω_{nA+1} + t
end if
```

In order to avoid short selling and to respect the constraints on the holding size of the assets, the procedure for the selection of a neighbor solution must be refined. Algorithm 2 describes the procedure of the selection of a neighbor-solution in detail.

Definition of thresholds

In order to define the sequence of thresholds, we compute the empirical distribution of the distance of the objective function evaluated at random points and its neighbors. Figure 1 shows this empirical distribution computed from 5000 random points. In this case the computed quantiles which determine the sequence of thresholds are $10^{-3}\left[\begin{array}{cccccc} 22.5 & 3.4 & 1.1 & 0.7 & 0.4 & 0 \end{array}\right]$.

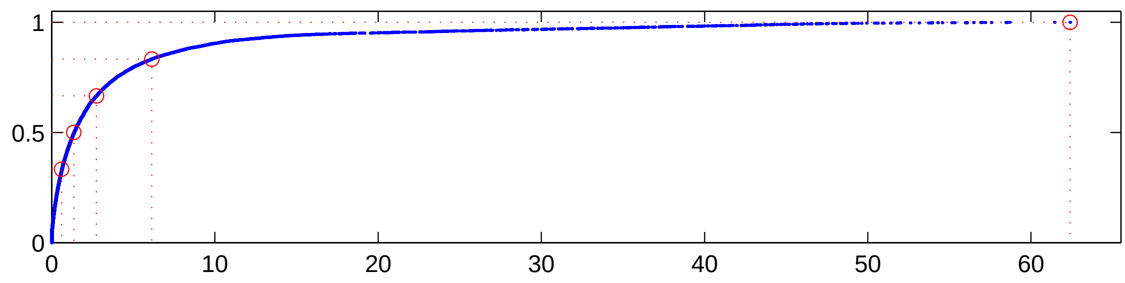

Figure 1: Empirical distribution of distance between x^0 and neighbors x^1.

Choosing $niter = 6$ and $steps = 1000$ we have determined all the parameters of our TA algorithm. Figure 2 illustrates how the algorithm searches its way to the solution. At the optimal solution the expected return and the variance are practically the same for the QP and TA algorithms. The optimal portfolio contains asset 3, 5 and 8 and cash (column 11). The weights of the assets in the optimal portfolio for both algorithms are given in Figure 3.

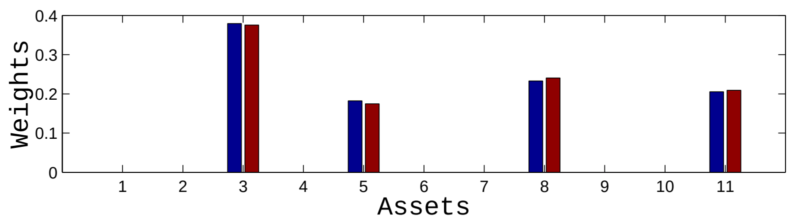

Figure 3: Composition of the optimal portfolio for QP (left bars) and TA (right bars).

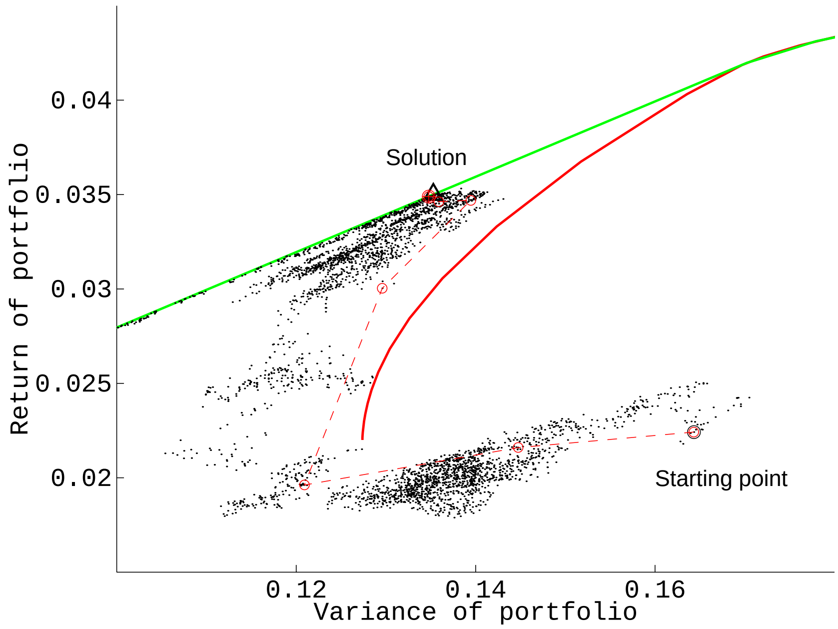

Figure 2: Working of the TA algorithm. Efficient frontier with cash (upper line) and without cash (lower curve).

4.2 Mean downside risk optimization

Our second illustration of the working of the TA algorithm is a non-convex optimization problem with integer variables and a variety of constraints such as holding and trading size.

In the following, the quantity of each asset in the portfolio is defined by an integer number. The generation of neighbors $x^1 \in \mathcal{N}_x^0$ to a given solution x^0 is again performed by drawing randomly two assets i and j. We then sell k_i assets i, transfer the amount to the cash and buy k_j assets j from cash. In order to make sure that each transfer is approximatively the same amount, the number of assets k_i and k_j to be transferred are defined as $k_i = \lceil \frac{p_{\max}}{p_i} \rceil$ and $k_j = \lceil \frac{p_{\max}}{p_j} \rceil$. This procedure is summarized in algorithm 3 where we omitted the details necessary to check for short selling and holding constraints.

Algorithm 3 Definition of neighborhood in case of integer variables.

1: Randomly select asset i to sell
2: $x_i = x_i - k_i$
3: cash = cash + $k_i \, p_i^0$
4: Randomly select asset j to buy
5: $x_j = x_j + k_j$
6: cash = cash − $k_j \, p_j^0$

Using the same data set as for the previous problem but considering an investment opportunity set of 20 assets (including cash) we now solve the mean-VaR problem defined in (7). To compute the capital v^s at the end of the planning period we use simulated prices p^s, computed as

$$p^s = p^0 \, r^s \qquad s = 1, \ldots, n_S$$

where the rate of return r^s has been defined in (2). We assume an initial capital of $v^0 = 800\,000$ and seek the portfolio which maximises the expected return given the following constraints: shortfall probability $\beta = 0.05$ for a VaR level of $750\,000$, minimum and maximum holding size for a particular asset $0.01\, v^0$ respectively $0.30\, v^0$ and a maximum of 9 assets in the portfolio. Figure 4 shows the results of the TA algorithm with the setting $niter = 6$ and $steps = 1500$.

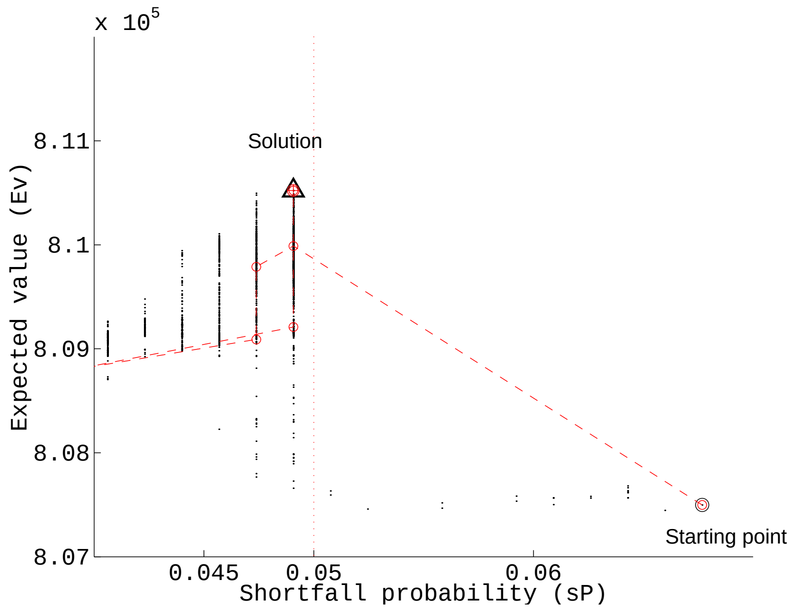

Figure 4: Search path of the TA algorithm in the β, $E(v)$ plane.

The composition of the portfolio, the TA algorithm found to be optimal, is given in figure 5. It contains the assets $\{2, 3, 6, 8, 10, 11, 14\}$ and cash. The minimum position is $28\,680$ (asset 3) and the maximum position is $212\,936$ (asset 14). Thus the constraints on the holding size and the number of assets in the portfolio are satisfied.

This optimal portfolio has an expected return of $E(v) = 810\,520$ with a shortfall probability of 0.049 for a VaR level of $749\,950$, which again satisfies the constraints.

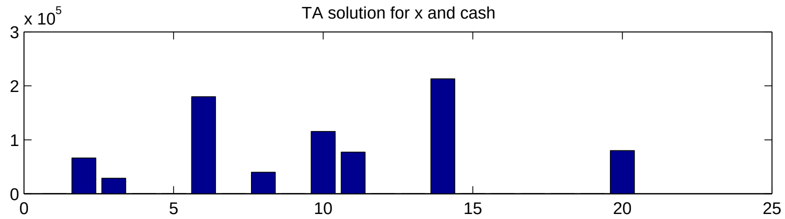

Figure 5: Optimal portfolio computed by TA for the mean-VaR problem.

In figure 4, we observe that the solutions lie in planes. The reason for this is the integer formulation of the problem. Figure 6 illustrates the working of the TA algorithm in the β, VaR plane and in figure 7, we see its working in the $E(v)$, VaR plane.

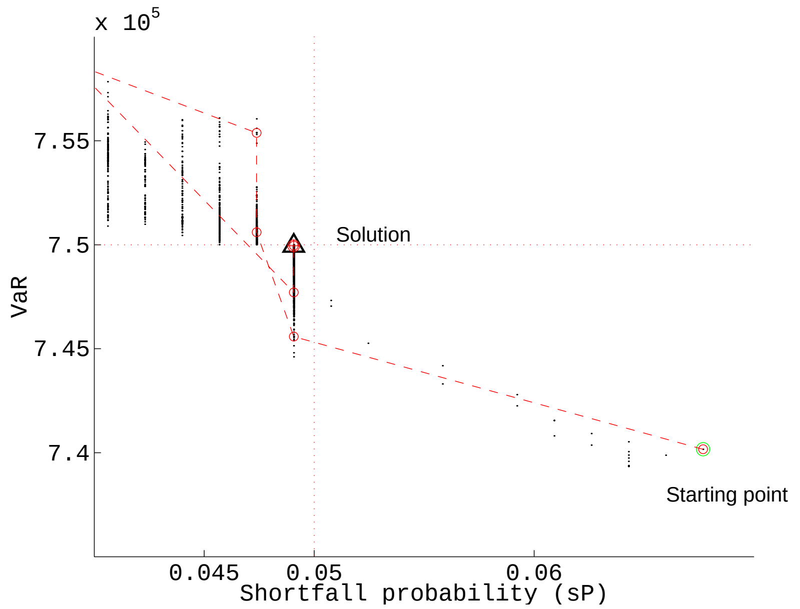

Figure 6: Search path of the TA algorithm in the β, VaR plane.

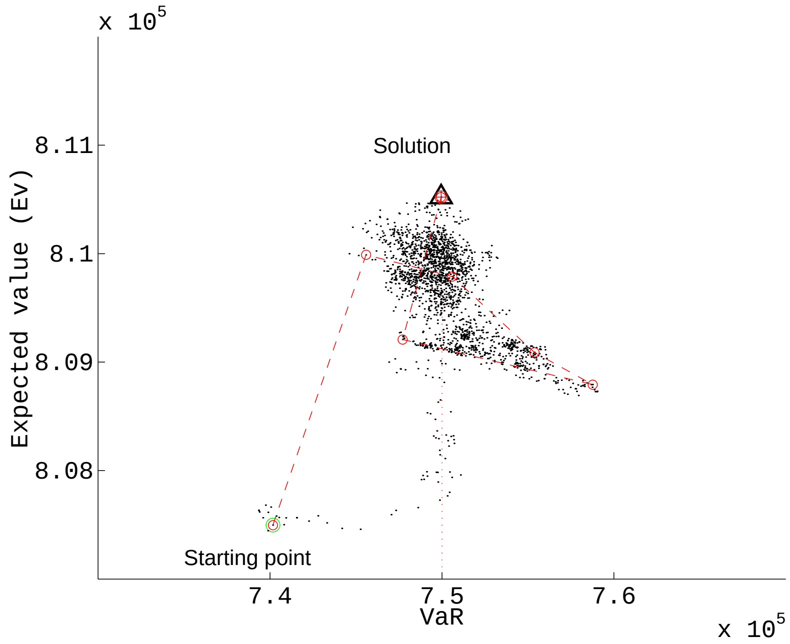

Figure 7: Search path of the TA algorithm in the $E(v)$, VaR plane.

5 Concluding remarks

In this paper, we attempted to illustrate how heuristic optimization algorithms like the threshold accepting method can be successfully applied to solve realistic non-convex portfolio optimization problems. We showed that, in the cases where these problems contain non-linear and non-convex constraints, the heuristic methods are the only reasonable way out. Examples of these situations can be problems where constraints on downside risk preferences are introduced, where the solutions are required to be integers, etc.

We mainly focus on the cases where the distribution of the asset future returns are modelled by equally weighted scenarios of past returns. The sensitivity of optimized portfolios with respect to alternative scenario generations procedures should be further investigated.

References

Arzac, V.S. and V.S. Bawa, 1977. Portfolio Choice and Equilibrium in Capital Markets with Safety-First Investors. *Journal of Financial Economics* 4, 277–288.

Chang, T.-J., N. Meade, J.E. Beasley and Y.M. Sharaiha, 2000. Heuristics for cardinality constrained portfolio optimization. *Computers & Operations Research* 27, 1271–1302.

Dembo, R.S. and D. Rosen, 1999. The Practice of Portfolio Replication: A Practical Overview of Forward and Inverse Problems. *Annals of Operations Research* 85, 267–284.

Dueck, G. and T. Scheuer, 1990. Threshold accepting: A general purpose algorithm appearing superior to simulated annealing. *Journal of Computational Physics* 90, 161–175.

Dueck, G. and P. Winker, 1992. New Concepts and Algorithms for Portfolio Choice. *Applied Stochastic Models and Data Analysis* 8, 159–178.

Holland, J.H., 1975. *Adaptation in Natural and Artificial Systems*. University of Michigan Press.

Kirkpatrick, S., C. Gelatt and M. Vecchi, 1983. Optimization by simulated annealing. *Science* 220, 671–680.

Koskosides, Y. and A. Duarte, 1997. A scenario-based approach for active asset allocation. *Journal of Portfolio Management*, 74–85.

Leibowitz, M.L. and S. Kogelman, 1991. Asset Allocation under Shortfall Constraints. *Journal of Portfolio Management* Winter, 18-23.

Lucas, A. and P. Klaassen, 1998. Extreme Returns, Downside Risk, and Optimal Asset Allocation. *Journal of Portfolio Management* 25, 71–79.

Mansini, R. and M.G. Speranza, 1996. Heuristic algorithms for the portfolio selection problem with minimum transaction lots. *European Journal of Operational Research* 114(2), 219–233.

Markowitz, H., 1952. Portfolio Selection. *Journal of Finance* 7, 77–91.

Palmquist, J., S. Uryasev and P. Krokhmal, 1999. Portfolio Optimization with Conditional Value-at-Risk Objective and Constraints. Research Report #99–14, Center for Applied Optimization, University of Florida.

Roy, A.D., 1952. Safety-First and the Holding of Assets. *Econometrica* 20, 431–449.

Speranza, M.G., 1996. A heuristic algorithm for portfolio optimization model applied to the Milan stock market. *Computers & Operations Research* 23, 433–441.

Winker, P., 2000. *Optimization Heuristics in Econometrics: Applications of Threshold Accepting*. Wiley, New York.