A Heuristic Approach to Portfolio Optimization

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Comments are welcome

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Abstract

Constraints on downside risk, measured by shortfall probability, expected shortfall, semi-variance etc., lead to optimal asset allocations which differ from the mean-variance optimum. The resulting optimization problem can become quite complex as it exhibits multiple local extrema and discontinuities, in particular if we also introduce constraints restricting the trading variables to integers, constraints on the holding size of assets or on the maximum number of different assets in the portfolio. In such situations classical optimization methods fail to work efficiently and heuristic optimization techniques can be the only way out. The paper shows how a particular optimization heuristic, called threshold accepting, can be successfully used to solve complex portfolio choice problems.

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Executive Summary

Mean-variance optimization is certainly the most popular approach to portfolio choice. In this framework, the investor is faced with a trade-off between the profitability of his portfolio — characterized by the expected return — and the risk, measured by the variance of the portfolio returns. Notwithstanding its popularity, this approach has been subject to a lot of criticism. Alternative approaches attempt to conform the fundamental assumptions to reality by dismissing the normality hypothesis in order to account for the fat-tailedness and the asymmetry of the asset returns.

Consequently, other measures of risk, such as Value at Risk (VaR), expected shortfall, mean absolute deviation, semi-variance and so on are employed which leads to problems that can not always be reduced to standard linear or quadratic programs. The resulting optimization problem often becomes quite complex as it exhibits multiple local extrema and discontinuities, in particular if we introduce constraints restricting the trading variables to integers, limits in the proportions held in a given asset, constraints on the maximum number of different assets in the portfolio, class constraints, etc.

In this paper, we illustrate how a heuristic optimization algorithm, called threshold accepting, can be successfully applied to solve realistic non-convex portfolio optimization problems arising in situations where we have to deal with downside risk and the constraints described above.

The working of the threshold accepting algorithm is first illustrated to solve a standard mean-variance optimization problem for which the solution is also computed with the quadratic programming algorithm which is used as a benchmark and thus provides some insight into the quality of the threshold accepting heuristic. Second the threshold accepting algorithm is used to solve a non-convex optimization problem where we maximize the future return for a given shortfall probability, i.e. we restrict the probability that the future portfolio value falls below a given VaR level.

From our results we conclude that the threshold accepting algorithm opens new perspectives in the practice of portfolio management as it allows to deal easily with all sort of constraints of practical importance, it provides useful approximations of the optimal solutions, it appears to be computationally efficient and is relatively easy to implement. We also observed that the algorithm is robust to changes in problem characteristics.
1 Introduction

The fundamental goal of an investor is to optimally allocate his investments among different assets. The pioneering work of (Markowitz, 1952) introduced mean-variance optimization as a quantitative tool which carries out this allocation by considering the trade-off between risk (measured by the variance of the future asset returns) and return. Assuming the normality of the returns and quadratic investor’s preferences allow the simplification of the problem in a relatively easy to solve quadratic program.

Notwithstanding its popularity, this approach has also been subject to a lot of criticism. Alternative approaches attempt to conform the fundamental assumptions to reality by dismissing the normality hypothesis in order to account for the fat-tailedness and the asymmetry of the asset returns. Consequently, other measures of risk, such as Value at Risk (VaR), expected shortfall, mean absolute deviation, semi-variance and so on are used, leading to problems that cannot always be reduced to standard linear or quadratic programs. The resulting optimization problem often becomes quite complex as it exhibits multiple local extrema and discontinuities, in particular if we introduce constraints restricting the trading variables to integers, constraints on the holding size of assets, on the maximum number of different assets in the portfolio, etc.

In such situations, classical optimization methods do not work efficiently and heuristic optimization techniques can be the only way out. They are relatively easy to implement and computationally attractive. The use of heuristic optimization techniques to portfolio selection has already been suggested by (Mansini and Speranza, 1999), (Chang et al., 2000) and (Speranza, 1996). This paper builds on work by (Dueck and Winker, 1992) who first applied a heuristic optimization technique, called Threshold Accepting, to portfolio choice problems. We show how this technique can be successfully employed to solve complex portfolio choice problems where risk is characterized by Value at Risk and Expected Shortfall.

In Section 2, we outline the different frameworks for portfolio choice as well as the most frequently used risk measures. Section 3 gives a general representation of the threshold accepting heuristic we use. The performance and efficiency of the algorithm is discussed in Section 4 by, first, comparing it with the quadratic programming solutions in the mean-variance framework and, second, applying the algorithm to problems minimizing the portfolio expected shortfall or VaR conditional to some return constraints. Section 5 concludes.
2 Approaches to the portfolio choice problem

2.1 The mean-variance approach

Mean-variance optimization is certainly the most popular approach to portfolio choice. In this framework, the investor is faced with a trade-off between the profitability of his portfolio, characterized by the expected return, and the risk, measured by the variance of the portfolio returns. The first two moments of the portfolio future return are sufficient to define a complete ordering of the investors preferences. This strong result is due to the simplistic hypothesis that the investors’ preferences are quadratic and the returns are normally distributed.

Denoting by \( x_i, \ i = 1, \ldots, n_A \), the amount invested in asset \( i \) out of an initial capital \( v^0 \) and by \( r_i, \ i = 1, \ldots, n_A \), the log-returns for each asset over the planning period, then the expected return on the portfolio defined by the vector \( x = (x_1, x_2, \ldots, x_{n_A})' \) is given as

\[
\mu(x) = \frac{1}{v^0} \sum_{i=1}^{n_A} E(r_i) x_i = \frac{1}{v^0} x' E(r).
\]

The variance of the portfolio return is

\[
\sigma^2(x) = x' Q x,
\]

where \( Q \) is the matrix of variances and covariances of the vector of returns \( r \).

Thus the mean-variance efficient portfolios, defined as having the highest expected return for a given variance and the minimum variance for a given expected return, are obtained by solving the following quadratic program

\[
\min_x \frac{1}{2} x' Q x \quad \sum_{j} x_j r_j \geq \rho v^0 \quad \sum_{j} x_j = v^0 \quad x^L_j \leq x_j \leq x^U_j \quad j = 1, \ldots, n_A.
\]

for different values of \( \rho \), where \( \rho \) is the required return on the portfolio. The vectors \( x^L_j, x^U_j, j = 1, \ldots, n_A \) represent constraints on the minimum and maximum holding size of the individual assets.

The implementation of the Markowitz model with \( n_A \) assets requires \( n_A \) estimates of expected returns, \( n_A \) estimates of variances and \( n_A(n_A-1)/2 \) correlation coefficients.
Several efficient algorithms exist to compute the mean-variance portfolios. Early successful parametric quadratic programming methods include the critical-line algorithm and the simplex method.

### 2.2 Scenario generation

An alternative approach to the above optimization setting is the scenario analysis where uncertainty about future returns is modeled through a set of possible realizations, called scenarios. Scenarios of future outcomes can be generated relying on a model, past returns or experts’ opinions.

A simple approach is to use empirical distributions computed from past returns as equiprobable scenarios. Observations of returns over $n_S$ overlapping periods of length $\Delta t$ are considered as the $n_S$ possible outcomes (or scenarios) of the future returns and a probability of $1/n_S$ is assigned to each of them.

Assume that we have $T$ historical prices $p^h, h = 1, \ldots, T$ of the assets under consideration. For each point in time, we can compute the realized return vector over the previous period of length $\Delta t$, which will further be considered as one of the $n_S$ scenarios for the future returns on the assets. Thus, for example, a scenario $r^s_j$ for the return on asset $j$ is obtained as

$$r^s_j = \log(p^{t+\Delta t}_j/p^t_j).$$

For each asset, we obtain as many scenarios as there are overlapping periods of length $\Delta t$, i.e. $n_S$. In this setting, problem (1) becomes

$$\min_{x \geq 0} \frac{1}{n_S} \sum_{s=1}^{n_S} \left( \sum_{j=1}^{n_A} r_j^s x_j - \frac{1}{n_S} \sum_{j=1}^{n_A} \sum_{s=1}^{n_S} r_j^s x_j \right)^2$$

$$\frac{1}{n_S} \sum_{j=1}^{n_A} \sum_{s=1}^{n_S} r_j^s x_j \geq \rho v^0$$

$$\sum_{j} x_j = v^0$$

$$x_j^l \leq x_j \leq x_j^u \quad j = 1, \ldots, n_A.$$
2.3 Mean downside-risk framework

If we denote by $v$ the future portfolio value, i.e. the value of the portfolio by the end of the planning period, then the probability

$$P(v < \text{VaR})$$

(4)

that the portfolio value falls below the VaR level, is called the shortfall probability. The conditional mean value of the portfolio given that the portfolio value has fallen below VaR, called the expected shortfall, is defined as

$$E(v \mid v < \text{VaR}).$$

(5)

Other risk measures used in practice are the mean absolute deviation

$$E(|v - Ev|)$$

and the semi-variance

$$E((v - Ev)^2 \mid v < Ev)$$

where we consider only the negative deviations from the mean.

Maximizing the expected value of the portfolio for a certain level of risk characterized by one of the measures defined above leads to alternative ways of describing the investor’s problem (e.g. (Leibowitz and Kogelman, 1991), (Lucas and Klaassen, 1998) and (Palmquist, Uryasev and Krokhmal, 1999)). Earlier related work had suggested a safety-first approach (see e.g. (Arzac and Bawa, 1977) and (Roy, 1952)).

For example, if the risk profile of the investor is determined in terms of VaR, a mean-VaR efficient portfolio would be the solution of the following optimization problem:

$$\max_x Ev$$

$$P(v < \text{VaR}) \leq \beta$$

$$\sum_j x_j = v^0$$

$$x^L_j \leq x_j \leq x^U_j$$

$$(j = 1, \ldots, n_A).$$

(6)

In other words, such an investor is trying to maximize the future value of his portfolio, which requires the probability that the future value of his portfolio falls below VaR not to be greater than $\beta$. 
If the uncertainty in the future asset returns is handled via scenario generation, the above optimization can be further explicited as follows:

\[
\min_x -\frac{1}{n_S} \sum_{s=1}^{n_S} v^s \\
\#\{s \mid v^s < \text{VaR}\} \leq \beta n_S \\
\sum_j x_j = v^0 \\
x_j^l \leq x_j \leq x_j^u \\
j = 1, \ldots, n_A.
\] (7)

Furthermore, it would be realistic to consider an investor who cares not only for the shortfall probability, but also for the extent to which his portfolio value can fall below the VaR level. In this case, the investor’s risk profile is defined via a constraint on the expected shortfall tolerated \( \nu \) if the portfolio value falls below VaR. Then the mean-expected shortfall efficient portfolios are solutions of the following program for different values of \( \nu \):

\[
\max_x \quad E v \\
E[v \mid v < \text{VaR}] \geq \nu \\
\sum_j x_j = v^0 \\
x_j^l \leq x_j \leq x_j^u \\
\nu \quad j = 1, \ldots, n_A.
\] (8)

Again if the future returns are generated by scenarios, the optimization problem becomes:

\[
\min_x -\frac{1}{n_S} \sum_{s=1}^{n_S} v^s \\
\frac{1}{\#\{s \mid v^s < \text{VaR}\}} \sum_{s \mid v^s < \text{VaR}} v^s \geq \nu \\
\#\{s \mid v^s < \text{VaR}\} \leq \beta n_S \\
\sum_j x_j = v^0 \\
x_j^l \leq x_j \leq x_j^u \\
\nu \quad j = 1, \ldots, n_A.
\] (9)

3 The threshold accepting optimization heuristic

Heuristic approaches prove useful in situations where the classical optimization methods fail to work efficiently. Heuristic optimization techniques like simulated annealing (Kirkpatrick et al., 1983) and genetic algorithms (Holland, 1975) are used with
increasing success in a variety of disciplines. The reason for their success is that they are relatively easy to implement and that the cost of computing power is no longer a matter of concern.

Threshold accepting (TA) was introduced by (Dueck and Scheuer, 1990) as a deterministic analog to simulated annealing. It is a refined local search procedure which escapes local minima by accepting solutions which are not worse by more than a given threshold. The algorithm is deterministic in the sense that we fix a number of iterations and explore the neighborhood with a fixed number of steps during each iteration. The threshold is decreased successively and reaches the value of zero in the last round.

The threshold accepting algorithm has the advantage of an easy parameterization, it is robust to changes in problem characteristics and works well for many problem instances. An extensive introduction to threshold accepting is given in (Winker, 2000).

Let us formalize our optimization problem as

\[ f : \mathcal{X} \to \mathbb{R} \]

where \( \mathcal{X} \) is a discrete set and where we may have more than one optimal solution defined by the set

\[ \mathcal{X}_{\text{min}} = \{ x \in \mathcal{X} | f(x) = f_{\text{opt}} \} \]  \hspace{1cm} (10)

with

\[ f_{\text{opt}} = \min_{x \in \mathcal{X}} f(x). \]  \hspace{1cm} (11)

The threshold accepting heuristic described in algorithm 1 will, after completion, provide us with a solution \( x \in \mathcal{X}_{\text{min}} \) or a solution close to an element in \( \mathcal{X}_{\text{min}} \). The complexity of the algorithm is \( \mathcal{O}(n_{\text{iter}} \times \text{steps}) \).

\textbf{Algorithm 1} Pseudo-code for the threshold accepting algorithm.

1: Initialize \( n_{\text{iter}} \) and \( \text{steps} \)
2: Initialize sequence of thresholds \( t_{h_r}, r = 1, 2, \ldots, n_{\text{iter}} \)
3: Generate starting point \( x^0 \in \mathcal{X} \)
4: \textbf{for} \( r = 1 \) to \( n_{\text{iter}} \) \textbf{do}
5: \hspace{0.5cm} \textbf{for} \( i = 1 \) to \( \text{steps} \) \textbf{do}
6: \hspace{1cm} Generate \( x^1 \in \mathcal{N}_{x^0} \) (neighbor of \( x^0 \))
7: \hspace{1cm} \textbf{if} \( f(x^1) < f(x^0) + t_{h_r} \) \textbf{then}
8: \hspace{1.5cm} \( x^0 = x^1 \)
9: \hspace{1cm} \textbf{end if}
10: \hspace{0.5cm} \textbf{end for}
11: \hspace{0.5cm} \textbf{end for}

The parameters of the algorithm are the number of iterations \( n_{\text{iter}} \), the number of steps per iteration \( \text{steps} \) and the sequence of thresholds \( t_{h} \). In practice, we start with
the definition of the objective function, which can be a non-trivial task if $f$ comprises several dimensions. Second, we construct a mapping $\mathcal{N}: \mathcal{X} \rightarrow 2^{\mathcal{X}}$ which defines for each $x \in \mathcal{X}$ a neighborhood $\mathcal{N}(x) \subseteq \mathcal{X}$. Third, we define the sequence of thresholds by exploring the neighborhood of randomly selected elements $x \in \mathcal{X}$.

These different steps of the implementation and parameterization of the algorithm will be illustrated with the application presented in the following section.

4 Application

The working of the TA algorithm is first illustrated to solve a standard mean-variance optimization problem for which the solution is also computed with the quadratic programming algorithm which will be used as a benchmark. Second we apply the TA algorithm to a non-convex optimization problem with integer variables and a variety of constraints such as holding and trading size.

4.1 Mean-variance optimization

In the following application we consider an investment opportunity set of ten assets from the Swiss Market Index (SMI) and cash. The annual mean return $r$ and the matrix of variances and covariances $Q$ are based on the closing prices of the last 90 trading days before June 30, 1999.

The mean-variance optimization problem has already been defined in (1). The following is a reformulation of the problem where the initial capital $v^0$ has been normalized to one:

$$\min_{\omega} \frac{1}{2} \omega' Q \omega$$
$$\omega' r \geq \rho$$
$$\omega' \omega = 1$$
$$\omega_j^l \leq \omega_j \leq \omega_j^u \quad j = 1, \ldots, n_A + 1.$$

The composition of the portfolio is defined by the shares $\omega_i = x_i/v^0$ and $\omega_{n_A+1}$ is the proportion of cash in the portfolio. The risk-free return of cash is $r_{n_A+1}$. 
Definition of objective function

The variance can now be minimized by exploring with the threshold accepting algorithm 1 the elements in the set $\mathcal{X}$ which satisfy the constraints. However, a better way is to accept solutions which violate the return constraint in the search process. This can be done by minimizing the following objective function

$$F(\omega) = V(\omega) + p (\rho - R(\omega))$$

where $p$ is a penalty function defined as

$$p = \begin{cases} \frac{V_{\text{max}} - V_{\text{min}}}{\rho - \overline{R}} & \text{if } \rho > R(\omega) \\ 0 & \text{otherwise.} \end{cases}$$

$V(\omega)$ and $R(\omega)$ denote respectively the variance and the return of a portfolio defined by $\omega$. The values for $V_{\text{max}}$, $V_{\text{min}}$ and $\overline{R}$ which define the scaling constant $(V_{\text{max}} - V_{\text{min}})/(\rho - \overline{R})$ are estimated from 1000 randomly drawn portfolios.

Definition of neighborhood

To generate a point $x^1$ in the neighborhood $\mathcal{N}_{x^0}$ of a given point $x^0$ we draw with a probability $1/(n_A + 1)$ two assets $i$ and $j$ out of all $n_A$ assets and cash. The amount of $i$ and $j$ in the portfolio is $\omega_i$, respectively $\omega_j$. We then sell a fraction $q$ of asset $i$, i.e. $q \omega_i$ and buy for the corresponding amount asset $j$. After this move the amount of $i$ and $j$ in the portfolio is $(1 - q)\omega_i$, respectively $\omega_j + q \omega_i$. The fraction $q$ is a fixed parameter.

**Algorithm 2** Definition of neighborhood.

1: Select two assets $i$ and $j$ with probability $1/(n_A + 1)$
2: $t = q \omega_i$
3: if $(\omega_i - t) \geq \omega_i^t$ then
4: $\omega_i = \omega_i - t$
5: else
6: $t = \omega_i$
7: $\omega_i = 0$
8: end if
9: if $(\omega_j + t) \leq \omega_j^u$ then
10: $\omega_j = \omega_j + t$
11: else
12: $\omega_{n_A+1} = \omega_{n_A+1} + t$
13: end if
In order to avoid short selling and to respect the constraints on the holding size of the assets, the procedure for the selection of a neighbor solution must be refined. Algorithm 2 describes the procedure of the selection of a neighbor-solution in detail.

**Definition of thresholds**

In order to define the sequence of thresholds, we compute the empirical distribution of the distance of the objective function evaluated at random points and its neighbors. Figure 1 shows this empirical distribution computed from 5000 random points. In this case the computed quantiles which determine the sequence of thresholds are $10^{-3} \begin{bmatrix} 22.5 & 3.4 & 1.1 & 0.7 & 0.4 & 0 \end{bmatrix}$.

![Figure 1: Empirical distribution of distance between $x^0$ and neighbors $x^1$.](image)

Choosing $n_{iter} = 6$ and $steps = 1000$ we have determined all the parameters of our TA algorithm. Figure 2 illustrates how the algorithm searches its way to the solution. At the optimal solution the expected return and the variance are practically the same for the QP and TA algorithms. The optimal portfolio contains asset 3, 5 and 8 and cash (column 11). The weights of the assets in the optimal portfolio for both algorithms are given in Figure 3.

![Figure 3: Composition of the optimal portfolio for QP (left bars) and TA (right bars).](image)
4.2 Mean downside risk optimization

Our second illustration of the working of the TA algorithm is a non-convex optimization problem with integer variables and a variety of constraints such as holding and trading size.

In the following, the quantity of each asset in the portfolio is defined by an integer number. The generation of neighbors $x^1 \in \mathcal{N}_x^{0}$ to a given solution $x^0$ is again performed by drawing randomly two assets $i$ and $j$. We then sell $k_i$ assets $i$, transfer the amount to the cash and buy $k_j$ assets $j$ from cash. In order to make sure that each transfer is approximately the same amount, the number of assets $k_i$ and $k_j$ to be transferred are defined as $k_i = \lceil \frac{p_{i \max}}{p_i} \rceil$ and $k_j = \lceil \frac{p_{j \max}}{p_j} \rceil$. This procedure is summarized in algorithm 3 where we omitted the details necessary to check for short selling and holding constraints.

Figure 2: Working of the TA algorithm. Efficient frontier with cash (upper line) and without cash (lower curve).
Algorithm 3 Definition of neighborhood in case of integer variables.

1: Randomly select asset $i$ to sell
2: $x_i = x_i - k_i$
3: cash = cash + $k_i p_i^0$
4: Randomly select asset $j$ to buy
5: $x_j = x_j + k_j$
6: cash = cash $- k_j p_j^0$

Using the same data set as for the previous problem but considering an investment opportunity set of 20 assets (including cash) we now solve the mean-VaR problem defined in (7). To compute the capital $v^a$ at the end of the planning period we use simulated prices $p^s$, computed as

$$p^s = p^0 r^s \quad s = 1, \ldots, n_S$$

where the rate of return $r^s$ has been defined in (2). We assume an initial capital of $v^0 = 800000$ and seek the portfolio which maximises the expected return given the following constraints: shortfall probability $\beta = 0.05$ for a VaR level of 750000, minimum and maximum holding size for a particular asset 0.01 $v^0$ respectively 0.30 $v^0$ and a maximum of 9 assets in the portfolio. Figure 4 shows the results of the TA algorithm with the setting $niter = 6$ and $steps = 1500$.

![Figure 4: Search path of the TA algorithm in the $\beta, E(v)$ plane.](image-url)
The composition of the portfolio, the TA algorithm found to be optimal, is given in figure 5. It contains the assets \{2, 3, 6, 8, 10, 11, 14\} and cash. The minimum position is 28 680 (asset 3) and the maximum position is 212 936 (asset 14). Thus the constraints on the holding size and the number of assets in the portfolio are satisfied.

This optimal portfolio has an expected return of \(E(v) = 810 520\) with a shortfall probability of 0.049 for a \(VaR\) level of 749 950, which again satisfies the constraints.

![Figure 5: Optimal portfolio computed by TA for the mean-VaR problem.](image)

In figure 4, we observe that the solutions lie in planes. The reason for this is the integer formulation of the problem. Figure 6 illustrates the working of the TA algorithm in the \(\beta, VaR\) plane and in figure 7, we see its working in the \(E(v), VaR\) plane.

![Figure 6: Search path of the TA algorithm in the \(\beta, VaR\) plane.](image)
5 Concluding remarks

In this paper, we attempted to illustrate how heuristic optimization algorithms like the threshold accepting method can be successfully applied to solve realistic non-convex portfolio optimization problems. We showed that, in the cases where these problems contain non-linear and non-convex constraints, the heuristic methods are the only reasonable way out. Examples of these situations can be problems where constraints on downside risk preferences are introduced, where the solutions are required to be integers, etc.

We mainly focus on the cases where the distribution of the asset future returns are modelled by equally weighted scenarios of past returns. The sensitivity of optimized portfolios with respect to alternative scenario generations procedures should be further investigated.

Figure 7: Search path of the TA algorithm in the $E(v)$, $\text{VaR}$ plane.
References


