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Trading Volume in Dynamically Efficient Markets

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Abstract

The classic Lucas asset pricing model with complete markets stresses aggregate risk and, hence, fails to investigate the impact of agents heterogeneity on the dynamics of the equilibrium quantities and measures of trading volume. In this paper, we investigate under what conditions non-informational heterogeneity, i.e. differences in preferences and endowments, leads to non trivial trading volume in equilibrium. Our main result comes in form of a non-informational no trade theorem which provides necessary and sufficient conditions for zero trading volume in a dynamically efficient, continuous time Lucas market model with multiple goods and securities.

Keywords : General Equilibrium, Trading Volume, heterogenous agents, multiple goods, incomplete markets, no-trade theorem.

JEL Codes : D51, D52, G11, G12

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1 Introduction

The classic Lucas asset pricing model with symmetric information and complete markets almost exclusively focuses on equilibrium prices and returns. As the unique equilibrium in this model is Pareto efficient, it can be solved for under the representative agent paradigm. Consequently, such a model stresses aggregate market risk and, hence, fails to investigate the impact of any form of heterogeneity in risk attitudes and/or endowments on the dynamics of the equilibrium quantities and measures of trading volume.

It is commonly thought that models which focus on market risk only, are unable to produce the large trading volume observed in most stock markets. In addition, and as shown by Wang (1994), heterogeneity in risk attitudes of agents has different implications for the co-movement of equilibrium prices and portfolios than heterogeneous information among agents. For these reasons, most models which explicitly derive the equilibrium trading volume link asset quantity directly to the flow of information and, thus, introduce differential or insider information.[1] But, since asymmetric information is the only motive for trade in these models, they have to circumvent the informational no trade theorems of Milgrom and Stokey (1982) and Holmström and Myerson (1984). This fundamental result states that given a Pareto efficient allocation, the arrival of additional information cannot lead to trade, and hence does not generate any incremental trading volume. The apparent paradox between information based models of trading volume and the informational no trade theorems is resolved through the introduction of so-called noise traders whose presence guarantees that the market clearing conditions, which impose informational equilibrium constraints, are automatically satisfied. Loosely speaking, in such models there always exists a counterparty which is willing to trade contingent on information signals received by an optimizing agent. As a consequence, additional information always causes trade. But it is important to note that without introducing liquidity traders the informational no-trade theorem holds and asymmetric information will not necessarily increase trading volume.

At the intuitive level, it is well understood that in homogeneous information models the degree of heterogeneity with respect to endowments, preferences and beliefs determines the equilibrium trading volume, but a formal study of this subject has not been undertaken so far. A notable exception is Judd *et al.* (2003) who discuss the properties of the equilibrium trading volume in a discrete time complete market economy with finite state space populated by

[1] Such models are either solved in a competitive setting (see e.g. Wang (1994)) or allow for strategic interactions among agents (see Pfleiderer (1984), Kyle (1985) and Foster and Vishwanathan (1990, 1993)).

agents endowed with heterogeneous preferences. They show that when the exogenous dividend process is an irreducible, stationary Markov chain, optimal consumption policies are time homogeneous, and conclude that no trading occurs after the initial period. Notice, however, that the stationary nature of the Markov chain process does not permit to introduce growth in the aggregate dividend, and might therefore be a strong restriction. It is not a priori evident that their results extend to a more general setting, namely that in a dynamically complete economy, preference heterogeneity alone is not sufficient to generate trading. The main objective of this paper is to investigate under what conditions non-informational heterogeneity leads to non trivial equilibrium trading volume. Our main result comes in form of a non-informational no trade theorem. It provides necessary and sufficient conditions for zero trading volume after the initial period in a dynamically efficient market model with multiple goods, homogeneous information and homogeneous beliefs.

The structure of the paper is as follows. Section 2 introduces our model of a continuous time, multi-good economy. Next, Section 3 introduces simplifying notation, and presents preliminary results. Then in Section 4, we present our main result and discuss a number of examples. Section 5 discusses extensions of the basic model to include heterogeneous beliefs and random endowments.

2 The Economy

We consider a continuous time, stochastic economy on the finite time interval $[0, T]$, which we proceed to describe in the following sections.

2.1 Information Structure

The uncertainty is represented by a probability space $(\Omega, \mathcal{F}, \mathbf{F}, P)$ on which is defined an $n-$dimensional Brownian motion B. The filtration

$$\mathbf{F} := \left\{ \mathcal{F}_t : t \in [0, T] \right\}$$

is the augmentation under P of the filtration generated by the Brownian motion and we let $\mathcal{F} = \mathcal{F}_T$ so that the true state of nature is completely determined by the paths of the Brownian motion up to the terminal date of the model.

All agents are endowed with the same information structure represented by $\mathbf{F}$ and the same beliefs represented by the probability measure P. All random processes to appear in the sequel are assumed to be progressively measurable with respect to the filtration $\mathbf{F}$ and all statements involving random quantities processes are understood to hold either almost surely or almost everywhere depending on the context.

2.2 Consumption Space and Goods Markets

There is a finite number of perishable consumption goods labelled $a \in \mathcal{A}$ for some finite set $\mathcal{A}$. The consumption space $\mathcal{C}$ is given by the set of non negative, vector valued consumption rate processes such that

$$\int_0^T |c_t^a| dt < \infty, \qquad a \in \mathcal{A}$$

Each of the $A := \text{card}(\mathcal{A})$ available consumption goods can be traded in a perfect spot market. The first consumption good will be taken as a numéraire throughout the paper and we will denote by p with $p^1 = 1$ the $A-$dimensional process of relative prices.

2.3 Securities

The financial market consists of a locally riskless savings account in zero net supply and $\mathbf{n}$ risky stocks in positive net supply each representing a claim to an exogenously specified stream of dividends denominated in one of the A available consumption goods. More precisely, we assume that for each consumption good $a \in \mathcal{A}$ there is a number n_a, with

$$\mathbf{n} := \sum_{a \in \mathcal{A}} n_a \leq n,$$

of traded securities whose dividends are labelled in consumption good a. The column vector of dividend rate processes associated to these securities is denoted by D^a and is assumed to be a non negative Itô process of the form

$$D_t^a = D_0^a + \int_0^t \left[\rho_s^a ds + \vartheta_s^a dB_s \right]$$

for some locally square integrable, vector valued drift ρ^a and matrix valued volatility ϑ^a. In what follows, we denote by D the $\mathbf{n}-$dimensional column vector obtained by stacking up the good specific dividend vectors $(D^a)_{a \in \mathcal{A}}$ and assume that its volatility matrix ϑ has full row rank.

The initial value of the savings account is normalized to one and we assume that in equilibrium its price process is given by

$$S_t^0 = \exp\left[\int_0^t r_s ds \right]$$

for some instantaneous interest rate process r such that the above integral is well defined. For each consumption good $a \in \mathcal{A}$, we denote by S^a the vector of prices of the stocks whose dividends are labelled in good a and assume that in equilibrium

$$S_t^a + \int_0^t p_s^a D_s^a ds = S_0^a + \int_0^t \left[\mu_s^a ds + \sigma_s^a dB_s\right]$$

for some vector valued drift μ^a and matrix valued volatility σ^a such that the above integrals are well defined. The security prices coefficients $(r, \{\sigma^a, \mu^a\})$, or equivalently the security price processes $(S^0, \{S^a\})$ as well as the vector p of relative goods prices are to be determined endogenously in equilibrium.

2.4 Trading Strategies

Trading takes place continuously and there are no market friction. Given the security prices, a trading strategy is a collection $\theta := (\theta^0, \{\theta^a\})$ of vector valued processes satisfying

$$\int_0^T \left(|\theta_t^0 r_t S_t^0| + \sum_{a \in \mathcal{A}} \left\{ |(\theta_t^a)^* \mu_t^a| + \|(\sigma_t^a)^* \theta_t^a\|^2 \right\} \right) dt < \infty$$

where the star superscript denotes transposition. The process θ^0 represents the number of shares of the savings account held in the portfolio and, for each consumption good $a \in \mathcal{A}$, the vector process θ^a represents the number of shares of each of the stock paying out in good a held in the portfolio.

A trading strategy θ is said to be admissible if the associated wealth process, which is defined by

$$W_t := \theta_t^0 S_t^0 + \sum_{a \in \mathcal{A}} (\theta_t^a)^* S_t^a, \tag{1}$$

is uniformly bounded from below by a constant. In what follows, we denote by Θ the set of all admissible trading strategies.

Remark 1 The requirement that the wealth process of an admissible trading strategy be bounded from below is standard in the asset pricing literature. It rules out the possibility of doubling strategies and thus implies that the set Θ is free of arbitrage opportunities. See Dybvig and Huang (1988).

2.5 Agent's Preferences and Endowments

The economy is populated by two price taking agents labelled $i \in \{1, 2\}$. The preferences of agent i over consumption plans in $\mathcal{C}$ are represented by a time additive expected utility functional:

$$U_i(c) := E\left[\int_0^T u_i(t, c_t) ds\right].$$

We assume that the utility function $u_i : [0,T] \times \mathbf{R}_+^A \to \mathbf{R}$ is continuously differentiable in its first argument, strictly increasing, strictly concave and three times continuously differentiable in its second argument with gradient

$$\nabla u_i(t,c) := \left[\frac{\partial u_i}{\partial c^1}(t,c), \ldots, \frac{\partial u_i}{\partial c^A}(t,c)\right]^*$$

mapping $(0,\infty)^A$ bijectively onto itself for every fixed $t \in [0,T]$. In order to guarantee that certain expectations can be differentiated under the integral sign, we further assume that

$$AE_t(u_i) := \limsup_{b\to\infty} \sup_{c\in\mathcal{M}(b)} \frac{c^* \nabla u_i(t,c)}{u_i(t,c)} < 1, \quad t \in [0,T]$$

where $\mathcal{M}(b)$ denotes the set of non negative vectors whose lowest coordinate is larger than the constant $b \in \mathbf{R}_+$. In the single good case, this condition is referred to as reasonable asymptotic elasticity and has proved crucial in the resolution of incomplete markets portfolio and consumption choice problems, see Kramkov and Schachermayer (1999).

Remark 2 As a result of the above assumptions, we have that for each $t \in [0,T]$ the gradient $\nabla u_i(t,\cdot)$ admits an inverse which we denote by $f_i(t,\cdot)$. In the single good case, a well-known necessary and sufficient condition for this property to hold is that the utility functions satisfy the Inada conditions.

Agent i is initially endowed with a portfolio consisting of $\nu_i^{ak} \geq 0$ shares of each of the available stocks, where ν_i^{ak} denotes the k^{th} stock paying dividends in good a. We assume, without loss of generality, that the agents' initial portfolios verify the identity

$$\nu_1^{ak} + \nu_2^{ak} = 1, \qquad (a,k) \in \mathcal{A} \times \{1, \ldots, n_a\}$$

so that the net supply of each of the stocks is normalized to one unit. For further reference we also denote by $\nu_i^a := (\nu_i^{a1} \cdots \nu_i^{an_a})^*$ the vector of number shares of stocks paying dividends in good a in the initial portfolio of agent i.

2.6 Feasible Consumption Plans

A consumption plan $c \in \mathcal{C}$ is said to be feasible for agent i if there exists an admissible trading strategy $\theta \in \Theta$ whose wealth process satisfies agent i's dynamic budget constraint

$$\begin{aligned} W_0 &= \theta_0^0 + \sum_{a\in\mathcal{A}} (\theta_0^a)^* S_0^a = W_0^i := \sum_{a\in\mathcal{A}} (\nu_i^a)^* S_0^a \\ dW_t &= \theta_t^0 dS_t^0 + \sum_{a\in\mathcal{A}} \left\{ (\theta_t^a)^* \left[dS_t^a + p_t^a D_t^a dt \right] - p_t^a c_t^a dt \right\} \end{aligned}$$

and has a non negative terminal value. In what follows we denote by $\mathcal{C}_i$ the set of consumption plans which are feasible for agent i.

2.7 Definition of Equilibrium

In what follows we denote by $\mathcal{E} := ((\Omega, \mathcal{F}, \mathbf{F}, P), \{u_i, \nu_i^a\}, \{D^a\})$ the primitives for the above continuous time economy. The concept of equilibrium that we use throughout this paper is similar to that of equilibrium of plans, prices and expectations introduced by Radner (1972) and is defined in the following:

Definition 1 *An equilibrium for the continuous time economy $\mathcal{E}$ is a set of security prices $(S^0, \{S^a\})$, a relative price process p and a set of consumption plans and admissible trading strategies $\{c_i, \theta_i\}$ such that*

1. *The consumption plan c_i maximizes U_i over the feasible set $\mathcal{C}_i$ and is financed by the admissible trading strategy $\theta_i \in \Theta$.*

2. *The securities and goods markets clear in the sense that*

$$\begin{aligned} \theta_{1t}^0 + \theta_{2t}^0 &= 0, \\ \theta_{1t}^a + \theta_{2t}^a &= \mathbf{1}_a, \\ c_{1t}^a + c_{2t}^a &= \mathbf{1}_a^* D_t^a \end{aligned}$$

   hold for all $a \in \mathcal{A}$ and $t \in [0, T]$ where $\mathbf{1}_a$ denotes an n_a-dimensional column vector of ones.

In our model the dividend processes of the traded securities are linearly independent since their volatility matrix has full rank. However, because there are less traded securities than there are Brownian motions, the equilibria for the economy $\mathcal{E}$ have incomplete financial markets in general. Furthermore, and as demonstrated by Cass and Pavlova (2004), the equilibrium may very well have incomplete financial markets even if there are as many traded securities as there are Brownian motions. Given this observation, and in order to facilitate our study, we further restrict ourselves to equilibria that are dynamically efficient in the following sense:

Definition 2 *An equilibrium for the continuous time economy $\mathcal{E}$ is said to be dynamically efficient, or simply efficient, if given the securities and goods prices, the associated consumption allocations are Pareto optimal.*

Note that in a static model this notion is also referred to as constrained Pareto optimality or Pareto optimality in the Diamond sense (see Magill and Quinzii (1996) for a discussion).

While the set of equilibria is in general very hard to characterize (see for example Cuoco and He (1994)), that of efficient equilibria is easier to analyze. Indeed, by the Pareto optimality of equilibrium allocations we have that there exists a strictly positive constant λ such that

$$\nabla u_1(t, c_{1t}) = \lambda \nabla u_2(t, c_{2t}).$$

Along with the goods market clearing condition, the above restriction implies that the individual consumption allocations solve the maximization problem

$$u(t, \lambda, \delta_t) := \max_{c_1 + c_2 = \delta_t} \Big\{ u_1(t, c_1) + \lambda u_2(t, c_2) \Big\} \tag{2}$$

where δ_t denotes the vector of good specific aggregate dividends. As a result, all of the efficient equilibria for the continuous time economy $\mathcal{E}$ can be supported by a representative agent endowed with the aggregate supply of securities and with utility function $u(t, \lambda, \cdot)$ even though the resulting financial markets might be incomplete. In order to facilitate the presentation of our main results, we briefly review this characterization in the next section.

3 Preliminary Results

3.1 A Useful Notation

In order to facilitate the presentation of our results, we now introduce a vectorial notation which will be used repeatedly in what follows.

For an arbitrary collection $(x^a)_{a \in \mathcal{A}}$ of vectors with $x^a \in \mathbf{R}^{n_a}$, we use the shorthand notation $\Phi(x^a)$ to denote the rectangular matrix

$$\Phi(x^a) := \begin{bmatrix} x^{1*} & & & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & x^{2*} & & \cdots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ \vdots & & \vdots & & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & x^{A*} & & \end{bmatrix} \in \mathbf{R}^{A \times \mathbf{n}}.$$

The linear operator Φ allows one to transform a collection of good specific vectors into a matrix which can then be used in consumption and portfolio computations. In particular, letting $\mathbb{I} := \Phi(\mathbf{1}_a)$ we have that the vector of good specific aggregate dividend processes is given by $\delta_t := \mathbb{I} D_t$.

3.2 Individual Optimality

Let the security and goods prices be given and assume that there are no arbitrage opportunities for otherwise the market could not be in equilibrium. As is well known (see e.g. Karatzas and Shreve (1998)), this assumption implies that there exists an almost surely square integrable, $n-$dimensional process κ such that

$$\mu_t^a - r_t S_t^a = \sigma_t^a \kappa_t, \qquad a \in \mathcal{A}.$$

Any almost surely square integrable process satisfying the above restrictions is referred to as a market price of risk or relative risk premium.

Denote by $\mathcal{K}$ the set of relative risk premia, and for every such process consider the non negative process defined by

$$\xi_t^\kappa := \exp\left[-\int_0^t r_s ds - \int_0^t \kappa_s^* dB_s - \frac{1}{2}\int_0^t \|\kappa_s\|^2 ds\right].$$

The following proposition shows that the set $\mathcal{S} := \{\xi^\kappa : \kappa \in \mathcal{K}\}$ coincides with the set of arbitrage free state price densities and provides a convenient necessary and sufficient condition for the optimality of a given consumption plan.

Proposition 1 *Assume that the security and goods prices are given, then the following assertions hold:*

1. *A consumption plan is feasible for agent i if and only if it satisfies the static budget constraint*

$$E\left[\int_0^T \xi_t^\kappa p_t^* c_t dt\right] \leq W_0^i \tag{3}$$

for all market price of risk processes $\kappa \in \mathcal{K}$.

2. *A feasible consumption plan is optimal for agent i if and only if*

$$c_t = f_i\left(t, y_i p_t \xi_t^{\kappa_i}\right)$$

for some strictly positive constant y_i and some process $\kappa_i \in \mathcal{K}$ such that equation (3) holds as an equality.

Proof. The assertions follow, respectively, from Theorem 8.5 and Theorem 9.3 in Karatzas et al. (1991) after some straightforward modifications. ■

3.3 A Characterization of Efficient Equilibria

Assume that there exists an efficient equilibrium for the economy $\mathcal{E}$, denote by $\{c_i\}$ the corresponding consumption allocations and let the representative agent's utility function be defined as in (2). Since consuming the aggregate output must be optimal for the representative agent, it follows from the second part of Proposition 1 that the process of marginal rates of substitution

$$p_t \xi_t := \frac{\nabla u(t, \lambda, \delta_t)}{\nabla_1 u(0, \lambda, \delta_0)} = \frac{\nabla u_i(t, c_{it})}{\nabla_1 u_i(0, c_{i0})} \tag{4}$$

identifies the vector of good specific equilibrium state prices. Moreover, Pareto optimality implies that the consumption allocations solve the representative agent's optimization problem and it follows that

$$c_{1t} := f_1\Big(t, \nabla u(t, \lambda, \delta_t)\Big), \qquad c_{2t} := \delta_t - c_{1t} = f_2\Big(t, \nabla u(t, \lambda, \delta_t)/\lambda\Big), \tag{5}$$

where $f_i(t, \cdot)$ denotes the inverse of agent i's gradient mapping. On the other hand, the definition of the set of state price densities and the absence of arbitrage opportunities implies that the process

$$M_t^a := \xi_t S_t^a + \int_0^t \xi_s p_s^a D_s^a ds$$

is a local martingale and assuming for the moment that this process is a real martingale we get that the equilibrium securities prices are given by

$$S_t^a = E\left[\int_t^T \frac{\xi_s p_s^a D_s^a ds}{\xi_t} \middle| \mathcal{F}_t \right]. \tag{6}$$

The following proposition justifies the assumption that the process M^a is a uniformly integrable martingale and provides a complete characterization of the set of efficient equilibria.

Proposition 2 *Assume that the collection $(r, p, \{S^a\}, \{c_i, \theta_i\})$ is an efficient equilibrium. Then the relative goods prices, the consumption allocations and the securities prices are respectively given by equations (4), (5) and (6) for some strictly positive constant λ.*

Proof. All there is to prove is that for each $a \in \mathcal{A}$ the local martingale M^a is in fact a uniformly integrable martingale. To this end, we start by observing that the wealth of agent i along the equilibrium path is given by

$$W_t^i := E\left[\int_t^T \frac{\xi_s p_s^* c_{is} ds}{\xi_t} \middle| \mathcal{F}_t \right]$$

where the vector $p\xi$ of good specific state prices is defined as in equation (4). Summing the above expressions over i and using the goods market clearing conditions, we deduce that the aggregate wealth in the economy is given by

$$R_t = W_t^1 + W_t^2 = E\left[\int_t^T \frac{\xi_s p_s^* \delta_s ds}{\xi_t} \middle| \mathcal{F}_t \right] \tag{7}$$

and it follows that the non negative process defined by

$$Q_t = \xi_t R_t + \int_0^t \xi_s p_s^* \delta_s ds$$

is a uniformly integrable martingale under the objective probability measure. Now let $a \in \mathcal{A}$ be given and fix an arbitrary $k \in \{1, \cdots, n_a\}$. As mentioned before the statement of the proposition, the process

$$M_t^{ak} := \xi_t S_t^{ak} + \int_0^t \xi_s p_s^a D_s^{ak} ds$$

is a local martingale and hence a global supermartingale since it is non negative. On the other hand, the absence of arbitrage opportunities and the definition of the vector of aggregate dividends imply that $M^{ak} \leq Q$. The process Q being uniformly integrable by construction, it follows from the reverse Fatou lemma that M^{ak} is a global submartingale. The process M^{ak} being both a supermartingale and submartingale, it is a martingale. In addition, M^{ak} is bounded from above by Q , hence uniformly integrable and the desired results follows. ■

4 Equilibrium Trading Volume

4.1 The No–Trade Theorem

We now turn to this paper's main topic and investigate conditions under which an efficient equilibrium generates trade.

Intuition strongly suggests that as soon as the agents populating the economy are sufficiently heterogeneous, their demand for the available securities should fluctuate over time, thus generating trading activity in both the goods market and the financial market. Our main result confirms this intuition and comes in the form of a no–trade theorem.

Theorem 1 *The following assertions are equivalent:*

1. *There exists an efficient equilibrium that has no–trade in the sense that both $\theta_{1t}^0 = \theta_{10}^0$ and $\theta_{1t}^a = \theta_{10}^a$ hold for all $a \in \mathcal{A}$ and there is no activity on spot market for the consumption goods.*
2. *There exists an efficient equilibrium in which the individual consumption policies satisfy*
$$\frac{c_{it}^a}{c_{i0}^a} = \frac{\mathbf{1}_a^* D_t^a}{\mathbf{1}_a^* D_0^a} = \frac{\delta_t^a}{\delta_0^a}, \qquad i \in \{1, 2\}$$
where δ_t^a denotes the aggregate output of good $a \in \mathcal{A}$ at time $t \in [0, T]$.

3. *There exists a diagonal matrix w with strictly positive, constant diagonal elements $w^a \in (0,1)$ such that*

$$\frac{\nabla u_1(t, w\delta_t)}{\nabla_1 u_1(0, w\delta_0)} = \frac{\nabla u_2(t, \delta_t - w\delta_t)}{\nabla_1 u_2(0, \delta_0 - w\delta_0)}, \tag{8}$$

and

$$E\left[\int_0^T \nabla u_1(t, w\delta_t)^* \Big\{ w\mathbb{I} - \Phi(\nu_1^a) \Big\} D_t dt \right] = 0, \tag{9}$$

where $\mathbb{I}$ is the rectangular matrix defined in Section 3.1 and $\delta = \mathbb{I}D$ denotes the vector of good specific aggregate dividends.

Proof. To establish the implication $1 \Rightarrow 2$, assume that there exists a no–trade efficient equilibrium. First note that the optimal holding of the money market account must be zero. Indeed, since $S_T^a = \mathbf{0}_{n_a}$ in the absence of arbitrage opportunities it follows from individual optimality that

$$0 = W_T^i = \theta_{iT}^0 S_T^0 + \sum_{a \in \mathcal{A}} (\theta_{iT}^a)^* S_T^a = \theta_{iT}^0 S_T^0$$

and observing that S_T^0 is strictly positive we conclude that $\theta_{i0} = \theta_{iT}^0 = 0$ for all $i \in \{1, 2\}$. On the other hand, applying Itô's lemma to (1) and using the dynamic budget constraint we obtain

$$p_t^* \Big\{ \Phi(\beta_a) D_t - c_{1t} \Big\} = 0,$$

where $\beta_a := \theta_{10}^a \in (0,1)^{n_a}$. Since, by definition of a no–trade equilibrium, there is no activity on the goods markets, the above identity and the goods market clearing conditions imply that the equilibrium consumption policies are linear and given by

$$c_{1t} = \Phi(\beta_a) D_t, \qquad c_{2t} = \big(\mathbb{I} - \Phi(\beta_a)\big) D_t. \tag{10}$$

Now, Pareto optimality of the equilibrium consumption allocations implies that the marginal utilities of the two agents are aligned in the sense that there exists a strictly positive constant λ such that

$$\nabla u_1\big(t, \Phi(\beta_a) D_t\big) = \lambda \nabla u_2\big(t, \big(\mathbb{I} - \Phi(\beta_a)\big) D_t\big).$$

Applying Itô's lemma to both sides of the above equation and identifying the volatility coefficients, we obtain that

$$\mathcal{H} u_1\big(t, \Phi(\beta_a) D_t\big) \Phi(\beta_a) \vartheta_t = \lambda \mathcal{H} u_2\big(t, \big(\mathbb{I} - \Phi(\beta_a)\big) D_t\big) \big(\mathbb{I} - \Phi(\beta_a)\big) \vartheta_t$$

where $\mathcal{H}$ denotes the matrix of second derivatives. Since the volatility matrix of the dividend processes has full rank by assumption, this in turn implies

$$\mathcal{H} u_1\big(t, \Phi(\beta_a) D_t\big) \Phi(\beta_a) = \lambda \mathcal{H} u_2\big(t, \big(\mathbb{I} - \Phi(\beta_a)\big) D_t\big) \big(\mathbb{I} - \Phi(\beta_a)\big).$$

The above system of equations yields the existence of strictly positive constants $(w^a)_{a\in\mathcal{A}}$ such that $\beta_a = w^a \mathbf{1}_a$ and plugging this back into (10) gives the condition in Assertion 2 after some straightforward simplifications.

To establish the implication $2 \Rightarrow 3$, assume that there exists an efficient equilibrium satisfying the conditions of Assertion 2 and let w denote the diagonal matrix with strictly positive diagonal elements defined by

$$w^a := \frac{c_{10}^a}{\mathbf{1}_a^* D_0^a}, \qquad a \in \mathcal{A}.$$

Using the Pareto optimality of the equilibrium allocations in conjunction with the assumed form of the consumption plans we obtain that there exists a strictly positive constant λ such that

$$\nabla u_1(t, w\delta_t) = \lambda \nabla u_2(t, \delta_t - w\delta_t). \tag{11}$$

Writing the first coordinate of this vectorial identity at time zero, then allows us to identify the Negishi weight as

$$\lambda = \frac{\nabla_1 u_1(0, w\delta_0)}{\nabla_1 u_2(0, \delta_0 - w\delta_0)}$$

and plugging this expression back into equation (11) gives the first condition in Assertion 3. On the other hand, the assumed form of the equilibrium allocations and the second part of Proposition 1 imply that

$$\nabla u_i(t, c_{it}) = y_i p_t \xi_t^i \tag{12}$$

for some strictly positive constant y_i and some arbitrage free state price density process $\xi^i := \xi^{\kappa_i} \in \mathcal{S}$ such that

$$W_0^i = \Phi(\nu_i^a) S_0 = E\left[\int_0^T \xi_t^i p_t^* w\delta_t dt\right] \tag{13}$$

where S denotes the $\mathbf{n}-$dimensional column vector obtained by stacking up the good specific securities price vectors $(S^a)_{a\in\mathcal{A}}$. Using (12) in conjunction with the Pareto optimality of the equilibrium allocations we deduce that

$$p_t \xi_t^1 = p_t \xi_t^2 = \frac{1}{y_1} \nabla u_1(t, w\delta_t) = \frac{1}{y_2} \nabla u_2(t, \delta_t - w\delta_t).$$

Using the first coordinate of the above equation at time zero to identify the constant y_1 and plugging the result into (13) with $i = 1$ then gives

$$\Phi(\nu_1^a) S_0 = E\left[\int_0^T \frac{\nabla u_1(t, w\delta_t)^* w\delta_t}{\nabla_1 u_1(0, w\delta_0)} dt\right]. \tag{14}$$

Since it is efficient, the equilibrium can be supported by a representative agent with utility function $u(t, \lambda, \cdot)$ as in equation (2) even if the resulting markets are incomplete. Thus, it follows from the definition of the representative agent's marginal utility and Proposition 2 that the equilibrium securities prices satisfy

$$S_0^a = E\left[\int_0^T \frac{\nabla_a u_1(s, w\delta_t) D_t^a}{\nabla_1 u_1(0, w\delta_0)} dt\right]. \tag{15}$$

Plugging this expression back into equation (14) and rearranging the terms gives the second condition in Assertion 3.

In order to establish the implication 3 ⇒ 1, and thus complete the proof of the theorem, we have to show that given a matrix w satisfying the conditions of Assertion 3 we can construct a no–trade efficient equilibrium. To this end, consider the trading strategies and consumption rates defined by

$$\begin{aligned}
\theta_{1t}^0 &= \theta_{2t}^0 = 0,\\
\theta_{1t}^a &= w^a \mathbf{1}_a = \mathbf{1}_a - \theta_{2t}^a,\\
c_{1t}^a &= w^a \delta_t^a = w^a \mathbf{1}_a^* D_t^a = \delta_t^a - c_{2t}^a,
\end{aligned}$$

and let the securities and relative goods prices be given by

$$\begin{aligned}
p_t &:= \frac{\nabla u_1(t, c_{1t})}{\nabla_1 u_1(t, c_{1t})} = \frac{\nabla u_2(t, c_{2t})}{\nabla_1 u_2(t, c_{2t})},\\
S_t^a &:= E\left[\int_t^T \frac{\nabla_1 u_1(s, c_{1s})}{\nabla_1 u_1(t, c_{1t})} p_t^a D_t^a dt \,\middle|\, \mathcal{F}_t\right].
\end{aligned}$$

Since (i) all markets clear, (ii) there is no trading volume on any of the open markets and (iii) the marginal utilities of the two agents are aligned, all there is to prove in order to establish that the collection $(p, \{S^a\}, \{c_i, \theta_i\})$ constitutes a no–trade efficient equilibrium is that the consumption allocations are optimal given the security prices. To this end, let ξ be the process defined by

$$\xi_t := \frac{\nabla_1 u_1(t, c_{1t})}{\nabla_1 u_1(0, c_{10})} = \frac{\nabla_1 u_2(t, c_{2t})}{\nabla_1 u_2(0, c_{20})} \geq 0.$$

Using the definition of c_i in conjunction with the definition of the securities prices and the second condition in Assertion 3 we have that

$$E\left[\int_0^T \xi_t p_t^* \left\{c_{it} - \Phi(\nu_i^a) D_t\right\} dt\right] = 0.$$

On the other hand, using the fact that for each $a \in \mathcal{A}$ the process

$$\xi_t S_t^a + \int_0^t \xi_s p_s^a D_s^a ds$$

is a martingale we deduce that the process ξ belongs to the set $\mathcal{S}$ of state price densities and since c_i is feasible by construction, it follows from the second part of Proposition 1 that the consumption plan c_i is optimal for agent i. ■

The results of the above theorem can be summarized as follows. Assertion 2 shows that in a no-trade efficient equilibrium, the consumption policies of each of the agents must exhibit the same growth rate as the corresponding good specific aggregate output and that, given the existence of an efficient equilibrium, this property is also sufficient for the existence of a no-trade equilibrium.

The third assertion of the theorem is the most important from a practical point of view as it provides necessary and sufficient conditions for the existence of a no-trade efficient equilibrium in terms of the model primitives. In the next section we review most of the classic forms of multi-goods utility functions and use Assertion 3 to determine what is the minimal level of non informational heterogeneity needed to generate non trivial trading volume.

Remark 3 Borch (1962), Wilson (1968) and Huang and Litzenberger (1985) have shown that a necessary and sufficient condition for the generic optimality of linear sharing rules in single good, static economies is that all agents have linear risk tolerance with identical cautiousness parameters. Our results can be viewed as a generalization of theirs to the case of multi-goods, dynamic economies.

To see this, consider the single good case with time independent utility functions. We start by observing that since consumption must be positive at all times market clearing implies that any linear sharing rule must have a zero intercept in order to be feasible. Thus, it follows Theorem 1 that given the existence of an efficient equilibrium, the generic optimality of linear sharing rules is equivalent to the generic existence of a no-trade equilibrium. Using Assertion 3, this is in turn equivalent to the fact that for any aggregate dividend process there exists a constant $w \in (0, 1)$ such that

$$\tau_1(wD_t) = \tau_2((1-w)D_t), \qquad t \in [0, T],$$

where τ_i denotes the relative risk tolerance of agent i. For the above equation to admit a solution in $(0, 1)$ regardless of the aggregate dividend process, it is necessary and sufficient that both agents have the same constant relative risk aversion parameter. We thus conclude that in a single good, continuous time economy a necessary and sufficient condition for the generic optimality of linear sharing rules is that both agents have the same constant relative risk aversion utility function.

Remark 4 It is well known from Hakansson (1969) and Cass and Stiglitz (1970), that linear sharing rules in the single good case are related to fund separation. Assertion 2 of Theorem 1 shows that the optimality of the linear sharing rule

$$c_{it}^a = (c_{i0}^a/\delta_0^a)\delta_t^a$$

is equivalent to the existence of a no-trade equilibrium. In the single good case, this implies that fund separation must hold in a no-trade equilibrium. More precisely, as $W_t^i = \frac{c_{i0}}{\delta_0} R_t$ in a no-trade equilibrium, where R corresponds to the value of the market portfolio defined in (7), any agent holds a constant fraction of the market portfolio and one-fund separation holds if and only if the equilibrium is a no-trade equilibrium.

Remark 5 A careful inspection of the proof of Theorem 1 reveals that the only place where the assumptions of continuous time and Itô process dynamics are used is in the proof of the implication 1 $\Rightarrow$ 2. It follows that, after suitable modifications of the basic model, the conditions of Assertion 3 are still sufficient for the existence of a no-trade efficient equilibrium in a discrete time economy with either finite or infinite horizon.

While sufficient, the conditions of Assertion 3 are far from being necessary in a discrete time model. Indeed, it can easily be shown that Assertion 3 remains sufficient for the existence of a no-trade equilibrium if we replace equation (8) by the weaker requirement that

$$\frac{\nabla u_1(t, \Phi(\beta_a)D_t)}{\nabla_1 u_1(0, \Phi(\beta_a)D_0)} = \frac{\nabla u_2(t, (\mathbb{I} - \Phi(\beta_a))D_t)}{\nabla_1 u_2(0, (\mathbb{I} - \Phi(\beta_a))D_0)}. \tag{16}$$

holds for some collection of non negative vectors $(\beta^a)_{a\in\mathcal{A}}$. In a recent paper, Judd *et al.* (2003) show that trading volume is generically zero in a discrete time, single good economy populated by heterogeneous agents. This seems to contradict our theorem. As they remark, however, their result relies on the strong distributional assumption of a stationary Markov chain process for the aggregate dividend. In that particular case, equilibrium consumption allocations inherit the time homogeneity properties of the dividend process and it follows that there always exists a solution to equation (16) irrespective of the choice of the utility functions. To illustrate this, let us consider, as in Judd *et al.* (2003), a single good economy with N states of the world and $J = N$ stocks paying linearly independent dividends. Equation (16) may be rewritten as

$$u_1'(t, \beta^* D_n) = \lambda u_2'(t, (\mathbf{1} - \beta)^* D_n) \qquad n \in \{1, N\} \tag{17}$$

where D_n is the vector of dividends in state n and λ is a strictly positive constant. When utility functions are time separable and discount rates are identical across agents as in Judd *et. al.* (2003), the system may be rewritten without the time dependency as

$$u_1'(\beta^* D_n) = \lambda u_2'((\mathbf{1}-\beta)^* D_n), \qquad j \in \{1, N-1\}$$
$$\lambda = \frac{u_1'(\beta D_N)}{u_2'((\mathbf{1}-\beta)^* D_N)}.$$

The marginal utilities $u_1'(\cdot)$ and $u_2'(\cdot)$ being strictly decreasing functions, there always is a solution to this system of equation. As a result, when the dividend process follows a stationary Markov chain, a no trade equilibrium can always be constructed irrespective of the choice of the utility functions.

Remark 6 Assertion 2 is sufficient for the existence of a no-trade equilibrium even if the equilibrium is not efficient. This follows since any market clearing allocation which satisfies Assertion 2, can be financed by a no-trade portfolio policy independently of whether the equilibrium is efficient or not.

4.2 Examples

We illustrate here the implications of Theorem 1 for some common classes of utility functions. For simplicity we assume throughout this section that there are only two consumptions goods ($A = 2$) and that there is only one security paying out in each of the two available consumption goods ($n_1 = n_2 = 1$).

Constant Elasticity of Substitution

As a first example, we consider the class of CES utility functions. Agents' utility functions display constant elasticity of substitution when they take the parametric form

$$u_i(t, c) := e^{-kt}\left(\alpha_{i1}\left[c^1\right]^{\rho_i} + \alpha_{i2}\left[c^2\right]^{\rho_i}\right)^{\frac{1}{\rho_i}}$$

where k is a non negative constant subjective discount rate which we assume equal across agents, $\rho_i \in (0,1)$ and $\alpha_{ia} > 0$ are fixed constants. Using the equivalent assertions of Theorem 1 we now show that, given such preferences, a no-trade efficient equilibrium exists if and only if the agents have the same elasticity of substitution.

Corollary 1 *Assume that agents have constant elasticity of substitution utility functions. Then a no-trade efficient equilibrium exists if and only if $\rho_1 = \rho_2$.*

Proof. According to the third assertion of Theorem 1, we have that there exists a no-trade efficient equilibrium if and only if there exist $(w^a) \in (0,1)^2$ such that

$$\frac{\alpha_{11}}{\alpha_{12}}\left[\frac{w^1 D_t^1}{w^2 D_t^2}\right]^{\rho_1 - 1} = \frac{\alpha_{21}}{\alpha_{22}}\left[\frac{(1-w^1)D_t^1}{(1-w^2)D_t^2}\right]^{\rho_2 - 1}$$

and the static budget constraint (9) holds. If the coefficients ρ_i differ across agents, then the above equation implies that the dividends are proportional almost everywhere. This contradicts the assumption that the volatility matrix of the dividend process has full rank, and thus implies that the above equation can only hold if $\rho_1 = \rho_2$. Assuming that this is indeed the case and solving the above equation for the constant w^1 we obtain

$$w^1 = g(w^2) := \left\{ 1 + \frac{1 - w^2}{w^2} \left[\frac{\alpha_{11}\alpha_{22}}{\alpha_{12}\alpha_{21}} \right]^{\frac{1}{\rho_1 - 1}} \right\}^{-1}.$$

Plugging this relation back into the static budget constraint of agent 1, we obtain that a no-trade efficient equilibrium exists provided that there exists a strictly positive constant $\phi \in (0,1)$ such that

$$h(\phi) := E\left[\int_0^T \nabla u_1 \left(t, GD_t\right)^* \left\{ G - \Phi\left(\nu_1^a\right) \right\} D_t dt \right] = 0$$

where G denotes the diagonal matrix with diagonal elements $g(\phi)$. Using well-known analytic arguments, as found for example in Detemple and Serrat (2003), it is easily shown that under our assumptions the function h is continuous on the interval $(0,1)$ with

$$h(0+) := \lim_{\phi \to 0} h(\phi) < 0 < h(1-) := \lim_{\phi \to 1} h(\phi).$$

This implies the existence of a point ϕ such that $h(\phi) = 0$ and it follows that there exists a no-trade efficient equilibrium. ■

Note that in this example the existence of a no-trade efficient equilibrium does not require that the agents have the same utility function. In particular, a no-trade efficient equilibrium can exist even though the agents attribute different weights to consumption in each of the goods ($\alpha_{1a} \neq \alpha_{2a}$).

Non Separable Cobb–Douglas Preferences

Agents have non separable Cobb–Douglas preferences, if their utility function take the parametric form

$$u_i(t,c) := e^{-kt} \left[c^1\right]^{\alpha_{i1}} \left[c^2\right]^{\alpha_{i2}}$$

where k is a non negative constant subjective discount rate which we assume equal across agents and $\alpha_{ia} \in [0,1]$ are constants such that $\alpha_{i1} + \alpha_{i2} < 1$. Note that this parametric form is the limit of the constant elasticity of substitution specification as the coefficient ρ_i goes to zero.

Corollary 2 *Assume that agents have non separable Cobb–Douglas preferences. Then a no-trade efficient equilibrium exists if and only if* $\alpha_{1a} = \alpha_{2a}$ *for all* $a \in \mathcal{A}$.

Proof. According to the third assertion of Theorem 1, we have that a no-trade efficient equilibrium exists if and only if

$$\left[\frac{D_t^1}{D_0^1}\right]^{\alpha_{11}-\alpha_{21}} = \left[\frac{D_t^2}{D_0^2}\right]^{\alpha_{22}-\alpha_{12}}$$

and there are constants $(w^a) \in (0,1)^2$ such that

$$\frac{\alpha_{12}\alpha_{21}}{\alpha_{11}\alpha_{22}}\left[\frac{1-w^2}{w^2}\right] = \left[\frac{1-w^1}{w^1}\right]$$

and the static budget constraint (9) holds. The dividend processes associated with the risky securities being linearly independent by assumption, the first of the above equations cannot hold unless we have $\alpha_{1a} = \alpha_{2a}$ and solving the second equation for the constant w^1 we find

$$w^1 = g(w^2) := \left\{1 + \frac{\alpha_{12}\alpha_{21}}{\alpha_{11}\alpha_{22}}\left[\frac{1-w^2}{w^2}\right]\right\}^{-1}.$$

Plugging this relation back into the static budget constraint of agent 1 and invoking an argument similar to that used in the proof of Corollary 1 then gives the existence of a no-trade efficient equilibrium. ■

Separable Cobb–Douglas Preferences

Agents have separable Cobb–Douglas preferences, if their utility function takes the parametric form

$$u_i(t,c) := e^{-kt}\left\{\frac{[c^1]^{\alpha_{i1}}}{\alpha_{i1}} + \frac{[c^2]^{\alpha_{i2}}}{\alpha_{i2}}\right\}$$

where k is a non negative constant subjective discount rate which we assume equal across agents and $\alpha_{ia} \in (-\infty, 1)\backslash\{0\}$ are constants. Using the equivalent assertions of Theorem 1, we now show that a no-trade efficient equilibrium exists if and only if the agents preferences exhibit the same relative risk aversion.

Corollary 3 *Assume that agents have separable Cobb–Douglas preferences. In this case, a no-trade efficient equilibrium exists if and only if* $\alpha_{1a} = \alpha_{2a}$ *for all* $a \in \mathcal{A}$.

Proof. According to the third assertion of Theorem 1, we have that there exists a no-trade efficient equilibrium if and only if

$$\left[\frac{D_t^1}{D_0^1}\right]^{\alpha_{11}-1} = \left[\frac{D_t^1}{D_0^1}\right]^{\alpha_{21}-1},$$

and there exist constants $(w^a) \in (0,1)^2$ such that

$$\frac{[w^2 D_t^2]^{\alpha_{12}-1}}{[(1-w^2)D_t^2]^{\alpha_{22}-1}} = \frac{[w^1 D_0^1]^{\alpha_{11}-1}}{[(1-w^1)D_0^1]^{\alpha_{21}-1}},$$

and the static budget constraint (9) holds. The first of these equations implies that $\alpha_{11} = \alpha_{21}$. On the other hand, the dividend processes associated with the risky securities being stochastic, the second of the above equation cannot hold unless we have $\alpha_{12} = \alpha_{22}$. Assuming that this is the case and solving the second equation for the non negative constant w^1 we find

$$w^1 = g(w^2) := \left\{ 1 + \left[\frac{1-w^2}{w^2}\right]^{\frac{\alpha_{12}-1}{\alpha_{11}-1}} \right\}^{-1}.$$

Plugging this relation back into the static budget constraint of agent 1 and invoking an argument similar to that used in the proof of Corollary 1 then gives the existence of a no-trade efficient equilibrium. ■

Remark 7 In the absence of non traded goods, this specification of the agents utility function is a special case, with identical discount rates, of that employed in Serrat (2001). In his Section 3.2 the author claims that in the absence of non tradable goods agents follow buy and hold strategies. Using the above results, we note that this claim is only valid provided that the subjective discount rate is the same for the two agents.

Log–Linear Preferences

Agents have log–linear preferences if their utility functions take the parametric form

$$u_i(t,c) := e^{-kt}\Big\{\alpha_{i1}\log(c^1) + \alpha_{i2}\log(c^2)\Big\}$$

where k is a non-negative constant subjective discount rate which we assume equal across agents and α_{ia} are strictly positive, agent specific constants. This specification of preferences has been used in numerous studies including those of Zapatero (1995) and Cass and Pavlova (2004).

Corollary 4 *Assume that agents have log–linear preferences. Then a no–trade efficient equilibrium exists for all* $(\alpha_{ia}) \in (0,\infty)^4$.

Proof. Using the equivalent assertions of Theorem 1 in conjunction with the log–linear structure of the agents' preferences, we deduce that a no-trade efficient equilibrium exists if and only if the two-dimensional system

$$
\begin{aligned}
\frac{\alpha_{12}\alpha_{21}}{\alpha_{11}\alpha_{22}} &= \left[\frac{1-w^1}{w^1}\right]\left[\frac{w^2}{1-w^2}\right],\\
1 &= \left[\frac{w^1}{\alpha_{11}w^1-\nu_1^1}\right]\left[\frac{\nu_1^2-\alpha_{12}w^2}{w^2}\right],
\end{aligned}
$$

admits a solution in $(0,1)^2$. Using the first equation to express w^1 as a function of w^2 and plugging the result in the second equation we obtain that the above system admits a unique solution, which is explicitly given by

$$
\begin{aligned}
w^2 &= \frac{\alpha_{12}\alpha_{21}\alpha_{22}\nu_1^2+\alpha_{11}\nu_1^1(\alpha_{22}\alpha_{21}+\alpha_{11}\alpha_{12})}{\alpha_{22}\alpha_{21}(\alpha_{11}+\alpha_{12})+\alpha_{11}\nu_1^1(\alpha_{12}\alpha_{11})},\\
w^1 &= \frac{\alpha_{22}\alpha_{21}w^2}{\alpha_{22}\alpha_{21}+\alpha_{12}\alpha_{11}(1-w^2)}.
\end{aligned}
$$

Using the fact that the non negative constants ν_1^a are smaller than one by assumption, it is then easily seen that the above solution belongs to $(0,1)^2$ and it follows that a no-trade efficient equilibrium exists. ■

Having established the existence of a no-trade efficient equilibrium, we can now recover its main characteristics from Proposition 2. In particular, equation (4) identifies the vector of good specific equilibrium state prices as

$$
p_t\xi_t = \left\{\frac{D_0^1}{D_t^1}, \frac{\alpha_{12}w_1D_0^1}{\alpha_{11}w_2D_t^2}\right\}^*
$$

and plugging this back into the pricing relations (6) shows that the equilibrium prices of the risky securities satisfy

$$
S_t^1 = \tfrac{1}{k}\left(1-e^{-k(T-t)}\right)D_t^1 = \frac{\alpha_{11}w^2}{\alpha_{12}w^1}S_t^2.
$$

With this particular form of utility function the price of the first stock is a linear function of the first dividend, and since the relative price of the second good is proportional to the ratio of dividends, the price of the second stock is also a linear function of the first dividend. It follows that the stock volatility matrix is degenerate and hence that markets are incomplete even though the dividends were assumed to be linearly independent. In Cass and Pavlova (2004) this situation is labelled as a Peculiar Financial Equilibrium. We stress here the fact that this type of equilibrium is not equivalent to the no trade equilibria we identified in Theorem 1. The two latter classes intersect in the log linear case, but peculiar financial equilibria are associated to the log-linear specification, whereas no-trade efficient equilibria may occur for any utility function satisfying the conditions of Theorem 1.

5 Discussion and Extensions

As demonstrated by the above examples, a no-trade efficient equilibrium can exist even if the agents do not have the exact same preferences. Given this result, one naturally wonders what other sources of heterogeneity could generate non trivial equilibrium trading volume. In order to partially answer this question, we now briefly discuss two extensions of the basic model: one incorporating heterogeneous beliefs and one where the agents receive random flows of endowments through time.

5.1 Heterogeneous Beliefs

Throughout the paper we have maintained the assumption that agents differ only through their utility functions and initial portfolios. In particular, we have assumed up to now that the two agents share the same beliefs about the current state of the economy and its future evolution.

Standard economic intuition strongly suggests that heterogeneity in beliefs is likely to increase exchanges among agents. In order to confirm this intuition, we consider an economy with a single consumption good and a single risky security ($A = \mathbf{n} = 1$). Agent i's beliefs are summarized by the density process Z^i of his equivalent subjective probability measure P^i with respect to the objective probability measure P. As a result, the preferences of agent i are represented by a time additive expected utility functional:

$$U_i(c) := E^i\left[\int_0^T u_i(t, c_t)dt\right] = E\left[\int_0^T Z_t^i u_i(t, c_t)dt\right].$$

where E^i is the expectation operator under the agent's probability measure P^i. The economy is otherwise identical to our previous description.

Now assume that there exists a no–trade efficient equilibrium. In any such equilibrium, the Pareto optimality of the consumption allocations and the fact that there is a single risky asset imply that

$$u_1'(t, wD_t)Z_t^1 = \lambda u_2'(t, (1-w)D_t)Z_t^2$$

holds almost everywhere for some strictly positive constants λ and $w < 1$. For this relation to hold, divergence in beliefs must exactly compensate the potential divergence in marginal utilities. While it might be possible to construct such beliefs structures, their economic relevance seems doubtful.

5.2 Random Endowments

In this section we present an extension of the results of Theorem 1 to the case where the agents differ not only in their preferences and initial portfolios but also in their intertemporal endowments. To accommodate such an extension of the basic model we assume throughout this section that there are multiple goods but a single traded security paying its dividends in each of the available goods ($n_a = 1, a \in \mathcal{A}$).

In addition to an initial portfolio of shares of the available securities, agents now receive a random flow of endowment in each of the available good. We denote by e_i^a the rate at which agent i receives his endowment in good a and assume that the corresponding vector of good specific endowments is a bounded Itô process of the form

$$e_{it} = e_{i0}^a + \int_0^t \left[\varsigma_{is} ds + \tau_{is} dB_s\right]$$

for some exogenously given drift process ς_i and volatility matrix τ_i. In such a setting, a consumption plan is said to be feasible for agent i if there exists an admissible trading strategy θ whose wealth process satisfies the dynamic budget constraint

$$\begin{aligned} W_0 &= \theta_0^0 + \sum_{a \in \mathcal{A}} (\theta_0^a)^* S_0^a = W_0^i := \sum_{a \in \mathcal{A}} (\nu_i^a)^* S_0^a \\ dW_t &= \theta_t^0 dS_t^0 + \sum_{a \in \mathcal{A}} \left\{ (\theta_t^a)^* \left[dS_t^a + p_t^a D_t^a dt \right] - p_t^a \left(c_t^a - e_t^a \right) dt \right\}, \end{aligned}$$

and has a non negative terminal value. In the following corollary we provide necessary and sufficient conditions for the existence of a no trade equilibrium for the above continuous time economy with random endowments. We state the results without proof, as they are simply obtained by replacing c_i^a by $c_i^a - e_i^a$ in the proof of Theorem 1.

Corollary 5 *The following assertions are equivalent*

1. *There exists an efficient equilibrium that has no-trade in the sense that both $\theta_{1t}^0 = \theta_{10}^0$ and $\theta_{1t}^a = \theta_{10}^a$ hold for all $a \in \mathcal{A}$ and there is no activity on the spot market for the consumption goods.*

2. *There exists an efficient equilibrium in which the individual consumption policies satisfy*
$$\frac{c_{it}^a - e_{it}^a}{c_{i0}^a - e_{i0}^a} = \frac{D_t^a}{D_0^a}, \qquad i \in \{1, 2\}.$$
where D^a is the dividend process associated with the only security paying out in good $a \in \mathcal{A}$.

3. *There exists a diagonal matrix ϕ with strictly positive, constant diagonal elements $(\phi^a)_{a\in\mathcal{A}}$ such that*

$$\frac{\nabla u_1(t,\phi D_t + e_{1t})}{\nabla_1 u_1(0,\phi D_0 + e_{10})} = \frac{\nabla u_2(t, D_t - \phi D_t + e_{2t})}{\nabla_1 u_2(0, D_0 - \phi D_0 + e_{20})},$$

and

$$E\left[\int_0^T \nabla u_1(t,\phi D_t + e_{1t})^* \Big\{(\phi - \Phi(\nu_1^a))\, D_t - e_{1t}\Big\} dt\right] = 0,$$

where D denotes the vector of good specific dividend processes and Φ is the linear operator defined in Section 3.1.

Note that, as in the case of heterogeneous beliefs considered in the previous section, it is always possible to construct the agents' endowment processes in such a way that, given the other primitives of the economy, there exists a no-trade efficient equilibrium. An example of such a construction, albeit in a slightly different setup, can be found in Constantinides and Duffie (1996).

6 Conclusion

In this paper we have investigated under what conditions non informational heterogeneity among agents leads to non trivial equilibrium trading volume. Our main result comes in the form of a non informational no-trade theorem which provides necessary and sufficient conditions for the existence of of a no-trade equilibrium in a continuous time economy with multiple goods and heterogenous agents. Our results are illustrated on a number of classical examples and relations with the literature on linear sharing rules are addressed.

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