Portf o Optimization with Conc e Transaction Costs

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Research F N° 103
December

FAME - Interna enter for Financial Asset Management and Engineering

IBF

HEC LAUSANNE

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Abstract

In this paper we study the optimal portfolio management for the constant relative-risk averse investor who maximizes an expected utility of his terminal wealth and who faces transaction costs during his trades. In our model the investor's portfolio consists of one risky and one risk-free asset, and we assume that the transaction cost is a concave function of the traded volume of the risky asset. We find that under such transaction cost formulation the optimal trading strategies and boundaries of the no-transaction region are different than those when transaction costs are proportional, i.e. when they are linear in the traded volume. When transaction costs are concave, we show that the no-transaction region is narrower than when transaction costs are proportional, and it is not a positive cone. Under our transaction cost formulation, when the investor's wealth is relatively high, the optimal trading strategy consists in bringing the post-trade portfolio position inside the no-transaction region, whereas proportional transaction costs induce the investor trading to the boundary of the no-transaction region. We also examine the impact of the risky asset volatility and the risk aversion parameter on the shape of the no-transaction region. When comparing different transaction cost structures, we show that the financial securities' market tends to be more liquid with concave transaction costs than with alternative cost specifications.

Keywords: concave transaction costs, optimal trading strategy

JEL Classification: C61, G11

*I thank Rajna Gibson, René Stulz, Marc Chesney, Kyung-Ha Cho and the participants of FAME PhD workshops for their helpful comments and suggestions. (E-mail: ad@fame.ch)

1 Introduction

In this paper we study the optimal portfolio management problem for a constant relative risk-averse investor who maximizes his expected utility of terminal wealth. In our model the investor's portfolio consists of one risky and one risk-free asset. Whenever the investor rebalances his portfolio in favor of one or the other asset class, he faces transaction costs. Transaction costs are assumed to be proportional to the volume of the risky asset traded. The proportionality rate, or the transaction cost rate, is assumed to depend on the traded volume of the risky asset.

This asset allocation problem is a variant of classical consumption - investment problem in modern finance. However, transaction costs add considerable complexity in the portfolio optimization problem. In the absence of transaction costs, Merton (1971) obtained the closed-form solution for the optimal portfolio holdings for a constant relative risk-averse investor. Namely, if the price of the risky asset (stock) follows a geometric Brownian motion and the risk-free asset instantaneously yields a constant rate of return r, then the optimal fraction of wealth to be invested in the stock is independent from time and is equal to:

$$\omega^M = \frac{1}{\gamma}\frac{\mu - r}{\sigma^2}$$

where γ is the relative risk aversion parameter, μ is the expected return on the stock and σ is the volatility of the stock returns. This solution implies that the investor should continuously rebalance his portfolio such that the fraction of his current wealth invested in the stock is equal to ω^M. In the risky wealth - risk-free wealth space, the latter condition means that the portfolio holdings should always be located on the so-called Merton line, i.e.

$$\frac{\text{Risky wealth}_t}{\text{Risk-free wealth}_t} = \frac{\omega^M}{1 - \omega^M}$$

This kind of adjustments can be accomplished in the absence of transaction costs. But when there are transaction costs (TC), the continuous trading would lead to ruin in a very short period of time. Therefore, there must be time intervals when the investor does not trade. It is said that his portfolio position belongs to the no-transaction (NT-) region during such time intervals. The identification of the no-transaction region as well as of optimal trading volumes was in fact the subject of a large body of literature devoted to the optimal portfolio management when there are transaction costs[1].

[1] A brief review of the transaction costs models is presented in section 2.

In this paper, we study the optimal portfolio management for the constant relative-risk averse investor who maximizes an expected utility of his terminal wealth and who faces transaction costs during his trades in a discrete time framework. In our model the investor's portfolio consists of one risky and one risk-free asset, and we assume that the transaction cost is a concave function of the traded volume of the risky asset. The concavity of transaction costs is modeled in the following way: We assume that the investor pays a higher fraction[2] of the traded amount of the risky asset as a transaction cost (TC) if this amount is below some exogenously specified level, and he pays a lower fraction as a TC for the amount which exceeds this level. In other words, the transaction cost rate is assumed to be a decreasing step function of the trading volume of the risky asset. We assume that the investor pays a fixed fraction δ_1 of the traded volume of the risky asset as the transaction cost if the absolute value of the trade does not exceed a certain level which we denote by λ. But, if the investor trades more than λ, then he receives a transaction cost rate discount and pays a lower fraction δ_2 for the trading volume which exceeds λ. We should notice that this cost structure can be considered as a combination of the following two structures: proportional transaction costs when the trade is "low" and proportional transaction costs with a fixed component when the trade is "high". However, the investor has an option to choose the appropriate cost structure trough the trading volume of the risky asset. We find that under such transaction cost structure the investor's optimal trading policy can differ significantly from the one when the transaction cost rate is constant and independent from the trading volume. Namely, the boundaries of the no-transaction region are different than those when transaction costs are proportional, i.e. when they are linear in the trading volume. When transaction costs are concave, we show that the no-transaction region is narrower than when transaction costs are proportional, and it is not a positive cone. Under our transaction cost formulation, when the investor's wealth is relatively high, the optimal trading strategy consists in bringing the post-trade portfolio position inside the no-transaction region, whereas proportional transaction costs induce the investor trading to the boundary of the no-transaction region. We also examine numerically the impact of the risky asset volatility and the risk aversion parameter on the shape of the no-transaction region. We show that the no-transaction region shifts down in the riskless-risky asset space when the risky asset volatility increases or when the relative risk aversion parameter increases. We also show that the

[2] We call this fraction as a transaction cost rate.

no-transaction region widens as time to maturity declines. When comparing different transaction cost structures, we show that the financial securities' market tends to be more liquid with concave transaction costs than with alternative cost specifications.

The rest of the paper is organized as follows. In section 2, we provide a literature review of the transaction costs models. In section 3, we describe the economy and the objective function of the investor. We also discuss the shape of the transaction cost function. In section 4, we present the results of portfolio optimization problem when transaction costs are proportional. In section 5, we introduce concave transaction costs and discuss some features of the value function. In section 6, we present numerical results and section 7 concludes the paper.

2 Literature review

In general, all the studies on portfolio optimization with transaction costs differ from each other either through the modelling of transaction costs structure or with respect to the objective function of investors. We would like to stress the attention on the fist source of difference since it is directly related to the subject of the given paper. Structures of transaction costs have been modelled in several distinct ways.

Under a first approach, the investor has to pay a fixed fraction of his current wealth at the time of the transaction. This is the so-called portfolio management fee approach. The investigation of models with such cost structure was done, for example, by Morton and Pliska (1995), Cadenillas and Pliska (1996), Atkinson and Willmot (1995). The main result of these studies is that as soon as the portfolio position leaves the no-transaction region, the investor trades in order to bring the portfolio back to the optimal interior point of the no-transaction region, and the latter is a positive cone.

In a second approach, the transaction costs are assumed to be proportional to the trading volume of the risky assets, where the proportionality rate is constant and less than one. Under this cost formulation, the optimal consumption-investment policy has been studied by many authors in the continuous as well as the discrete time framework. Constantinides (1979) considers a discrete-time version of the proportional transaction cost model when there are one risky and one risk-free asset in the economy. He finds that the no-transaction region is a positive cone and the optimal investment policy is "*simple*", that is characterized by two reflecting barriers a and b ($a \leqq b$) such that the investor does not trade when the ratio of his risky wealth to the riskless wealth belongs to the interval $[a, b]$ and trades to the nearest boundary in the opposite case. In the continuous-time framework, when the price of the risky asset follows a geometric Brownian motion, Constantinides (1986), Davis and Norman (1990), and Dumas and Luciano (1991) investigate the problem for the investor maximizing his expected utility of the future consumption stream, whereas Assaf, Taksar and Klass (1983) study optimal trading strategies for investors maximizing the asymptotic growth rate of the portfolio value. These studies prove that the optimal investment policy is "simple" as well, that is, there are no trades when the portfolio position lies within the no-transaction region, and infinitesimal trades when the position reaches the boundary of the no-transaction region. In the later case the continuous rebalancing is optimal during some time intervals. Genotte and Jung (1994) solve numerically the discrete-time model with proportional transaction costs and analyze optimal trading strategies for a CRRA investor with a finite horizon. Boyle and Lin (1997) consider the model with two securities in the discrete time framework when the investor maximizes his expected utility of final wealth. For the power utility function case they derive the explicit analytical solutions for the boundaries of the no-transaction region and for the optimal trading volumes. Since their paper is of particular interest for our investigation, we present its main results in Section 3.

There is also a number of papers that include the fixed cost component in the structure of transaction costs. Eastham and Hasting (1988) study a finite horizon problem when the investor derives utility from consumption. They show that with a fixed cost component in transaction costs, the optimal trading strategies allow for a finite number of trades on finite time intervals. Øksendal and Sulem (1999) and Zakamouline (2002) study optimal portfolio selection with both fixed and proportional transaction costs. Their results indicate that the boundaries of the no-transaction region are wealth-dependent, and the optimal trading strategies consist in bringing the post-trade portfolio position inside the NT-region. Korn (1998) adds a fixed cost component to the proportional transaction cost and applies a formal optimal stopping approach and an approach using quasi-variational inequalities to solve the consumption-investment problem. Duffie and Sun (1990) combine three different approaches we have mentioned above to model transaction costs. Namely, at the time of the transaction the investor has to pay a fixed fraction of the portfolio value plus a proportional cost for withdrawal of cash for consumption plus a fixed cost as the transaction cost. They argue that a fixed cost component influences the solution significantly. When a fixed component is equal to zero, it is optimal to transact at fixed deterministic intervals of time. In the opposite case, the length of trading time intervals depends on the total wealth at the beginning of each interval.

Konno and Wijyanayake (2001) study the portfolio optimization problem under concave transaction costs and the minimal transaction volume constraints. The authors propose an algorithm for calculating a globally optimal solution of a portfolio construction/rebalancing problem for an investor who minimizes portfolio risk for a given level of the expected return. In the paper, the authors consider the absolute deviation of portfolio returns as a measure of risk, and thus reduce the problem to the one of linear optimization. They report that the proposed algorithm is very efficient with respect to the computation time. However, in their paper, Konno and Wijyanayake concentrate solely on the efficiency of the algorithm and do not study, on the contrary to our paper, optimal trading strategies and the boundaries of the no-transaction region. As we discuss in Section 6, the computation time becomes a real problem when we try to solve our dynamic programming problem in the multi-period setting. Unfortunately, we cannot apply the algorithm proposed by Konno and Wijyanayake since the objective function in our paper is not linear, but, as we show in Section 5, it is piece-wise concave and not differentiable everywhere. Also, due to that fact that we cannot obtain an explicit analytical formula for the value function, the identification of the no-transaction region and of the optimal trading strategies requires to solve the optimization problem at each discretization point of a sub-set of possible portfolio allocations. Therefore, even if the algorithm we build is computationally very efficient for the last period optimization problem, the computational time becomes already an issue when we try to solve the dynamic programming problem for two periods.

All modelling approaches described above (except the last one) have one common feature. Namely, the transaction cost rate, which is the fraction of either a total portfolio value or a trading volume the investor has to pay as a transaction cost, was assumed to be a fixed constant for any value of the trading volume. In this paper we study how the deviation from linear to concave transaction costs affects the optimal trading strategies of the constant relative risk-averse investor.

3 The Economy

We consider the economy in which there are two assets available for the investment: one risky asset Y which pays no dividends and one risk-free asset X. We assume that the investor has a power utility function

$$U(W) = \frac{W^\gamma}{\gamma}, \quad \gamma < 1 \tag{0}$$

and his investment horizon is finite and is equal to T[3]. We also assume that the investor observes the prices of the assets only at discrete times $t = 0, 1, ..., T$, and, as a consequence, he can trade only at these times. Put differently, we assume that it is too costly for the investor to monitor the prices of the assets continuously in time, and thus he revises his portfolio positions periodically at some discrete times (for example, once a day, or once a month, or once a year). We assume that the price of the risky asset evolves as:

$$\frac{Y_{t+1}}{Y_t} = \mu_t, \quad t = 0, 1, ..., T-1 \tag{1}$$

where Y_t is the price of the risky asset at time t and μ_t is its total rate of return over the period $(t, t+1)$, and $\{\mu_t\}_{t=0,...,T-1}$ are assumed to be i.i.d. discrete random variables with a finite number of states $\{w_i\}_{i=1}^n$ which can occur with probabilities $\{p_i\}_{i=1}^n$ respectively. Therefore, under the above assumption the price of the risky asset $\{Y_t\}_{t=1,...,T}$ is a Markov chain. The price of the riskless asset is supposed to grow at a constant rate r, that is:

$$\frac{X_{t+1}}{X_t} = r, \quad t = 0, 1, ..., T-1. \tag{2}$$

The portfolio position of the investor at time t is denoted by the vector (x_t, y_t), where x_t and y_t are the dollar values of the holdings in the riskless and the risky asset respectively. We assume that at time $t = 0$ the investor starts his business being endowed with $x_0 \geq 0$ and $y_0 \geq 0$. It is also assumed that whenever the investor buys or sells the risky asset he pays the transaction cost TC at the expense of riskless asset. For instance, if at time t he decides to buy or sell the amount v_t of the risky asset[4], then the post-trade holdings are:

$$\begin{aligned} y_{t^+} &= y_t + v_t \\ x_{t^+} &= x_t - v_t - TC_t \end{aligned} \tag{3}$$

We have mentioned in the Introduction that there are different approaches to model transaction costs, but at this stage we do not specify

[3]The case $\gamma = 0$ corresponds to the logarithmic utility function: $U(W) = \ln W$.

[4]Here and in the rest of the paper v_t represents the dollar amount of the traded risky asset, i.e. number of stocks multiplied by the stock price.

the nature of TC_t. We develop this issue in the sections below. In the formula above a positive (negative) value of v_t means that the investor buys (sells) the risky asset. To make things simple, in the model we assume that the transaction cost does not depend on the direction of trade: the investor pays the same transaction cost if he buys or sells the same volume of the risky asset[5]. Given the price dynamics of the risky and the risk-free assets (1)-(2), the portfolio position at time $t+1$ is then

$$\begin{aligned} y_{t+1} &= y_{t^+}\mu_t = (y_t + v_t)\mu_t \\ x_{t+1} &= x_{t^+}r = (x_t - v_t - TC_t)r \end{aligned} \tag{4}$$

In addition to that, we assume that short-selling of the risky asset and borrowing cash are not allowed. This restriction implies that at any point of time the trading volume of the risky asset v_t should be such that

$$y_{t^+} \geq 0 \quad \text{and } x_{t^+} \geq 0 \qquad\qquad t = 0, 1, ..., T-1 \tag{5}$$

In the given economy the problem of the investor is to choose an optimal trading strategy $(v_0, v_1, ..., v_{T-1})$ in order to maximize his expected utility of terminal wealth, i.e.

$$J(0, x_0, y_0) := \underset{\{v_0, v_1, ..., v_{T-1}\}}{\text{Max}} E_0[U(x_T + y_T)] \tag{6}$$

where x_0, and y_0 are the initial holdings in the risk-free and the risky asset respectively, E_0 denotes time zero unconditional expectation, and $J(0, x_0, y_0)$ is the initial value function of the investor.[6]

The portfolio optimization problem (6) can be solved by applying a dynamic programming technique. Therefore, we define the value function in a recursive way, i.e.

$$J(t, x_t, y_t) = \underset{v_t}{\text{Max}} E_t[J_{t+1}(t+1, x_{t+1}, y_{t+1})], \quad t = 0, 1, ...T-1 \tag{7}$$

[5]One can also consider the asymmetric cost structure, but this will no change our results qualitatively.

[6]Notice that the maximization problem (6) is based on the fact that at terminal date T the investor derives utility from the total portfolio capitalization at that time T. One can however consider the problem when the utility is derived from the portfolio liquidation value:

$$\text{Max } E_0[U(x_T + y_T - TC_T)]$$

where TC_T is the transaction cost associated with the selling of the value y_T of the risky stock.

with a terminal condition

$$J(T, x_T, y_T) = U(x_T + y_T) = \frac{1}{\gamma}(x_T + y_T)^\gamma \tag{7.1}$$

where y_{t+1} and x_{t+1} are given by (4), and E_t denotes time t conditional expectation. Notice, that due to the Markovian property of the stock price evolution, the time t conditional expectation is based only on the current portfolio holdings x_t and y_t. In the maximization problem at hand, v_t is a control parameter, and thus, as proposed by Boyle and Lin (1997), it is convenient to introduce the following function:

$$\varphi(t, x_t, y_t, v_t) = E_t J(t+1, x_{t+1}, y_{t+1}) \tag{8}$$

Thus, the value function (7) can be rewritten as

$$J(t, x_t, y_t) = \underset{v_t}{\text{Max}}\ \varphi(t, x_t, y_t, v_t) \tag{9}$$

Therefore, due to the dynamic programming principle, the maximization problem (6) is reduced to the searching for the optimum of the function $\varphi(t, x_t, y_t, v_t)$ with respect to the trading volume v_t at every date $t = 0, 1, ..., T-1$. Along with the determination of optimal trading volumes at each point of time t, we are interested in the identification of those portfolio positions (x_t, y_t) from which it is optimal not to trade. It is said, that in the latter case portfolio belongs to the no-transaction region (or continuation region, since the investor continues to hold his portfolio without changes). Formally, the no-transaction region can be defined in the following way: at time t, portfolio position (x_t, y_t) belongs to the NT (continuation) region if and only if $v_t = 0$ is the solution to the optimization problem (9). Put it differently, the continuation region G_t is

$$G_t = \{(x_t, y_t) : \ \varphi(t, x_t, y_t, v_t) \leqq \varphi(t, x_t, y_t, 0) \quad \text{for all } v_t \text{ such that (5) holds}\}.$$

Clearly, the shape of the no-transaction region, as well as of the optimal trading strategies depend on the structure of transaction costs. For example, when transaction costs are proportional, the boundaries of the no-transaction region depend only on time, but not on the investor's wealth. On the other hand, the presence of a fixed component in transaction costs results in the dependence of the boundaries on the investor's wealth. In the following sections, we discuss this dependency in more details.

3.1 Transaction costs

Very generally, transaction costs are considered as comprising of two parts, an asset exchange or brokerage fee and a liquidity or marketability cost. The transaction cost associated with the transaction volume is depicted in Fig.1. Up to a certain level of the transaction volume (point A on Fig.1) the transaction cost is a concave function of the transaction volume. This is because the unit transaction cost, e.g. unit brokerage fee, is relatively large when the transaction volume is small and it gradually decreases as the transaction volume increases. The extract from the transaction fees charged by the New York Stock Exchange (Table 1) and prices for securities trading with UBS e-banking via Internet (Table 2) support the concavity of the transaction cost function.

However, beyond point **A** (Fig.1), the unit transaction cost may increase due to the "illiquidity" effect. That is, if for a certain security the transaction order is high, then there can be not enough supply (demand) of this security, and thus the unit transaction cost will increase (the transaction cost function becomes convex).

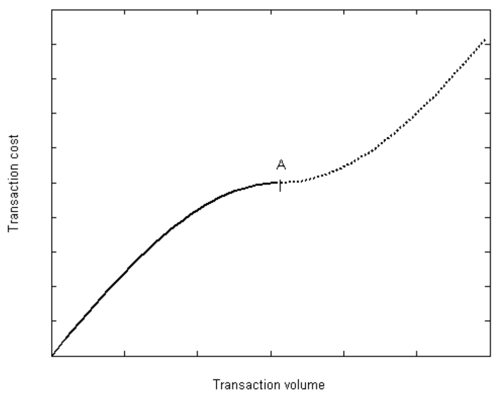


Fig. 1. Transaction cost function.

In the paper we assume that transaction volume is always "moderate" and does not go beyond this critical point. Let us notice that in a single-agent economy it is problematic to model the mentioned "illiquidity" effect for the following reason. No rational investor will trade beyond point **A** because he can always split his order into smaller ones to reduce transaction costs. To model the "illiquidity" effect one would need to introduce another agent, let say a market maker, who would be able to recognize split orders.

Such a modelling is not considered in the paper. Therefore, in our model the transaction cost function is assumed to be concave.

Table 1. Transaction fees charged by the New York Stock Exchange (an extract for equity transactions for the year 2003).

NYSE: Regular Session Trading	
Equity Transactions	
Per Share Charge – per transaction	
System orders under 2,100 Shares(4)	No Charge
Floor Executed Trades and System Trades greater than 2,099 Shares	
First 5,000 Shares	$0.0023
5,001 to 690,000 Shares	$0.0001
Subsequent Shares	No Charge

Table 2. Prices for securities trading with UBS e-banking via Internet

			Stock Exchange Transactions			
		Bonds, Eurobonds, Notes	SWX, virt-x (SMI)	virt-x (EU) Germany, GB, USA, Italy, Canada	Rest of Europe	Asia and other countries
Minimum prices (per order)		CHF 35	CHF 35	CHF 75	CHF 100	CHF 110
Trade value in CHF						
up to	10 000	0,70%	0,95%	1,50%	1,70%	1,90%
up to	15 000	0,70%	0,95%	1,50%	1,70%	1,90%
up to	25 000	0,70%	0,95%	1,50%	1,70%	1,90%
up to	50 000	0,65%	0,85%	1,45%	1,60%	1,80%
up to	100 000	0,55%	0,80%	1,35%	1,50%	1,70%
up to	200 000	0,40%	0,65%	1,25%	1,40%	1,60%
up to	300 000	0,40%	0,50%	1,15%	1,20%	1,55%
up to	400 000	0,40%	0,40%	1,15%	1,15%	1,35%
up to	1 000 000	0,30%	0,30%	0,95%	0,95%	1,15%
from	1 000 001	0,20%	0,20%	0,75%	0,75%	0,95%

The assumption usually made in the TC models is that transaction costs are proportional in purchases and sales of the risky security. That is, the transaction cost function is assumed to be liner. As we already mentioned in section 2, this assumption allowed many authors to obtain analytical solutions of the optimal trading strategies and of the no-transaction region. In the following section we present the main results of the study of Boyle and Lin (1997), who consider proportional transaction costs. We present this study since we use a similar economy setting. Then, in section 5, we present our modelling of the transaction cost function. Namely, we intend to capture the concavity of the transaction cost function by a linear approximation.

4 Proportional Transaction Costs

Boyle and Lin (1997) solve the investment problem in the economy described above when there are proportional transaction costs. Namely, the transaction cost paid at any trading time t is proportional to the trading volume v_t of the risky asset and is equal to

$$TC_t = \delta|v_t| \qquad t = 0, 1, ..., T-1 \tag{10}$$

where $0 < \delta < 1$ is the transaction cost rate and it is assumed to be invariant with respect to the trading volume. In this framework, the dynamics of the investor's portfolio holdings is:

$$\begin{aligned} y_{t+1} &= (y_t + v_t)\mu_t \\ x_{t+1} &= (x_t - v_t - \delta|v_t|\ )r \end{aligned} \tag{11}$$

Therefore, the objective of the investor is to define the optimal trading strategy $\{v_0, v_1, ..., v_{T-1}\}$ such that his expected utility derived from terminal wealth will be maximized (6). The imposed no-short selling and no borrowing condition (5) implies

$$-y_t \leqq v_t \leqq \frac{x_t}{1+\delta} \qquad t = 0, 1, ..., T-1. \tag{12}$$

Boyle and Lin solve this portfolio optimization problem by applying a dynamic programming technique. The value function and the terminal condition are defined by equations (7) and (7.1) respectively. Namely[7]:

$$J^P(t, x_t, y_t) = \underset{v_t}{\text{Max}} E_t[J^P(t+1, x_{t+1}, y_{t+1})], \quad t = 0, 1, ...T-1$$

and

$$J^P(T, x_T, y_T) = U(x_T + y_T) = \frac{1}{\gamma}(x_T + y_T)^\gamma$$

[7]Below we use supperscript P to denote that functions refer to the case of proportional transaction costs.

where y_{t+1} and x_{t+1} are given by (11). For the convenience of the analysis, they introduce function:

$$\varphi^P(t, x_t, y_t, v_t) = E_t J^P(t+1, x_{t+1}, y_{t+1}) \tag{13}$$

Thus, the value function J^P can be rewritten as

$$J^P(t, x_t, y_t) = \underset{v_t}{\text{Max}}\ \varphi^P(t, x_t, y_t, v_t) \tag{14}$$

Using the definition of the no-transaction region presented in the previous section, we write:

$$G_t^P = \{(x_t, y_t) :\ \varphi^P(t, x_t, y_t, v_t) \leqq \varphi^P(t, x_t, y_t, 0) \quad \text{for all } v_t \text{ satisfying (12)}\}.$$

Below, we present the main results of the study of Boyle and Lin (1997):

1. Since $U(W)$, defined by (0), is concave, differentiable and homogeneous function of degree γ, then $J^P(t, x_t, y_t)$ is also concave, differentiable and homogeneous function of the same degree. As a consequence, $\varphi^P(t, x_t, y_t, v_t)$ is concave with respect to v_t and is homogeneous of degree γ with respect to all its variables. Concavity and homogeneity are crucial properties of the function φ^P which enable the authors to obtain the explicit analytical solutions for the optimal trading volumes v_t as well as for the bounds of the no-transaction region.

2. There exist a_t and b_t, such that $a_t \leqq b_t$ and the no-transaction region at time t can be described as

$$G_t^P = \{(x_t, y_t) : a_t \leqq y_t/x_t \leqq b_t\}$$

It implies that the no-transaction region is a positive cone. This result is in line with all previous studies with proportional transaction costs. However, as we show later, this shape of the no-transaction region will not be preserved when transaction costs are concave.

3. At time t, portfolio (x_t, y_t) belongs to the no-transaction region if and only if the function $\varphi(t, x_t, y_t, v_t)$ attains its maximum at $v_t^* = 0$, and thus the bounds of the no-transaction region can be defined in the following way:

$$a_t = \min\ \left\{\frac{y_t}{x_t} :\ \frac{\partial \varphi^{P-}(t, x_t, y_t, 0)}{\partial v_t} \geq 0,\ \frac{\partial \varphi^{P+}(t, x_t, y_t, 0)}{\partial v_t} \leqq 0\right\} \tag{15}$$

$$b_t = \max\ \left\{\frac{y_t}{x_t} :\ \frac{\partial \varphi^{P-}(t, x_t, y_t, 0)}{\partial v_t} \geq 0,\ \frac{\partial \varphi^{P+}(t, x_t, y_t, 0)}{\partial v_t} \leqq 0\right\}$$

where $-(+)$ indicate the left hand side (right hand side, respectively) derivative. The above formulas for the boundaries of the no-transaction region are due to the global concavity of function φ^P with respect to the trading volume v_t.

4. If at time t portfolio (x_t, y_t) lies outside the no-transaction region, then the optimal transaction value is

$$v_t = \begin{cases} v_{t^+} = \frac{x_t a_t \; - \; y_t}{1+(1+\delta)a_t} & \text{if } \frac{y_t}{x_t} < a_t \\ v_{t^-} = \frac{x_t b_t \; - \; y_t}{1+(1-\delta)b_t} & \text{if } \frac{y_t}{x_t} > b_t \end{cases} \tag{16}$$

This value brings the post trade portfolio position to the nearest boundary of the no-transaction region, that is

$$\frac{y_{t^+}}{x_{t^+}} = \begin{cases} a_t & \text{if } \frac{y_t}{x_t} < a_t \\ b_t & \text{if } \frac{y_t}{x_t} > b_t \end{cases}$$

5. The value function is a piece-wise linear utility function with respect to U, that is there exists a certain sequence of increasing numbers $q_0, q_{1,...}, q_s$ such that

$$J^P(t, x, y) = \sum_{i=1}^{n} U(\alpha_{ij} x_t + \beta_{ij} y_t) p_i$$

where index j indicates that the ratio y_t/x_t belongs to the interval $[q_j, q_{j+1})$, $j = 0, ...s-1$, and p_i is the probability of the state w_i. The piece-wise linearity property of the value function was obtained due to its homogeneity.

The piece-wise linearity allows us to define the indirect utility function $J^P(t, x_t, y_t)$ for any t, an thus from formula (13) we obtain the expression for the function $\varphi^P(t, x_t, y_t, v_t)$. Then, formulas (15) and (16) yield analytical solutions for the boundaries of the no-transaction region and the optimal trading volumes, respectively.

In the following section we show that many of the "nice" features of the function $\varphi(t, x_t, y_t, v_t)$, in particular its global homogeneity and global concavity, are not preserved when transaction costs become concave. Therefore, we cannot solve our portfolio optimization problem analytically, but we solve it numerically.

5 Non-constant transaction cost rate

We propose a model that uses a transaction costs structure which has not been sufficiently explored in the existing literature. Namely, we assume that the transaction cost function is concave in the trading volume of the risky asset[8]. The justification of this modelling approach was already presented in section 3. We intend to capture the concavity of transaction costs by a piece-wise linear approximation.

We assume that during his trades the investor pays transaction costs which depend on the trading volume. That is, if the absolute value of the trading volume of the risky asset v_t is below a certain level, which we denote by λ, the investor pays $\delta_1|v_t|$ as a transaction cost. For trading volumes which are above λ, the investor receives a discount with respect to the transaction cost rate and pays lower fraction δ_2 of the trading volume which exceeds λ (Fig.2).

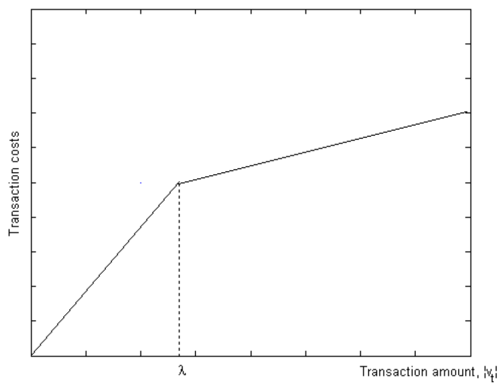


Fig. 2. Transaction cost function

We would like to notice that with a concave structure of transaction costs the investor has no incentives to split his orders because of non-increasing marginal transaction cost rate. In fact, the way we model transaction costs implies that the transaction cost rate is a decreasing step function. To formalize this modelling of the transaction costs, we write:

$$TC_t = \begin{cases} \delta_1|v_t| & \text{if } |v_t| \leqq \lambda \\ \delta_1\lambda + \delta_2(|v_t| - \lambda) & \text{if } |v_t| > \lambda \end{cases} \qquad \delta_1 > \delta_2 \qquad (17)$$

[8] As we mentioned in section 2, Konno and Wijyanayake (2001) study the portfolio optimization problem under concave transaction costs. However, their results do not provide any inside into the optimal trading strategies of the investor who faces concave transaction costs.

or equivalently

$$TC_t = \delta_1|v_t| - (\delta_1 - \delta_2)\ \max(|v_t| - \lambda, 0). \tag{18}$$

Analyzing the transaction cost specification provided by (17), we observe that the transaction costs are proportional when the trading volume is "low" ($|v_t| \leqq \lambda$), and they are proportional with a fixed component (the fixed component equals to$(\delta_1 - \delta_2)\lambda$) when the trading volume is high ($|v_t| > \lambda$). For these two cost specifications, the optimal trading strategies and the shapes of the no-transaction region are well known. In the previous section we presented the main results obtained by Boyle and Lin (1997) for the proportional transaction costs model. Zakamouline (2002) and Øksendal and Sulem (1999) study the optimal portfolio problem having both fixed and proportional transaction costs. They show that the boundaries of the no-transaction region depend on the investor's current wealth and that the optimal trading strategies consist in bringing the post-trade portfolio position strictly inside the no-transaction region if the initial portfolio position is outside the NT-region.

In our transaction costs framework, we study the effect of the transaction cost rate discount on the optimal trading strategies and to compare the results with the ones when transaction costs are proportional and when the fixed component is added to proportional transaction costs. Also, under given transaction cost structure (17) or (18), we want to identify the no-transaction region.

We consider the same optimization problem (6) and apply the dynamic programming technique. The investor's value function is defined as

$$J(T, x_T, y_T) = U(x_T + y_T) = \frac{(x_T + y_T)^\gamma}{\gamma}\ , \qquad \gamma < 1$$

and

$$J(t, x_t, y_t) = \underset{v_t}{\text{Max}}\ E_t[J(t+1, x_{t+1}, y_{t+1})] \qquad t = 0, 1, ..., T-1,$$

where, according to out modelling of transaction costs (18),

$$\begin{aligned} y_{t+1} &= (y_t + v_t)\mu \\ x_{t+1} &= (x_t - v_t - \delta_1|v_t| + (\delta_1 - \delta_2)\max\ (|v_t| - \lambda, 0)\ )r \end{aligned} \tag{19}$$

Therefore, we look for the optimal trading strategies $\{v_0, v_1, ..., v_{T-1}\}$ such that at any point of time the no-short selling and no-borrowing conditions are satisfied[9]. That is, in our framework condition (5) can be rewritten as

$$-y_t \leqq v_t \leqq ub(x_t) = \begin{cases} \frac{x_t}{1+\delta_1} & \text{if } \frac{x_t}{1+\delta_1} \leqq \lambda \\ \frac{x_t - \lambda(\delta_1-\delta_2)}{1+\delta_2} & \text{otherwise} \end{cases} \tag{20}$$

We will call the strategies which satisfy this inequality as *admissible* and $ub(x_t)$ stands for the upper boundary of the interval.

As we discussed in section 3, the application of the dynamic programming allows us to reduce the problem to a static optimization. As Boyle and Lin (1997), we introduce the following function

$$\varphi(t, x_t, y_t, v_t) = E_t[J(t+1, x_{t+1}, y_{t+1})]$$

and thus the value function can be expressed as

$$J(t, x_t, y_t) = \operatorname*{Max}_{v_t} \varphi(t, x_t, y_t, v_t)$$

where function $\varphi(t, x_t, y_t, v_t)$ is maximized over the interval of admissible trading strategies.

We should notice that the problem at hand is much more complicated than one described in the section above and solved analytically by Boyle and Lin (1997). For example, and what is significant, we loose the global concavity property of the function $\varphi_t(x_t, y_t, v_t)$[10], and hence the boundaries of the no-transaction region cannot be defined like in (15) in general. Also, $\varphi_t(x_t, y_t, v_t)$ is not homogeneous with respect to its variables[11], thus we do not expect from the no-transaction region to be a positive cone in general.[12]

[9] The no-short selling constraints can be relaxed. Since we solve our problem numerically, this relaxation will require much more computational time because one will have to solve the optimization problem on sufficiently large intervals of possible trading strategies for each possible portfolio allocation. Moreover, one will have to extend the discretization of the possible portfolio allocations from the subset of $\mathcal{R}^2_+$ to the subset of $\{(x, y) \in \mathcal{R}^2 : x + y \geq 0\}$.

[10] One can easily show that function $\varphi_t(x_t, y_t, v_t)$ is not concave with respect to the trading volume v_t at points $\pm\lambda$.

[11] By examining (19), one can see that x_{t+1} is not homogeneous with respect to x_t and v_t. Therefore, function $\varphi_{T-1}(x_{T-1}, y_{T-1}, v_{T-1})$

$$\varphi_{T-1}(x_{T-1}, y_{T-1}, v_{T-1}) = E_{T-1}[J_T(x_T, y_T)] = E_{T-1}[U(x_T + y_T)]$$

is not homogeneous. Moving backwards, one can show that the result holds for any time t.

[12] In fact, homogeneity of $\varphi_t(x_t, y_t, v_t)$ garantees that the no-transaction region is a positive cone. Indeed, if portfolio (x_t, y_t) lies on the boundary of the NT-region, i.e. $v_t^* = 0$ maximizes $\varphi_t(x_t, y_t, v_t)$, then for any $\phi > 0$ we obtain that the portfolio $(\phi x_t, \phi y_t)$ is also located on the boundary of the NT-region, because $\phi v_t^* = 0$ will maximize $\varphi_t(\phi x_t, \phi y_t, \phi v_t)$.

By now, we could not find the way to define the boundaries of the no-transaction region as well as optimal trading volumes by explicit analytical formulas in general. However, for the last period optimization problem some analytical results can be obtained. At the beginning of this section we mentioned that the transaction costs structure we consider can be decomposed into pure proportional transaction costs for small trading volumes and proportional with a fixed component for bigger trading volumes. Therefore, function $\varphi(T-1, x_{T-1}, y_{T-1}, v_{T-1})$ can be presented as:

$$\varphi(T-1, x_{T-1}, y_{T-1}, v_{T-1}) = \begin{cases} \varphi^P(T-1, x_{T-1}, y_{T-1}, v_{T-1}) & |v_t| \leqq \lambda \\ \varphi^{FP}(T-1, x_{T-1}, y_{T-1}, v_{T-1}) & |v_t| > \lambda \end{cases}$$

where $\varphi^P(T-1, x_{T-1}, y_{T-1}, v_{T-1})$ corresponds to the case with pure proportional TC (the transaction cost rate is equal to δ_1) and is defined by (13), and $\varphi^{FP}(T-1, x_{T-1}, y_{T-1}, v_{T-1})$ corresponds to the case when the fixed component $(\delta_1 - \delta_2)\lambda$ is added to the proportional TC (with the TC-rate δ_2), i.e.

$$\varphi^{FP}(T-1, x_{T-1}, y_{T-1}, v_{T-1}) = E_{T-1}U\left[y_T + x_T\right] \ , \ \ |v_t| > \lambda$$

where

$$\begin{aligned} y_T &= (y_{T-1} - v_{T-1})\mu_{T-1} \\ x_T &= (x_{T-1} - v_{T-1} - (\delta_1 - \delta_2)\lambda - \delta_2|v_{T-1}|)r \end{aligned}$$

When the wealth of the investor is relatively small at the beginning of the last period optimization, and thus he cannot trade big volumes of both the risky asset and the risk-free asset due to the no-short-selling and no-borrowing constraints, then the results of the previous section fully apply. Intuitively, if the investor is precluded from trading the risky asset in amounts which exceed the critical value λ, then only a higher transaction cost rate applies to all admissible trades, and hence we are in the framework of proportional transaction costs. In this case the boundaries of the no-transaction region and the optimal trading strategies are driven by function $\varphi^P(T-1, x_{T-1}, y_{T-1}, v_{T-1})$. The following proposition formalizes the above statement.

Proposition 1 *At time $t = T-1$, function $\varphi(t, x_t, y_t, v_t) \equiv \varphi^P(t, x_t, y_t, v_t)$ for all admissible values of v_t and for all portfolio positions (x_t, y_t) which belong to the quadrant $[0, \lambda(1+\delta_1)] \times [0, \lambda]$. Therefore, for these portfolio positions the optimal trading strategies and the boundaries of the no-transaction region are defined by (16) and (15) respectively.*

Proof. If $(x_{T-1}, y_{T-1}) \in [0, \lambda(1+\delta_1)] \times [0, \lambda]$ then $x_{T-1}/(1+\delta_1) \leqq \lambda$ and the no-short sale and the no- borrowing conditions (20) imply:

$$-\lambda \leqq v_{T-1} \leqq \lambda$$

In this case, since the absolute value of the trading volume cannot exceed λ, the investor is facing only higher transaction cost rate δ_1. Using the fact that functions J and J^P are of the same functional form at maturity, and are equal to the utility function, the result follows. ■

One can show that both functions φ^P and φ^{FP} are concave on the intervals on which they are defined, but function φ is not concave at points $\pm\lambda$, as we already said. Therefore, function φ is piece-wise concave, and hence its local maximum does not need to coincide with a global maximum. Also, due to the presence of the fixed component, function $\varphi_t(x_t, y_t, v_t)$ is not homogeneous, and thus the boundaries of the no-transaction region will depend on the investor's wealth. Another consequence of the presence of the fixed component (of course for big trades) is the discontinuity of the optimal trading volume $v_t^* = v_t^*(x_t, y_t)$. All these "unpleasant" features of function $\varphi(t, x_t, y_t, v_t)$, and hence of the value function, make it difficult to handle the problem analytically. Therefore, the rest of the analysis is conducted by solving the portfolio optimization problem numerically.

6 Numerical simulations

In this section we apply the numerical procedure for the derivation of the boundaries of the no-transaction region and optimal trading volumes. Since we could not obtain analytical expressions for them, the numerical simulation is an important tool to handle the problem. Moreover, a simple program (we use MatLab 6.5) gives the numerical value for the optimal trading volume for any portfolio position in few seconds. However, the identification of the no-transaction region requires a substantial amount of computational time. This is because one has to make a discretization in the $\mathcal{R}_+^2$ space, i.e. riskless - risky wealth space, and to compute the optimal trading volume at each point of the discretization.

By definition, those points (portfolio allocations), at which the optimal trading volume v^* is equal to zero, belong to the no-transaction region. Since we are interested only in the boundaries of the no-transaction region, we build a routine which allows us to identify those boundaries.

To model the uncertainty about the stock price evolution, we assume that:

$$\mu = \begin{cases} u & \text{with Probability } p \\ d = 1/u & \text{with Probabilitry } (1-p) \end{cases}$$

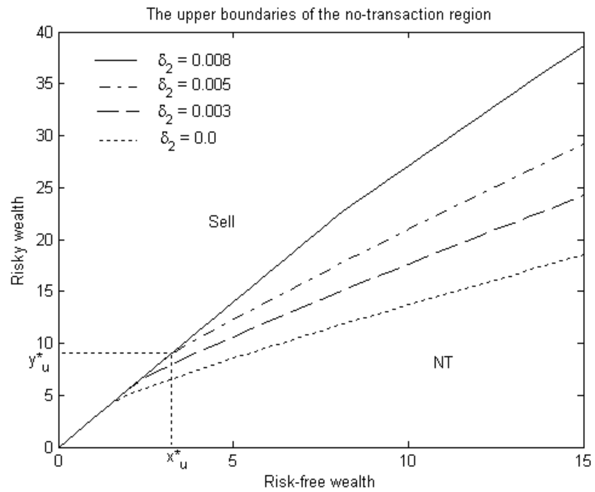
The simulations were done using Cox Ross Rubinstein parametrization:

$$u = e^{\sigma\sqrt{h}}, d = e^{-\sigma\sqrt{h}}, p = \frac{e^{\nu h} - d}{u - d}, r = e^{\theta h}$$

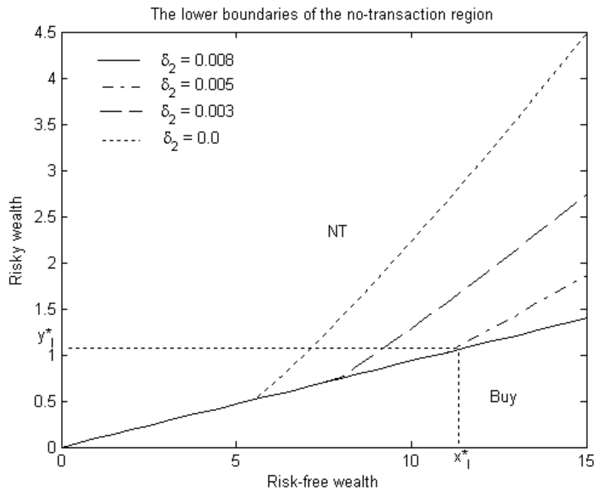
where the length of one period is $h = 0.25$ and $\nu = 0.1, \sigma = 0.25, \theta = 0.05$. These numbers imply that the length of one period is three months, the volatility of the risky asset's returns is 25% (annualized) and the risk-free rate is 5.13% (annualized). The critical transaction value is $\lambda = 1$ and the higher transaction cost rate is $\delta_1 = 0.01$. We do not fix the parameter of the relative risk aversion γ and the value of the lower transaction cost rate δ_2 because we wand to study their impact on the optimization results. We also look at the impact of the risky asset volatility σ on the location of the no-transaction region.

A. The boundaries of the no-transaction region

We begin with the last period optimization problem. In Fig. 3, we plot the boundaries of the no-transaction region for different values of the transaction cost discount, i.e. different values of δ_2 parameter. It turns out that, to the contrary of the model with proportional transaction costs, the no-transaction region is not a positive cone in general. We can see that it remains to be a positive cone when the investor's wealth is relatively small, but then, at certain wealth levels, the slopes of the boundaries of the no-transaction region change. Namely, the slope of the upper boundary, which separates the "sell" region from the no-transaction region, decreases (see figures 3a and 4a). This means that not having the transaction cost discount, at some portfolio positions the investor would be better off by not trading, but with a discount he prefers to sell some amount of the risky asset. To the contrary, the slope of the lower boundary of the no-transaction region increases when δ_2 declines (see figures 3b and 4b). That is, the area (portfolio allocations) from which the investor will prefer to buy the risky asset widens. In other words, the lower the value of δ_2, or the higher the transaction cost discount, the bigger is a set of portfolio position from which the investor will make nonzero trades, and hence the narrower is the no-transaction region. When the investor's wealth increases, the boundaries of the no-transaction region quickly converge to those of the model of proportional transaction costs (TC rate is δ_2) with the fixed component $(\delta_1 - \delta_2)\lambda$.

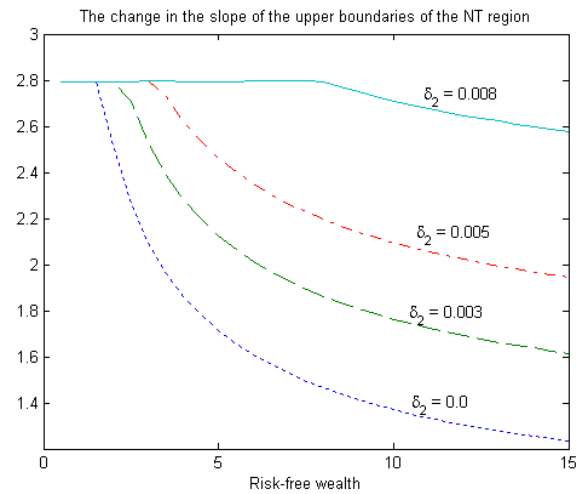


a)

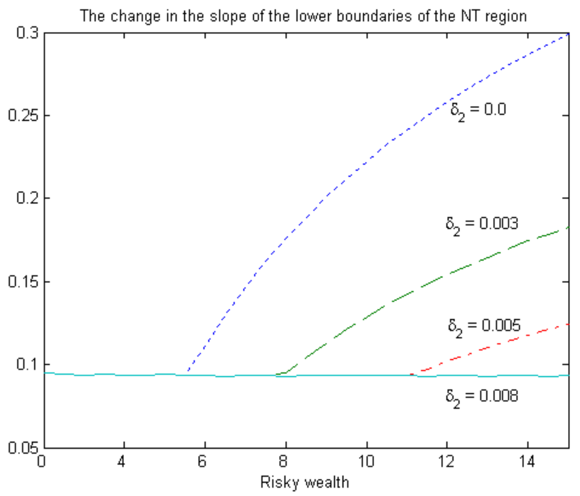


b)

Fig. 3. The upper and the lower boundaries of the no-transaction region for one period optimization problem for different values of the smaller transaction cost rate. The other parameters are: $\lambda = 1$, $\gamma = -2$, $\delta_1 = 0.01$.



a)



b)

Fig. 4. The change in the slope of the upper and the lower boundaries of the no-transaction regions for one period optimization problem for different values of the smaller transaction cost rate. The other parameters are: $\lambda = 1, \gamma = -2, \delta_1 = 0.01$.

Those portfolio allocations at which the boundaries of the no transaction region change their slope, that is (x_u^*, y_u^*) for the upper boundary and (x_l^*, y_l^*) for the lower boundary respectively, can be defined as:

$$(x_u^*, y_u^*) := \underset{x}{\text{Min}}\underset{y}{\text{Max}}\{(x,y) \in R_+^2 : \varphi^P(T-1, x, y, 0) =$$
$$= \underset{v\in[-y,-\lambda]}{\text{Max}} \varphi^{FP}(\text{T-1}, x, y, v)\}$$

$$(x_l^*, y_l^*) := \underset{x}{\text{Min}}\ \underset{y}{\text{Min}}\{(x,y) \in R_+^2 : \varphi^P(T-1, x, y, 0) =$$
$$= \underset{v\in[\lambda,ub(x)]}{\text{Max}} \varphi^{FP}(\text{T-1}, x, y, v)\}$$

The numerical solution of the last period optimization problem indicates that when the risk-free wealth $x_{T-1} \le x_u^*(T-1)$, then the upper boundary of the no-transaction region is defined by pure proportional transaction cost structure (with TC rate δ_1), i.e. it is driven by function $\varphi^P(T-1, x_{T-1}, y_{T-1}, v_{T-1})$. Therefore, for these values of the risk-free wealth, the optimal trading volume is continuous in the vicinity of the upper boundary of the NT region. Similarly, when $x_{T-1} \le x_l^*(T-1)$, the lower boundary of the NT region is also defined by $\varphi^P(T-1, x_{T-1}, y_{T-1}, v_{T-1})$ and the trading volume is continuous near the lower boundary. On Fig.5 we depict the relationship between the investor's risky wealth and the optimal trading volume when the risk-free wealth is fixed and equals to 1, 2.5 and 3.2. In that example $x_u^* = 3.2$. We can see that the optimal trading volume is positive at the very low levels of the risky wealth, that is when the portfolio position is below the lower boundary of the no-transaction region, then it is equal to zero when the portfolio lies inside the no-transaction region, and finally it becomes negative when the portfolio moves to the "sell" region. Notice, that the optimal trading volume of the risky asset is a continuous function until a certain level of the risky wealth, and then it makes a jump. In the selling region, for a fixed risk-free wealth $\bar{x}$, this jump happens at the maximum level of the risky wealth when

$$\underset{v\in[-\lambda,0]}{\text{Max}} \varphi^P(T-1, \bar{x}, y, v) = \underset{v\in[-y,-\lambda]}{\text{Max}} \varphi^{FP}(T-1, \bar{x}, y, v)\}$$

Similarly, in the buying region the jump happens at the minimum level of the risky wealth such that:

$$\underset{v\in[0,\lambda]}{\text{Max}} \varphi^P(T-1, \bar{x}, y, v) = \underset{v\in[\lambda,ub(\bar{x})]}{\text{Max}} \varphi^{FP}(T-1, \bar{x}, y, v)\}$$

In Fig.6, we plot the optimal post-trade portfolio allocations, as well as the upper boundary of the no-transaction region for fixed values of the risk-free wealth. As we discussed in the previous section, at low wealth levels the investor is constrained to trade big amounts, and thus he faces only the higher transaction cost rate δ_1. Therefore, his optimal behavior can be explained by proportional transaction cost models. According to those models, the optimal post-trade portfolio allocations should lie on the closest boundary of the no-transaction region. That is what we see on Fig. 6 when the risky wealth is low. On the contrary, at high levels of the risky wealth, it is optimal for the investor to make trades which bring his post-trade portfolio position inside the no-transaction region, like in the case when the fixed component is added to proportional transaction costs (for instance, see Zakamouline (2002)). Such a behavior of the investor can be explained in the following way: Optimally, the investor would wish his portfolio to be located on the so-called Merton's line[13], which lies inside the no-transaction region and corresponds to the zero-transaction-cost case. However, the presence of non-zero transaction costs induces the investor to balance costs, i.e. transaction costs, and benefits from being closer to the Merton's line. Hence, the transaction cost discount allows the investor to get closer to the Merton's line, and thus he trades inside the NT-region. On the other hand, small investor cannot follow this strategy because they will pay higher transaction cost (relative to their wealth) than big investors (again, relative to their wealth).

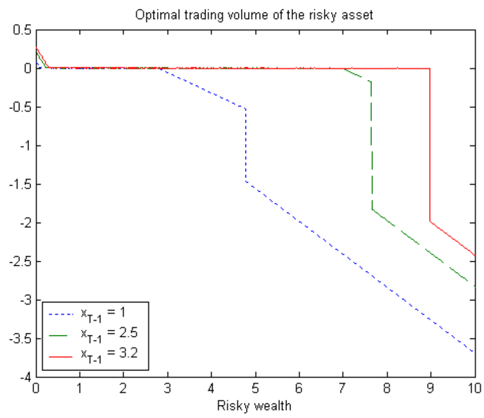


Fig. 5. The relationship between the risky wealth and the optimal trading volume of the risky asset when the value of the risk-free wealth is equal to 5. The other parameters are: $\lambda = 1$, $\gamma = -2$, $\delta_1 = 0.01$, $\delta_2 = 0.005$.

[13]The boundaries of the no-transaction region will converge to the Merton's line when the transaction cost rates approache zero.

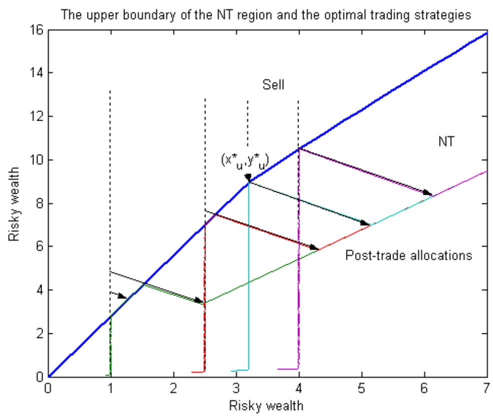


Fig. 6. The upper and the lower boundaries of the no-transaction region and the post-trade portfolio allocations. The parameters are: $\lambda = 1$, $\gamma = -2$, $\delta_1 = 0.01$, $\delta_2 = 0.005$.

Next, we look at how the length of the optimization period affects the no-transaction region. On Fig. 7 we plot the no-transaction region for the one- and the two-period optimization problem and in Table 3 we present the results for three periods optimization for fixed values of the risk-free wealth. We can see that the no-transaction region tends to widen as we approach maturity. This result agrees with findings of Boyle and Lin (1994), Genotte and Jung (1994) and Zakamouline (2002). Also, Panel B of Table 3 shows that the boundaries of the no-transaction region are flattering when the number of periods to maturity increases.

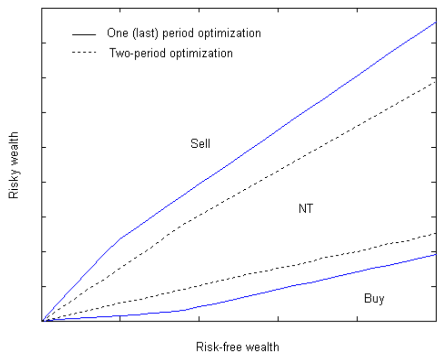


Fig.7. The boundaries of the NT region for one- and two-period optimization problem for $\lambda = 1$, $\delta_1 = 0.01$, $\delta_2 = 0.005$, $\gamma = -1$.

Table 3.

Panel A: for different values of the risk-free wealth the boundaries of the no-transaction region are defined by the corresponding risky wealth

	Risk-free wealth			
	X=1	X=5	X=10	X=15
One period				
Lower Boundary	0.094	0.467	0.934	1.862
Upper boundary	2.791	12.3079	20.943	29.150
Two periods				
Lower Boundary	0.327	1.651	3.291	4.956
Upper boundary	1.351	6.742	13.117	19.375
Three periods				
Lower Boundary	0.426	2.130	4.259	6.388
Upper boundary	1.108	5.544	11.088	16.632

Panel B: for different values of the risk-free wealth the boundaries of the no-transaction region are defined by the ratio of the risky wealth to the corresponding risk-free wealth

	Risk-free wealth			
	X=1	X=5	X=10	X=15
One period				
Lower Boundary	0.094	0.093	0.093	0.124
Upper boundary	2.791	2.462	2.094	1.943
Two periods				
Lower Boundary	0.329	0.330	0.329	0.330
Upper boundary	1.351	1.348	1.312	1.292
Three periods				
Lower Boundary	0.426	0.426	0.426	0.426
Upper boundary	1.108	1.109	1.109	1.109

Let us notice that due to substantial computational time, we restrict our attention to only the one-, two- and three-period optimization problems. However, the derived properties of the optimal trading strategies and the boundaries of the no-transaction region should hold for an arbitrarily large number of periods. This is because our model is a combination of both pure proportional and proportional with a fixed component transaction costs models, and the results of those two models hold in general.

B. Sensitivity analysis

In this sub-section we show that on average the upper boundary of the no transaction region is more sensitive to the changes in the volatility and the CRRA parameter than the lower boundary. This is due to the fact that the investor's relative risk aversion, when measured by the value function ,$RRAV$, is not constant and it increases with the investor's wealth. In figure 8, we plot the relative risk aversion, which is defined as $RRAV_x = -\frac{yJ_{yy}}{Jy}$ for fixed values of the risk-free wealth. We can see that risk aversion increases with the investor's holdings in the risky asset. Therefore, the investor is more risk averse near the upper boundary of the NT-region, and less risk averse near the lower boundary. Also, the investor's risk aversion decreases with his risk-free wealth. Note, that when markets are frictionless, i.e. when there are no transaction costs, no taxes, etc., the investor's relative risk aversion, when measured by the value function, does not depend on investor's wealth, and it equals to the γ (Merton (1971)). However, such a friction as the transaction cost makes the investor's RRA, when measured by the value function, wealth-dependent. For example, similarly to transaction costs, the presence of default risk also induces the $RRAV$ to be wealth-dependent (Chang and Sundaresan (1999)).

In Fig. 9, we present the impact of the investor's relative risk aversion parameter $|\gamma|$ on the boundaries of the no-transaction region. As the risk aversion increases, the no-transaction region shifts down towards the risk-free wealth. That is, the area from which the investor optimally sells the risky asset widens and from which he optimally buys the risky asset narrows (fig. 9a). In general, the upper boundary appears to be more sensitive to changes in the risk aversion than the lower boundary (fig. 9b). This difference in sensitivities due to the dependance of the $RRAV$ from the investor's wealth. That is, near the upper boundary the investor is wealthier, and thus more risk averse, than near the lower boundary. We can also see that the sensitivity of the upper boundary to changes in γ remains constant at low wealth levels, i.e. when only proportional transaction costs drive the investor's trading strategies, and then it monotonically decreases with wealth. On the other hand, the sensitivity of the lower boundary displays different behavior. It also remains constant at low wealth levels, then, up to a certain wealth level, it increases, and then it decreases as wealth tends to infinity. Both the sensitivity of the upper boundary and the sensitivity of the lower boundary converge to those with pure proportional transaction costs ($\delta_1 = \delta_2 = 0.005$) when the investor's wealth tends to infinity.

The impact of the risky asset volatility σ on the location of the no-transaction region (fig. 10a) is similar to the effect of the risk aversion parameter: an increase in the volatility shifts the no-transaction region down towards the riskless asset. This move of the no-transaction region towards the risk-free wealth is due to the risk-aversion of the investor. Also, due to the fact that $RRAV$ increases with wealth, the upper boundary is more sensitive to changes in the volatility than the lower boundary. As the investor's wealth tend to infinity, the sensitivities of both boundaries tend to those with pure proportional transaction costs (fig. 10b).

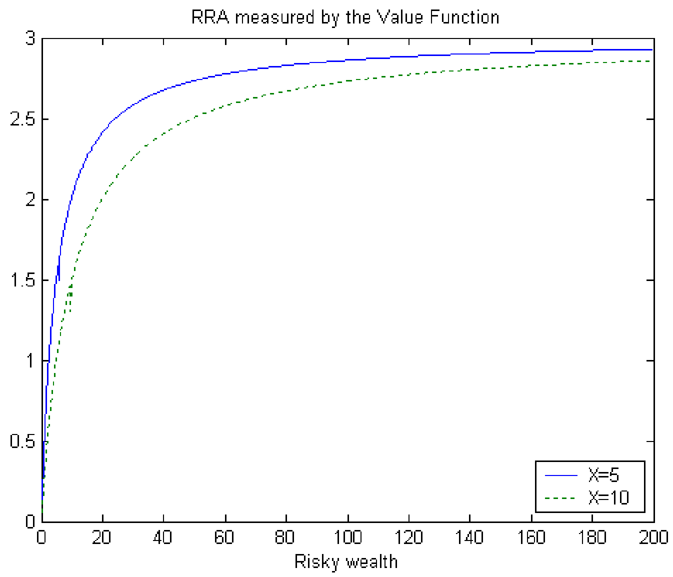


Fig.8. The relative risk aversion when measured by the value function when $\lambda = 1, \gamma = -2, \delta_1 = 0.01, \delta_2 = 0.005$.

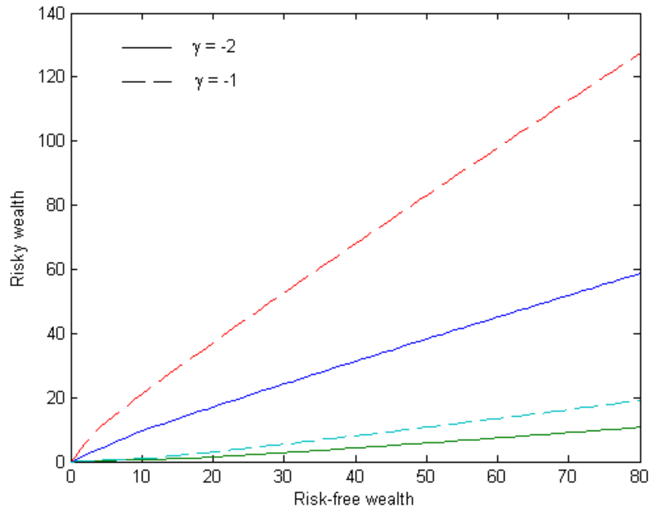


Fig.9a. The upper and the lower boundaries of the no-transaction region for one period optimization problem for different values of the relative risk aversion parameter. The other parameters are: $\lambda = 1$, $\delta_1 = 0.01$, $\delta_2 = 0.005$.

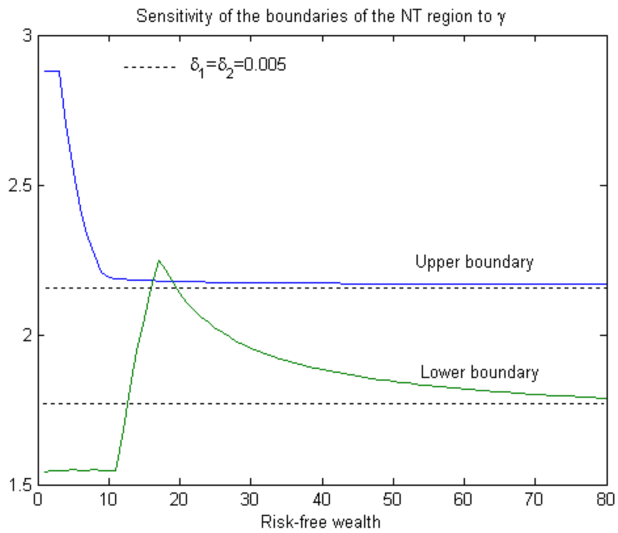


Fig.9b. The relative change in the boundaries of the NT region when γ changes from -2 to -1.

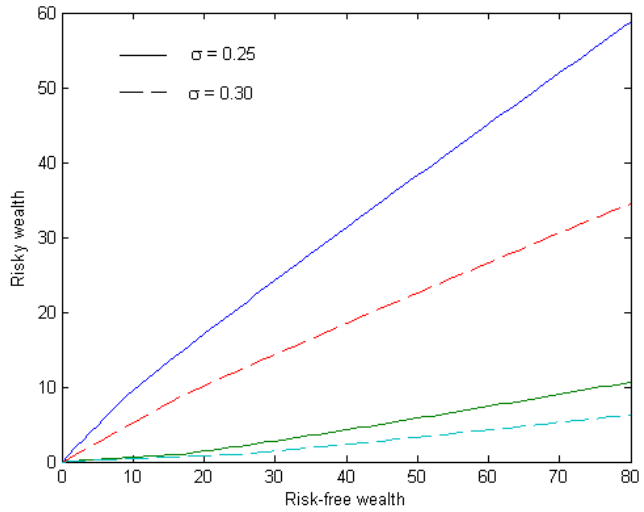


Fig.10a. The upper and the lower boundaries of the no-transaction region for one period optimization problem for different values of the risky asset volatility. The other parameters are: $\lambda = 1$, $\gamma = -2$, $\delta_1 = 0.01$,$\delta_2 = 0.005$.

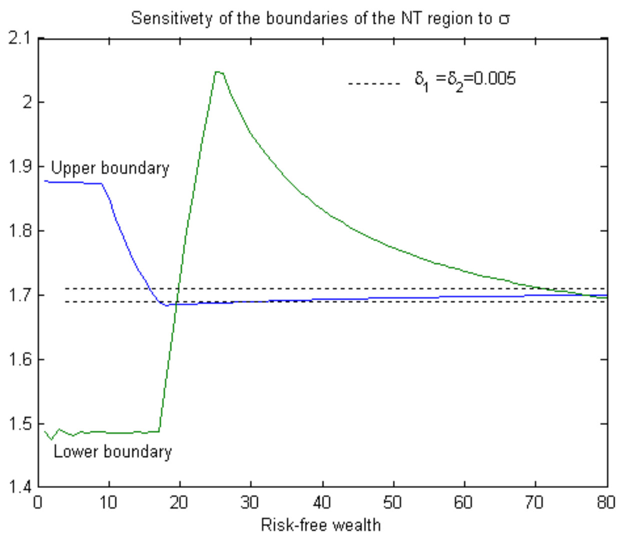
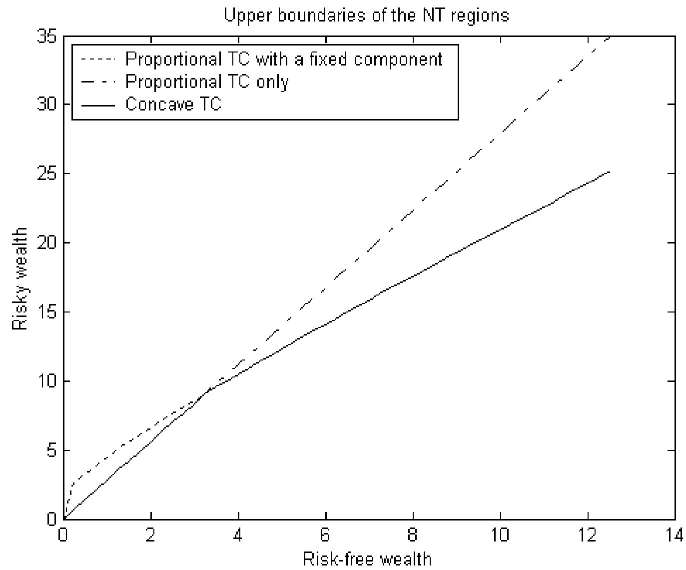


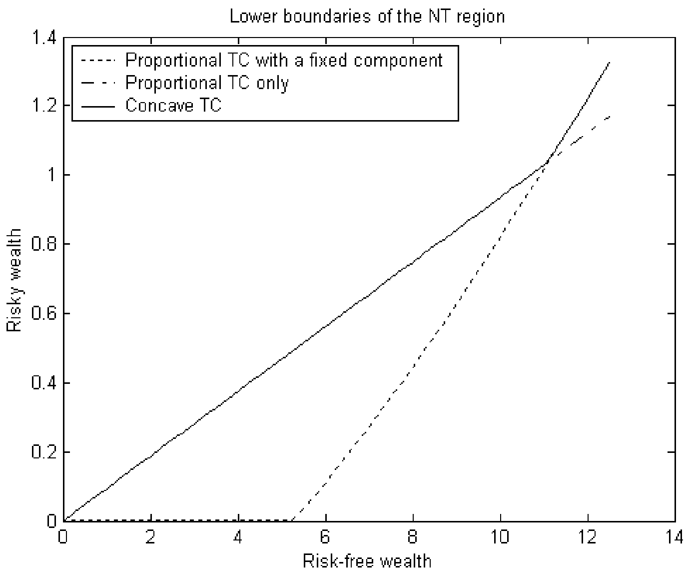
Fig.10b. The relative change in the boundaries of the NT region when σ changes from 0.25 to 0.30.

C. Boundaries of the no transaction region and different TC structures.

In this sub-section, we would like to compare the boundaries of the no-transaction region for different specifications of the transaction cost structure. Namely, we consider concave transaction costs, pure proportional transaction costs and proportional transaction costs with a fixed component. As we already mentioned, concave transaction costs which we consider is a combination of the last two cost structures. Moreover, in our model the investor has an option to choose the cost structure by choosing the trading volume of the risky asset. On Figure 11a and 11b we plot the upper and lower boundaries respectively for the mentioned transaction costs specifications. As we can see, under concave transaction costs the no-transaction region is the narrowest. This means that under concave transaction costs the investor trades more often than under the other TC formulation. For example, a shift from pure proportional to concave transaction costs (with the same transaction cost rate for small trading volumes) is beneficial for big investors (since they receive TC discount) and it does nor affect small investors since they have to pay the same rate in both cases. On the other hand, a shift from proportional transaction costs with a fixed component to concave transaction costs (with the same marginal TC rate for high trading volumes) is beneficial for small investors, since for small trading volumes the proportional transaction costs are below the costs with a fixed component. In other words, concave transaction costs make the securities market the most liquid. This result can have implications for a design of an optimal fee structure on a market place.



a)



b)

Fig. 11. a)The upper and b) the lower boundaries of the no-transaction region for different transaction cost specifications.

7 Conclusions

All studies with the proportional transaction cost structure postulate that the no-transaction region is a positive cone and optimal trading strategies consist of having the post-trade portfolio allocations at the nearest boundary of the no-transaction region. On the other hand, studies which combine proportional transaction cost with a fixed component report that the no-transaction region is wealth-dependent and the optimal trading consists in bringing the post-trade portfolio position inside the NT region. In the paper we show that when transaction costs are concave, that is when the transaction cost rate decreases with the trading volume of the risky asset, the optimal behavior of the investor can be largely explained by the above two models. Namely, at low wealth levels the investor face only proportional transaction costs and trades to the nearest boundary of the NT region. When the wealth increases, the investor prefers proportional TC with a fixed component to pure proportional TC, and hence his optimal trades go mostly inside the no-transaction region.

We also show that under concave transaction costs the no-transaction region lies inside the NT regions with the two alternative cost specifications. This means that concave transaction costs imply more frequent trading, and therefore they make the securities' market more liquid.

One of the limitations of the paper is that we cannot obtain analytical solutions. Therefore, we solve the problem numerically. Another limitation is that we solve the problem only for a small number periods (maximum three). This is due to the substantial computation time required. However, since our model is a combination of the other two models, which are solved for an arbitrary large number of periods, we claim that our results hold in general.

For the further research, one can think about introducing another agent in the model, a "market maker", and study the impact of the "illiquidity effect" on the optimal behavior of the investor. Also, one can think about the optimal choice of the transaction cost rates δ_1 and δ_2 imposed by financial institutions or intermediaries. This is an important issue, since we show that the investors' demand for the risky asset is sensitive to the variations in the transaction cost rates. Moreover, imposing a convex structure of transaction cost, a financial intermediary can substantially increase trading activities.

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