On the Informational Content of Changing Risk for Dynamic Asset Allocation

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Abstract

The informational content of changing risk for dynamic asset allocation is analyzed in order to investigate its importance in determining expected index returns. We consider a class of optimal dynamic strategies taking into account both changing risk and expected returns that vary accordingly to changing risk. We compare their risk adjusted performance to that of a buy and hold strategy under different hypotheses on the form of conditionally expected returns. The statistical evidence in favour of expected returns varying accordingly to changing risk is elusive. On the other hand, we find some evidence of a superior unconditional risk adjusted performance of volatility based trading rules compared to buy and hold strategies. This suggests that changing risk conveys information useful to improve performance.

JEL Classification: G0, G1

1 Introduction

A key issue in the financial literature is the predictability of excess returns.

Several papers in this area relate changing monthly market premia to firm size, term structure spreads, dividend yields, book to market values and further economic variables that are found to predict excess returns in a statistically significant way; cf. for instance Ferson and Harvey [11] and Evans [10]. More recently, asset allocation strategies implied by intertemporal portfolio models where monthly risk premia are related to interest rates and dividend yields seem to produce a significant economic performance; cf. Brennan, Schwartz and Lagnado [7] and Bielecki, Pliska and Sherris [3]. On the other hand, performance evaluation models incorporating the conditioning information of economic variables that predict returns show that most investment funds are not able to beat such a dynamic benchmark; cf. Bansal and Harvey [1].

A second direction of research on returns predictability focuses on volatility and further related risk measures. This is a quite natural approach since almost all available asset pricing models imply a clear equilibrium relation between risk premia, volatility and risk aversion; cf. Merton [19] and [20].

Campbell [8], Glosten, Jagannathan and Runkle [12] and Whitelaw [24] observe monthly market volatility changing with information variables like interest rates; however most of the evidence reported produced a negative or insignificant relation between monthly expected returns and volatility. On the other side, Li [18] finds monthly stock market risk premia to be related to the variance of further economic variables but not to their own variance; market volatility seems to be not the only factor influencing the predictive part of market risk premia. Other authors like He, Kan, Ng and Zhang [13] and Li [17] focuse on market prices of risk by modelling time varying monthly reward-to-variability variables and find them to capture a larger fraction of predictable monthly excess returns than changing volatilities of returns.

If changing risk is not a good explanatory variable for changing expected returns an obvious question is whether it pays an investor to be in the market at times of high risk. To answer this question we consider the optimization problem of an investor with constant relative risk aversion which invests in a market with changing risk. We compare the risk adjusted unconditional performance of the implied dynamic optimal strategies to that of a buy and hold investment rule. Specifically, we specify some models of changing risk and returns where risk premia can change as a function of volatility and try to exploit changing reward to variance ratios by investing more in the risky asset when the reward to variance ratio is high, respectively more in the conditionally riskless asset when it is low.

Although the economic and the empirical intuitions for these dynamic strategies seem to be quite evident, the practical implementation of these trading rules encounters some difficulties due to the elusive relation between changing risk and changing expected returns. In fact, while conditional variances of security returns can be well approximated by some type of conditional heteroscedasticity models like those of the GARCH (cf. for instance [5] and [9]) family, the predictable part of returns derived from volatility predictions is small. Furthermore, influential observations and leverage points caused by market crashes and (or) market rallies affect the conditional mean equation when regressing daily returns on volatilities[1], thereby making the statistical inference on estimated conditional means difficult to interpret or inconclusive. Therefore, rather than attempting to link changing returns to risk in a pure statistical way we measure the impact of changing risk on unconditional financial performance.

We investigate if volatility pays in dynamic trading for four main stock indices (the Dow Jones Industrials, the S&P500, the FTSE 100 and the Nikkei) with two related but clearly distinct approaches.

In the first part of the paper we look at a simple Bayesian trading strategy maximizing conditional expected quadratic utility under two competing hypotheses on the relation between expected returns and volatility: constant expected returns and expected returns that are proportional to volatilities. We interpret these two hypotheses as some kind of limit cases for the type of re-

[1] We illustrate this point in more detail in the applied part of the paper.

lation that one can realistically expect to hold in real data between changing expected returns and changing risk, without however claiming that one of the two specifications has to be exactly satisfied in reality.

Our Bayesian approach tries to take into account (in a simple way) the high estimation risk related with the estimation of expected returns under the union of the two competing hypotheses. The performance evaluation approach followed in this part of the paper is rather standard and practice oriented. That is because without introducing further assumptions it is not possible to embed the optimization problem of a conditional mean variance optimizer in a corresponding unconditional optimal portfolio problem. As a consequence, we adopt standard performance measures likes Jensen's alphas and Sharpe ratios to assess the ability of the Bayesian strategy to beat buy and hold unconditionally.

In the second part of the paper we solve an intertemporal stochastic volatility Merton's [19] model based on an exponential M-GARCH specification (cf. Nelson [22]) of market excess returns. The model is rich enough to take into account the main empirical features of the relation between changing expected returns and changing risk (as for instance asymmetries) and at the same time simple enough to yield analytical solutions. The unconditional performance of the implied optimal strategy is then assessed using the certainty equivalent implied by the explicit model solution, after having calibrated the model to our data set. While this comparison is by construction model dependent, it yields a performance evaluation of trading rules that is theoretically well defined since it takes into account the whole unconditional return distribution of a trading strategy and not only the first two unconditional moments. Moreover, this approach allows us to take explicitly into account optimal hedging positions for asymmetries and thereby investigate the practical relevance of intertemporal hedging in the present situation.

We find a modest but consistent evidence that strategies taking into account changing risk outperform buy and hold strategies for the four stock indices and over the sample period under scrutiny. In a in-sample and an out-of-sample performance comparison the Bayesian approach yields better performances for the four major stock indices analyzed with unconditional out-of-sample Jensen's alphas between 0.5% and 1% on a yearly basis. Moreover, a calibration of our stochastic volatility Merton model to the Dow Jones data set suggests that further improvements are not easily realizable even when the data generating mechanism of returns and volatilities is known by the investor. Finally, the contribution of intertemporal hedging to the performance of the optimal strategy is found to be negligible.

The reminder of the paper is organized as follows. In Section 2 our Bayesian approach is introduced and the arising dynamic strategy is evaluated against buy and hold. Section 3 defines the Merton model used in our analysis and derives the explicit solution and the optimal strategy for the implied Merton's problem. This section is concluded by the calibration of the model to the Dow data on which a performance analysis using certainty equivalents is based. Section 4 summarizes and concludes.

2 A Bayesian Approach

2.1 Preliminaries

Let r_t and r_{ft} be the return at time t of a stock index and of a conditionally riskless asset, respectively, over the period $(t-1, t]$.

We consider a manager investing in the index and in the conditionally riskless asset. Our manager is interested in the *unconditional* performance of a trading rule $(w_t)_{t=0,..,T-1}$ over the given holding period $[0, T]$; w_t is the percentage of cash she is ready to invest in the risky asset at time $t-1$ depending on the available market information I_{t-1}.

Obviously, different dynamic strategies $(w_t)_{t=0,..,T-1}$ are available, one of which is a "static"[2] buy and hold strategy with fixed weights $w_t = w$, $w \in \mathbb{R}$

[2] We refer to this strategy as "static" in the sense that the fraction of wealth allocated in the risky asset is constant through time. Changing prices require at each period a reallocation of wealth in order to maintain these fixed weights, unless in the case where the whole wealth is allocated in one asset.

for $t = 0, .., T-1$. However, depending on the kind of information available to the manager at time $t-1$ and on the underlying market structure, several other strategies are natural, as for instance a combination of a conditionally optimal mean variance rule with some kind of intertemporal hedging position (cf. for instance Merton [19] and [20]). In this and the next section we restrict ourself to a standard conditional (myopic) mean variance optimizer and abstract from the possibility of intertemporal hedging, to which we devote a specific discussion in Section 3.

Let $E(r_t|I_{t-1})$ and $Var(r_t|I_{t-1})$ be the time $t-1$ conditional expectation and conditional variance, respectively, of the index return. The optimal choice of a conditional expected quadratic utility optimizer having at her disposal the information set I_{t-1} at time $t-1$ is:

$$w_t = \frac{1}{\lambda} \cdot \frac{E(r_t - r_{ft}|I_{t-1})}{Var(r_t - r_{ft}|I_{t-1})} \quad , \tag{1}$$

where λ is a parameter of risk aversion that we assume constant over the holding period $[0, T]$.

As a consequence, a situation where expected excess returns pay for risk in a way that is increasing with and proportional to market returns variances:

$$E(r_t - r_{ft}|I_{t-1}) = \gamma \cdot Var(r_t - r_{ft}|I_{t-1}) \quad ; \quad \gamma > 0 \quad , \tag{2}$$

is equivalent to the optimality of a buy and hold strategy for our conditional mean variance optimizer. However, such a quadratic growth of conditionally expected excess returns with respect to conditional volatilities seems to be a quite unrealistic structural hypothesis in light of the observed form of the (weak) empirical relation between these two magnitudes. Figure 1 illustrates this feature further by a scatter plot of the Dow Jones Industrial daily index returns plotted against the conditional volatilities estimated with a GARCH(1,1) model[3] for the period from January 1994 to January 1999

Insert Figure 1 about here

[3] We discuss these estimates in more detail in the next sections.

Notice that the dispersion of returns is higher for higher volatilities in the diagram confirming the well-known ability of GARCH volatility estimates to explain the time variant volatility structure of index returns. On the other side, this same graph shows clearly that the predictable part of daily excess returns is small, thereby producing an elusive link between returns and volatilities (with some hints of a positive relation between the two). Moreover, a simple quadratic regression of excess returns on estimated volatilities produces often a negative point estimate for the parameter of the quadratic volatility term, suggesting expected returns increasing at a less than quadratic rate as a function of volatility. Similar structures are obtained for other indices like the S&P 500, the FTSE and the NIKKEI.

In view of the weak empirical evidence of a quadratic compensation for volatility risk, alternative specifications where expected returns pay at lower growth rates as a function of volatility seem to be more realistic. On the other hand, the elusive empirical relation between these magnitudes suggests us to consider very simple specifications, for which we have a clear economic intuition. In fact, the use of flexible specifications can easily produce an economically counterintuitive estimated relation between expected returns and volatilities[4].

The easiest specification one could try is one where volatility does not pay at all:

$$E\left(r_t - r_{ft} | I_{t-1}\right) = \alpha_1 \quad , \quad \alpha_1 > 0 \quad . \tag{3}$$

Alternatively, to model expected returns that are relatively slowly increasing with volatility one could try a linear specification of the form:

$$E\left(r_t - r_{ft} | I_{t-1}\right) = \alpha_0 \cdot \sqrt{Var\left(r_t - r_{ft} | I_{t-1}\right)} \quad , \quad \alpha_0 > 0 \quad . \tag{4}$$

We interpret (3) and (4) as two limit-case specifications that determine a lower and upper bound for the functional forms that one can realistically expect to describe the growth of expected returns with volatility in real data, without

[4]For instance, the risk free expected return implied by a linear regression of returns on a constant and estimated volatilites is often negative and irrealistically large in absolute value.

however claiming the "true" underlying relation between expected returns and volatility to be either of the form (3) or (4) (cf. again the illustration presented in Figure 1).

For any given level of conditional volatility the optimal allocation of a conditional mean variance optimizer will then lie between the optimal mean variance allocations implied by the hypotheses (3) and (4), which are, respectively (using the optimality condition (1)):

$$w_t^1 = \frac{1}{\lambda} \cdot \frac{\alpha_1}{Var\left(r_t - r_{ft} | I_{t-1}\right)} \quad , \tag{5}$$

and

$$w_t^0 = \frac{1}{\lambda} \cdot \frac{\alpha_0}{\sqrt{Var\left(r_t - r_{ft} | I_{t-1}\right)}} \quad . \tag{6}$$

In such a situation a margin to beat buy and hold strategies may theoretically exist, since we are investing less in the risky asset when the reward to variance ratio is low, respectively more when it is high. However, this gain may be partially offset by the loss of time diversification. In the next two sections these two basic hypotheses are combined in a Bayesian framework and the implied optimal trading strategy of a Bayesian conditional mean variance optimizer is evaluated in terms of financial risk adjusted unconditional performance on real data.

2.2 A Simple Bayesian Strategy

While the specifications (3) and (4) are economically clearly interpretable hypotheses that can be each easily implemented in standard econometric models, they are difficult to disentangle using data sets consisting of a realistic number of daily observations. As observed, the elusive empirical relation between excess returns and volatilities makes a precise estimation of conditional expected returns a difficult task in applications. As a consequence, any statistical testing procedure trying to accept one of these two competing models or a further hypothesis "near" to (3) or (4) will have a poor power.

Based on this motivation, we treat (3) and (4) as two limit-case competing models that cannot be really well identified, based on realistic data sets of observations, and assigne to each of these two competing models an a priori probability determined at time 0, in a very Bayesian spirit; cf. also Jorion [14][5].

We consider parametric models for the dynamics of $r_t - r_{ft}$ where conditional means and variances can be parameterized by some corresponding functions m_t and h_t evaluated at some unknown parameter vector $\vartheta = (\alpha, \beta)'$:

$$m_t(\vartheta) = E(r_t - r_{ft}|I_{t-1}) \quad , \quad h_t(\beta) = Var(r_t - r_{ft}|I_{t-1}) \quad .$$

Following the discussion in the last section, we consider dynamic investment policies under the two competing hypotheses (3) and (4) on m_t:

$$\mathcal{H}_0 \; : \; m_t^0 = \alpha_0 \sqrt{h_t} \quad , \quad h_t = h_t(\beta) \quad , \tag{7}$$

$$\mathcal{H}_1 \; : \; m_t^1 = \alpha_1 \quad , \quad h_t = h_t(\beta) \quad , \tag{8}$$

and assume they to hold at time zero with a given *a priori* probability π and $(1-\pi)$, respectively[6]. As outlined, $\mathcal{H}_1$ is consistent with investors ignoring short term fluctuations in volatility, possibly because of transaction costs and model misspecifications making dynamic policies ineffective, while $\mathcal{H}_0$ corresponds to a situation where investors are rewarded for taking risk by excess returns that are proportional to the risk being taken.

On the other hand, defining the conditional error:

$$\varepsilon_t(\vartheta) = r_t - r_{ft} - m_t(\vartheta) \quad , \tag{9}$$

we model conditional variances with an asymmetric GARCH(1,1) process, cf. again Glosten, Jagannathan and Runkle [12]:

$$h_t = \beta_1 + \beta_2 \epsilon_{t-1}^2 + \beta_3 h_{t-1} + \beta_4 I_{\{\epsilon_{t-1} \leq 0\}} \epsilon_{t-1}^2 \quad , \tag{10}$$

[5] In a related paper Barberis ([2]) shows rhat parameter uncertainty can produce an important impact on the portfolio allocation, depending on the investor's time horizon.

[6] Remark that we assume an identical volatility structure under $\mathcal{H}_0$ and $\mathcal{H}_1$.

where $I_{\{\epsilon_{t-1}\leq 0\}}$ is the indicator function of the event $\{\epsilon_{t-1} \leq 0\}$.

A Bayesian mean variance optimizer maximizes expected quadratic utility at time t, based on *predictive* conditional densities of $r_t - r_{ft}$; see again Jorion [14] as a basic reference.

We assume our manager has at her disposal a set $Y = (r_t - r_{ft})_{t=-N,..,-1}$ of observations that can be used at time 0 to estimate the process for excess market returns under $\mathcal{H}_0$ and $\mathcal{H}_1$, respectively. Based on the realization Y, the corresponding *a posteriori* probabilities for these two hypotheses can be computed.

The optimization problem of our Bayesian optimizer is:

$$\max_{w_t} \left[w_t E^* \left(r_t - r_{ft} | I_{t-1} \right) - \frac{\lambda}{2} \cdot w_t^2 Var^*(r_t - r_{ft}|I_{t-1}) \right] \quad , \tag{11}$$

where $E^* \left(\cdot | I_{t-1} \right)$ and $Var^*(\cdot|I_{t-1})$ denote conditional expectations and variances based on the predictive densities implied by Bayes formula, given the a priori probability π. Straightforward computations give[7]:

$$E^* \left(r_t - r_{ft} | I_{t-1} \right) = \pi^* \cdot m_t^0 + (1-\pi^*) m_t^1 \quad , \quad Var^*(r_t - r_{ft}|I_{t-1}) = h_t \quad ,$$

where

$$\pi^* = \frac{\pi l_{\mathcal{H}_0}(Y)}{\pi l_{\mathcal{H}_0}(Y) + (1-\pi) l_{\mathcal{H}_1}(Y)} \tag{12}$$

is the posterior probability of $\mathcal{H}_0$ and $l_.(Y)$ is the likelihood of Y at time 0 under the hypothesis $\mathcal{H}_.$.

The solution to the optimization problem (11) is[8]:

$$w_t = \frac{1}{\lambda} \cdot \frac{\pi^* \cdot m_t^0 + (1-\pi^*) m_t^1}{h_t} \quad . \tag{13}$$

As intuitively obvious, the assumption of an identical volatility process under $\mathcal{H}_0$ and $\mathcal{H}_1$ implies an identical a priori and a posteriori conditional volatility process. As a consequence, this assumption models a situation were our

[7] Compared to a full Bayesian approach we neglect estimation errors that can arise *within* one of the two competing model choices.

[8] Notice that as a particular case one gets the standard optimal rule of an investor *fully believing* to $\mathcal{H}_0$ and $\mathcal{H}_1$, when fixing a priori probabilities given by $\pi = 0, 1$, respectively.

manager is primarily concerned with estimation errors deriving from the mis-specifications of the conditional mean equation and less with those caused by volatility estimates, which are by some orders of magnitude less influential than conditional means estimates[9].

Assuming conditionally normal errors

$$h_t^{-\frac{1}{2}} \cdot \varepsilon_t \backsim i.i.d\, \mathcal{N}(0,1) \quad ,$$

we can easily approximate the optimal weights in (12) by computing the odd ratio $\frac{\pi^*}{1-\pi^*}$ implied by our competing hypotheses:

$$\frac{\pi^*}{1-\pi^*} = \frac{\pi}{1-\pi} \cdot \frac{l_{\mathcal{H}_0}(Y)}{l_{\mathcal{H}_1}(Y)} = \frac{\pi}{1-\pi} \cdot \left(\frac{\widehat{\sigma}_0}{\widehat{\sigma}_1}\right)^{-N} \quad , \tag{14}$$

where $\widehat{\sigma}_.$ is the estimated standard deviation of residuals under $\mathcal{H}_.$ in the OLS regressions:

$$r_t - r_{ft} = \dot{m_t} + u_t \quad ; \quad u_t \backsim \mathcal{N}(0, h_t) \quad . \tag{15}$$

The next section tests our Bayesian methodology on real data using excess return series for four main stock indices.

2.3 Empirical Evidence

We consider daily observations from June 1989 to January 1999 for four main stock indices: the S&P 500, the Dow Jones Industrials, the FTSE 100 and the Nikkei.

For estimation and out of sample evaluation purposes we split this sample in two subperiods of approximative equal size (period I and period II), the first going from June 1989 to December 1993, the second from January 1994 to January 1999.

[9] In a sense this assumption is also linked to the common practice in the literature of estimating conditional means and volatilities by a two step regression where in the first step the volatility parameters are estimated neglecting the contribution of conditionally expected excess returns. This precisely because of the elusive relation between returns and volatilities that renders estimated conditional means negligible in the estimation of the conditional variance equation for practical purposes.

In keeping with the common practice in the literature we estimate model (7) and (10), respectively (8) and (10) (our two basic hypotheses) using a two step procedure. In the first step the asymmetric GARCH(1,1) model (10) is fitted ignoring the contribution of expected excess returns in defining the $(\varepsilon_t)_{t=0,..,T}$ process in (9). Conditionally on these volatility estimates[10] we then regress index excess returns on a constant (under $\mathcal{H}_1$), or on the estimated GARCH volatilities (under $\mathcal{H}_0$).

The asymmetric GARCH(1,1) Quasi Maximum Likelihood parameter estimates obtained by such a procedure for model (10) are presented in Table 1 with the corresponding Bollerslev-Wooldridge [6] robust standard errors in parentheses.

Insert Table 1 about here

The daily series of returns and volatility estimates implied by these GARCH(1,1) parameter estimates are then analyzed in order to investigate the linkage between volatilities and expected returns. A scatter plot illustrating the empirical form of the relation observed for the Dow over the period January 1994 to January 1999 was already presented in Figure 1. As noted, the predictable component of returns emerging from this preliminary analysis is small. Indeed, if we regress excess returns on the series of volatility estimates there is some tendency to find positive slopes, however a not significative one[11].

This weak evidence of compensation for volatility risk is consistent with the existence of trading policies producing a higher risk adjusted performance than buy and hold strategies. However, an active volatility policy of the form (1) still needs a specification of conditional means. The elusive relation between the latter and volatilities suggests us to consider simple specifications for which a clear economic intuition exists. In the following the two (empirical) limit-case bench-

[10] We also used a GARCH in mean specification to estimate all parameters of interest simultaneously and found the parameters in the conditional mean equation to be not significant in the period I and II as well. Since larger sample sizes are not meaningful for our kind of application we preferred to avoid introducing additional noise in the volatility estimates.

[11] These results are consistent with those obtained by a GARCH in mean specification.

marks defined by the hypotheses $\mathcal{H}_0$ and $\mathcal{H}_1$ in (7) and (8) are considered. Since we cannot expect our data to provide a clear statistical evidence in favor of one of the two hypotheses they are combined in the bayesian framework (13) using the relative likelihoods in (14) obtained by running the OLS regressions[12] (15). Moreover, we use a diffuse prior $\pi = 0.5$ to compute the a posteriori probability (12).

The estimated parameters, under $\mathcal{H}_0$ and $\mathcal{H}_1$ respectively, and the implied a posteriori weights (12) for period I and II are summarized in Table 2 below.

Insert Table 2 about here

Both the Dow Jones and the S&P 500 yield positive estimated expected returns as a function of volatility under $\mathcal{H}_0$ and $\mathcal{H}_1$. A small negative unconditional estimated expected return occurs for the FTSE in period I under $\mathcal{H}_0$ while the Nikkei produces a negative estimated relation between expected returns and volatilities under $\mathcal{H}_0$ and $\mathcal{H}_1$ for both periods analyzed, due to the repeated large losses of the Japanese stock market over our investigation period.

It is interesting to notice that the optimal a posteriori weights (12) in Table 2 are somehow between 0.3 and 0.5 for the S&P 500, the FTSE and the Nikkei, indicating a slightly more likely hypothesis $\mathcal{H}_1$ for these indices over both investigation periods. On the other side, $\mathcal{H}_0$ is more likely for the Dow Jones and specifically highly plausible in period I. Nevertheless, the inference on the competing hypotheses $\mathcal{H}_0$ and $\mathcal{H}_1$ implied by the observed odd ratios is - as expected - largely inconclusive. Finally, observed relative likelihoods are relatively stable over the two subsamples.

Given the parameter estimates obtained for the conditional variance equation (10) and the estimated conditional expected returns (7) and (8) under

[12] As a futher check, a regression under the union $\mathcal{H}_0 \vee \mathcal{H}_1$ of our two competing hypotheses yields a negative estimated intercept for all indices analyzed with exeption of the S&P 500 both for period I and II, implying volatility ranges where expected excess returns are negative. Moreover, the numerical values of these estimated intercepts are by some orders of magnitude unrealistic.

our competing hypotheses, we can compute a series of returns implied by the bayesian optimal strategy given in (13).

In order to compare the unconditional performance of our dynamic Bayesian strategy to that of a "static" buy and hold stategy[13], we use Sharpe ratios and adjusted Jensen's alphas[14] as performance measures. Table 3 summarizes the in-sample (full sample period) and out-of-sample (period II) results obtained when testing our bayesian strategy.

Insert Table 3 about here

In Table 3 the bayesian strategy performs better[15] than buy and hold both in the in- and out-of-sample results. As expected, the performance margins in sample are higher[16].

In the out of sample analysis some evidence of a superior performance of the Bayesian strategy arises for all indices analyzed; on an annual basis the adjusted Jensen's alpha's range from 0.5% for the FTSE to about 0.75% and 1% for the Dow and the S&P 500, respectively.

It is interesting to discuss the results obtained for the Nikkei where while buy and hold systematically lost money the Bayesian strategy produced a positive performance. Obviously, this result is a direct consequence of the large losses of the Nikkei over period I and II resulting in negative parameter estimates α_0 and α_1, which in turn caused the Bayesian strategy to systematically shorten the

[13]For the buy and hold strategy we choose as weight on the risky asset $w = 1$, i.e. the whole wealth is invested in the index and no reallocation is needed through time. Notice that the Sharpe ratio of a strategy is unchanged if the weights w_t are rescaled by a multiplicative factor, hence in particular the Sharpe ratio of buy and hold is independent of the weight w choosen.

[14]The Sharpe ratio is defined as the ratio of average excess returns to sample volatility. The adjusted Jensen's alpha is defined as the intercept of a regression of portfolio's excess returns on index excess returns, scaled in order to make the unconditional volatility of portfolio's returns identical to that of index returns. In future versions of this paper we will compare these measures with other based on portfolio's performance over a given horizon.

[15]Relative to the performance measures used.

[16]When developing the same performance analysis with degenerate priors $\pi = 0, 1$, no evidence of a superior performance over buy and hold was found.

index in period II. This explains the huge Jensen's alpha in Table 2 amounting to about 9% on annual basis. From this perspective, a performance comparison that would be more fair and more consistent with the philosophy of our approach is one where the index is "sold" and the position held rather than one where the index is bought and held. In this case the implied Sharpe ratio of the Bayesian strategy is still twice that of such a reversed buy and hold strategy (0.04098 against 0.01894). The implied Jensen's alpha is unchanged.

The Sharpe ratio and the Jensen's alpha are performance measures taking into account only the first two unconditional moments of returns. Hence, higher order moments like skewness and kurtosis are neglected. On the other hand, while our Bayesian policy is based on a pure conditional mean variance optimization, we expect the unconditional skewness and kurtosis of the implied excess returns to be not higher than those of the index returns. Indeed, in a conditionally normal asymmetric GARCH(1,1) model of the form (10) unconditional skewness and kurtosis are generated by the pure dynamics of conditional volatility which, by construction, our Bayesian optimizer tries to optimally anticipate. In order to verify our intuition Table 4 presents the estimated skewness and kurtosis coefficients obtained for the Bayesian and the buy and hold strategy in the out of sample period II.

Insert Table 4 about here

The results in Table 4 indicate that the Bayesian strategy does not generally increase the third and fourth moment of the unconditional distribution of returns, compared to buy and hold. As a consequence, the superior unconditional mean variance performance presented in Table 3 should not come at the expense of other risk dimensions.

3 An Intertemporal Stochastic Volatility Merton's Model

In this section we present an intertemporal stochastic volatility Merton's [19] model, based on a continuous time limit of an exponential M-GARCH specification of conditional expected returns and volatilities. The model allows for an explicit solution of the implied optimal portfolio rules and the corresponding value function, and is rich enough to take into account the main empirical features of the relation between changing expected returns and changing risk, as for instance asymmetries.

The explicit expression obtained for the value function of an agent maximizing the expected indirect power utility of end of period wealth is used to compute the certainty equivalent of the dynamic portfolio rules under investigation. This yields a performance comparison of dynamic strategies that is clearly defined but also, by construction, model dependent.

Although in the context of a well defined model a margin of improvement over buy and hold should realistically exist, we do not expect a very large possibility of improvement using "active" volatility based strategies[17]. In order to further assess the feasibility of dynamic strategies exploiting changing risk, we therefore try to quantify the margins of improvement that are realistically available within a theoretical model calibrated to real data[18].

[17] In fact, such strategies require in some way a forecast of daily future excess returns whose predictable part is small, as showed by the empirical evidence above.

[18] While we focus in this part on quantifying the margins of improvement over buy and hold, a theoretically important aspect of the model solution is the complete characterization of the origin and the relevance of intertemporal hedging in a stochastic volatility framework.

3.1 The Basic Model

Let us start by specifying a discrete time exponential M-GARCH process (cf. Nelson [21]) for the joint dynamics of returns and volatilities[19]:

$$r_t = \left(\mu(\sigma_t) - \frac{1}{2}\sigma_t^2\right) + \sigma_t z_t \quad , \tag{16}$$

$$\ln \sigma_t^2 = \omega + \delta \ln \sigma_{t-1}^2 + \vartheta z_{t-1} + \gamma(|z_{t-1}| - E|z_{t-1}|) \quad , \tag{17}$$

where $z_t \backsim i.i.d.(0,1)$ and $\mu(\sigma_t) - \frac{1}{2}\sigma_t^2$ is a possible specification for expected returns[20] as a function of volatility σ_t, defined by a given function $\mu : (0,\infty) \longrightarrow \mathbb{R}$. Equation (17) models the dynamics of volatilities as an exponential GARCH process in which asymmetries in conditional log variances are described by the terms ϑz_{t-1} and $\gamma(|z_{t-1}| - E|z_{t-1}|)$, where typically $\vartheta < 0$ and $\gamma > 0$. The combination of these two parts introduces in the GARCH dynamics an asymmetric response of log volatilities with respect to both the sign and the magnitude of unexpected returns, thereby incorporating a kind of "leverage" effect; cf. Black [4].

Let S_t be the price at time t of the risky asset (the index level in our case), where by definition $r_t = \log(S_t/S_{t-1})$. Assuming (only for simplicity of exposition) a normal distribution of z_t, we follow Nelson [22] in deriving the continuous time limit of the discrete time process (16) and (17), which is given by[21]:

$$\frac{dS_t}{S_t} = \mu(\sigma_t)dt + \sigma_t dZ_{1,t} \tag{18}$$

$$d\ln \sigma_t^2 = \beta(\alpha - \ln \sigma_t^2)dt + \psi dZ_{2,t} \tag{19}$$

[19] In this part we model directly index returns rather than excess returns. Given the assumption of a deterministic interest rate introduced below (and needed to obtain explicit model solutions) this is equivalent to modelling excess index returns.

[20] The motivation for isolating the component $-\frac{1}{2}\sigma_t^2$ in this conditional mean specification derives from the fact that this term will cancel out when taking the continuos time limit of (16). This continuous time limit process is the one that will be used to define an intertemporal Merton's model with stochastic volatility later on.

[21] For completeness, we provide in the Appendix a proof of a more general statement that covers the case of an asymmetric distribution of z_t.

where $Z = (Z_{1,t}, Z_{2,t})'_{t\geq 0}$ is a driftless Brownian Motion in $\mathbb{R}^2$ such that:

$$dZ_t dZ_t' = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} dt \quad . \tag{20}$$

The continuous time and the discrete time parameters are related by the equations:

$$\beta = 1 - \delta \quad , \quad \alpha = \frac{\omega}{1-\delta} \quad , \quad \rho\psi = \vartheta \quad , \quad \psi\sqrt{\frac{1-\rho^2}{1-\frac{2}{\pi}}} = \gamma \quad . \tag{21}$$

Notice that the continuous time limit (18) and (19) yields an investment opportunity set where returns and volatilities are locally correlated, precisely because of the asymmetries arising in the discrete time evolution of log volatilities[22]. Specifically, we observe that given $\vartheta < 0$ and $\gamma > 0$ the implied correlation ρ between the Brownian motions Z_1 and Z_2 is negative[23].

We now consider an economic agent investing in a risky asset, with price S_t at time t and dynamics given by (18) and (19), and in a riskless asset with price B_t at time t whose dynamics are:

$$\frac{dB_t}{B_t} = rdt \quad ,$$

where r is a constant riskless rate. Denoting by w_t the fraction of current wealth W_t allocated to the risky asset and assuming a zero consumption rate, the implied wealth dynamics are:

$$dW_t = W_t[w_t(\mu(\sigma_t) - r) + r]dt + W_t w_t \sigma_t dZ_{1,t} \quad . \tag{22}$$

Let η be a constant time preference rate, $V(\cdot)$ an indirect utility of wealth and T a final date.

[22] For a Merton investor such a correlation induces optimal allocations where hedging positions are taken to hedge against intertemporal movements in the stochastic opportunity set.

[23] In the more general formulation presented in the Appendix this correlation is also related to the moment $E(z_t|z_t|)$ that describes asymmetries in the distribution of z_t. By definition, under normality this moment is 0.

The investor's objective is to maximize the expected discounted indirect utility of terminal wealth:

$$J(W_t, \xi_t, t) = \max_{w_t} E_t \left(e^{-\eta(T-t)} V(W_T) \right) \quad , \tag{23}$$

under the dynamics (18) and (19) for $(S_t)_{t\geq 0}$ and $(\xi_t)_{t\geq 0} := \left(\log \sigma_t^2\right)_{t\geq 0}$. The Hamilton Jacobi Bellman (HJB) equation for this problem reads:

$$\begin{aligned} \eta J &= \max_w \left\{ [w(\mu(\sigma) - r) + r] W J_W + \frac{1}{2} w^2 W^2 \sigma^2 J_{WW} + \psi \rho w W \sigma J_{W,\xi} \right\} \\ &\quad + J_t + \beta(\alpha - \xi) J_\xi + \frac{\psi^2}{2} J_{\xi,\xi} \quad , \end{aligned} \tag{24}$$

with the boundary condition $J(W, \xi, T) = V(W)$. The implied optimal portfolio allocation is given by:

$$w_t = -\frac{J_W}{W J_{W,W}} \cdot \frac{\mu(\sigma_t) - r}{\sigma_t^2} - \frac{J_{W,\xi}}{W J_{W,W}} \cdot \frac{\psi \rho}{\sigma_t} \quad . \tag{25}$$

The first term on the right hand side of (25) is the standard conditionally optimal "mean variance" allocation in Merton's model. The second part is an hedging position for intertemporal movements in the investment opportunity set, due to asymmetries in the return distributions that induce a local correlation in the common dynamics of returns and variances.

Specifically, for positive[24] $J_{W,\xi}$ and since $\psi\rho = \vartheta$ we see that the hedging position is negative (positive) if and only if ϑ is. This is obvious, since the sign of the asymmetry parameter ϑ completely determines the direction of the local correlation between returns and variances. Finally, the second asymmetry parameter γ in (17) enters only indirectly in (25), through the solution for the value function J.

[24] This is the case in the explicit model presented below given the set of estimated parameters and over the range of observed estimated volatilities.

3.2 Explicit Solutions for Power Indirect Utility

The HJB equation (24) has an explicite solution for a power indirect utility function of wealth:

$$V(W) = \frac{W^{1-p}}{1-p} \quad , \tag{26}$$

and a linear specification of reward to volatility ratios with respect to log variances:

$$\frac{\mu(\sigma_t) - r}{\sigma_t} = d_0 + d_1 \xi_t \quad . \tag{27}$$

In this case the model is equivalent to that of Kim and Omberg ([16]), who model directly the reward to volatility ratio as an Ornstein-Uhlenbeck process.

We review briefly the solution methods.

Substituting these specifications and the optimal rules (25) in (24), the HJB equation of our optimizing investor looks explicitly:

$$\begin{aligned} \eta J \quad = \quad & J_t - \frac{1}{2}\frac{J_W^2}{J_{WW}}(d_0 + d_1\xi)^2 - \psi\rho\frac{J_W J_{W,\xi}}{J_{WW}}(d_0 + d_1\xi) \\ & + rWJ_W + \beta(\alpha - \xi)J_\xi + \frac{\psi^2}{2}J_{\xi,\xi} - \frac{\psi^2\rho^2}{2}\frac{J_{W,\xi}^2}{J_{WW}} \quad , \end{aligned} \tag{28}$$

subject to the boundary condition:

$$J(W, \xi, T) = \frac{W^{1-p}}{1-p} \quad . \tag{29}$$

We look for a solution of the form:

$$J(W, \xi, t) = \frac{W^{1-p}}{1-p} \exp(S(\xi, t)) \quad , \tag{30}$$

which is suggested by the solution for the standard constant volatility Merton's model; cf. Merton [19]. The partial differential equation (PDE) implied by (28) and (29) for S is:

$$\frac{\psi^2}{2} \cdot S_{\xi\xi} + \frac{\psi^2}{2} A \cdot S_\xi^2 + (B + C\xi) \cdot S_\xi + D + E\xi + \frac{F}{2} \cdot \xi^2 + S_t = 0, \tag{31}$$

subject to the boundary condition $S(\xi,T)=0$; the constants A, B, C, D, E and F are related to the model parameters by the formulas:

$$\begin{aligned}A &= 1+\frac{(1-p)}{p}\rho^2\ ,\ B=\beta\alpha+\frac{1-p}{p}\psi\rho d_0\ ,\\ C &= \frac{1-p}{p}\psi\rho d_1-\beta\ ,\ D=r(1-p)+\frac{1-p}{2p}d_0^2-\eta\ ,\\ E &= \frac{1-p}{p}d_0d_1\ ,\ F=\frac{1-p}{p}d_1^2\ . \qquad (32)\end{aligned}$$

Solving this PDE amounts to giving the solution of problem (23) for the given specifications (26) and (27). Notice that when $p>1$ then one has

$$A>0,\quad F<0\quad ,\quad \Delta:=C^2-\psi^2AF>0\quad . \qquad (33)$$

In the next proposition we report the explicit solution of (23) for the case $p>1$[25], since this interval of values for p contains the set of risk aversion parameters that yield realistic risky allocations in applications with real data. The solution given here can be seen to correspond to that of Kim and Omberg ([16]) after sobstituting the constants A, B, C, D, E, and F with their definitions in terms of the original parameters. We prefer to give the solution in terms of the constants A, B, C, D, E, and F since the computation of the indirect utility for a mean-variance policy considered later will necessitate the solution of a PDE in the form of (31) with slightly different constants. [26].

Proposition 1 *Let $p>1$; it then follows:*

$$J(W,\xi,t)=\frac{W^{1-p}}{1-p}\exp\left(a(t)+b(t)\xi+\frac{1}{2}c(t)\xi^2\right)\quad , \qquad (34)$$

with an optimal allocation (25) given by:

$$\begin{aligned}w_t &= \frac{d_0+d_1\ln(\sigma_t^2)}{p\sigma_t}+\psi\rho\frac{b(t)+c(t)\ln(\sigma_t^2)}{p\sigma_t}\\ &= \frac{(d_0+\psi\rho b(t))+(d_1+\psi\rho c(t))\ln(\sigma_t^2)}{p\sigma_t}\quad . \qquad (35)\end{aligned}$$

[25] The case $p>1$ corresponds to the "normal, well-behaved solutions" in Kim and Omberg[16]

[26] All proofs are given in the Appendix.

where:

$$c(t) = \frac{\sqrt{\Delta}}{\psi^2 A}\left(\frac{1-\zeta\exp(2\sqrt{\Delta}(T-t))}{1+\zeta\exp(2\sqrt{\Delta}(T-t))} - \frac{C}{\sqrt{\Delta}}\right) \quad ,$$

$$b(t) = \beta_0\left(1-\sqrt{\frac{F+2Cc(t)+\psi^2 Ac(t)^2}{F}}\right) + \beta_1 c(t) \quad ,$$

$$\begin{aligned} a(t) &= \alpha_0(T-t) + \alpha_1 c(t) - \beta_0\beta_1\left(\frac{2\sqrt{\Delta}\delta}{\sqrt{-FA\psi^2}}\frac{\exp(\sqrt{\Delta}(T-t))}{1+\zeta\exp(2\sqrt{\Delta}(T-t))} - 1\right) \\ &\quad -\alpha_2 \ln\left(\frac{1+\zeta\exp(2\sqrt{\Delta}(T-t))}{1+\zeta}\right) \quad . \end{aligned}$$

The parameters ζ, β_0, β_1, α_0, α_1, α_2, α_3 *are given by:*

$$\zeta = \frac{1-\frac{C}{\sqrt{\Delta}}}{1+\frac{C}{\sqrt{\Delta}}} \quad , \quad \beta_0 = \frac{FB-CE}{\Delta} \quad , \quad \beta_1 = \frac{BC-\psi^2 AE}{\Delta} \quad ,$$

$$\begin{aligned} \alpha_0 &= D + B\beta_0 - \frac{B\beta_1 C}{\psi^2 A} + \frac{\psi^2 A}{2}\left(\beta_0^2 + \frac{\beta_1}{\psi^2 A}\left(\frac{2\beta_1 C^2}{\psi^2 A} - \beta_1 F - 2\beta_0 C\right)\right) \\ &\quad + \frac{\sqrt{\Delta}-C}{2A} + \sqrt{\Delta}\beta_1\left(\beta_0 + \frac{B-C\beta_1}{\psi^2 A}\right) \end{aligned}$$

$$\alpha_1 = \frac{\psi^2 A}{2}\left(\frac{\beta_1^2}{\psi^2 A} + \frac{\beta_0^2}{F}\right) \quad , \quad \alpha_2 = \frac{1}{2}\left(\frac{1}{A} + \beta_1\left(\beta_0 + \frac{B-C\beta_1}{\psi^2 A}\right)\right)$$

Obviously, since in real data we expect $\psi\rho = \vartheta < 0$, a pure "mean variance" allocation of the form:

$$\widetilde{w}_t = \frac{d_0 + d_1 \ln \sigma_t^2}{p\sigma_t} \quad , \tag{36}$$

will not be generally optimal[27]. A natural question is if, given realistic model parameters, hedging positions taking asymmetries into account matter for practical purposes. The next proposition derives the expected indirect utility of

[27] A case where this trading rule is optimal also when $\vartheta < 0$ arises when the reward to volatility ratio does not depend on σ_t^2, that is when $d_1 = 0$. This is easily seen when setting $d_1 = 0$ in the constants C, E and F in (32) and then solving the implied PDE for S. Intuitively, in this case "conditional mean variance" allocations are optimal because the intertemporal correlation of returns and variances does not affect the conditional opportunity set and specifically reward to variability ratios.

terminal wealth in our model for a trading rule of the form (36) that does not hedge for asymmetries.

Proposition 2 *The indirect utility*

$$\widetilde{J}(W, \xi, t) = E_t\left(e^{-\eta(T-t)}\frac{W_T^{1-p}}{1-p}\right) \tag{37}$$

of the policy (36) is given by the functional form (34) for the value function J, computed setting $A = 1$ in (32).

A comparison of the value function (35) and the indirect utility (37) for a given parameter choice in the model can provide insights about differences in the certainty equivalents of these two strategies, and consequently on the relevance of intertemporal hedging for asymmetries in the present setting; cf. also the next section.

3.3 Calibration of the Model

We calibrate the dynamics of the stochastic volatility Merton's model of the previous sections to the Dow Jones Index for the whole sample period 1989-1999.

The parameters of the continuous-time model (18) and (19) implied by the estimated parameters of the exponential asymmetric M-GARCH process (16) and (17) are:

$$\begin{aligned} \alpha &= -9.574 \quad , \quad \beta = 0.0470 \quad , \quad \psi = 0.1404 \quad , \\ \rho &= -0.6590 \quad , \quad d_0 = -0.0770 \quad , \quad d_1 = -0.0119 \quad . \end{aligned}$$

We set the instantaneous interest rate equal to $r = 0.0002$ (corresponding to an annual interest rate of approximative 5%) and fix a time preference rate $\eta = 0.0002$. Finally, the risk aversion parameter used is $p = 4$.

The dependence between expected returns and volatilities implied by our parameter choice is plotted in Figure 2.

Insert Figure 2 about here

A slight payment for volatility risk in estimated expected returns is observed: estimated expected returns increase as a function of volatility and are between 0.00027 and 0.00035 for in sample estimated volatilities ranging from 0.006 to 0.012.

Figure 3 presents the optimal allocation w_t in (35) (including hedging for asymmetries) and the "mean variance" position $\widetilde{w}_t$ (36) in dependence of the current volatility level[28].

Insert Figure 3 about here

As expected, both positions are decreasing functions of volatility. For any volatility level, $\widetilde{w}_t$ is less that w_t, because of the negative hedging position for asymmetries induced by the negative estimated correlation between returns and volatilities. However, the differences between the two positions that are implied by the given parameter choice are small and specifically never larger than 2%.

Figure 4 plots $\widetilde{w}_t$ and w_t as a function of (decreasing) time to maturity for an investment horizon of 250 days and a current volatility level 0.0085.

Insert Figure 4 about here

It is interesting to notice that the difference in the two positions is almost constant (amounting to about 2%) for the first 160 days. After this date, intertemporal hedging positions are monotonically relaxed until maturity.

We conclude this section by a performance analysis making use of the certainty equivalent implied by the model solution (34). Three competing dynamic strategies are compared: the optimal strategy (35), the pure "mean variance" rule (36) and a "static" buy and hold allocation over a time horizon of $T-t = 500$ business days. Initial current wealth W_t is normalized to 1.

[28] Remark that the given choice for the parameter α implies a long term expected volatility level of 0.00834.

For these three trading strategies we compute the corresponding certainty equivalent CE_t at time t defined by:

$$V(CE_t) = E_t\left(e^{-\eta(T-t)}V\left(W_T\right)\right) \quad , \tag{38}$$

where W_T is the terminal wealth produced at time T by the strategy under scrutiny and V is the power utility function.

The expectations on the right hand side of (38) were already computed in (34) and (37) for the optimal and the "pure mean variance" strategy. For a buy and hold rule this same expectation has to be computed by simulation[29] since the PDE implied for this case by the Feynman Kac formula cannot be easily solved analytically.

As expected[30], one cannot observe important differences in certainty equivalent units between the optimal strategy (35) and the pure "mean variance" rule (36). We therefore concentrate on a comparison between buy and hold and a pure mean variance allocation by plotting in Figure 5 the relative certainty equivalent of these two strategies as a function of volatility.

Insert Figure 5 about here

Differences amount to about 1.3% (on an annual basis) in favour of the mean variance allocation and are quite stable as a function of volatility (as one would expect for such a relatively long time horizon).

4 Conclusions

For all stock indices under scrutiny, we find a modest but consistent evidence of a superior performance of dynamic strategies taking into account varying risk in a simple "conditional mean variance" way. Moreover, using explicit solutions in an intertemporal Merton's model it seems that larger margins of improvement

[29] For each value of σ_t, a Monte Carlo simulation consisting of 100000 repetitions using the method of antithetic variables was used.

[30] Remember the similar optimal weights obtained in Figure 3.

are not easily available, even when knowing the underlying data generating process. Finally, for practical purposes intertemporal hedging may be negligible at least for the class of models considered in the paper.

5 Appendix

Proof. (**The continuous time limit process in (18) and (19)**) Let $y_t = \log(S_t)$. The discrete time process (16) and (17) can be then rewritten as:

$$y_t = y_{t-1} + \left(\mu(\sigma_t) - \frac{1}{2}\sigma_t^2\right) + \sigma_t z_t \quad ,$$

$$\ln(\sigma_t^2) = \ln(\sigma_{t-1}^2) - \beta(\ln(\sigma_{t-1}^2) - \alpha) + g(z_{t-1}) \quad ,$$

where:

$$g(z) = \rho\psi z + \psi\left(\frac{1-\rho^2}{1-\lambda^2}\right)^{\frac{1}{2}}(|z| - \lambda) \quad , \quad \lambda = E(|z|) \quad .$$

The new parameters are defined in an obvious way, accordingly to (21). Remark that for a normally distributed random variable z_t one has $\lambda = \sqrt{\frac{2}{\pi}}$ (as for the case presented in the main text). However, we do not assume normality in this proof. Consider now, the sequence of processes

$$\left\{X_t^h = (y_t, \ln\sigma_{t+h}^2) : t = 0, h, 2h, ...\right\} \quad , \quad h \to 0 \quad ,$$

defined by:

$$y_t = y_{t-h} + h\left(\mu(\sigma_t) - \frac{1}{2}\sigma_t^2\right) + h^{\frac{1}{2}}\sigma_t z_t \quad ,$$

$$\ln(\sigma_{t+h}^2) = \ln(\sigma_t^2) - h\beta(\ln(\sigma_t^2) - \alpha) + h^{\frac{1}{2}} g(z_t) \quad .$$

Then:

$$\nu(X_t^h) = \frac{1}{h}E_t(X_{t+h}^h - X_t^h) = \begin{pmatrix} \mu(\sigma_{t+h}) - \frac{1}{2}\sigma_{t+h}^2 \\ -\beta(\ln(\sigma_{t+h}^2) - \alpha) \end{pmatrix} \quad ,$$

and:

$$\begin{aligned} \Omega(X_t^h) &= \frac{1}{h}Var_t(X_{t+h}^h - X_t^h) \\ &= Var_t\begin{pmatrix} \sigma_{t+h} z_{t+h} \\ g(z_{t+h}) \end{pmatrix} \\ &= \begin{pmatrix} \sigma_{t+h}^2 & \left(\rho\psi + \psi\left(\frac{1-\rho^2}{1-\lambda^2}\right)^{\frac{1}{2}}\kappa\right)\sigma_{t+h} \\ \left(\rho\psi + \psi\left(\frac{1-\rho^2}{1-\lambda^2}\right)^{\frac{1}{2}}\kappa\right)\sigma_{t+h} & \psi^2\left(1 + 2\rho\left(\frac{1-\rho^2}{1-\lambda^2}\right)^{\frac{1}{2}}\kappa\right) \end{pmatrix} , \end{aligned}$$

where $\kappa = E(z|z|)$. Hence, covariances among returns and volatilities originate in this model from the asymmetry term $\rho\psi$ and the "asymmetric" moment κ of the conditional distribution of returns. Applying a standard Theorem by Stroock and Varadhan [23], the sequence of processes $X_t^h = (y_t, \ln \sigma_{t+h}^2)$ converges in distribution to the solution of the stochastic differential equation:

$$dX_t = \nu(X_t)dt + \Omega^{\frac{1}{2}}(X_t)d\widetilde{Z}_t \quad ,$$

as $h \to 0$, where $\left(\widetilde{Z}_t\right)_{t\geq 0}$ is a standard Brownian motion in $\mathbb{R}^2$. The equations (18) and (19) for the process $\left(S_t, \ln \sigma_t^2\right)'_{t\geq 0}$ follow by Ito's Lemma. ■

Proof. (**Proposition 1**) A solution of (31) exists in the form:

$$S(\xi, t) = a(t) + b(t)\xi + \frac{1}{2}c(t)\xi^2 \tag{39}$$

Substituting (39) in (31) we get a system of nonlinear ordinary differential equations for a, b and c:

$$\begin{aligned} \dot{c}(t) + F + 2Cc(t) + \psi^2 Ac(t)^2 &= 0 \quad , \\ \dot{b}(t) + E + Cb(t) + Bc(t) + \psi^2 Ab(t)c(t) &= 0 \quad , \\ \dot{a}(t) + D + Bb(t) + \frac{\psi^2 A}{2}b(t)^2 + \frac{\psi^2}{2}c(t) &= 0 \quad , \end{aligned} \tag{40}$$

with the boundary conditions: $c(T) = b(T) = a(T) = 0$. The integral of the first differential equation is given by

$$\int_0^{c(t)} \frac{du}{F + 2Cu + \psi^2 Au^2} = T - t \quad .$$

This integral is standard and for $\Delta = C^2 - \psi^2 AF > 0$ (cf. (33)) we obtain:

$$\frac{1}{2}\ln\left(\frac{1 - \frac{\psi^2 Ac(t)+C}{\sqrt{\Delta}}}{1 + \frac{\psi^2 Ac(t)+C}{\sqrt{\Delta}}}\right) = \sqrt{\Delta}(T-t) + \frac{1}{2}\ln\left(\frac{1 - \frac{C}{\sqrt{\Delta}}}{1 + \frac{C}{\sqrt{\Delta}}}\right) \quad ,$$

which implies immediately the solution of c. Using the triangularity of the system (40), we could insert the solution for c in the second equation and solve for b. However it is much simpler to solve for b as a function of c, which by abuse of notation we denote as $b(c)$. This is possible since c is strongly increasing on

$[0,T]$, as we directly deduce from the differential equation for c. Writing:

$$\dot{b} = \frac{db}{dc} \cdot \dot{c} = -(A + 2Cc + \psi^2 Ac^2) \cdot \frac{db}{dc} \quad ,$$

and inserting this expression in the differential equation for b, we get:

$$(F + 2Cc + \psi^2 Ac^2) \cdot \frac{db}{dc}(c) - (C + \psi^2 Ac) \cdot b(c) = E + Bc \quad , \tag{41}$$

with the boundary condition $b(0) = 0$. Notice that this is a linear differential equation with an homogeneous part which is separable and admits a solution ($b_{HOM}(c)$, say) of the form:

$$b_{HOM}(c) = const \cdot \sqrt{\frac{F + 2Cc + \psi^2 Ac^2}{F}} \quad , \quad const \in \mathbb{R} \quad ,$$

for all c such that[31]:

$$\frac{F + 2Cc + \psi^2 Ac^2}{F} > 0 \ .$$

Furthermore, the function:

$$c \longmapsto \beta_0 + \beta_1 c \quad ,$$

with β_0 and β_1 as given in Proposition 1, defines a particular solution of (41). Using the boundary condition $b(0) = 0$ this implies:

$$b(c) = \beta_0 \left(1 - \sqrt{\frac{F + 2Cc + \psi^2 Ac^2}{F}} \right) + \beta_1 c \quad .$$

The solution for b is given by $b(t) = b(c(t))$.

The third differential equation in (40) is solved by the integral:

$$a(t) = D(T-t) + B \int_t^T b(s)ds + \frac{\psi^2 A}{2} \int_t^T b(s)^2 ds + \frac{\psi^2}{2} \int_t^T c(s)ds \quad ,$$

whose computation inserting the solutions for $b(t)$ and $c(t)$ is straightforward , although somewhat cumbersome. The change of variable from t to c is acould be useful. For example (using the boundary condition $c(T) = 0$):

$$\int_t^T b(s)ds = \int_{c(t)}^{c(T)} \left(\frac{b(c)}{\dot{c}} \right) dc = \int_0^{c(t)} \frac{b(u)du}{F + 2Cu + \psi^2 Au^2} \quad ,$$

[31] Note that this is precisely the relevant domain fo the solution of the above ordinary differential equation for c.

which is standard. The other integrals can be computed in a similar way. ■

Proof. (**Proposition 2**) Under the portfolio rules (36) the current wealth dynamics are given by:

$$\frac{dW_t}{W_t} = \left(\frac{1}{p}(d_0 + d_1\xi_t)^2 + r\right) dt + \left(\frac{d_0 + d_1\xi_t}{p}\right) dZ_{1,t} \quad . \tag{42}$$

The goal is therefore to compute:

$$\widetilde{J}(W_t, \xi_t, T - t) = E_t(e^{-\eta(T-t)} \frac{W_T^{1-p}}{1-p})$$

under the dynamics (42) and (19) for the state vector $X_t = (W_t, \xi_t)'$. By the Feynman-Kac Theorem (cf. for instance [15], p. 268) $\widetilde{J}$ solves the PDE:

$$\begin{aligned} \widetilde{J}_\tau &= (\mathcal{A} - \eta)\widetilde{J} \\ \widetilde{J}(W, \xi, 0) &= \frac{W^{1-p}}{1-p} \end{aligned} \tag{43}$$

where $\tau := T - t$ and $\mathcal{A}$ is the generator of the Markov process $(X_t)_{t\geq 0}$:

$$\begin{aligned} \mathcal{A} &= \frac{(d_0 + d_1\xi)^2 + pr}{p} W \frac{\partial}{\partial W} + \beta(\alpha - \xi)\frac{\partial}{\partial \xi} + \frac{(d_0 + d_1\xi)^2}{2p^2} W^2 \frac{\partial^2}{\partial^2 W} \\ &\quad + \frac{\rho\psi}{p}(d_0 + d_1\xi) W \frac{\partial^2}{\partial W \partial \xi} + \frac{\psi^2}{2}\frac{\partial^2}{\partial^2 \xi} \quad . \end{aligned}$$

Guessing a solution of (43) in the form:

$$\widetilde{J}(W, \xi, \tau) = \frac{W^{1-p}}{1-p} \exp(\widetilde{S}(\xi, \tau)) \quad ,$$

we see that $\widetilde{S}$ solves the PDE (31) with $A = 1$ in (32). ■

6 Tables

TABLE 1

	DOW	**S&P500**	**FTSE**	**NIKKEI**
Period I				
β_1	$1.00E-07$ $(2.45E-07)$	$4.65E-07$ $(3.37E-07)$	$2.43E-06$ $(1.15E-06)$	$1.59E-06$ $(4.80E-07)$
β_2	0.00500 (0.014833)	0.000437 (0.012530)	0.046438 (0.019927)	0.019855 (0.01950)
β_3	0.986862 (0.019746)	0.964186 (0.023624)	0.895002 (0.025989)	0.906119 (0.015572)
β_4	0.009946 (0.010114)	0.038480 (0.020752)	0.054563 (0.02271)	0.145437 (0.033423)
Period II				
β_1	$1.08E-06$ $(6.17E-07)$	$9.07E-07$ $(6.74E-07)$	$9.94E-07$ $(6.17E-07)$	$3.65E-06$ $(2.71E-06)$
β_2	0.004729 (0.015980)	0.061847 (0.021090)	0.013478 (0.014767)	0.013513 (0.013675)
β_3	0.947693 (0.016540)	0.926051 (0.019050)	0.928784 (0.019388)	0.902219 (0.032330)
β_4	0.062690 (0.028342)	-0.008330 (0.025524)	0.085806 (0.020465)	0.134983 (0.051880)

TABLE 2

	DOW	**S&P500**	**FTSE**	**NIKKEI**
Period I				
α_0	0.018999	0.020818	0.009397	-0.026201
α_1	0.000185	0.000140	-1.82E-5	-0.000577
π_*	0.913093	0.500000	0.435968	0.313237
Period II				
α_0	0.072088	0.005761	0.037900	-0.000259
α_1	0.000523	8.81E-05	0.000278	-0.000991
π_*	0.638954	0.398795	0.471414	0.427120

TABLE 3

	DOW	**S&P500**	**FTSE**	**NIKKEI**
Full Sample Period				
SR Buy and Hold	0.04162	0.01271	0.01844	-0.029493
SR Bayesian Strategy	0.04816	0.01953	0.02307	0.02783
Jensen's alpha	0.00012	7.00E-5	9.00E-5	0.00012
Period II				
SR Buy and Hold	0.05778	0.01129	0.03054	-0.01894
SR Bayesian Strategy	0.05993	0.01550	0.03253	0.04098
Jensen's alpha	3.00E-5	4.00E-5	2.00E-5	0.00036

TABLE 4

	DOW	**S&P500**	**FTSE**	**NIKKEI**
Skewness Buy and Hold	-0.7099	-0.2086	-0.0940	-0.1550
Skewness Bayesian Strategy	-0.6549	-0.0342	-0.1892	0.0539
Kurtosis Buy and Hold	10.721	4.8012	5.1795	6.0626
Kurtosis Bayesian Strategy	5.4516	4.0797	3.5776	8.2407

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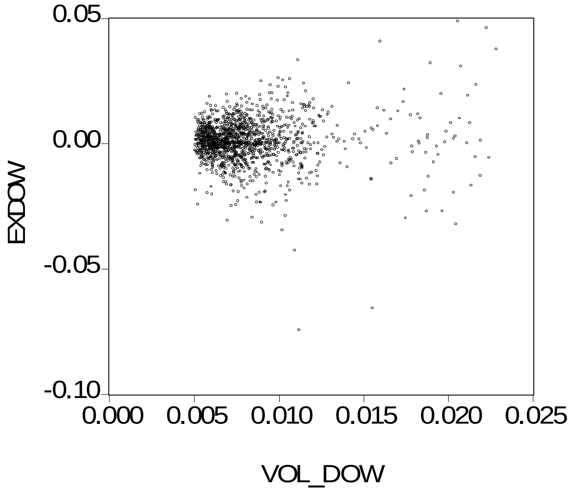

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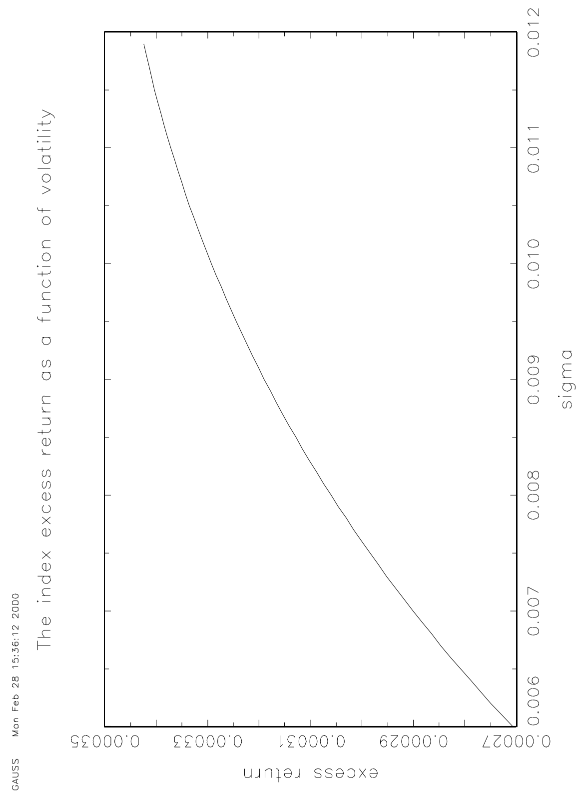

GAUSS Mon Feb 28 15:36:12 2000
The index excess return as a function of volatility
excess return
0.00027
0.00029
0.00031
0.00033
0.00035
sigma
0.006
0.007
0.008
0.009
0.010
0.011
0.012

Figure 2:

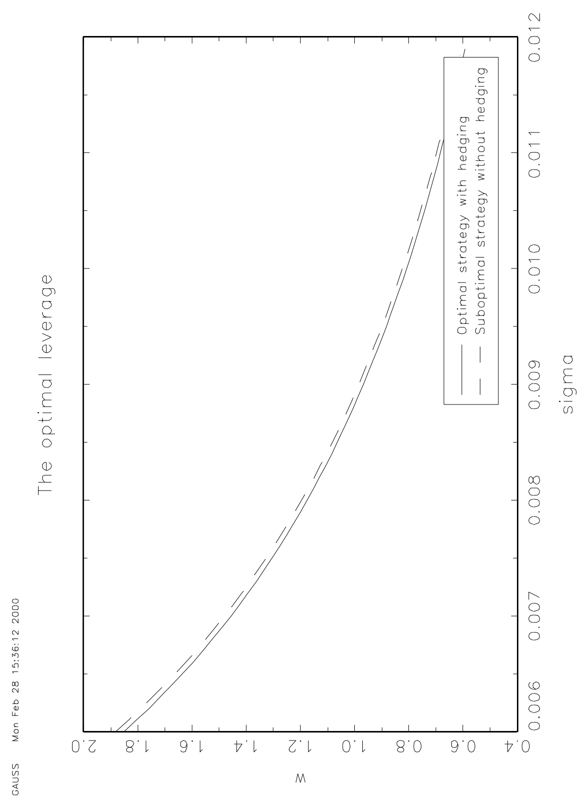

GAUSS Mon Feb 28 15:36:12 2000
The optimal leverage
Optimal strategy with hedging
Suboptimal strategy without hedging
sigma
w
0.006
0.007
0.008
0.009
0.010
0.011
0.012
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4

Figure 3:

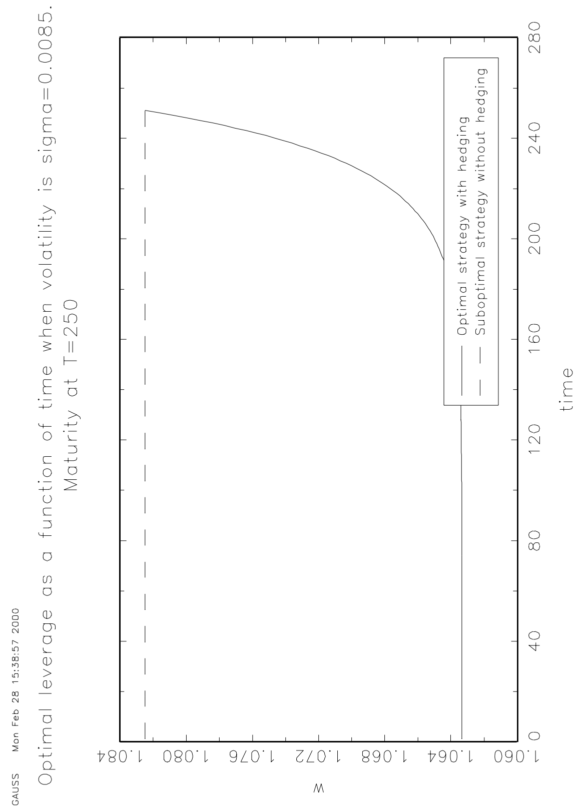
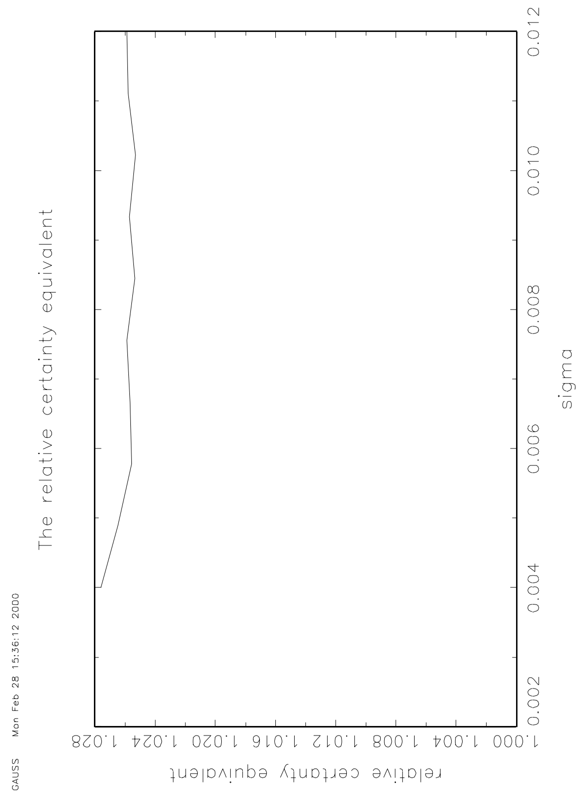

GAUSS Mon Feb 28 15:38:57 2000
Optimal leverage as a function of time when volatility is sigma=0.0085.
Maturity at T=250
Optimal strategy with hedging
Suboptimal strategy without hedging
w
time
1.084
1.080
1.076
1.072
1.068
1.064
1.060
0
40
80
120
160
200
240
280

Figure 4:



GAUSS Mon Feb 28 15:36:12 2000
The relative certainty equivalent
relative certainty equivalent
1.000 1.004 1.008 1.012 1.016 1.020 1.024 1.028
sigma
0.002 0.004 0.006 0.008 0.010 0.012

Figure 5: