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Interpolation with circular basis functions.

Simon Hubbert and Stefan Müller

Abstract

In this paper we consider basis function methods for solving the problem of interpolating data over N distinct points on the unit circle. In the special case where the points are equally spaced we can appeal to the theory of circulant matrices which enables an investigation into the stability and accuracy of the method. This work is a further extension and application of the research of Cheney, Light and Xu ([7] and [10]) from the early nineties.

1 Introduction

Radial basis functions provide highly useful and flexible interpolants to multivariate functions on $\mathbb{R}^d$, see the textbook [2]. Recently, guided by potential applications in the field of physical geodesy, the method has been successfully transferred to problems set on the surface of the 2−sphere where so-called spherical basis functions are employed, see [4]. In this paper we consider the case where the data are known to lie on the unit circle $S^1 \subset \mathbb{R}^2$, and so we consider an interpolation method using *circular basis functions*. A circular kernel $\Psi : S^1 \times S^1 \to \mathbb{R}$ is defined by $\Psi(x, y) = \psi(x^T y)$, where ψ is a univariate function on $[-1, 1]$ and $x^T y$ is the Euclidean dot product of the points $x = (\cos\theta, \sin\theta)$ and $y = (\cos\phi, \sin\phi)$. For a fixed point $x \in S^1$ the value of $\Psi(x, y)$ depends upon the angular distance between x and y, which is defined as

$$\begin{aligned} g(x,y) &= \cos^{-1}(x^T y) = \cos^{-1}\cos(\theta-\phi) \\ &= \begin{cases} |\theta-\phi| & \text{if } |\theta-\phi| \le \pi; \\ 2\pi - |\theta-\phi| & \text{if } |\theta-\phi| > \pi; \end{cases} \end{aligned} \tag{1.1}$$

The circular interpolation problem is as follows. Given values $\{f_i\}_{i=1}^N$ of a function $f : S^1 \to \mathbb{R}$ at a set of distinct points

$$X = \{x_i = (\cos\theta_i, \sin\theta_i), \ \text{where } \ \theta_i \in [0, 2\pi), \ 1 \le i \le N\},$$

find a sequence of numbers $\{\alpha_i\}_{i=1}^N$ such that the function

$$s_f(x) = \sum_{j=1}^N \alpha_j \Psi(x, x_j) = \sum_{j=1}^N \alpha_j \psi \circ \cos\Big(g(x, x_j)\Big) \tag{1.2}$$

satisfies the interpolation conditions

$$s_f(x_i) = f_i \qquad 1 \le i \le N. \tag{1.3}$$

We shall find it useful to recast the interpolation problem in terms of polar angles. Specifically, we set $\varphi = \psi \circ \cos$ and notice that, for any $x = (\cos\theta, \sin\theta) \in S^1$, the interpolant (1.2) can be written as

$$s_f(\theta) = \sum_{j=1}^{N} \alpha_j \varphi(\theta - \theta_j). \tag{1.4}$$

We shall call the function φ the circular basis function (CBF).

Clearly finding the interpolant is equivalent to solving the linear system $A\alpha = f$, where $A \in \mathbb{R}^{N\times N}$ is the interpolation matrix defined by

$$A_{ij} = \varphi(\theta_i - \theta_j) \quad 1 \le i, j \le N. \tag{1.5}$$

Thus a unique interpolant exists for any f if and only if A is non-singular.

Definition 1.1. *A circular basis function φ is said to be*

(*i*) **Strictly positive definite** (SPD) on S^1 whenever its associated matrix (1.5) is positive definite on $\mathbb{R}^N \setminus \{0\}$ for all sets of distinct points on S^1.

(*ii*) **Almost strictly positive definite** (ASPD) on S^1 whenever its associated matrix (1.5) is positive definite on the hyperplane

$$Z = \{\alpha = (\alpha_1, \ldots, \alpha_N)^T \in \mathbb{R}^N \setminus \{0\} : \sum_{i=1}^{N} \alpha_i = 0\}$$

for all sets of distinct points on S^1.

We remark that if φ is SPD on S^1 then it is also ASPD, however the converse does not hold.

If we employ a CBF φ that is SPD on S^1 then the interpolant (1.4) is unique since the interpolation matrix is positive definite and hence non-singular. In the case where φ is ASPD then it is well-known (see [2] Theorem 2.2) that the extra condition $\varphi(0) \le 0$ is sufficient to guarantee uniqueness.

In general we shall consider CBFs with the following form

$$\varphi(\theta) = \sum_{k=0}^{\infty} a_k \cos k\theta, \quad \text{where} \quad \sum_{k=0}^{\infty} |a_k| < \infty, \quad \theta \in [0, 2\pi). \tag{1.6}$$

Using the seminal work of Schoenberg [9] we can formulate the following result.

Theorem 1.2. *If φ is SPD on S^1 then it has the form* (1.6) *where*

$$a_k \geq 0 \quad \text{for} \quad k \geq 0. \tag{1.7}$$

If φ is ASPD but not SPD on S^1 then it also has the form (1.6) *where*

$$a_0 \leq 0 \ \text{ and } \ a_k \geq 0 \quad \text{for} \quad k \geq 1. \tag{1.8}$$

The complete characterization of CBFs of the form (1.6) that are SPD or ASPD on S^1 remains an open problem. A sufficient condition however [11] is that the expansion coefficients a_k be positive rather than nonnegative in (1.7) and (1.8) respectively.

Working with infinite expansions (1.6) is not desirable when we come to implement the CBF method on a computer. Specifically we would like some examples of CBFs that have a closed form. We have the following well-known expansion (see [8])

$$\frac{1-\rho\cos\theta}{1+\rho^2-2\rho\cos\theta} = \sum_{k=0}^{\infty} \rho^k \cos k\theta, \quad \text{for } \ \rho \in (0,1).$$

In view of Theorem 1.2 and the fact that $\rho^k > 0$ for $k \geq 0$ the CBF

$$\varphi(\theta) = \frac{1-\rho\cos\theta}{1+\rho^2-2\rho\cos\theta} \quad \text{for } \ \rho \in (0,1), \tag{1.9}$$

is SPD on S^1. For a second example we take

$$\varphi(\theta) = -\sqrt{2-2\cos\theta} = -\frac{4}{\pi} + \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{\cos k\theta}{(k-\frac{1}{2})(k+\frac{1}{2})}. \tag{1.10}$$

The expansion of this function was computed in [5] and it is clearly ASPD.

2 Stability of interpolation

In order to examine the CBF interpolation method we shall investigate the case where the interpolation nodes are equally spaced, that is

$$x_l = (\cos\theta_l, \sin\theta_l), \quad \text{where } \ \theta_l = \frac{2\pi l}{N}, \ \ 0 \leq l \leq N-1.$$

In this framework the interpolation matrix is given by

$$A_{lm} = \varphi\Big(\frac{2\pi}{N}(l-m)\Big), \quad 0 \leq l, m \leq N-1.$$

An important property of this matrix is that it is a *circulant* since each row is a cyclic shift of the row above it.

Circulant matrices are extremely well studied, see for instance the textbook [3] which is entirely devoted to their properties; for the benefit of the reader we briefly compose the necessary results.

A circulant matrix $C = circ(c_0, c_1, \ldots, c_{N-1}) \in \mathbb{R}^{N\times N}$ has the form

$$\begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{N-1} \\ c_{N-1} & c_0 & c_1 & \cdots & c_{N-2} \\ c_{N-2} & c_{N-1} & c_0 & \cdots & c_{N-3} \\ \vdots & \vdots & \vdots & & \vdots \\ c_1 & c_2 & c_3 & \cdots & c_0 \end{pmatrix}. \tag{2.1}$$

The elements of each row of C are identical to the previous row but are moved one position to the right and are wrapped around.

Theorem 2.1. *The eigenvalues of the matrix C given by* (2.1) are

$$\lambda_j = \sum_{l=0}^{N} c_l e^{-\frac{2\pi i}{N} jl} \qquad j = 0, 1, \ldots, N-1, \tag{2.2}$$

and to each eigenvalue λ_j there corresponds the eigenvector

$$u^{(j)} = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 \\ e^{-\frac{2\pi i}{N} j} \\ e^{-\frac{2\pi i}{N} 2j} \\ \vdots \\ e^{-\frac{2\pi i}{N}(N-1)j} \end{pmatrix} \qquad j = 0, 1, \ldots, N-1. \tag{2.3}$$

Proof. Theorem 3.2.2 [3] □

The matrix of eigenvectors given by $U = (u^{(0)}, u^{(1)}, \ldots, u^{(N-1)})$ such that

$$U_{lm} = \frac{1}{\sqrt{N}} e^{-\frac{2\pi i}{N} lm} \quad \text{for } 0 \le l, m \le N-1, \tag{2.4}$$

is known as the *Fourier matrix* due to its connection with the discrete Fourier transform. It is unitary and we quote the following key result.

Theorem 2.2. *The circulant matrix $C = circ(c_0, c_1, \ldots, c_{N-1}) \in \mathbb{R}^{N\times N}$ is diagonalized by the Fourier matrix* (2.4), that is $C = U^* D U$, where

$$D = \mathrm{diag}(\lambda_0, \lambda_1, \ldots, \lambda_{N-1}) \tag{2.5}$$

is the diagonal matrix of the eigenvalues of C (not necessarily distinct). Furthermore, if C is non-singular then its inverse is a circulant with $C^{-1} = U^* D^{-1} U$, where D^{-1} is the diagonal matrix of the recipricals of the eigenvalues of C.

Proof. See Chapter 3.2 [3]. □

Applying Theorem 2.1 to the interpolation matrix allows us to deduce that its eigenvalues $\{\lambda_j : 0 \le j \le N-1\}$ are given by

$$\lambda_j = \sum_{l=0}^{N-1} \varphi\left(\cos\left(\frac{2l\pi}{N}\right)\right) e^{-\frac{2\pi i}{N} lj} \qquad j = 0, 1, \ldots, N-1.$$

Furthermore, since A is symmetric, each eigenvalue is real and so

$$\lambda_j = \sum_{l=0}^{N-1} \varphi\left(\cos\left(\frac{2l\pi}{N}\right)\right) \cos\left(\frac{2l\pi}{N} j\right) \qquad j = 0, 1, \ldots, N-1.$$

Substituting the cosine expansion of φ (1.6) into this formula gives

$$\begin{aligned} \lambda_j &= \sum_{l=0}^{N-1} \sum_{k=0}^{\infty} a_k \cos\left(\frac{2\pi}{N} kl\right) \cos\left(\frac{2\pi}{N} jl\right) \\ &= \sum_{k=0}^{\infty} a_k \sum_{l=0}^{N-1} \cos\left(\frac{2\pi}{N} kl\right) \cos\left(\frac{2\pi}{N} jl\right). \end{aligned} \tag{2.6}$$

In order to continue with our development we need the following lemma.

Lemma 2.3. *Let $f : \mathbb{N}_0 \to \mathbb{R}$ and $j \in \{0, 1, \ldots, N-1\}$ then we have the following identities*

$$\sum_{k=0}^{\infty} f(k) \sum_{l=0}^{N-1} \cos\frac{2\pi jl}{N} \cos\frac{2\pi kl}{N} = \begin{cases} N \sum_{r=0}^{\infty} f(rN), & \text{if } j = 0; \\ \frac{N}{2} \sum_{r=0}^{\infty} f(rN+j) + f(rN+N-j), & \text{otherwise.} \end{cases}$$

$$\sum_{k=0}^{\infty} f(k) \sum_{l=0}^{N-1} \sin\frac{2\pi jl}{N} \sin\frac{2\pi kl}{N} = \begin{cases} 0, & \text{if } j = 0; \\ \frac{N}{2} \sum_{r=0}^{\infty} f(rN+j) - f(rN+N-j), & \text{otherwise.} \end{cases}$$

$$\sum_{k=0}^{\infty} f(k) \sum_{l=0}^{N-1} \cos\frac{2\pi jl}{N} \sin\frac{2\pi kl}{N} = 0.$$

Proof. The proof of the lemma makes use of the elementary identities $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$. We shall demonstrate the first identity by considering two separate cases depending upon the value of j.

Case 1. If $j = 0$ then $\cos\frac{2\pi jl}{N} = 1$ and so, using the cosine identity, the left hand side of the first identity of the lemma is

$$\sum_{k=0}^{\infty} f(k) \sum_{l=0}^{N-1} \cos\frac{2\pi kl}{N} = \frac{1}{2}\sum_{k=0}^{\infty} f(k) \sum_{l=0}^{N-1} (e^{\frac{2\pi ikl}{N}} + e^{-\frac{2\pi ikl}{N}}). \tag{2.7}$$

For any integer K we have that

$$\sum_{l=0}^{N-1} e^{\frac{2\pi iK}{N}l} = \begin{cases} N, & \text{if } K = 0 \mod N \\ 0, & \text{if } K \neq 0 \mod N. \end{cases} \tag{2.8}$$

Hence we can set $k = rN$ in (2.7), where $r \in \mathbb{N}_0$ and immediately deduce that

$$\sum_{k=0}^{\infty} f(k) \sum_{l=0}^{N-1} \cos\frac{2\pi kl}{N} = N\sum_{r=0}^{\infty} f(rN).$$

Case 2. If $j \neq 0$ then we can use the cosine identity again to deduce that

$$\sum_{k=0}^{\infty} f(k) \sum_{l=0}^{N-1} \cos\frac{2\pi jl}{N}\cos\frac{2\pi kl}{N}$$

$$= \frac{1}{4}\sum_{k=0}^{\infty} f(k) \sum_{l=0}^{N-1} \Big(e^{\frac{2\pi ilj}{N}} + e^{-\frac{2\pi ilj}{N}}\Big)\Big(e^{\frac{2\pi ilk}{N}} + e^{-\frac{2\pi ilk}{N}}\Big)$$

$$= \frac{1}{4}\sum_{k=0}^{\infty} f(k)\Big(\underbrace{\sum_{l=0}^{N-1} e^{\frac{2\pi il}{N}(k-j)} + e^{-\frac{2\pi il}{N}(k-j)}}_{Sum\ (i)} + \underbrace{\sum_{l=0}^{N-1} e^{\frac{2\pi il}{N}(k+j)} + e^{-\frac{2\pi il}{N}(k+j)}}_{Sum\ (ii)}\Big).$$

Using (2.8) we can deduce that sum (i) contributes the value $2N$ whenever $k = rN + j$ and $r \in \mathbb{N}_0$. Similarly, sum (ii) contributes the value $2N$ whenever $k = rN + N - j$ and $r \in \mathbb{N}_0$. These simple observations allow us to deduce that

$$\sum_{k=0}^{\infty} f(k) \sum_{l=0}^{N-1} \cos\frac{2\pi jl}{N}\cos\frac{2\pi kl}{N} = \frac{N}{2}\sum_{r=0}^{\infty} f(rN + j) + f(rN + N - j)$$

as required. The proof of the other identities follow in a similar fashion. □

We can now immediately use the first identity of the lemma to show that (2.6) becomes

$$\begin{aligned} \lambda_0 &= N\Big(a_0 + \sum_{r=1}^{\infty} a_{rN}\Big) \\ \text{and } \lambda_j &= \frac{N}{2}\Big(a_j + \sum_{r=1}^{\infty}(a_{rN+j} + a_{rN-j})\Big) \quad j = 1, 2, \ldots, N-1. \end{aligned} \tag{2.9}$$

We remark that an equivalent expression for the eigenvalues can be found in [7], however we have provided here an alternative derivation of the result.

If N is even, $N = 2p$ say, where $p \in \mathbb{N}$, then we can group the eigenvalues together in p pairs

$$\mathcal{E}_{even} = \Big\{(\lambda_0, \lambda_{2p}), [\lambda_1, \lambda_{2p-1}], \ldots [\lambda_{p-1}, \lambda_{p+1}]\Big\}$$

whereas for N odd, $N = 2p + 1$, we write

$$\mathcal{E}_{odd} = \Big\{\lambda_0, [\lambda_1, \lambda_{2p-1}], \ldots, [\lambda_p, \lambda_{p+1}]\Big\}.$$

Using (2.9), we can deduce that the eigenvalues paired in the square brackets above are equal. It is useful to have expressions for the eigenvalues as it enables us to predict the condition number $\text{cond}(A) = \frac{\lambda_{max}}{\lambda_{min}}$ of the linear system, where λ_{max} and λ_{min} denotes the largest and smallest eigenvalues of A in absolute value.

• **Example 1.** Suppose that we wish to interpolate with the CBF (1.9). In this case we know the cosine expansion coefficients, hence we can use (2.9) to deduce that the eigenvalues of its interpolation matrix are given by

$$\begin{aligned} \lambda_0 &= N\Big(\sum_{r=0}^{\infty} \rho^{Nr}\Big) = \frac{N}{1-\rho^N} \\ \lambda_j &= \frac{N}{2}\Big(\rho^j + (\rho^j + \rho^{-j})\sum_{r=1}^{\infty} \rho^{rN}\Big) = \frac{N}{2}\frac{\rho^j + \rho^{N-j}}{1-\rho^N}, \quad (1 \le j \le N-1). \end{aligned} \tag{2.10}$$

If we use this basis function to interpolate data at an even number of points then λ_0 is the largest eigenvalue and $\lambda_{N/2}$ is the smallest and so the condition number is given by

$$\text{cond}(A) = \left(\frac{\frac{N}{1-\rho^N}}{\frac{2N\rho^{\frac{N}{2}}}{2(1-\rho^N)}}\right) = \Big(\frac{1}{\rho}\Big)^{\frac{N}{2}} \quad \rho \in (0, 1). \tag{2.11}$$

• **Example 2.** Suppose that we wish to interpolate with CBF (1.10). Again, we know the expansion coefficients and so we can use (2.9) to deduce that the eigenvalues of its interpolation matrix are given by

$$\lambda_0 = \frac{2N}{\pi}\Big(-4+\sum_{k=0}^{\infty}\frac{1}{Nk-\frac{1}{2}}-\frac{1}{Nk+\frac{1}{2}}\Big)$$
$$= -\frac{8N}{\pi}-\frac{2}{\pi}\Big(\sum_{k=0}^{\infty}\frac{1}{k-\frac{1}{2N}}-\frac{1}{k+\frac{1}{2N}}\Big).$$

We now use the following identity (see [1] (1.2.5))

$$\frac{1}{\pi}\sum_{k=1}^{\infty}\Big(\frac{1}{k+x}-\frac{1}{k-x}\Big)=\cot\pi x-\frac{1}{\pi x} \tag{2.12}$$

with $x=\frac{1}{2N}$ to yield that

$$\lambda_0 = -2\cot\frac{\pi}{2N}. \tag{2.13}$$

For $j\in\{1,\dots,N-1\}$ we have that

$$\lambda_j = \frac{2N}{\pi}\Big(\frac{1}{2j-1}-\frac{1}{2j+1}\Big)+\frac{1}{\pi}\sum_{k=1}^{\infty}\Big(\frac{1}{k+\frac{2j-1}{2N}}-\frac{1}{k-\frac{2j-1}{2N}}\Big)-\Big(\frac{1}{k+\frac{2j+1}{2N}}-\frac{1}{k-\frac{2j+1}{2N}}\Big).$$

Employing (2.12) once more, with $x=\frac{2j-1}{2N}$ and $x=\frac{2j+1}{2N}$ respectively, we find that

$$\lambda_j = \cot\frac{(2j-1)\pi}{2N}-\cot\frac{(2j+1)\pi}{2N},\quad 1\le j\le N-1. \tag{2.14}$$

If we use this basis function to interpolate data at an even number of points then λ_0 is the eigenvalue with the largest absolute value and $\lambda_{N/2}$ is the smallest and so, applying some elementary trigonometric identities, the condition number is given by

$$\mathrm{cond}(A)=\Big(\frac{1+\cos\frac{\pi}{N}}{1-\cos\frac{\pi}{N}}\Big)=\cot^2\Big(\frac{\pi}{2N}\Big). \tag{2.15}$$

3 The interpolation process

Suppose we wish to interpolate either $\cos m\theta$ or $\sin m\theta$ at N equally spaced points on the circle using a suitable CBF φ. If we consider the following vectors $f_+ = (1, e^{\frac{2\pi i}{N}m},\dots,e^{\frac{2\pi i}{N}m})$ and $f_- = (1, e^{-\frac{2\pi i}{N}m},\dots,e^{-\frac{2\pi i}{N}m})$, then the right hand side for our interpolation problem for $\cos m\theta$ and $\sin m\theta$ can be written as

$$f_{cos}=\frac{1}{2}(f_+ + f_-)^T \qquad \text{and} \qquad f_{sin}=\frac{1}{2i}(f_+ - f_-)^T$$

respectively. Furthermore, using Theorem 2.2, the unique coefficients that solve the interpolation problem are given by

$$\alpha_l^{(cos)}=(U^*D^{-1}Uf_{cos})_l,\quad \alpha_l^{(sin)}=(U^*D^{-1}Uf_{sin})_l,\quad l=0,1,\dots,N-1,$$

for cosine and sine data respectively.

As a first step we consider the matrix-vector product Uf_{cos}. We note that the j^{th} entry of the resulting vector is given by

$$(Uf_{cos})_j = \frac{1}{2}(u^{(j)})^T(f_+ + f_-) = \frac{1}{2\sqrt{N}}\Big(\underbrace{\sum_{k=0}^{N-1} e^{-\frac{2\pi ik}{N}(j-m)}}_{sum\ 1} + \underbrace{\sum_{k=0}^{N-1} e^{-\frac{2\pi ik}{N}(j+m)}}_{sum\ 2}\Big). \quad (3.1)$$

Let us consider two separate cases depending upon the value of m.

CASE I: Assume that $m = 0 \mod N$

Remark 3.1. *In this special case the target data are as follows*

$$f_l = \cos m\left(\frac{2\pi l}{N}\right) = 1, \quad l = 0, 1, \ldots, N-1,$$

for the cosine and

$$f_l = \sin m\left(\frac{2\pi l}{N}\right) = 0, \quad l = 0, 1, \ldots, N-1,$$

for the sine. Hence the resulting interpolants will provide an approximation to the functions $f(\theta) = 1$ and $f(\theta) = 0$ respectively. Thus, care must be taken in choosing a suitable value of N for accurate approximations to $f(\theta) = \cos m\theta$ and $f(\theta) = \sin m\theta$ when $m \neq 0$.

In order to investigate the interpolant we can use (2.8) to deduce that (3.1) simplifies to

$$(Uf_{cos})_j = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} e^{-\frac{2\pi ik}{N}j} = \begin{cases} \sqrt{N}, & \text{if } j = 0 \\ 0, & \text{otherwise.} \end{cases} \quad (3.2)$$

In particular, if we write the standard basis for $\mathbb{R}^N$ as

$$\hat{e}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ \hat{e}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ \hat{e}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \ \hat{e}_{N-1} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}, \quad (3.3)$$

then

$$Uf_{cos} = \sqrt{N}\hat{e}_0. \quad (3.4)$$

In the case of $f(\theta) = \sin m\theta$ we find that $Uf_{sin} = 0$, this is as expected in view of Remark 3.1.

Taking the intermediate step gives $D^{-1}Uf_{cos} = \frac{\sqrt{N}}{\lambda_0}\hat{e}_0$. Hence we can finally deduce that the coefficients of the resulting interpolant are given by

$$\begin{aligned}\alpha_l^{(cos)} &= (U^* D^{-1} U f_{cos})_l = \frac{\sqrt{N}}{\lambda_0}(u^{(l)})^T \hat{e}_0 \\ &= \frac{1}{\lambda_0}\left(1, e^{\frac{2\pi i}{N}l}, e^{\frac{2\pi i}{N}2l}, \ldots, e^{\frac{2\pi i}{N}(N-1)l}\right)^T \hat{e}_0 = \frac{1}{\lambda_0}.\end{aligned} \tag{3.5}$$

Hence for N equally spaced points on S^1 the interpolant to $\cos m\theta$ (with $m = 0 \mod N$) is given by

$$s(\theta) = \frac{1}{\lambda_0}\sum_{l=0}^{N-1}\varphi\left(\theta - \frac{2\pi}{N}l\right). \tag{3.6}$$

CASE II: Assume that $m \neq 0 \mod N$.

Consider *sum* 1 from equation (3.1). We know from (2.8) that this contributes only if

$$j \in \{1, \ldots, N-1\}, \text{ such that } j - m = 0 \mod N. \tag{3.7}$$

It is clear that, for any positive integer $m \neq 0 \mod N$, such an integer j exists. Similarly, for the contribution of *sum* 2 we must consider

$$j' \in \{1, \ldots, N-1\}, \text{ such that } j' + m = 0 \mod N. \tag{3.8}$$

Using these two conditions we can deduce that $j' = N - j$. With this established we have

$$Uf_{cos} = \frac{1}{2}\sqrt{N}(\hat{e}_j + \hat{e}_{N-j}), \text{ where } j - m = 0 \mod N, \tag{3.9}$$

and where $\{\hat{e}_0, \hat{e}_1, \ldots, \hat{e}_{N-1}\}$ is the standard basis for $\mathbb{R}^N$ (3.3). Similarly for *sine* data we have

$$Uf_{sin} = \frac{1}{2i}\sqrt{N}(\hat{e}_j - \hat{e}_{N-j}), \text{ where } j - m = 0 \mod N. \tag{3.10}$$

Taking the intermediate step gives

$$\begin{aligned}D^{-1}Uf_{cos} &= \frac{\sqrt{N}}{2}\left(\frac{\hat{e}_j}{\lambda_j} + \frac{\hat{e}_{N-j}}{\lambda_{N-j}}\right) \\ &= \frac{\sqrt{N}}{2\lambda_j}(\hat{e}_j + \hat{e}_{N-j}) \quad \text{since } \lambda_j = \lambda_{N-j}.\end{aligned}$$

For sine data we have $D^{-1}Uf_{sin} = \frac{\sqrt{N}}{2i\lambda_j}(\hat{e}_j - \hat{e}_{N-j})$.

We can now finally deduce that the coefficients of the resulting interpolant are given by

$$
\begin{aligned}
\alpha_l^{(cos)} &= (U^* D^{-1} U f_{cos})_{(l)} = \frac{\sqrt{N}}{2\lambda_j} \overline{u^{(l)}}^T (\hat{e}_j + \hat{e}_{N-j}) \\
&= \frac{\sqrt{N}}{2\lambda_j} \left(1, e^{\frac{2\pi i}{N} l}, e^{\frac{2\pi i}{N} 2l}, \ldots, e^{\frac{2\pi i}{N}(N-1)l}\right)^T (\hat{e}_j + \hat{e}_{N-j}) \\
&= \frac{\sqrt{N}}{2\lambda_j} \left(e^{\frac{2\pi i}{N} jl} + e^{\frac{2\pi i}{N}(N-j)l}\right) = \frac{1}{\lambda_j} \left(e^{\frac{2\pi i}{N} jl} + e^{-\frac{2\pi i}{N} jl}\right) \\
&= \frac{1}{\lambda_j} \cos\left(\frac{2\pi l j}{N}\right) \qquad l = 0, 1, \ldots, N-1.
\end{aligned}
\tag{3.11}
$$

For the sine data we have $\alpha_l^{(sin)} = \frac{1}{\lambda_j} \sin(\frac{2\pi l j}{N})$ for $l = 0, 1, \ldots, N-1$.
Hence, for N equally spaced points on S^1, the CBF interpolant to $\cos m\theta$ (with $m \neq 0 \mod N$) is given by

$$
s(\theta) = \frac{1}{\lambda_j} \sum_{l=0}^{N-1} cos\left(\frac{2\pi}{N} lj\right) \varphi\Big(\theta - \frac{2\pi}{N} l\Big), \tag{3.12}
$$

and the interpolant to $\sin m\theta$ is given by

$$
s(\theta) = \frac{1}{\lambda_j} \sum_{l=0}^{N-1} sin\left(\frac{2\pi}{N} lj\right) \varphi\Big(\theta - \frac{2\pi}{N} l\Big), \tag{3.13}
$$

where, in both cases, $j - m = 0 \mod N$.

3.1 Lagrange Interpolants

Suppose that we wish to compute the Lagrange interpolants at N equally spaced points, that is, we seek a function

$$
L_\varphi(\theta) = \sum_{l=0}^{N-1} c_l \varphi\left(\theta - \frac{2\pi l}{N}\right),
$$

which satisfies the following Lagrange interpolation conditions

$$
L_\varphi(0) = 1 \text{ and } L_\varphi\Big(\frac{2\pi l}{N}\Big) = 0 \text{ for } l = 1, 2, \ldots, N-1. \tag{3.14}
$$

The obvious appeal of the Lagrange interpolants is that they enable an easy solution to the general interpolation problem; for any function f on S^1 whose values are only known at the interpolation points, its unique CBF interpolant can be written immediately as

$$s(\theta) = \sum_{k=0}^{N-1} f_k L_\varphi\left(\theta - \frac{2\pi k}{N}\right) \quad \text{where } f_k = f\Big(\frac{2\pi k}{N}\Big), \quad k = 0, 1, \ldots, N-1.$$

The system above is perfectly conditioned since the corresponding interpolation matrix

$$A_{kl}^{(lgr)} = L_\varphi\Big(\frac{2\pi(l-k)}{N}\Big) = \delta_{kl}, \qquad (0 \le k, l \le N-1),$$

is the identity. This is in contrast to the case where the translates of the basis function alone are used to construct the interpolant. In this case, as we have observed in the previous section, the conditioning is extremely dependent upon the decay of the expansion coefficients of the CBF in question.

A useful way to compute the Lagrange interpolant is to consider the following function

$$F(\theta) = \frac{1}{N} \sum_{m=0}^{N-1} \cos m\theta,$$

which satisfies the Lagrange conditions (3.14). In addition we can use the expressions (3.6) and (3.12) to deduce that the Lagrange coefficients are given by

$$c_l = \frac{1}{N} \sum_{m=0}^{N-1} \frac{\cos\left(\frac{2\pi l m}{N}\right)}{\lambda_m} \tag{3.15}$$

where the λ_m are given by (2.9).Thus we can conclude that

$$s(\theta) = \sum_{k=0}^{N-1} f_k \sum_{l=0}^{N-1} \Big(\frac{1}{N} \sum_{m=0}^{N-1} \frac{\cos\left(\frac{2\pi l m}{N}\right)}{\lambda_m}\Big) \varphi\Big(\theta - \frac{2\pi(k-l)}{N}\Big).$$

In particular the Lagrange function is

$$L_\varphi(\theta) = \sum_{l=0}^{N-1} \Big(\frac{1}{N} \sum_{m=0}^{N-1} \frac{\cos\left(\frac{2\pi l m}{N}\right)}{\lambda_m}\Big) \varphi\Big(\theta + \frac{2\pi l}{N}\Big). \tag{3.16}$$

We remark that the above development is similar to a discovery made by Kushpel and Levesley in [6]. Their paper addresses, amongst other things, interpolation on the $d-$dimensional torus at gridded data using sk-splines. In one dimension the $1-$torus is just the circle and the definition of an sk-spline coincides with our notion of interpolation using an ASPD CBF. Furthermore, it can be shown that their representation of a cardinal sk-spline, [6] Lemma 17, coincides with our representation of the Lagrange function (3.16).

• **Example 1.** If we consider interpolating using the CBF (1.9) then, substituting (2.10) into (3.15), we find that the Lagrange coefficients of L_φ are given by

$$c_l = \frac{1-\rho^N}{N^2}\left(1 + 2\sum_{m=1}^{N-1} \frac{\cos\left(\frac{2\pi l m}{M}\right)}{\rho^m + \rho^{N-m}}\right), \quad l = 0, 1, \ldots, N-1.$$

Whether it is possible to express this sum in a closed form remains to be worked out. The authors believe this to be difficult.

• **Example 2.** If we consider interpolating using the CBF (1.10) then, after some algebraic manipulation, we find that the Lagrange coefficients have a much simpler form. they are given by

$$c_0 = \frac{\cos\frac{\pi}{N}}{2\sin\frac{\pi}{N}}, \qquad c_1 = c_{N-1} = \frac{-1}{4\sin\frac{\pi}{N}}$$

and

$$c_l = 0 \qquad \text{for} \quad l = 2, \ldots, N-2.$$

Thus, the Lagrange function is

$$L_\varphi(\theta) = \frac{\sqrt{2 - 2\cos(\theta - \frac{2\pi}{N})}}{4\sin\frac{\pi}{N}} - \frac{\cos\frac{\pi}{N}\sqrt{2-2\cos\theta}}{2\sin\frac{\pi}{N}} + \frac{\sqrt{2-2\cos(\theta + \frac{2\pi}{N})}}{4\sin\frac{\pi}{N}}.$$

3.2 Accuracy of interpolation

The interpolants that we have derived so far have a rather simple form. However, to investigate their accuracy it is useful, as we shall see, to consider their representations in terms of trigonometric functions (i.e., their Fourier series). Fortunately, these expansions are not too difficult to derive. We have to hand three representations namely (3.6), (3.12) and (3.13). The procedure to derive the trigonometric expansions from these simple expressions is as follows:

- Substitute the cosine expansion for the basis function.
- Employ the cosine difference formula $\cos(A - B) = \cos A \cos B + \sin A \sin B$.
- If necessary, exchange the order of summation so that the sum involving the index l (running from 0 to $N-1$) is isolated.
- Appeal to Lemma 2.3 to enable further simplification.
- Employ the appropriate expression for λ_j (2.9) where $j - m = 0 \mod N$.

We illustrate this procedure by considering the case $f(\theta) = \cos m\theta$ where $m \neq 0 \mod N$.

$$\begin{aligned}
s(\theta) &= \frac{1}{\lambda_j}\sum_{l=0}^{N-1}\cos\frac{2\pi jl}{N}\varphi\left(\theta - \frac{2\pi l}{N}\right) \\
&= \frac{1}{\lambda_j}\sum_{l=0}^{N-1}\cos\frac{2\pi jl}{N}\sum_{k=0}^{\infty}a_k\cos\Big(k\left(\theta-\frac{2\pi l}{N}\right)\Big) \\
&= \frac{1}{\lambda_j}\sum_{l=0}^{N-1}\cos\frac{2\pi jl}{N}\sum_{k=0}^{\infty}a_k\Big(\cos k\theta\cos\frac{2\pi kl}{N} + \sin k\theta\sin\frac{2\pi kl}{N}\Big) \\
&= \frac{1}{\lambda_j}\sum_{k=0}^{\infty}a_k\Big(\cos k\theta\sum_{l=0}^{N-1}\cos\frac{2\pi jl}{N}\cos\frac{2\pi kl}{N} + \sin k\theta\sum_{l=0}^{N-1}\cos\frac{2\pi jl}{N}\sin\frac{2\pi kl}{N}\Big) \\
&= \frac{N}{2\lambda_j}\sum_{r=0}^{\infty}a_{rN+j}\cos(rN+j)\theta + a_{rN+N-j}\cos(rN+N-j)\theta \qquad \text{by Lemma 2.3} \\
&= \frac{a_j\cos j\theta + \sum_{r=1}^{\infty}a_{rN+j}\cos(rN+j)\theta + a_{rN-j}\cos(rN-j)\theta}{a_j + \sum_{r=1}^{\infty}a_{rN+j}+a_{rN-j}}.
\end{aligned}$$

In summary we have demonstrated that the trigonometric expansion of the CBF interpolant to $\cos m\theta$ is given by

$$s(\theta) = \frac{a_j\cos j\theta + \sum_{r=1}^{\infty}a_{rN+j}\cos(rN+j)\theta + a_{rN-j}\cos(rN-j)\theta}{a_j + \sum_{r=1}^{\infty}a_{rN+j}+a_{rN-j}}. \tag{3.17}$$

Similarly, one can show that the CBF interpolant to $\sin m\theta$ is given by

$$s(\theta) = \frac{a_j\sin j\theta + \sum_{r=1}^{\infty}a_{rN+j}\sin(rN+j)\theta + a_{rN-j}\sin(rN-j)\theta}{a_j + \sum_{r=1}^{\infty}a_{rN+j}+a_{rN-j}}. \tag{3.18}$$

Remark 3.2. *Let N be a fixed positive integer and $m \in \{1, 2, \ldots, N-1\}$. In this case we can escape from the modular arithmetic conditions on the integer j in the above expressions* (3.17) and (3.18) and simply set $j = m$.

Let $N > 2$ be a fixed positive integer and, for simplicity, let us suppose we want to interpolate $f(\theta) = \cos m\theta$, where $m \in \{1, 2, \ldots, N-1\}$ at equally spaced points. In this case the pointwise error, $s(\theta) - \cos m\theta$, is given by

$$\frac{a_m\cos m\theta + \sum_{r=1}^{\infty}a_{rN+m}\cos(rN+m)\theta + a_{rN-m}\cos(rN-m)\theta}{a_m + \sum_{r=1}^{\infty}a_{rN+m}+a_{rN-m}} - \cos m\theta. \tag{3.19}$$

Under the assumption that the CBF expansion coefficients satisfy $a_m > 0$ for $m \geq 1$ we can deduce the following upper bound:

$$\|s(\theta) - \cos m\theta\|_{L_\infty(S^1)} = \sup_{\theta\in[0,2\pi)}|s(\theta)-\cos m\theta| \leq \frac{2}{a_m}\sum_{r=1}^{\infty}a_{rN+m}+a_{rN-m}. \tag{3.20}$$

Alternatively, we can use (2.9) to deduce that the denominator of expression (3.19) is simply $2\lambda_m/N$ and so the pointwise error can be expressed precisely as

$$\frac{(a_m - \frac{2\lambda_m}{N})\cos m\theta + \sum_{r=1}^{\infty} a_{rN+m}\cos(rN+m)\theta + a_{rN-m}\cos(rN-m)\theta}{\frac{2\lambda_m}{N}}.$$

If we are interested in the *root mean square* (RMS) error then we consider

$$\|s(\theta) - \cos m\theta\|_{L_2(S^1)}^2 = \frac{1}{2\pi}\int_0^{2\pi} (s(\theta) - \cos m\theta)^2 d\theta.$$

For a function $f \in L_2(S^1)$ that has a trigonometric expansion $f(\theta) = \sum_{k=1}^{\infty} c_k \cos k\theta$ we can use the orthonormality of the system $\{\sqrt{2}\cos k\theta\}_{k\geq 1}$ to deduce Parseval's identity $\|f(\theta)\|_{L_2(S^1)}^2 = \frac{1}{2}\sum_{k=1}^{\infty} c_k^2$. In view of the above expansion for $s(\theta) - \cos m\theta$, we can deduce that

$$\|s(\theta) - \cos m\theta\|_{L_2(S^1)}^2 = N^2\Big(\frac{(a_m - \frac{2\lambda_m}{N})^2 + \sum_{r=1}^{\infty} a_{rN+m}^2 + a_{rN-m}^2}{8\lambda_m^2}\Big). \qquad (3.21)$$

• **Example 1.** If we consider interpolating using (1.9) then we can use the fact that the cosine expansion coefficients ($a_m = \rho^m$ for $\rho \in (0,1)$) are monotonically decreasing as m increases, to continue the upper bound (3.20) as follows

$$\begin{aligned}\|s(\theta) - \cos m\theta\|_{L_\infty(S^1)} &< \frac{4}{a_m}\sum_{r=1}^{\infty} a_{rN-m} \\ &= \frac{4}{\rho^{2m}}\left(\rho^N + \rho^{2N} + \rho^{3N}\ldots\right).\end{aligned}$$

Hence, we can deduce that

$$\|s(\theta) - \cos m\theta\|_{L_\infty(S^1)} = O(e^{-cN}), \qquad c = -\log\rho > 0,$$

that is the interpolant converges at an exponentially fast rate.

Furthermore, we can also compute the right hand side of (3.21) exactly. In particular, using (2.9), it can be shown that

$$\|s(\theta) - \cos m\theta\|_{L_2(S^1)}^2 = \frac{\rho^{2N}}{(\rho^m + \rho^{N-m})^2}\Big(\frac{\rho^{2m} + \rho^{-2m}}{1+\rho^N} + 1\Big).$$

• **Example 2.** We recall that the cosine expansion coefficients of CBF (1.10) are given by

$$a_0 = -\frac{4}{\pi} \quad \text{and} \quad a_m = \frac{2}{\pi}\frac{1}{(m-\frac{1}{2})(m+\frac{1}{2})}, \quad m \geq 1. \qquad (3.22)$$

We note that this sequence is also monotonically decreasing and so we can bound the interpolation error as follows

$$\|s(\theta) - \cos m\theta\|_{L_\infty(S^1)} < \frac{4}{a_m}\sum_{r=1}^{\infty} a_{rN-m}$$
$$= \frac{\pi(4m^2-1)}{2}\Big(\frac{1}{(N-\frac{2m+1}{2})(N-\frac{2m-1}{2})} + \frac{1}{4(N-\frac{2m+1}{4})(N-\frac{2m-1}{4})}\cdots\Big).$$

Hence we can deduce that $\|s(\theta) - \cos m\theta\|_{L_\infty(S^1)} = O(N^{-2})$, that is, this interpolant converges quadratically.

The $RMS-$error profile for interpolation using (1.10) can also be computed exactly, for this we shall require some simplifying notation. In particular we set $x = \frac{2m+1}{N}$, $y = \frac{2m-1}{N}$, and observe that

$$\left(\frac{1}{y} - \frac{1}{x}\right) = \frac{N\pi}{2}a_m, \qquad \text{and } x - y = \frac{2}{N}. \tag{3.23}$$

Also, using (2.14) and elementary trigonometric identities we can write

$$\lambda_m = \cot \pi y - \cot \pi x = \frac{\sin\frac{\pi}{N}}{\sin \pi x \sin \pi y}, \tag{3.24}$$

and

$$\lambda_m^2 = (\cot \pi y - \cot \pi x)^2 = \csc^2 \pi y + \csc^2 \pi x - \frac{2\cos\frac{\pi}{N}}{\sin \pi x \sin \pi y}$$
$$= \csc^2 \pi y + \csc^2 \pi x - 2\lambda_m \cot\frac{\pi}{N}.$$

Hence

$$\csc^2 \pi y + \csc^2 \pi x = \lambda_m^2 + 2\lambda_m \cot\frac{\pi}{N}. \tag{3.25}$$

We can now analyze the $RMS-$error profile. We begin by considering the numerator of (3.21) and we compute:

$$\sum_{r\geq 1} a_{rN+m}^2 + a_{rN-m}^2$$
$$= \left(\frac{2}{N\pi}\right)^2 \sum_{r\geq 1}\left(\frac{1}{r+x} - \frac{1}{r+y}\right)^2 + \left(\frac{1}{r-x} - \frac{1}{r-y}\right)^2$$
$$= \left(\frac{2}{N\pi}\right)^2 \sum_{r=-\infty}^{\infty}\left\{\left(\frac{1}{r+x} - \frac{1}{r+y}\right)^2 - \left(\frac{1}{y} - \frac{1}{x}\right)^2\right\}$$
$$= \left(\frac{2}{N\pi}\right)^2 \sum_{r=-\infty}^{\infty}\left(\frac{1}{(r+x)^2} + \frac{1}{(r+y)^2} - \frac{2}{(x-y)}\left(\frac{1}{r+y} - \frac{1}{r+x}\right)\right) - a_m^2$$

Using the following identities ([1] Theorem 1.2.2)

$$\sum_{-\infty}^{\infty} \frac{1}{(n+\alpha)^2} = \pi^2 \csc^2 \pi\alpha \qquad \text{and} \qquad \sum_{-\infty}^{\infty} \frac{1}{n+\alpha} = \pi \cot \pi\alpha,$$

we find that $\sum_{r\geq 1} a_{rN+m}^2 + a_{rN-m}^2$ simplifies to

$$\left(\frac{2}{N\pi}\right)^2 \left(\pi^2 \left(\csc^2 \pi y + \csc^2 \pi x\right) - N\pi \left(\cot \pi y - \cot \pi x\right)\right) - a_m^2$$

$$= \left(\frac{2}{N}\right)^2 \left(\lambda_m^2 + 2\lambda_m \cot \frac{\pi}{N} - \frac{N}{\pi}\lambda_m\right) - a_m^2.$$

The above equality is due the substitution of (3.25) and (3.24) respectively. The numerator is now

$$\left(a_m - \frac{2\lambda_m}{N}\right)^2 + \left(\frac{2}{N}\right)^2 (\lambda_m^2 + 2\lambda_m \cot \frac{\pi}{N}) - \frac{4}{N\pi}\lambda_m - a_m^2$$

$$= \frac{8\lambda_m^2}{N^2} + \frac{8\lambda_m}{N^2}\left[\cot \frac{\pi}{N} - \frac{N}{2}\left(a_m + \frac{1}{\pi}\right)\right],$$

and, substituting for a_m (3.22) and rearranging, this simplifies to

$$\frac{8\lambda_m^2}{N^2}\left[1 + \frac{\cot \frac{\pi}{N} - \frac{N}{2\pi}\left(\frac{4m^2+7}{4m^2-1}\right)}{\lambda_m}\right]$$

and so the $RMS-$error profile for interpolation using CBF (1.10) is given by

$$\|s(\theta) - \cos m\theta\|_{L_2(S^1)}^2 = 1 + \frac{\cot \frac{\pi}{N} - \frac{N}{2\pi}\left(\frac{4m^2+7}{4m^2-1}\right)}{\cot\left(\frac{2m-1}{2N}\pi\right) - \cot\left(\frac{2m+1}{2N}\pi\right)}. \tag{3.26}$$

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Addresses:

Simon Hubbert
School of Economics, Mathematics and Statistics,
Birkbeck College,
Malet Street,
London, WC1E 7HX,
England

Stefan Müller
Georg-August-Universität Göttingen,
Institut für Numerische und Angewandte Mathematik,
Lotzestrasse 16-18,
37083 Göttingen,
Germany.