Dynamic Learning, Herding and Guru Effects in Networks

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Abstract

It has been widely accepted that herding is the consequence of mimetic responses by agents interacting locally on a communication network. In extant models, this communication network linking agents, by and large, has been assumed to be fixed. In this paper we allow it to evolve endogenously by enabling agents to adaptively modify the weights of their links to their neighbours by reinforcing 'good' advisors and breaking away from 'bad' advisors with the latter being replaced randomly from the remaining agents. The resulting network not only allows for herding of agents, but crucially exhibits realistic properties of socio-economic networks that are otherwise difficult to replicate: high clustering, short average path length and a small number of highly connected agents, called "gurus". These properties are now well understood to characterize 'small world networks' of Watts and Strogatz (1998).

Keywords: Mimetic decision rules; Evolving networks; Small world networks; Clustering coefficient; Reinforcement learning.

JEL Classification Codes : C63, D83, D85, O33

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1. Introduction

In general, in the course of decision making agents can respond to global signals and/or simply observe their neighbours' behaviour in a more local setting. While it is not surprising that agents responding to a common signal can coordinate and herd, what is increasingly being studied is the category of phenomena that arises when agents form networks for communication and interaction which result in coherent patterns of behaviour at an aggregate level. It is now widely held that mimetic responses to neighbours result in herd behaviour, see, Sharfstein and Stein (1990), Kirman (1983), Bikhchandani *et. al.* (1992, 1998 etc) and that the properties of herding can crucially arise from the structures of local interaction and more generally from the social and communication networks that agents' belong to (Kirman, 1983, 1997; Cont and Bochaud, 2000). In the economics literature, Föllmer (1974) followed by Kirman (1983) were amongst the first to suggest the use of random graphs to provide a general framework for economic interactions.

In graph theory, nodes can represent agents and edges are connective links with a function (fixed or time varying) measuring the intensity or frequency of interaction between agents. To date, herding models primarily rely on mimetic functions that determine the decision rule of agents' with fixed links to their nearest neighbours. In contrast to this, there are networks where every player can interact with every other player randomly with a given probability. As network formation in a socio-economic setting does not conform to these two extreme assumptions of a regular graph and a random graph structure, Kirman (1997), Kirman and Vignes (1991) and Vriend (1995) have proposed that some form of reinforcement learning which causes agents to make and break links determines how the networks themselves evolve. This paper develops such a model of dynamic learning in networks where agents make a binary decision, to buy or sell a unit of an asset, and adaptively modify the weights of their links to their neighbours by reinforcing 'good' advisors and breaking away from 'bad' advisors with the latter being replaced randomly from the remaining agents. Experiments are devised to distinguish between herding that arises solely from the mimetic quality of the given decision rule and when there is also adaptive identification of those agents by others in the system who have some inherent superiority to give good advice on how to play the game for a given global reward scheme. Such players who can give 'good' advice but not necessarily play the game well are called 'gurus' in the paper. With no prior knowledge of who such gurus are and what their distribution in the population is, it is a remarkable feature of the simple threshold based reinforcement learning rule of link formation to 'good' advisors that it enables a large of number agents to find and be linked to the gurus in what emerges as star or hub formations.

These network properties correspond to those of "small-world networks", Watts and Strogatz (1998) and Watts (2002), which are known to characterize real world socio-economic and communications networks.[1] These include the above mentioned star or hub formations when some agents in the system have a disproportionately large number of links to them *and* between themselves which lead to significant interconnectedness in the network. This is measured by the clustering coefficient which, in a random graph, is no greater than the initial probability of any two agents being connected. The small world phenomenon is known to arise from the process of random reconnections that make short cuts between a given agent and a far flung one, resulting in the average shortest path between randomly chosen agents to be "small" and bounded by the logarithm of the total number of nodes in the system.[2] In regular networks while everybody is highly interconnected locally, the distance in terms of average links needed between a given agent and another agent randomly selected from the system is high.

Many network construction procedures have been proposed to 'evolve' networks with properties of small world networks. The initial point has been either the regular network (as in the original Watts and Strogatz (1998) proposal) or the random network (see, Davidsen *et. al.* 2002) with different rationale given for the dynamics by which random reconnections are made. Many of these have been critiqued (see, Jackson and Rogers, 2004) as being some what contrived in the manner in which the small world properties arise. In the economics literature, to date, there have been very few studies on properties and implications of endogenously evolving communication networks. In particular, few consider if the end results of the dynamics being postulated in the learning from neighbours framework have realistic features of small world networks. By and large, herding models that have learning from neighbours and postulate local feedback effects on the decision rule of agents' are hampered by fixed links to neighbours.[3]

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[1] Examples of socio-economic networks include the world wide web, co-author relationships among academics, trade networks, criminal associations, airline routing etc. See also Newman (2002).

[2] It is this property that gives rise to the notion of a 'small world'. The most popular manifestation of this is known as "six degrees of separation" coined by the Stanley Milgram (1967) stating that most pairs of people in the United States can be connected through a path of only about six acquaintances.

[3] The very large literature on local interaction economic models (see, Durlauf and Young, 2000) that comes under the rubric of social dynamics for the study of the diffusion of innovation, information or norms is based on the Liggett/Ising framework originally adapted for economic analysis by Blume (1993). This framework treats local feedback effects as a stochastic process in which the probability that a given person adopts one of two possible actions, say **A** or **B** in a given period of time, is assumed to be an increasing function of the number of his

The rest of the paper is organized as follows. Section 2 sets out the model of herding in a simple asset market. This motivates the framework behind the experiments that evolve realistic communication network structures that influence trader behaviour. Section 3 gives some results from network theory that help to distinguish between the different network topologies. In particular, we give an easy 'look up' table on how the small world networks have connectivity properties which straddle the polar extremes of random networks, regular purely deterministic networks and a third category of networks called scale free networks, Barabesi and Albert (1999). Section 4 reports the results of the experiments. The conditions under which guru effects and star/hub formations emerge are carefully documented here. We also discuss here the conditions in our model that enable gurus to maximize and propagate their impact on the rest of the system. It is also found that once stable star/hub formations arise, this reduces the shortest average path length between any two random agents. The hub formation enhances the cohesiveness of the system by reducing the shortest average path length between agents relative to random graphs as network size increases and the network connections become sparse.

All experiments can be run by the reader using the 'Herding Simulator' on the Centre for Computational Finance and Economic Agent (CCFEA) website.[4]

2. The Model and Experimental Framework

2.1 Characteristics of agents and their spatial location

The model consists of a network of *N* agents who at each time period have to either buy or sell a unit of an asset. The *N* agents are initially placed on a random graph with the initial probability *p* that any two agents (i, j) are connected is independent of *i* and *j*. Agent i's immediate or first order neighbours, k_i in number, are defined by those links starting from *i* to other agents. The set of *i*'s neighbours is denoted by Ξ_i. The links between agents have weights $w_{i,j}$ associated with it, which represents the strength of the advice that agent i will take from agent *j*. The initial weights, $w_{i,j}$, for each agent's k_i links is randomly assigned according to a uniform distribution with a finite support (W_{min}, W_{max}), $0 < W_{min} < w_{i,j} < W_{max} < 1$.

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neighbours who have adopted it. There is also an idiosyncratic factor that reflects agents' preference for **A** or **B** irrespective of other agents. Many of these models assume a fixed network structure though endogenously changing networks have been considered in Mailath *et. al.* (1997) and Jackson and Rogers (2004). The impact of the spatial interconnectedness or the network structure on the diffusion dynamics has been analysed on a case by case basis by Goyal and Janssen (1996) and Chwe (2000), rather than under conditions when the network structure is altered by the actions taken by the agents.

[4] The CCFEA website is http://www.essex.ac.uk/ccfea/

Each agent decides to buy or sell based on a weighted average of the advice given by its neighbours, and also its own recommendation that reinforces the discounted sum of correct decisions from the past. As will be seen, this forces an inherently majoritarian outcome for actions taken. Note also agents evaluate the investment advice given by their neighbours. The idea behind experientially driven learning is simple in that those neighbours who give correct advice are reinforced while the influence of others is reduced incrementally. When a certain threshold is reached, links are cut from 'bad' advisors and a new ones are sought. The importance of this is that if there are agents in the system who have some inherent quality that make them capable of giving good advice (as judged by the common reward scheme) more often than others, then the question is whether and when the dynamic process of reinforced learning can lead to the discovery of such 'gurus' solely by a process of local interactive learning. Thus, what is significant to learning to play the game is that agents cannot explicitly strategize to win; they can improve their game only by a process of link formation that enables them to find 'good' advisors/neighbours. We adhere to assumption that agents are always bound to use their own recommendation and have no means of breaking away from that. Hence, there is a pseudo link $w_{i,,i} =1$ and it is useful to define a pseudo neighbourhood set that includes the agent himself, $\Xi_i^* = \Xi_i + 1$.

All agents are identical except for memory $M_i \in [0, M_{max}]$ and agents are uniformly distributed over this range of integers. Zero memory agents give advice on a random basis while others do so on the basis of up to M_i periods of their own past decisions to be specified below.

Experiments under two reward schemes are conducted with the agents not being aware of the global reward scheme except whether decisions made and advice taken were correct or not. In one set of experiments agents are rewarded or penalized by a purely random scheme which is exogenous to their actions, Krause (2003). The second set of experiments have an endogenous 'price' process arising from buy/sell orders and agents are rewarded only for being part in the minority, viz. to go against the herd. In the random reward scheme in terms of memory, no agent has any inherent reason to do better than others. On the other hand, in the minority game set up, the zero memory agents have the best chance of giving correct advice and hence have the potential to be gurus.

2.2 The Decision Rule

The decision rule of agents to buy or sell is done in two steps if $p>0$, viz. agents are linked to others. At time t, agents first generate a number, $f_{it}(t+1)$, which evaluates their past performance from which follow their recommendations to buy or sell at $t+1$. Agents only have access to the recommendations of their neighbours and do not observe their actions. In Step 2, the actual decision to buy or sell at $t+1$ is then based on the weighted sum of the contemporaneous recommendations agents receive from their neighbours.

As will be seen, memory only affects how agents form recommendations. Zero memory agents simply recommend a buy or a sell at random with each zero memory agent making a different random draw from an independent and identically distributed process. Those with memory on the other hand will make recommendations that follow time trends based on what in fact are the correct decisions from the past.

The decision of an agent i at time t is denoted by $\gamma_i(t)$, and can take a value of +1 if the agent buys, or a value of –1 if the agent sells. The outcome or utility of this decision at each t is denoted by $u(\gamma_i(t))$, and can take a value of +1 if the agent wins , or a value of –1 otherwise. To make a recommendation to buy or sell at $t+1$, the agents generate a number $f_{it}(t+1)$ which evaluates their own past M_i decisions and outcomes as follows:

$$f_{it}(t+1) = \begin{cases} \sum_{\tau=1}^{M_i} \lambda^{\tau} [u(\gamma_i(t-\tau))\, \gamma_i(t-\tau)] & if\ M_i > 0 \\ 0 & if\quad M_i = 0. \end{cases} \qquad (1)^5$$

Here, M_i is the length of memory of agent i. When agents have memory, $M_i > 0$, a discount factor $\lambda \in (0,1)$ applies to past decisions, so that recent outcomes have a greater influence than older ones. In (1) , in the absence of memory $M_i = 0$, agents simply set $f_{it}(t+1) = 0$. The rule in (1) above results in the following advice or recommendations:

$f_i(t+1) > 0$ implies a recommendation given at t to buy at $t+1$;

$f_{it}(t+1) < 0$ implies a recommendation given at t to sell $t+1$;

$f_{it}(t+1) = 0$ will result in a random buy or sell decision.

Note, as all agents can evaluate if their decisions were correct or not, all agents with memory produce the same sequence of +1s and –1s, for each t, from the product $u(\gamma_i(t))\gamma_i(t)$ in (1) up to the length of their memory. Agents with memory, therefore, produce time trends in their

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[5] Note, that at each t $u(\gamma_i(t)).\gamma_i(t)$ in (1) yields products of 2- tuples {(+1,+1), (-1, +1), (+1, -1), (-1, -1)} where the second term of each 2-tuple signifies the agent's decision to a buy or sell and the first term signifies if this was decision was correct or not.

recommendations based on the same global data. Zero memory agents produce random viz. serially uncorrelated recommendations

In keeping with the literature on herding on account of learning from neighbours, the decision rule to buy or sell is mimetic in that an agent's action $\gamma_i\,(t+1)$ is based on a weighted sum of forecasts/recommendations $f_{it}\,(t+1)$ in (1) that its neighbours give it at t.

$$\gamma_i(t+1)=\begin{cases}+1 \quad (buy) \qquad if \qquad \sum_{j\in\Xi_i*} f_{j_t}(t+1)w_{ij}>0\\ -1 \quad (sell) \qquad if \qquad \sum_{j\in\Xi_i*} f_{j_t}(t+1)w_{ij}<0\end{cases} \qquad (2).$$

Note the decision rule (2) also includes an agent's own recommendation with the pseudo fixed weight of $w_{ij} = 1$. For zero memory agents it is certainly a case of 'do as I say, not as I do' as though the advice they give to buy or sell is random, they are no different from all other agents in following the mimetic (weighted) majority structure of the decision rule in (2). Note, if there are no links to neighbours as in (2), agents simply do what they forecast in (1). The important aspect of the above decision making process in a dynamic context is how agents endogenously chose the members of the set Ξ_i of neighbours.

2.3 Endogenous link formation: Dynamic weights

We distinguish between static and dynamic weights. We could obviously determine a network exogenously and then investigate the dynamics of the decisions arising on this network. The simplest form would be to generate a random network and having assigned the weights randomly, they remain fixed over time. We refer to such a model as having *static weights*.

Under dynamic weights, agents are allowed to modify the weights of the links after each round. If the recommendation that agent *i* receives from agent *j* at time *t* is correct, agent *i* will increase the weight of the link w_{ij} by a rate of increment R^+. In contrast, an incorrect recommendation from *j* means that agent *i* will reduce the weight of that link by a rate of reduction R^-. All agents have the same rates for the adjustment of weights.

There is a maximum threshold W_{max} after which a weight cannot be further increased. Similarly, there is a minimum threshold W_{min}, after which the agent breaks the link with agent *j*, and randomly selects another agent *m* in the network, which is not already one of its neighbour, and assigns a random value to this new link w_{im} from $[W_{\min}, W_{\max}]$.

2.4 Global Reward Schemes

A global reward scheme which is the same for all agents deems whether to buy ($\gamma_i(s) = +1$) or to sell ($\gamma_i(s) = -1$) is the correct decision at each time step s. We investigate two reward schemes, random rewards and minority rewards. In neither case do agents know what the reward scheme is. They are simply told if the action they took at each period is 'correct' or not.

Random Reward Scheme : In this reward scheme the winning outcomes are determined independent of the behaviour of individual agents and hence bears no correlation with the numbers of agents who buy and sell. The scheme is implemented as follows : when heads are realized in a random flip of a coin, viz. with probability *.5*, all buyers, $\gamma_i(t) = +1$, receive 1 point (viz. $u(+1) = +1$) and all sellers are penalized by -1. The opposite is the case when tails are realized. Note the random draws of the reward scheme are independent of the draws for any of the zero memory agents.

Minority Game Rewards: There is an endogenous reward scheme that arises from observing the aggregate number of buys or sells at each t and then rewarding those in the minority by *1* point and others are penalized by *-1*. It should be clear that the minority game reward scheme penalizes herd behaviour that follows from (2).

In an asset market, being in the minority is rewarding for the seller in that he can get a higher price when the majority of traders are buying and vice versa when an agent buys and the majority are selling. The El-Farol or minority game with its contrarian structure was made famous by Arthur (1994) as the prototype of a game for which there is no homogenous and systematic way for determining the winning strategy.[6] However, in Arthur (1994) and in the formulations of the minority game made popular by Challet and Zhang (1997) network interconnections do not feature.[7]

In the random reward scheme in terms of memory, no agent has any inherent capacity to give better recommendations in (1) than others. This is because irrespective of how agents give advice each agent can be right with probability half as the rewards are determined exogenously to what agents do.

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[6] Markose (2003,2004) shows why the contrarian structure of the game makes it an undecidable problem with a non-computable fixed point.

[7] The minority game with network interconnections studied in Paczuski *et. al.* (2003) specifies each agent's binary/Boolean action at t+1 to be a function of the actions taken by its K neighbours at t. From a total of 2^{2^K} Boolean functions of K variables, each agent's strategy is drawn randomly and is held fixed except for the worst performer who has 'failed' in the long run. Paczuska *et. al.* (2003) find herding of upto 0 .67 for large N. However, the K neighbours that each of the N players have is fixed and do not evolve unlike what we propose in this paper.

On the other hand, in the minority game set up, those with memory and with the same discount factor applied to what is correct in the past will be perpetuate the same again in (1). This will produce majoritarian trends via (2) and hence those with memory will almost always give the wrong recommendation for being in the minority. As zero memory agents give advice on a random basis in (1), they have a greater chance of giving the correct advice in the minority game and hence have the potential to be gurus.

2.5 The Herding Coefficient

The herding phenomenon in all experiments with the two reward schemes can be captured at each t by a simple time varying herding function $\frac{N_{bt}}{N} \in [0, 1]$. Here, N_{bt} is the number of agents who have bought at time t and N is the total number of agents. When this function is close to a half, the market shows no herding. When $\frac{N_{bt}}{N}$ is close to zero, the market is herding in the direction of selling and vice versa when $\frac{N_{bt}}{N} = 1$.

It is useful to have a single average measure of herding in the system over the length of time T which is irrespective of the direction of herding. This is given by the herding coefficient :

$$\sigma = \frac{2}{N}\sqrt{\left(\frac{1}{T}\sum_{t=1}^{T}\left(N_{bt} - \frac{N}{2}\right)^2\right)}, \sigma \in (0, 1). \qquad (3)$$

This measure is given in terms of deviations of aggregate market behaviour at each t from the benchmark of half of the population. Clearly, $\sigma = 0$, implies that on average there is no herding in the market while σ close to 1 implies that the market behaviour is perfectly synchronized whether in the direction of buying or selling.

3. Network Connectivity

To understand the consequences on herding of dynamic network formation and endogenous determination of neighbours, we will now briefly present some graph theoretic concepts. An important aspect of the connectivity of the links in our framework and in communication networks in general relates to the directedness of links. In *directed networks* communication flows only in one direction, the reciprocal communication flow is not guaranteed and if it is present it does not have to be of equal importance. In contrast to this, in *undirected networks* communication flows in both directions at an equal rate. In a directed network the total number of possible connections is N(N-1) while in a undirected network there are only half as many. In this paper we are only concerned with directed networks as each agent can initiate and break off links to another agent. To understand how this process can result in small world networks observed in socio-economic systems as opposed to the polar theoretical cases of random and regular networks or the constructed scale free network, three main connectivity properties of networks are involved. These are discussed below.

The starting point of our analysis is the random graph studied extensively by Erdös and Renyi (1960). In a random graph denoted by $G_{N,p}$ where N is the number of nodes and p is the identical and independent probability that any pair of agents (i,j) are linked from i to j or from j to i. All agents on average have the same number of links to them and from them. In a regular graph, agents are located on the edges of a Latin square and are connected to their four (eight) neighbours on the edge of each square (including the diagonal links). Other regular structures with larger neighbourhoods are easily imaginable. A key property of regular networks is that all agents have the same number of links. Properties of scale-free networks are observed after a process of 'construction' that involves preferential attachment, see e.g. Barabasi and Albert (1999). Barabasi and Albert add new nodes to the network sequentially, and postulate that the probability that a new node is linked to an existing node is higher, the more links an existing node already has.

3.1 Degree Distribution and the Influence of any Particular Node

The degree of a node is the number of edges connected to it. In directed graphs, there is the *in-degree*, number of edges pointed to it , and *out- degree*, number of edges pointing away from it. Note, the out-degree of an agent in a network defined by those edges starting from i gives the number of its first order neighbours, k_i. In our context, the in-degree of an agent would be the number of other agents connected to it. That is, the number of agents that are taking advice from it and the measure of influence of an agent.

The average in-degree denoted by z, of a node for the random graph $G_{N,p}$ is the average number of edges connected to any randomly selected node. This is given by

$$z = \frac{N(N-1)p}{N} = (N-1)\,p \cong Np . \quad (4)$$

The last approximation is for large N. To understand how influential agents in a given network are, we study the in-degree distribution. The probability π_q that a node in an Erdös-Rényi random graph has an in-degree exactly q is given by the Binomial distribution

$$\pi_q = \binom{N-1}{q} p^k (1-p)^{N-1-q} . \qquad (5)$$

$$= \frac{(N-1)!}{q!(N-1-q)!} p^q (1-p)^{N-1-q} .$$

In the limit where N>>qz , this becomes the Poisson Distribution -

$$\pi_q = \left(\frac{z^q e^{-z}}{q!} \right) . \qquad (6)$$

Both distributions are strongly peaked about the mean z and have a tail that decays rapidly as 1/q!. The degree distributions of real world networks are very different from Binomial or Poisson distributions. In particular, some agents are found to have a disproportionately large number of incoming links while the others have very few. This is more in keeping with the degree distribution of scale-free networks whereby the distribution has a 'fat tail' and tails decay hyperbolically. For any given number for the degree q, scale free network degree distribution follows a power law, for positive constants A and α :

$$\pi_q^{scale-free} = Aq^{-\alpha} .$$

Note, in contrast to the above, the degrees of regular graphs are fixed and identical for all nodes, hence the distribution collapses to this degree.

3.2 Clustering and Interconnectedness

Clustering in networks measures how interconnected each agent's neighbours are. Specifically, there should be an increased probability that two of an agent's neighbours are also neighbours of one another. For each agent with k_i neighbours the total number of all possible directed links between them is given by $k_i (k_i-1)$. Let E_i denote the actual number of links between agent i's k_i neighbours viz. those of i's k_i neighbours who are also neighbours. The clustering coefficient C_i for agent i is given by

$$C_i = \frac{E_i}{k_i(k_i - 1)} \quad . \qquad (7)$$ [8]

The clustering coefficient of the network as a whole is the average of all C_i's and is given by

$$C = \frac{\sum_{i=1}^{N} C_i}{N} \quad . \qquad (8)$$

Note that the clustering coefficient for a random graph is

$$C^{random} = p.$$

This is because in a random graph the probability of node pairs being connected by edges are by definition independent, so there is no increase in the probability for two agents to being connected if they were neighbours of another agent than if they were not.

3.3 Average Path Length

A useful measure of the distance between two agents is given by the number of directed edges that separate them and this is referred to as their path length. In a random graph, the average shortest path length between all (i,j) pairs denoted by ℓ^{random}, is given by

$$\ell^{random} = \frac{\log N}{\log Np} \quad . \qquad (9)$$

If we keep the average number of degrees constant, i.e. $Np = z$, we see that the average path length increases logarithmically with the size N of the network. Random networks have quite a short path length which is due to the fact that many "shortcuts" between nodes arise from the random nature of the connections. As already discussed, in small world networks, the possibility of random reconnections enable two randomly chosen nodes in a network to have short path lengths. Regular networks miss these shortcuts and hence the average path length between an agent and a far flung one will be significantly longer. The exact path length depends crucially on the form of the network generated. Scale-free networks show an average path length which in most cases is also proportional to the logarithm of the network size, but the details depend on the way the preferential attachment is modelled.

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[8] Numerically , E_i is calculated as follows. Denote the NxN adjacency matrix $A = (a_{ij})^N$ with a_{ij}=1 if there is a link between i and j and a_{ij}=0, if not. Agent i's k_i neighbours $\Xi_i = \{ \forall j , j \neq i, \text{ s.t } a_{ij} = 1\}$. Denoting a_{ij}= 1 by a_{ij}^1, E_i in (9) for a directed graph is calculated as $E_i = \sum_{j\in\Xi_i}\sum_{m\in\Xi_i} a_{jm}^{\ 1}$, $j \neq m$.

To conclude this section : the characteristics that the small world networks have in common with those observed in socio-economic systems and how they interpolate between the three above mentioned classes of random, regular and scale-free networks can be found along the diagonal elements of the Table 1. In principle, the adaptive learning process driving an endogenous link formation such as the one described in the learning from neighbours model of Section 2 must result in network connectivity properties of small world networks given along the diagonal of Table 1.

Table 1: Properties of Networks: Diagonal Elements Characterize Small World Networks

Properties of Networks	Clustering Coefficient	Average Path Length	Degree Distribution
Regular	***High***	**High**	**Equal and fixed In-degrees to each node**
Random	**Low**	***Low***	**Exponential/ Poisson**
Scale Free/Power Law	**Low**	**Variable**	***Fat Tail Distribution***

4. Results Of Agent Based Network Simulations

The computational experiments are geared toward understanding herding as a response to - (i) common global signals when agents are in isolation with no links, (ii) mimetic response to neighbours using the rule in equation (2) with static links, and (iii) endogenous dynamic network formation specifically leading to high clustering and 'guru' effects.

We initially set up a random network with $N=100$ agents with a probability of p that any two nodes are connected. In most cases we chose $p = 0.2$ implying that on average all agents have 20 in-degrees and out-degrees. The initial weights w_{ij} for the out degrees or links that start from agent i to agent j are assigned independently from a uniform distribution on $[0.1, 0.9]$. In the case (iii) above when dynamic link formation is allowed, a correct recommendation from agent i to agent j leads j to increase the given weight by $R^+ = 0.2$, the so called rate of increments. We also assume that an incorrect recommendation reduces the w_{ij} by $R^- = 0.4$, unless otherwise stated. The benchmark is the case of zero memory, $M_i = 0$, for all agents, which in terms of the decision rule in (1) implies that all agents take the decision to buy or sell on a random basis. When memory is given, memory is assigned to agents from a uniform distribution on *[0, 10]*. In this case, the discount factor for the forecast rule in (1) is $\lambda = 0.9$. The network structure has been investigated after 1000 time steps and it has been found that in most cases no significant changes are observed after that.

Experiments for each of the cases (i) – (iii) above were done for the random reward scheme and the with the minority game. Table 2 summarizes the results of the simulations for the herding coefficient, σ, defined in (3), the clustering coefficient, C, given in (9), the number of gurus, and the average shortest path length, ℓ.

Table 2: Summary of Results: For different reward functions

(*With and without memory; Static and dynamic links* ;$R^-=-0.4$, $R^+=0.2$) ; $\lambda = 0.9$; $N=100$; $T=1000$)

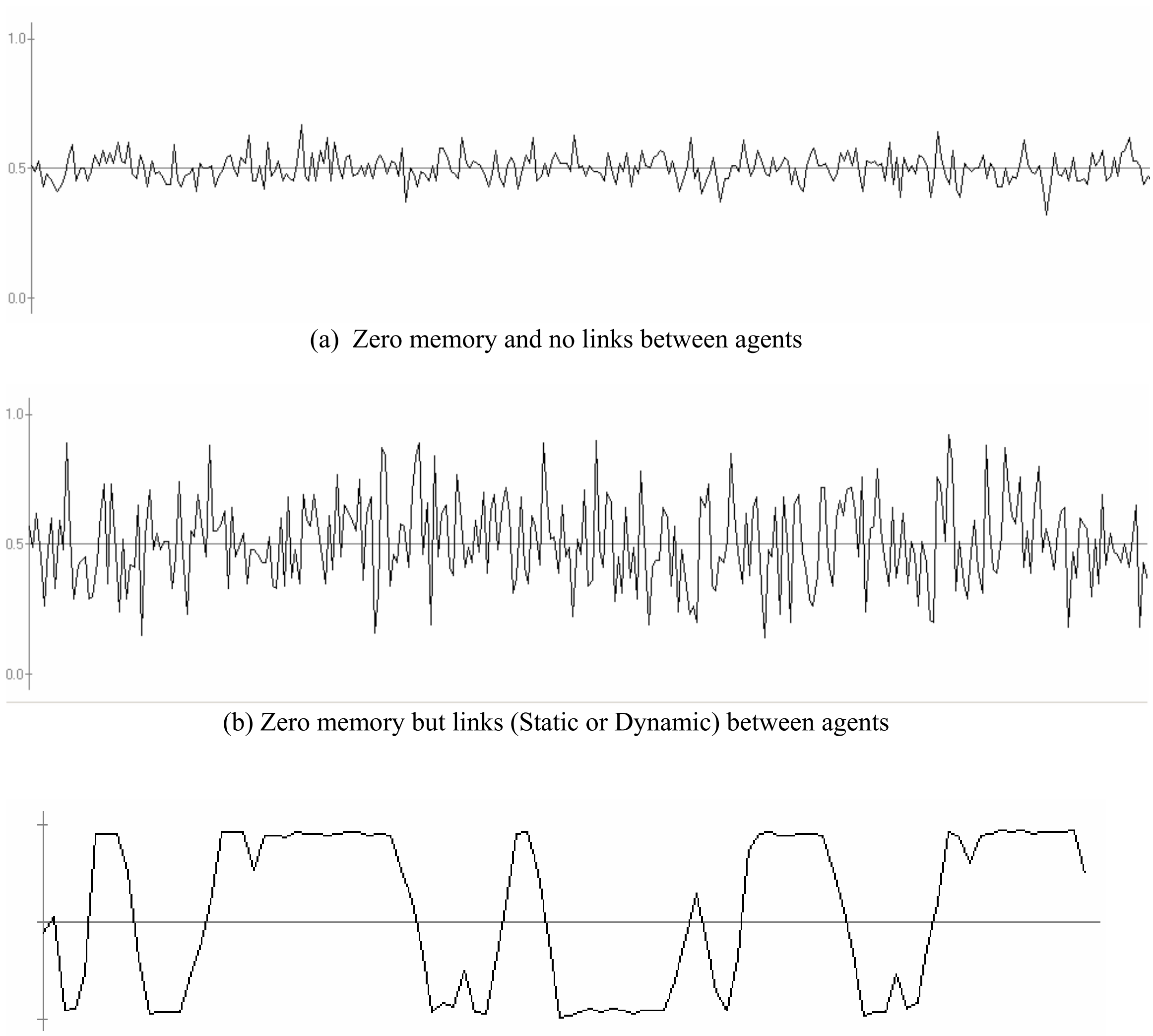
Parameters				**Results**			
Reward function	*Memory*	*Probability of interaction (p)*	*Weights*	*Herding coefficient:* σ	*Clustering Coefficient:* ***C***	*Number of Gurus*	*Average path length:* ℓ
Random Rewards	$M_i = 0$	0	-	0.09	-	0	-
		0.2	Static	0.34	0.2	0	≈1.5
	$M_i \in [0,10]$	0	-	0.74	-	0	-
		0.2	Static	0.93	0.2	0	≈1.5
		0.2	$R^- < R^+$	0.93	0.2	0	≈1.5
		0.2	$R^- > R^+$	0.93	0.2	0	≈1.5
Minority Rewards	$M_i = 0$	0	-	0.09	-	0	-
		0.2	Static	0.34	0.2	0	≈1.5
	$M_i \in [0,10]$	0	-	0.70	-	0	-
		0.2	Static	0.97	0.2	0	≈1.5
		0.2	$R^- < R^+$	0.96	0.21	0	≈1.5
		0.2	$R^- > R^+$	0.91	0.57	10	≈1.25
		0.1	$R^- > R^+$	0.66	0.84	10	≈1

Note: $\ell^{random} = 1.54$ for *p=0.2* and $\ell^{random} = 2.00$ for *p= 0.1.*

4.1 Impact of memory and links for herding: No clustering and gurus

It is amply clear from Table 2 that chief prerequisite for herding is memory followed by the presence of links between agents. In the absence of any links agents will base their decisions only on their own recommendations, which in the absence of memory will be random. Consequently, on average over time the herding coefficient is small with σ = *.09*. For this case of zero memory and no links Figure 1(a) shows the typical time series of the fraction of agents choosing $\gamma_i(t) = 1$, *viz.* $\frac{N_{bt}}{N}$.

Figure 1: **Herding : Fraction of agents who are buying**



(a) Zero memory and no links between agents

(b) Zero memory but links (Static or Dynamic) between agents

(c) Agents with memory but no links between them

Even without memory the presence of links increases herding of agents. The origin of this behaviour is the mimicking by an agent of its neighbours', Γ_i , recommendations . With an average of 20 neighbours with *p=0.2*, many agents will be influenced by the same recommendations, resulting in similar decisions being taken across agents, thus increasing the herding coefficient, σ, to 0.34. Figure 1(b) shows a typical time series for this scenario.

The introduction of memory to agents with the 100 agents uniformly distributed over memory lengths from [0,10] implies that there is on average 10 agents of each memory length. In the case when agents with these memory classes are introduced with a common discount factor of λ *=0.9*, their behaviour in isolation with zero links gives the impact of the pure common global signal component from the forecasting rule (1) on herding. The role of memory in the forecasting rule in (1) introduces trends or persistence into the decision of individuals as well as increased herding as illustrated in the flat portions of the time series for $\frac{N_{bt}}{N}$ in Figure 1 (c). In both reward schemes, agents with memory and operating in isolation on the same global data leads to about 70% - 74% of herding over time, see Table 2.

On combining memory with the local interaction links in their static form with *p=0.2,* the herding coefficient, σ increases to over 90% in both reward schemes. Krause (2004) has analysed at length the feature of the mimetic function (2) that imposes a majoritarian outcome from 'learning from neighbours' and hence produces herding even with an exogenous random reward scheme.

What is important to note is that in all the above cases the network remains unchanged from the initial random network with $p = 0.2$ and we could thus not observe any clustering or the emergence of gurus. Hence, the clustering coefficient σ remains identical to $p = 0.2$ of the random network and so does the average path length ℓ at about *1.5*. Interestingly, in the case of random rewards even on allowing reinforcement learning and dynamic link formation, the network properties remain unchanged from the initial random network. Despite agents being allowed to break away from bad advisors, no superior agents are identified. The explanation for this finding is that there does not exist a superior agent. The random reward structure means that regardless of the strategy followed, success will always be random, hence no agent can maintain a superior performance over time. Although agents break links, they will do so on a purely random basis which does not cause systematic deviations of the network structure from that of a random network.

To conclude, it is clear from Table 2 as well as from more extensive simulations undertaken, that while herding can be caused either when agents respond to memory based information of a common global signal or from mimicking other agents, network structural changes from the initial random network take place only once we introduce dynamic learning of link formation within an endogenously driven reward scheme such as the one for the minority game.

4.2 Rates of Adjustment and Clustering Coefficient in the Minority Reward Scheme

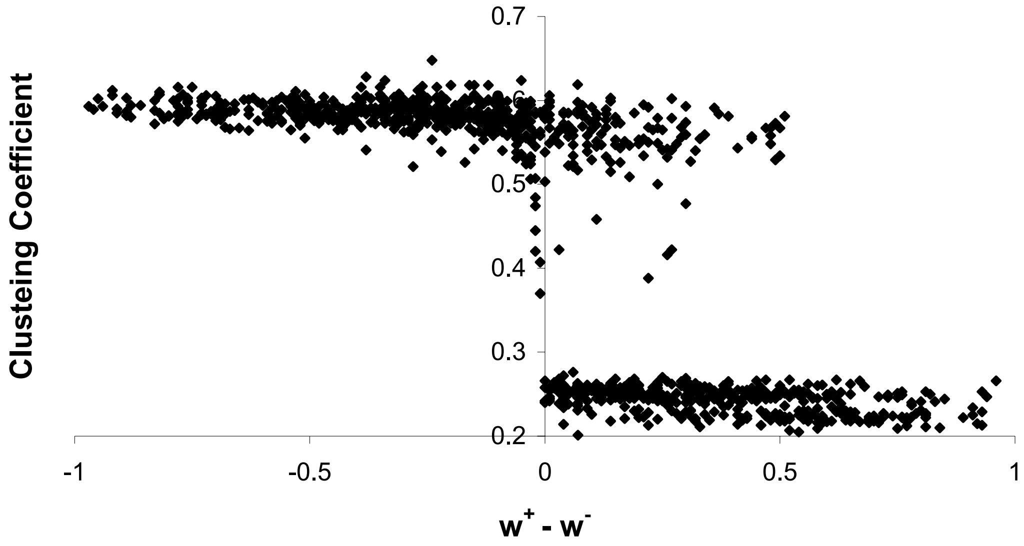
We now turn to how properties of small world networks with high clustering coefficients and guru/star effects emerge starting from a random network. In most exercises devised to date to produce small world properties for the network topology, the experimenter sets in an *ad hoc* fashion the proportion of an agent's k_i neighbours from which it breaks away to form random reconnections. What is remarkable in our framework, is that the agents are endogenously set up to break away from those who provide 'bad' advice. In the random reward scheme discussed above, no subset of agents appeared to be able to give better advice than others. In contrast, when the minority reward scheme is in operation, though agents do not know who has memory between 0 and 10, by the adaptive process of breaking away from bad advisers and making new random reconnections, it is remarkable that zero memory agents emerge as gurus with the network becoming highly clustered and the in-degree distribution displaying a 'fat tail'. We will now investigate the conditions under which the random network shows the small world properties of clustering when agents play under the minority reward scheme.

Just as in previous exercises (see, Davidsen *et. al.* 2002) for the generation of clustering in networks which found that the process was sensitive to the proportion of neighbours from whom agents break off links to find new randomly selected neighbours, in our model the conditions under which the network shows significant clustering is likewise governed by the relative rates of increment and decrement driving dynamic link formation.

A large number of simulations were run, varying the parameters of the model such as p, the probability of links, the discount factor λ and the rates R^+ and R^-. It was found that the critical condition for the network to become highly clustered under the minority reward scheme is that the rate of reduction R^- on the weights w_{ij} linking any two (i,j) agents has to be greater than the rate of increment R^+. In other words, the system inertia had to be sufficiently overcome for the random network to be altered significantly.

Figure 3 illustrates the results of 1000 simulations, of 1000 time periods each, where the R^+ and R^- were randomly chosen for each simulation. As before, the probability p was set to 0.2 and the discount factor to 0.9. Similar results were found when varying those parameters. The difference between the rate of reduction and rate of increment is plotted along the X- axis, and the clustering coefficient of the resulting network after the 1000 time periods is plotted along the Y-axis.

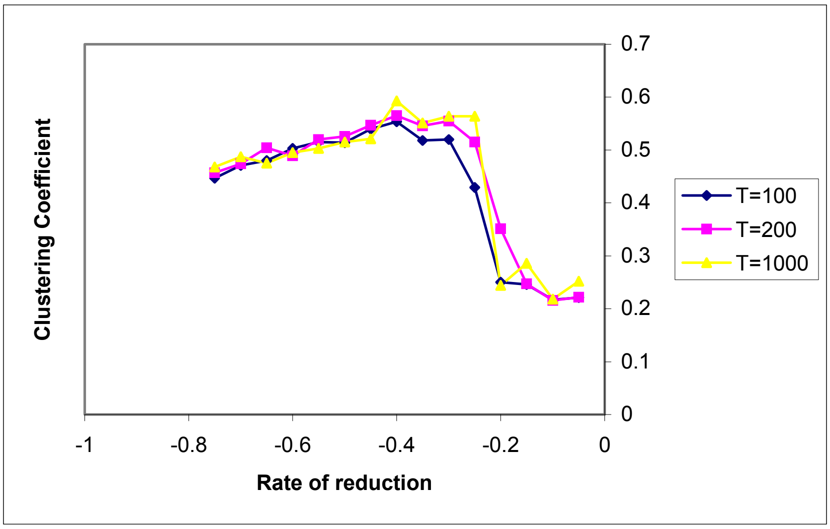
Figure 3 : Impact of Rates of Adjustment (R^+, R^-) On Clustering Coefficient



The vertical zero axis in Figure 3 indicates the points where $R^- = R^+$. It can be seen that to ensure agents will find the gurus, and that the system achieves a high clustering coefficient, the rate of reduction, R^- has to be greater in absolute terms than the rate of increment, R^+. Otherwise, the clustering coefficient stays close to the original value, that is, the Erdös-Rényi probability of the random network. It is worth noting that there are a few points to the right of vertical axis where a high clustering coefficient is obtained, even though the $|R^-| < |R^+|$. This arises from the random nature of these experiments.

Experiments were conducted to find if there was a rate of reduction R^- relative to a given rate of increment R^+ which maximizes the clustering coefficient and how many time periods were needed for its stationarity. As can be seen from Figure 4 with the R^+ set at *0.2*, the clustering coefficient C is maximized when $R^- = -0.4$. In other words the rate at which agents break away from 'bad' advisers and make random connections to others should be twice as fast as the rate at which they strengthened their links to 'good' advisers. However, if random sampling was speeded beyond this with $R^- < -0.4$, the clustering coefficient falls marginally. To achieve the maximum clustering coefficient $C = 0.60$, approximately 1000 time periods are needed.

Figure 4 Maximum and Stationary Clustering Coefficient (For fixed Rate Increment = +0.2, Varying Rate of Reduction)



4.4 Guru Effects and Star Formation

The clustering phenomenon in the network topology is closely related to the capacity for adaptive learning process to take place with $|R^-| > |R^+|$ and hence for agents to find the zero memory agents who give best advice in a minority game. The first manifestation that most agents identify the zero memory agents as 'gurus' can be seen from the histogram for the in-degrees obtained after T=1000. The histogram in Figure 5a represents the degree distribution of a non-clustered network. This is the typical degree distribution of the initial Erdös-Rényi random graph, and also the degree distribution of a network after 1000 time steps using dynamic weights in the case $|R^-| < |R^+|$. Under such conditions, as shown in previous section, the network does not get any more clustered than the random graph.

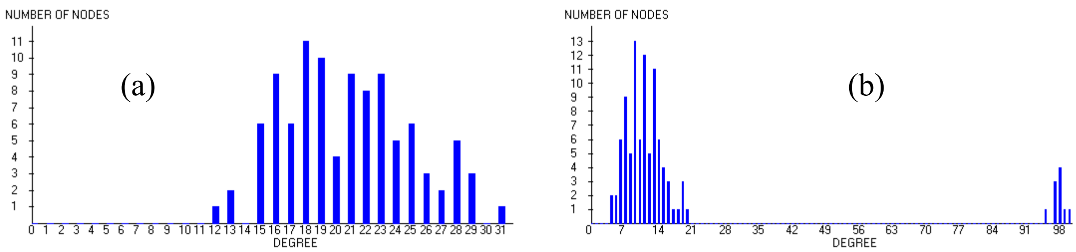
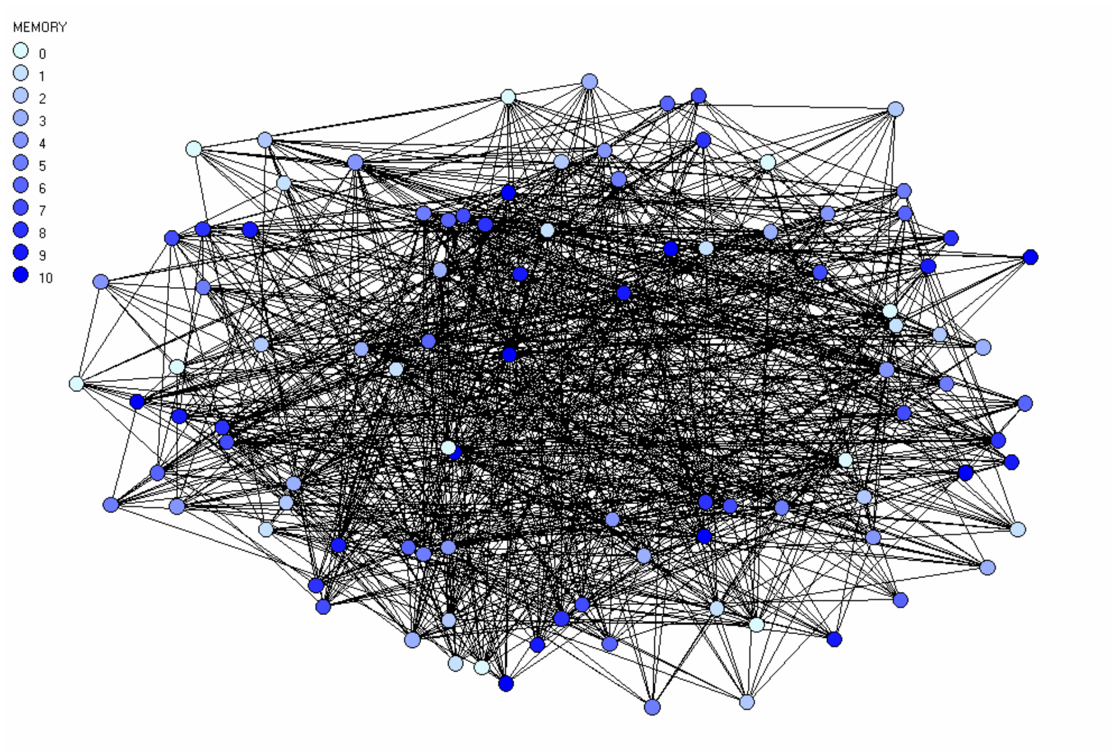


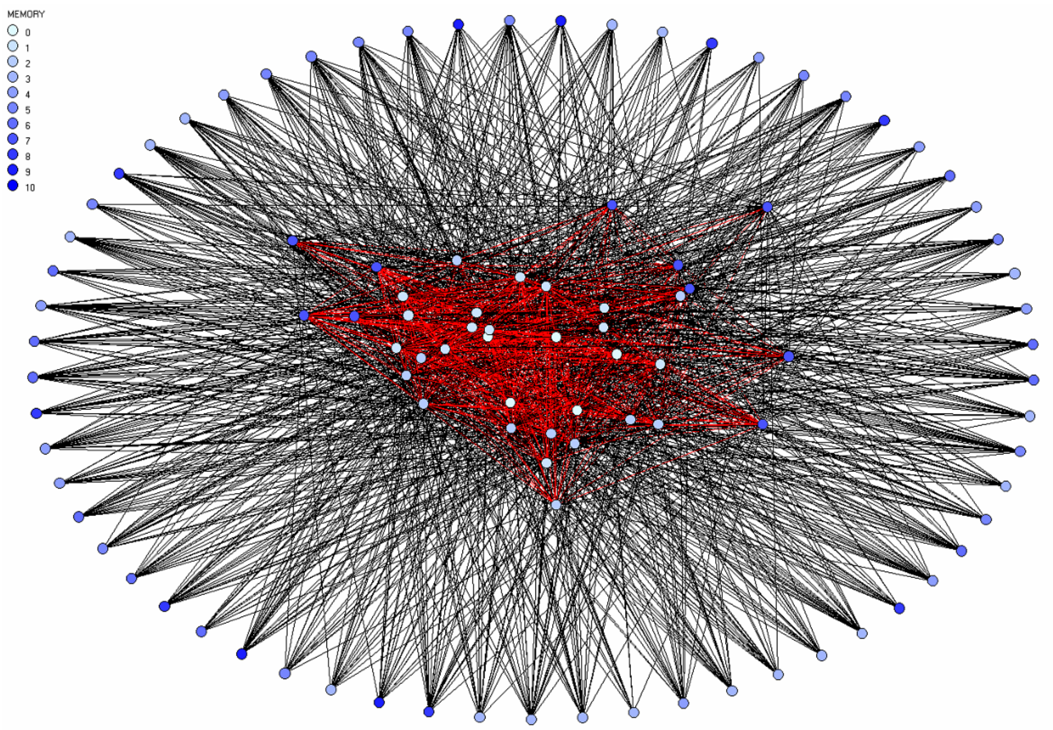
Figure 5 : Typical histograms (a) $R^+ = 1$, $R^- = -0.1$, C = 0.21; (b) $R^+ = 0.2$, $R^- = -0.4$ and C = 0.57

In contrast, if $|R^-| > |R^+|$, the network becomes highly clustered, and the degree distribution displays a "fat tail", as shown in Figure 5b. That is, a small number of nodes/agents have a high number of in-

Figure 6: Structure of networks



6(a) Random non-clustered network



6(b) Clustered network with gurus

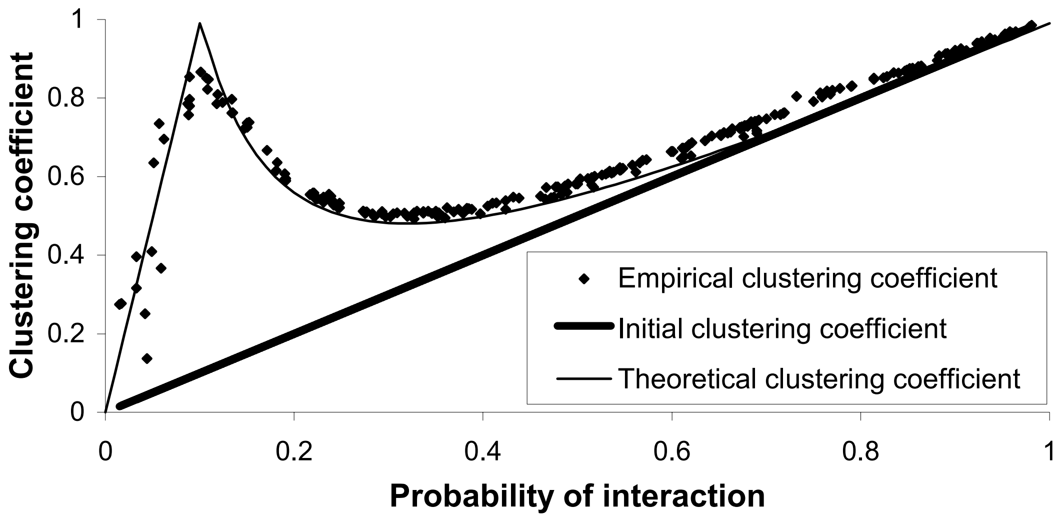
degrees. Indeed, almost 98% of all agents have found the gurus and have out degrees to them. The histogram shows a two tiered hierarchy in terms of influence or in-degrees. The same information with more evidence on the identity of the gurus can be given in terms of the star/hub formations in the network topology. The pictures 6a and 6b above show a representation of the network topology where the nodes are placed according to their in-degrees, with the highest degree nodes in the centre. The legend indicates the colour coding of the memory class – darker the colour, the greater the memory. The initial random network is shown in Figure 6a, and the resulting network after 98% of agents are connected to the gurus, nodes with the lightest blue viz. zero memory agents, is shown in Figure 6b. Magnification of the hub of Figure 6b highlights that gurus almost exclusively seek connections among themselves. The significance of this will be dealt with the following sections.

4.6 Effect of Initial Interaction probability p on clustering coefficient

Recall, starting from the initial random network with the clustering coefficient, $C=p$, a network becomes clustered beyond that of a random graph only if $C > p$. Recall also that $p(N-1)$ gives the average number of out-degrees or neighbours k_i from whom an agent takes advice. Remarkably, it is found that this number relative to the potential gurus or zero memory agents in the system is what determines whether clustering coefficient $C \geq p$. Note that with 100 agents uniformly distributed in terms of memory lengths ranging from zero to ten, there are on average ten zero memory agents who are the potential gurus in the minority game structure.

The results of 200 simulations for the clustering coefficient of the model varying only p are shown in Figure 7. The diagonal line represents the initial clustering coefficient, which is equal to the probability p, by definition, and therefore has a slope of 1. The dots on the graph represent the final clustering coefficient after dynamic learning has taken place. It can be seen that for values of $p > 0.3$, the resulting clustering coefficient does not differ from that of a random network. For values $p<0.3$, the simulated clustering coefficient peaks at *84%* for $p=0.1$. In other words, $C^{max} \approx .84$ for N=100 and uniformly distributed over $M_i \in [0,10]$ occurs for $p=0.1$. The reason for this is that for this value of p all agents on average have about 10 neighbours they take advice from and this also coincides with the number of zero memory agents in the system. Therefore, all agents are only connected to the gurus, since the number of available out degrees on average is equal to the number of gurus in the system. Since the clustering coefficient measures the number of neighbours of agents that are also neighbours between themselves, we see how it is maximal when all agents on average only have gurus as neighbours and with the gurus exclusively connected to themselves. For any value $p < 0.1$, the agents can only be connected to a sub-set of the gurus, since they do not have enough out degrees to link to all of them, and that in turn, causes the clustering coefficient to reduce. Similarly, when $p > 0.1$, agents have on average more out-degrees than available gurus in the system, and with those additional connections they randomly link to other agents. This again results in a lower clustering coefficient. The theoretical clustering coefficient for a given N, p and memory classes uniformly distributed over $M_i \in [0, M_{max}]$ has been derived in the Appendix. Figure 7 also plots the values of the clustering coefficients for different values of p for the empirical/simulated network and the theoretical result.

Figure 7 : Maximum Clustering Coefficient



4.7 Influence of Gurus On Herding and Path Length

Having identified the initial condition $p = 0.1$ which maximizes the clustering coefficient for N=100, $M_i \in [0,10]$, we will see why this is also the case when the zero memory agents or the gurus in the system with the minority reward scheme have the maximum impact. From Table 2, for $p = 0.2$, the clustering coefficient is only *57%* as opposed to *C= 84%* when gurus with their random recommendations have maximal impact on the decisions of agents. Figure 8a shows how herding continues unabated with $\sigma > 0.90$ when $p=.2$ and $C=0.57$ while Figure 8b shows that with $C= 0.87$, maximal connections to gurus produces greater randomness in the binary choice and ameliorates herding substantially to $\sigma = 0.66$.

Further our experiments show that in isolation without links, the zero memory class of agents spend more time in the minority than any other memory class. Indeed, when the network remains unclustered, there is much variability across the memory classes as to time spent in the minority. However, once the network becomes maximally clustered, the fraction of time in the minority is similar for all memory classes. It is not that zero memory gurus lose their superiority in playing the game, it is that other agents linked to the gurus become as good as the gurus at playing the minority game.

Figure 8a
Dynamic Learning in Minority Game : Herding With Clustering C= 0.57
(p= 0.2; R^- =-0.4, R^+ =0.2 ;T= 1000)

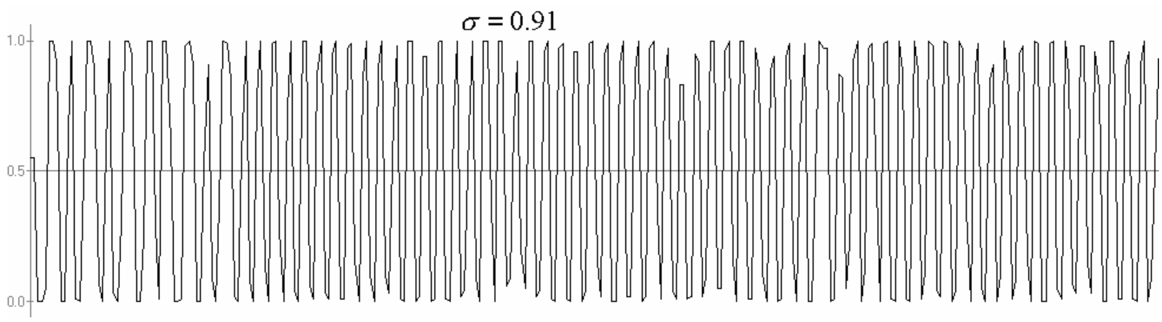
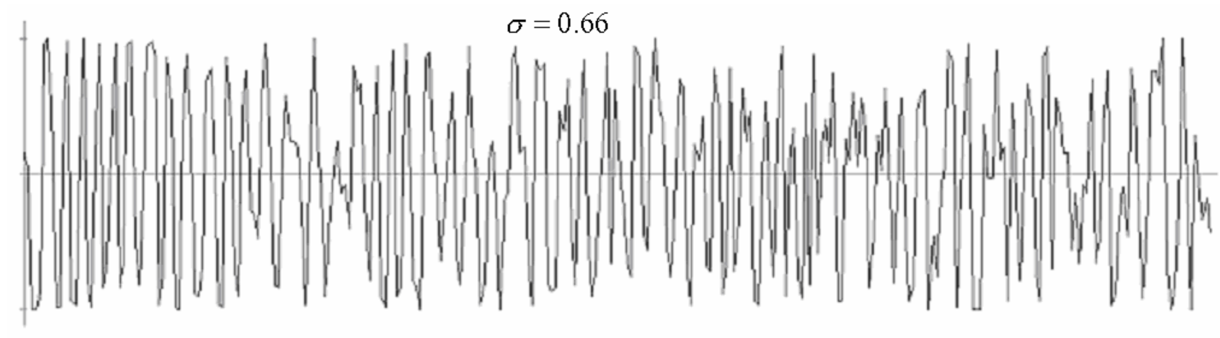


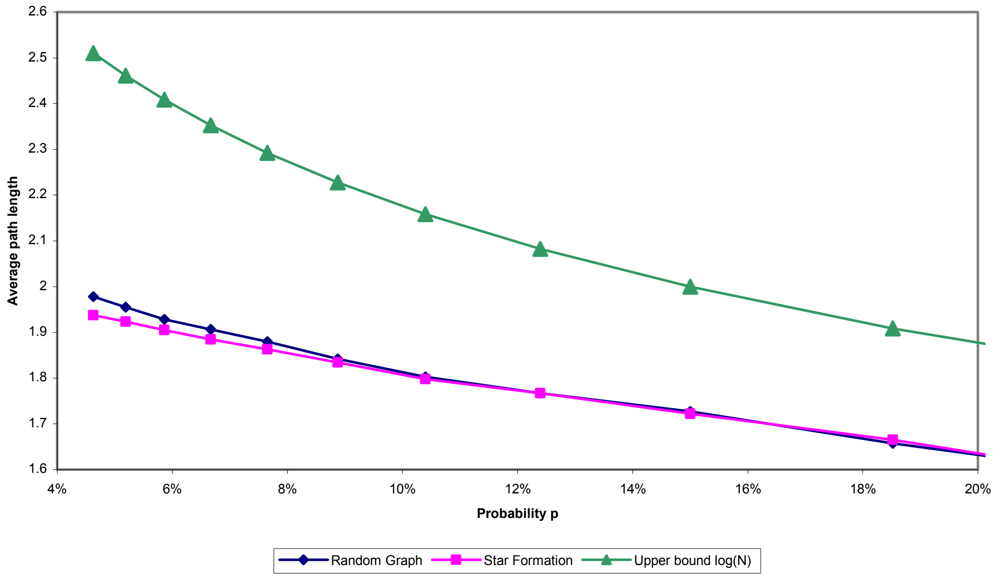
Figure 8b
Dynamic Learning in Minority Game : Herding With Clustering C= 0.84
(p= 0.1; R^- =-0.4, R^+ =0.2 ;T= 1000)



Finally, high clustering with gurus providing the hub in the system, reduces the average shortest path length to even less than ℓ^{random} for the random network with the same *N* and *p*, see Table 2.

Figure 9 shows how the average path length of the network behaves as we increase the number of nodes *N* of the network, keeping the mean degree constant at z=15. Since the mean degree of a random graph is $z = N \cdot p$, as *N* increases *p* must decrease to keep *z* constant. We can see how for *p* smaller than 10% , the average path length of the network with star formation and guru effects becomes shorter than the average path length of a random network. This is because as population size *N* increases though the falling *p* implies that the density of network connections are now sparser, all agents with memory find the gurus with zero memory and the hubs work as shortcuts. Thus, networks in which the gurus have been found with hub formations succeed in providing greater cohesiveness in connectivity than the random network where *p* falls relative to a growing *N*.

Figure 9: Path Length in Random and Clustered Network



5. Conclusion

To date, herding models with mimetic behaviour from following neighbours almost always assume that an agent's neighbours are fixed. In other words, little analysis has gone into the study of whether the network topologies arising from the process of 'learning from neighbours' resemble the small world network type features of real world socio- economic networks. This paper takes a random network as the initial point and is concerned with dynamic network formation by an experientially driven process to see if the small world network properties of clustering and star/hub formations will emerge. The paper contrasts clustering which represents the network topology of the underlying communication network with herding which represents aggregate behaviour with regard to a binary decision problem.

Some interesting and intriguing results follow. Simple adaptive threshold based behaviour that results in new random links to be formed to replace 'bad' connections, can lead individuals with no prior knowledge of the distribution of some critical characteristic in the population of agents to find the 'gurus' who possess this characteristic. Significantly, agents have no capacity to form explicit strategies to win the game. They can do so only by forming links to good advisors. Given the global reward scheme, gurus have an inherent quality of playing the game better than other classes of agents. Within a minority game reward scheme, zero memory agents become gurus while those with memory are disadvantaged in that they will almost always produce the wrong majoritarian advice as trend followers. Remarkably as we saw in Figures 6a –6b, the dynamic process of link formation produces the star/hub formations in the network topology often found in real world networks.[9] Highly skewed in-degree distributions are produced reflecting the influence of the gurus who then propagate their characteristics best when the clustering coefficient for the system is maximal. We are able to give numerical approximations for a formula for the clustering coefficient for any given initial *N*, *p* and distribution of memory classes given that clustering $C > C^{Random}$ takes place in the first instance. The latter critically depends on the rate of attrition of links to be greater than the rate of reinforcement. Further, when the clustering coefficient $C > C^{Random}$ we find that the shortest average path length becomes less than that for random networks.

To conclude, in the literature a number of constructions have been given to generate the properties of small world networks. All constructions rely on finding a critical rate at which new random reconnections are being made in the system: either too much or too little, the network fails to cluster. This paper shows how an endogenous and heterogeneous process of random reconnections by agents in the system result in agents discovering 'useful' neighbours. In so doing, the model and experiments of the paper go some way in integrating the herding and clustering phenomena.

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[9] Experiments with a majority global reward scheme was played with memory 0 and memory 1 classes of players. As can be expected the hierarchy in terms of degree distribution was reversed with memory 1 agents in the centre of the star/hub formation.

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Appendix : Theoretical Clustering Coefficient at Stationary Point With Star Formations

We derive here the theoretical clustering coefficient plotted in Figure 7 for any network of N agents, p, initial link probability and uniform distribution of memory classes with finite support [0, M], under the following assumptions.

(i) The sufficient condition identified in Section 4.2 exists for agents in the system to identify zero memory agents as 'gurus' in the minority reward scheme.

(ii) The system is at the stationary point of the dynamic learning process.

Recall, the clustering coefficient of a network is the average of the individual clustering coefficients

$$C = \frac{\sum_{i=1}^{n} C_i}{N}.$$

The individual C_i each for each agent requires evaluation on average how many of each of i's neighbours are also neighbours. For this we first determine the probability p_G of any node being connected to a guru :

$$p_G = \begin{cases} 1 & if \quad N_G \le z \\ \frac{z_{out}}{N_G} & otherwise \end{cases} \qquad \text{(A.1)}$$

Here, N_G denotes the number of gurus in the system, and the mean out degree of a node is $z_{out} = (N-1)p$. Note, given assumption 1, the number of zero memory agents coincides with the total number of gurus there can be in the system .

The probability of any node being connected to a non-guru or two non-guru nodes being connected is p_{NG} is given by

$$p_{NG} = \begin{cases} p & if \quad N_G = 0 \quad \text{(if there are no gurus)} \\ 0 & if \quad N_G \ge z_{out} \quad \text{(if there are more gurus than out links)} \\ \frac{(z_{out} - N_G)^+}{(N - N_G)} & otherwise \end{cases} \quad (A.2).$$

Note that the number of excess links, after having linked to the gurus is $(z_{out} - N_G)^+$, and the number of non-gurus is given by $(N - N_G)$. The premise here is that at stationary point, agents are linked to non-gurus only if they have more out links than there are gurus in the system.

Using p_G and p_{NG} in (A.1) and (A.2), we evaluate the theoretical clustering coefficient as follows.

$$C^{\#} = \frac{\left[z^G\left(z^G - 1\right) + z^G \cdot z^{NG}\right] \cdot p_G + \left[z^{NG}\left(z^{NG} - 1\right) + z^G \cdot z^{NG}\right] \cdot p_{NG}}{z_{out}\left(z_{out} - 1\right)} \qquad \text{(A3)}$$

Here, $z^G = \min(z_{out}, N_G)$ denotes average number of neighbours of any agent i who are gurus ;

$z^{NG} = \max(z_{out} - N_G, 0)$ is the average number of neighbours of any agent i who is not a guru.

Note, $z_{out} = z^G + z^{NG}$. In (A.3), the term $z^G(z^G -1)p_G$ gives the average number of *connected* edges between neighbours of gurus who are also gurus while $(z^G z^{NG})\, p_G$ gives the same for neighbours of non-gurus who are gurus. In (A.3), the term $z^{NG}(z^{NG} -1)p_G$ gives the average number of *connected* edges between neighbours of non-gurus who are also non- gurus while $(z^G z^{NG})\, p_G$ gives the same for neighbours of gurus who are non-gurus. The sum of this divided by the average total number of all possible edges in the network $z_{out}(z_{out} - 1)$ gives the theoretical clustering coefficient for the network in stationary state when all gurus have been discovered.

The above formula is verified for the two limiting cases.

If there are no gurus in the system, N_G=0, the clustering coefficient *C* must be equal to the Erdös-Rényi probability $C = p$ .If number of gurus is equal to the average degree of the nodes, then the clustering coefficient must be: $C \approx 1$.