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Ken Burdett and Melvyn Coles

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Ken Burdett
Department of Economics
University of Pennsylvania

Melvyn Coles
Department of Economics
University of Essex

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Abstract

This paper investigates equilibria in a labor market where heterogeneous firms post wage/tenure contracts and risk-averse workers, both employed and unemployed, search for new job opportunities. Different firms, even those with the same productivity, typically offer different contracts. Equilibrium finds workers never quit from higher productivity firms to lower productivity firms, but turnover is inefficiently low as employees with large tenures at low productivity firms may reject job offers from more productive firms. A worker who quits to a more productive firm may take a wage cut in anticipation of better wage promotion prospects. Wages within a firm might also increase by a discrete amount at the end of an initial "probationary" spell.

Keywords: wage/tenure contracts, on-the-job search, heterogeneous firms

Wage/Tenure Contracts with Heterogeneous Firms

For the typical worker, life in the labor market is full of change. While employed a worker may change employer, or move to unemployment, whereas a worker when unemployed may get a job or become a non-participant. Further, the wages faced continue to change through time. Investigating how and why these changes occur during the lives of workers has been the focus of a great deal of work during the last twenty years. Indeed, it has become a major element of modern labor economics. Within this context three stylized facts are generally agreed upon. First, a typical worker changes jobs several times during his/her lifetime and a significant percentage of job changes involve no interim unemployment. Clearly, on-the-job search by workers is a common phenomenon. Second, at least for younger workers, the wage received by a worker is positively correlated with job tenure and total work experience. Of course, identifying by how much a worker's wage increase is due to tenure and how much is due to experience is no small task. Indeed, there is a large and flourishing literature on this topic (see, for example, Altonji and Skakotko (1987), Topel (1991), Dustman and Meghir (2005)). Finally, job-to- job changes typically involve a jump in the wage received. Although most wage jumps are positive, there is a significant number of negative jumps where wages decrease in a job-to-job change.

Among those who have tried to explain these stylized facts, two big ideas have dominated. First, most have used the seminal work of Becker (1964) and Mincer (1974) in which workers accumulate both general and firm specific human capital as they work for employers. Second, a number have used the arguments proposed by Jovanovic (1979,1984) that when a firm hires a worker the quality of the match is unknown. The participants, however, learn about the quality through time. There is now a significant and insightful literature that have used these two ideas to explain the stylized facts presented above (recent studies include Kambourov and Manovskii (2007) and Moscarini (2005)). An objective of the present study is to take an alternative approach where both human capital accumulation and learning play no role by assumption. We construct and analyze an equilibrium search model of the labor market where in equilibrium the model constructed predicts, amongst other things, the stylized facts described above.

Recent years have seen a number of significant developments in the study of equilibrium in labor markets where firms post wages and workers, both unemployed and employed, search for better jobs. Perhaps the most used framework in this area was developed by Burdett and Mortensen (1998), hereafter B/M. Within the B/M framework it can be shown that even if both workers and firms are homogeneous, the equilibrium distribution of wages is non-degenerate. As workers are assumed to search while employed, the dispersed wage distribution implies an employed worker may change jobs as new job opportunities arrive. Hence, in equilibrium there is job-to-job movements by workers. This, and other resulting predictions, has led to a great deal of empirical work.

Three restrictions play a crucial role in the B/M framework: (A) employed workers receive job offers as well as unemployed workers, (B) employers do not respond to offers from other firms received by its employees, and (C) the offers made by any employer can be completely described by a positive real number - the wage it is willing to pay any employee at any tenure.

Restriction (A) appears harmless in that it is satisfied in most labor markets. Restriction (B) is clearly not satisfied in the academic labor market in the US, but may well hold in other types of labor markets - it's hard to tell. Postel-Vinay and Robin (2002a, 2002b) and Cahuc, Postel-Vinay and Robin (2006) replace restriction (B) by assuming that if an employee does contact another firm, the two firms (the worker's current employer and the firm contacted) partake in a Bertrand wage bidding contest for the services of the worker. The firm with the higher productivity wins this contest. If the worker's current employer is less productive than the other firm, the worker changes employer. Otherwise the worker stays put but may still enjoy a wage increase. In this framework firm heterogeneity not only induces wage dispersion across firms, but there is also within-firm wage dispersion where each employee's wage depends on outside offers received in the past. Thus, within-firm wages are also positively correlated with tenure.

In a recent contribution Burdett and Coles (2003) (hereafter B/C) extends the

B/M analysis by instead dropping restriction (C) while maintaining restrictions (A) and (B). In particular, B/C assumes firms offer contracts where the wage paid depends upon a worker's tenure. Given workers are risk-averse, it was shown that the wage/tenure contract which maximizes firm profits implies the wage paid increases with tenure. Further, in equilibrium different firms offer different contracts - some more desirable to workers than others. This implies that employed workers who receive outside offers may change jobs. A useful interpretation of this outcome is that firms implement an internal promotion scheme, where (otherwise identical) employees are promoted by seniority. Thus more senior workers are higher up in the firm's wage hierarchy. By promising a junior employee higher wages in the future, an internal promotion scheme reduces a junior employee's incentive to quit. The employer can then set low wages at short tenures and so extract the junior employee's search rents. Restriction B implies the firm ignores employee outside offers; i.e., a firm never promotes a junior worker over equally productive, more senior, employees. Instead an employee who receives a preferred offer is simply allowed to leave. Postel-Vinay and Robin (2004) argue such a rigid pay structure may be profit maximizing as "outside offer matching" increases the return to on-the-job search and so encourages greater on-the-job search activity. With on-the-job search and no "outside offer matching", a company wage structure which promotes by seniority (or tenure) weakens worker quit incentives and so increases firm profit.

In the present study we extend the B/C study by assuming that firms have different productivities. Allowing for firm heterogeneity with search frictions and on-the-job search leads to a rich set of possible equilibrium outcomes. Here we mention four. First as in B/C, different firms with the same productivity typically offer different wage/tenure contracts. This implies job to job movements by workers between firms with the same productivity as well as between firms with different productivities. Second, we show that more productive firms make more desirable offers than less productive firms. This implies workers never quit from more productive firms to less productive firms. Nevertheless, we shall show employees with large tenures at low productive firms may reject outside job offers from more productive firms.

Thus, turnover is inefficient. This arises as hiring firms offer wages below marginal product, while firms design wage tenure contracts to reduce employee quit incentives. Together these factors imply quit turnover is too low - too many employees remain in low productivity firms. Third, if a worker quits a firm to a more productive one it may involve a reduction in wage. This occurs as a worker does not only compare current wages when considering an outside job offer - internal promotion prospects also play an important role. More productive firms offer better internal prospects which can more than compensate for a lower initial wage. Finally, we show that although wages increase with tenure at a firm, this increase may not be smooth. In particular we identify market equilibria where the distribution of contract offers has a mass point in the interior of its support. A numerical example establishes that higher productivity firms might then offer a contract with an initial probationary spell. During the probationary period wages are low (but increasing). On completion of this spell, the worker's wage "jumps" up by a discrete amount.

In the next Section the basic framework is outlined and the optimal search strategy of a worker is described. The optimal wage/tenure contract of a representative type i firm is derived. We then show that in equilibrium the wage/tenure contracts offered by all type i firms can be fully described by a baseline salary scale. Sections 3 and 4 describe market equilibrium, where from Section 4 onwards we restrict attention to only two types of firms. Section 5 provides numerical examples to illustrate the richness of possible equilibrium outcomes. Section 6 concludes.

1.**THE BASIC FRAMEWORK**

Many aspects of the labor market model presented below are kept as simple as possible to ease the burden of exposition. Generalizations in several ways is straightforward but would complicate the presentation.

Time is continuous and only steady states are considered. A unit mass of both workers and firms participate in a labor market. Workers are homogeneous but there are n types of firms. A type i firm generates revenue p_i per unit of time from each worker it employs, $i = 1, 2, ..n$, where $p_1 < p_2 ... < p_n$. Proportion α_i of all firms are type i.

Workers are either unemployed or employed and obtain new job offers at Poisson rate λ, independent of their employment status. Any job offer is fully described by the wage contract offered by the firm. This specifies the wage the worker receives as a function of tenure at that firm, i.e., an offer is a function $w(.) \geq 0$ defined for all tenures $t \geq 0$. All new hires are offered the same contract and this contract is binding. There is no recall should a worker quit or reject a job offer. For simplicity assume firms and workers have a zero rate of time preference.

The objective of any firm is to maximize steady state flow profit. Workers are strictly risk averse and finitely lived, where any worker's life is described by an exponential random variable with parameter $\delta > 0$. Thus workers discount the future at rate δ. δ also describes the flow of new (unemployed) workers into the market and so ensures a unit mass of workers in a steady state. The objective of any worker is to maximize total expected lifetime utility. Unemployed workers obtain b per unit of time and $p_1 > b > 0$. Following previous studies that investigate contracts between employers and workers, we assume there is no capital market where a worker can borrow or save.

Workers

As a worker cannot borrow or save, a worker who obtains income $w \geq 0$ at any instant of time obtains flow utility $u(w)$ by immediately consuming it. We impose the following restriction.

A1: u is strictly increasing, strictly concave, twice differentiable and

$$\lim_{w \to 0^+} u(w) = -\infty \ .$$

A1 is useful as it ensures the constraint $w \geq 0$ is never binding in an optimal contract. As discussed further in B/C, we could relax A1 and instead assume the flow value of being unemployed is sufficiently large.

An unemployed worker's expected lifetime payoff is indicated by V_u. Given an optimal quit strategy, let $V(t|w(.))$ denote a worker's expected lifetime payoff when employed with tenure t at a firm offering contract $w(.)$. Thus, $V_0 = V(0|w(.))$ denotes the value of accepting a job offer $w(.)$. As firms may offer different contracts, let $F(V_0)$ denote the proportion of firms in the market whose job offer, if accepted, yields a worker an expected lifetime payoff no greater than V_0. Search is random in that given a job offer, $F(V_0)$ is the probability the offer has value no greater than V_0. Let $\underline{V}$ and $\overline{V}$ denote the infimum and supremum of this distribution function.

Standard arguments imply the expected payoff to an unemployed worker, V_u,can be written as

$$\delta V_u = u(b) + \lambda \int_{V_u}^{\overline{V}} [V_0 - V_u] dF(V_0). \tag{1}$$

An unemployed worker accepts a job offer if and only if it has value $V_0 \geq V_u$.

Now consider the value of being employed with contract $w(.)$. As formally established in B/C, an optimal contract implies it is never optimal for a worker to quit into unemployment.[1] Thus, given an optimal contract, standard arguments imply $V(t|w(.))$ is given by

$$\delta V(t|w(.)) - \frac{dV(t|w(.))}{dt} = u(w(t)) + \lambda \int_{V(t|.w())}^{\overline{V}} [V_0 - V(t|w(.))] dF(V_0). \tag{2}$$

Note, an employed worker with tenure t quits on receiving an outside offer if and only if the value of that offer $V_0 > V(t|w(.))$. As a worker may also die at any time, a worker, employed at a firm with tenure s and contract $w(.)$, leaves at rate $\delta + \lambda[1 - F(V(s|w(.))]$. This implies the probability a new hire will still be an employee with tenure t can be written as

$$\psi(t|w(.)) = e^{-\int_0^t [\delta + \lambda(1 - F(V(s|w(.)))] ds}. \tag{3}$$

Firms

Let $G(V)$ denote the steady-state number of workers in the economy with expected lifetime utility strictly less than V. Suppose a type i firm posts $w(.)$ and recall $V_0 = V(0|w(0))$ is the value of accepting this job offer. If $V_0 < V_u$ all potential employees

[1] Suppose an optimal contract implies the worker quits into unemployment at tenure $T \geq 0$. Thus at tenure T, the firm's continuation profit is zero and the worker obtains V_u. The same contract but which instead offers $w(t) = b$ for all tenures $t \geq T$ is strictly profit increasing - the worker obtains the same payoff V_u at T but, by not quitting, the firm's continuation payoff is strictly positive (as $b < p_i$). This latter contract then makes greater expected profit which contradicts optimality of the original contract.

prefer being unemployed to accepting this job offer and so such an offer yields zero profit. Suppose instead $V_0 \geq V_u$. $\lambda G(V_0)$ then describes this firm's hiring rate and its steady state flow profit can be written as

$$\Omega_i = [\lambda G(V_0)] \left[\int_0^\infty \psi(t|w(.))[p_i - w(t)]dt \right],$$

Note, the firm's steady state flow profit equals its hiring rate $\lambda G(V_0)$ multiplied by the expected profit per new hire.

To determine the contract that maximizes Ω_i we use a two step procedure. First, we determine a firm's wage/tenure contract that maximizes it's expected profit flow conditional on offering a new hire lifetime payoff V_0. Such a contract is termed an optimal contract. Assuming such an optimal contract exists, let $w_i^*(t|V_0)$ denote it and let $\Pi_i^*(0|V_0)$ denote the firm's maximized payoff per new hire. A type i firm's steady-state flow profits can then be written as

$$\Omega_i^*(V_0) = \lambda G(V_0) \Pi_i^*(0|V_0).$$

A firm i's optimization problem then reduces to choosing a starting payoff V_0 to maximize $\Omega_i^*(V_0)$.

A useful preliminary insight is that because the arrival rate of offers is independent of a worker's employment status, an unemployed worker will always accept a contract which offers $w(t) = b$ for all t. Further, as $b < p_i$ by assumption, a firm can always obtain strictly positive profit by offering this contract. Thus without loss of generality we assume (a) all firms make strictly positive profit; $\Omega_i > 0$, (b) $\underline{V} \geq V_u$ (as an offer $V_0 < V_u$ attracts no workers and so makes zero profit), and (c) $\overline{V} < u(p_n)/\delta$ (as no firm pays a worker more than the worker's expected value). To simplify further the exposition, we impose the following restriction on F.

A2: F has a connected support $[\underline{V}, \overline{V}] \subseteq [V_u, u(p_n)/\delta]$.

Unlike B/C we do not rule out mass points in F. Indeed, a surprising feature of the analysis is that equilibrium may yield mass points in the interior of the support of F.

Given productivity p_i, a firm's optimal contracting problem is formally defined as

$$\max_{w(.)} \int_0^\infty \psi(t|w(.))[p_i - w(t)]dt \tag{4}$$

subject to (a) $w(.) \geq 0$, (b) $V(0|w(.)) = V_0$ and (c) the optimal quit strategies of workers which determine ψ (as defined by (3)). Theorem 1 describes the optimal contract.

Theorem 1

For any F satisfying A2 and any starting payoff $V_0 \in [\underline{V}, V_i^\infty)$ with V_i^∞ defined by (9), the optimal wage-tenure contract w_i^* and corresponding worker and firm payoffs V_i^* and Π_i^* are solutions to the dynamical system $\{w, V, \Pi\}$ where

(a) V and Π evolve according to the differential equations

$$\frac{dV}{dt} = -u'(w)\frac{d\Pi}{dt} \tag{5}$$

$$[\delta + \lambda(1-F)]\Pi - \frac{d\Pi}{dt} = [p_i - w], \tag{6}$$

(b) w is determined by

$$u(w) + u'(w)[p_i - w - [\delta + \lambda(1-F)]\Pi] = \delta V - \lambda \int_V^{\overline{V}} [1 - F(x)]dx, \tag{7}$$

subject to the boundary conditions:

(i) the initial condition $V(0) = V_0$, and

(ii) $\lim_{t\to\infty}\{w(t), V(t), \Pi(t)\} = (w_i^\infty, V_i^\infty, \Pi_i^\infty)$, where

$$w_i^\infty = \max\{\overline{w}, p_i\} \tag{8}$$

$$\delta V_i^\infty = u(w_i^\infty) + \lambda \int_{V_i^\infty}^{\overline{V}} [V_0 - V_i^\infty]dF(V_0) \tag{9}$$

$$\Pi_i^\infty = (p_i - w_i^\infty)/\delta. \tag{10}$$

and $\overline{w}$ is the solution to $u(\overline{w})/\delta = \overline{V}$.

Proof

See Appendix A.

Unlike B/C, the conditions described in Theorem 1 allow mass points in F. The structures, however, are closely related. Indeed differentiating (7) with respect to t yields, after some simplification, the differential equation for w

$$\frac{-u''(w)}{u'(w)^2}\frac{dw}{dt} = \lambda F'(V)\Pi \tag{11}$$

as derived in B/C. Thus (7) is the integrated solution of that differential equation. As discussed in detail in B/C, the optimal contract is the saddle path to the differential equation system (5), (6),(11) for $\{w, V, \Pi\}$ where $(w_i^\infty, V_i^\infty, \Pi_i^\infty)$ is the limiting stationary point. That saddle path determines the wage paid w, the value of employment V and the firm's continuation profit Π at each tenure t, and the initial condition (i) ensures it yields value V_0 at $t = 0$. As equilibrium F may contain mass points, (11) does not always apply. Instead (7) applies for any F and so identifies the (discrete) change in optimal w across mass points.

Theorem 1 also describes the limiting wage w_i^∞ at long tenures. Note that $\overline{w}$ is the solution to $u(\overline{w})/\delta = \overline{V}$ and, in equilibrium, will describe the highest wage paid in the market. (8) implies high productivity firms - those with $p_i > \overline{w}$ - have $w_i^\infty = \overline{w}$; i.e., high productivity firms raise wages with tenure until they match the highest wage in the market. Conversely, low productivity firms - those with $p_i < \overline{w}$ - have $w_i^\infty = p_i$, i.e., low productivity firms raise wages with tenure until they pay marginal product.

Figure 1 depicts the optimal contract offered by the type i firm which offers the lowest starting payoff of any type i firm (denoted $\underline{V}_i$). If $\underline{V}_i < V_i^\infty$ (which is true in equilibrium), the conditions of Theorem 1 imply the wage paid increases with tenure and converges to w_i^∞. We refer to this optimal contract as the *type i baseline salary scale,* and denote it as $w_i(t)$. Theorem 1 also yields a corresponding path for V, denoted by $V_i(.)$ and continuation profit Π, denoted $\Pi_i(.)$. As the wage paid along the baseline salary scale increases with tenure, then the value of employment, $V_i(t)$, is also increasing in tenure. Note, if there is a mass point in F, then the conditions of Theorem 1 imply w increases by a discrete amount across that mass point, while $V(.)$ remains continuous (but it is not continuously differentiable). In Figure 1, F is assumed to be differentiable.

Optimality of the type i baseline salary scale yields a major simplification. Suppose a type i firm instead wishes to offer starting payoff $V_0 \in [\underline{V}_i, V_i^\infty]$. Optimality of the type i baseline salary scale implies the optimal contract yielding V_0, is the

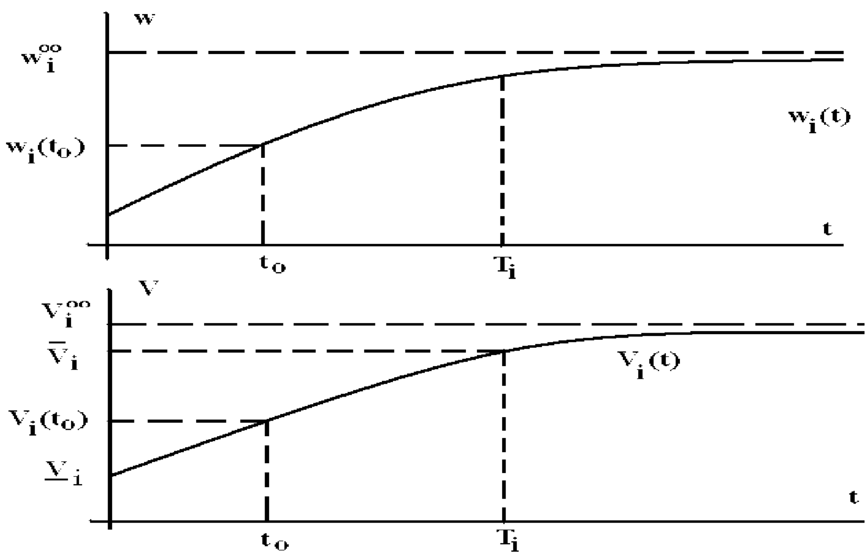

Figure 1: Type i baseline salary scale

starting point t_0 on the baseline salary scale where $V_i(t_0) = V_0$, and corresponding wage payments $w_i(t_0 + t)$ along the baseline salary scale at tenures $t \geq 0$.

With homogeneous firms, B/C shows that all starting points $t_0 \geq 0$ are offered in equilibrium. This does not necessarily occur with heterogeneous firms. For low productivity firms, those with $p_i < \overline{w}$, an upper bound exists, say T_i as depicted in Figure 1, such that all type i firms only offer starting points $t_0 \leq T_i$. To reduce inefficient quits, type i firms raise wages with tenure and pay marginal product at (arbitrarily) long tenures. Equilibrium, however, implies a finite upper bound $T_i < \infty$ exists on starting offers as offering $w(.) = p_i$ for all t yields zero profit.

2. MARKET EQUILIBRIUM

A moment's reflection establishes that workers do not care about the particular wage tenure contract offered, only the value V_0 obtained by accepting it. To proceed we need to transform the equations obtained above into value space (V).

Recall that at salary point t_0 on baseline salary scale i, the worker (a) earns current wage $w = w_i(t_0)$, (b) enjoys expected lifetime value $V = V_i(t_0)$ while (c) the firm obtains continuation profit $\Pi = \Pi_i(t_0)$. As $V_i(.)$ is a continuous increasing function of t_0, its inverse function, denoted by $\widehat{t}_i(V)$, is well defined for all $V \in [\underline{V}_i, V_i^\infty]$. Note, $\widehat{t}_i$ describes the salary point on the type i baseline salary scale which yields lifetime payoff V. $\Pi_i(\widehat{t}_i(V))$ then describes the firm's continuation profit given the worker enjoys current value V, and we let $\widehat{\Pi}_i(V)$ denote this payoff. Similarly define $\widehat{w}_i(V) \equiv w_i(\widehat{t}_i(V))$ which then describes the wage paid by a type i firm given the worker enjoys current value V. Claim 1 now identifies $\widehat{\Pi}_i(V)$ and $\widehat{w}_i(V)$.

Claim 1

Fix an F satisfying A2. For $V \in [\underline{V}, V_i^\infty]$, $\widehat{\Pi}_i(V)$ evolves according to the differential equation

$$\frac{d\widehat{\Pi}_i}{dV} = -\frac{1}{u'(\widehat{w}_i)} \tag{12}$$

subject to the boundary condition $\widehat{\Pi}_i = \Pi_i^\infty$ at $V = V_i^\infty$, and $\widehat{w}_i$ satisfies

$$u(\widehat{w}_i) + u'(\widehat{w}_i)[p_i - \widehat{w}_i - [\delta + \lambda(1-F)]\widehat{\Pi}_i] = \delta V - \lambda \int_V^{\overline{V}} [1 - F(x)]dx. \tag{13}$$

Further, $\widehat{t}_i$ is the solution to the differential equation

$$\frac{d\widehat{t}_i}{dV} = \frac{1}{u'(\widehat{w}_i)[p_i - \widehat{w}_i - [\delta + \lambda(1-F)]\widehat{\Pi}_i]}$$

with initial condition $\widehat{t}_i(\underline{V}_i) = 0$ and, as $\widehat{t}_i$ is strictly increasing for $V < V_i^\infty$, implies

$$[p_i - \widehat{w}_i - [\delta + \lambda(1-F)]\widehat{\Pi}_i] > 0 \text{ for all } V < V_i^\infty.$$

Proof

Claim 1 follows directly from Theorem 1 and the definitions of $\widehat{t}_i, \widehat{w}_i, \widehat{\Pi}_i$.

When F is differentiable, (13) implies $\widehat{w}_i$ evolves according to

$$\frac{d\widehat{w}_i}{dV} = \left[\frac{u'(\widehat{w}_i)}{-u''(\widehat{w}_i)}\right] \frac{\lambda F'(V)\widehat{\Pi}_i}{\left[p_i - \widehat{w}_i - [\delta + \lambda(1-F)]\widehat{\Pi}_i\right]}. \tag{14}$$

The approach below uses (14) and (12) to jointly determine $\widehat{w}_i, \widehat{\Pi}_i$ whenever F is differentiable, but uses (13) to determine $\widehat{w}_i$ whenever F is not differentiable; i.e. across mass points.

By construction, a type i firm's optimized steady state flow profit by offering $V_0 \in [\underline{V}, V_i^{\infty}]$ is

$$\Omega_i^*(V_0) = \lambda G(V_0)\widehat{\Pi}_i(V_0).$$

We can now formally define a Market Equilibrium.

A **Market Equilibrium** requires identifying the following:
(1) A distribution of job offers by firm type i, denoted $F_i(V)$, where $F(V) = \Sigma_i \alpha_i F_i(V)$ satisfies A2;
(2) optimal job search by workers,
(3) a distribution of expected lifetime payoffs G consistent with offers F and steady state turnover and
(4) a steady-state profit condition where for all firm types:

$$\begin{aligned} &\Omega_i^*(V_0) = \overline{\Omega}_i > 0 \text{ if } dF_i(V_0) > 0, \\ &\Omega_i^*(V_0) \leq \overline{\Omega}_i, \text{otherwise,} \end{aligned} \tag{15}$$

The definition of a Market Equilibrium requires the constant profit condition; that all offers with $dF_i > 0$ make the same profit $\overline{\Omega}_i$, while all other offers make no greater profit. We begin with some standard results.

Lemma 1

A Market Equilibrium implies:
(a) $\underline{V} = V_u$;
(b) steady state unemployment $U = \delta/(\lambda + \delta)$;
(c) $1 - G$ and F do not have a mass point at $\overline{V}$.

Proof

Lemma 1(a) is a well-known result; the lowest value offer in the market equals the value of unemployment. The proof uses simple contradiction arguments : $\underline{V} < V_u$ is inconsistent with strictly positive profit (firms offering starting value $V_0 < V_u$ make zero profit), while $\underline{V} > V_u$ is inconsistent with the constant profit condition (offering $V_0 = \underline{V}$ is dominated by offering $V_0 = V_u$ as both offers only attract the unemployed and offering $V_u < \underline{V}$ generates greater profit per hire). Lemma 1(b) follows as unemployed workers accept their first job offer and thus steady state turnover implies

$\delta = (\lambda+\delta)U$. Lemma 1c is an important technical result which also holds in B/C; we prove it in Appendix A. This completes the proof.

We next show a Market Equilibrium implies higher productivity firms offer contracts with higher starting payoffs. Thus, workers never quit from a high to a low productivity firm. Lemma 2 establishes the necessary preliminary step.

Lemma 2

For any i, j with $p_i < p_j$, and for any $V_0 \in [\underline{V}, V_i^{\infty})$:

(i)

$$\widehat{w}_i(V_0) > \widehat{w}_j(V_0)$$

(ii)

$$\frac{d\widehat{\Pi}_i(V_0)}{dV_0} < \frac{d\widehat{\Pi}_j(V_0)}{dV_0} < 0.$$

Proof

See Appendix.

Lemma 2(i) states that for the same (starting) value V_0, a higher productivity firm pays a lower initial wage than a lower productivity firm. This result occurs as higher productivity firms offer higher wages at long tenures and so, to yield the same V_0, the higher productivity firms pay a lower (starting) wage. Lemma 2(ii) establishes that for the same (starting) value V_0, the marginal loss in profit by increasing V_0 is strictly smaller for the higher productivity firm. This reflects that a worker with a higher expected lifetime payoff has a lower quit rate and the marginal return to retaining that worker is greater for the high productivity firm. Theorem 2 now establishes the segmentation result.

Theorem 2

A market equilibrium implies $\overline{V}_i = \underline{V}_{i+1}$; i.e. the support of F can be partitioned into n sets, where type i firms offer $V_0 \in [\overline{V}_{i-1}, \overline{V}_i]$ with $\underline{V} \leq \overline{V}_1 \leq \overline{V}_2 \leq .... \leq \overline{V}_n = \overline{V}$.

Proof

A firm with productivity $p = p_i$ chooses V_0 to maximize

$$\Omega_i^*(V_0) = \lambda G(V_0)\widehat{\Pi}_i(V_0).$$

First note that higher productivity firms must make strictly greater profit per hire for the same V_0 (they can offer the optimal contract of a lower productivity firm and enjoy strictly higher revenues). Hence for any increasing $G(.)$, Lemma 2(ii) implies the marginal return to increasing V_0 is strictly greater for a higher productivity firm. Thus optimality implies a higher productivity firm chooses a higher V_0. Connectedness of F (assumption A2) then implies the Theorem. This completes the proof of Theorem 2.

Identifying a Market Equilibrium now reduces to solving a sequence of constant profit conditions

$$\lambda G(V_0)\widehat{\Pi}_i(V_0) = \overline{\Omega}_i \text{ for } V_0 \in [\overline{V}_{i-1}, \overline{V}_i]$$

with G consistent with steady state turnover. As characterizing G is cumbersome, we demonstrate how to do it only for the two types case.

3. THE TWO TYPE CASE

Let $\widehat{N}_i(V)$ denote the number of type i employees with current lifetime payoff strictly less than V, $i = 1, 2$. Thus $G(V) = U + \widehat{N}_1(V) + \widehat{N}_2(V)$ for $V > \underline{V} = V_u$. We distinguish between three regions:

Region 1 is defined for $V \leq \overline{V}_1$;

Region 2 (the intermediate region) where $V \in [\overline{V}_1, V_1^\infty]$.

Region 3 is defined for $V \geq V_1^\infty$;

In Region 1 type 1 firms post offers $V \in [\underline{V}, \overline{V}_1]$ and $\widehat{N}_1(.)$ evolves according to Claim 2(i) below. In this region there are no type 2 employees; i.e. $d\widehat{N}_2 = 0$. In Region 2 type 2 firms post offers $V \in [\overline{V}_1, V_1^\infty]$ and type 1 firms raise wages with tenure to marginally reduce quits to such outside offers. Claim 2(ii) describes the resulting $\widehat{N}_i(.)$ dynamics. In Region 3, type 2 firms post offers with value $V \in [V_1^\infty, \overline{V}]$, $d\widehat{N}_1 = 0$ as all type 1 employees have strictly lower payoffs and Claim 2(iii) describes $\widehat{N}_2(.)$ in this region.

Claim 2

Steady state turnover in a Market Equilibrium implies:

(i) In Region 1 $[\underline{V}, \overline{V}_1)$, $\widehat{N}_2 = 0$ and $\widehat{N}_1$ satisfies

$$\frac{d\widehat{N}_1}{dV} = \left[U\lambda\alpha_1 F_1(V) - [\delta + \lambda(1-\alpha_1 F_1(V))]\widehat{N}_1\right]\frac{d\widehat{t}_1}{dV}$$

subject to $\widehat{N}_1(\underline{V}) = 0$;

(ii) In Region 2 $[\overline{V}_1, V_1^\infty]$,

$$\frac{d\widehat{N}_1}{dV} = \left[\alpha_1\lambda U - \delta\widehat{N}_1 - \alpha_2\lambda\int_{\underline{V}}^{V}[1-F_2(x)]d\widehat{N}_1(x)\right]\frac{d\widehat{t}_1}{dV}, \tag{16}$$

$$\frac{d\widehat{N}_1}{dV}/\frac{d\widehat{t}_1}{dV} + \frac{d\widehat{N}_2}{dV}/\frac{d\widehat{t}_2}{dV} = \lambda U[\alpha_1 + \alpha_2 F_2(V)] - [\delta + \lambda\alpha_2[1-F_2(V)]]\left[\widehat{N}_1 + \widehat{N}_2\right], \tag{17}$$

subject to $\widehat{N}_2(\overline{V}_1) = 0$.

(iii) In Region 3, $(V_1^\infty, \overline{V}]$, $d\widehat{N}_1/dV = 0$ and

$$\frac{d\widehat{N}_2}{dV} = \left[-\lambda\alpha_2(1-F_2(V)) + [\delta + \lambda\alpha_2(1-F_2(V))](\overline{N}_2 - \widehat{N}_2)\right]\frac{d\widehat{t}_2}{dV},$$

where $\overline{N}_2 = \widehat{N}_2(\overline{V})$ and $\overline{N}_2 - \widehat{N}_2 \equiv 1 - G$.

Proof

See Appendix A.

We can now identify equilibrium F. Given the constant profit condition

$$\lambda G(V_0)\widehat{\Pi}_i(V_0) = \overline{\Omega}_i \text{ for all } V_0 \in [\overline{V}_{i-1}, \overline{V}_i]$$

for each i, differentiation with respect to V_0 yields:

$$\frac{dG}{dV}\widehat{\Pi}_i + G\frac{d\widehat{\Pi}_i}{dV} = 0 \text{ for all } V_0 \in [\overline{V}_{i-1}, \overline{V}_i].$$

Now substitute out $d\widehat{\Pi}_i/dV$ using Claim 1, substitute out dG/dV using claim 2 and note $G = \overline{\Omega}_i/[\lambda\widehat{\Pi}_i]$ (by the constant profit condition). Performing suitable algebraic manipulations, Appendix B now determines the equilibrium job offer density, F', in each region and so yields the following equilibrium characterization.

Region 1: Equilibrium in this region implies job offer density:

$$\frac{dF}{dV} = \left[\frac{-u''(\widehat{w}_1)}{u'(\widehat{w}_1)^2}\right] \frac{2\delta\left[p_1 - \widehat{w}_1 - [\delta + \lambda(1-F)]\widehat{\Pi}_1\right]}{\overline{\Omega}_1}. \tag{18}$$

This and Claim 1 yield an autonomous differential equation system for $(\widehat{w}_1, \widehat{\Pi}_1, F)$, where the constant profit condition implies $G = \overline{\Omega}_1/[\lambda\widehat{\Pi}_1]$. This system thus determines equilibrium behaviour within Region 1. Note that the boundary $V = \underline{V}$ occurs where $G = U$ (i.e. $\widehat{N}_1, \widehat{N}_2 = 0$). The boundary with region 2, where $V = \overline{V}_1$, occurs where $F = \alpha_1$ (*i.e.*, where $F_1 = 1$).

Region 2: Let $\phi = d\widehat{N}_1/dV$ denote the density of type 1 employees in this region. Appendix B establishes the equilibrium job offer density is

$$\frac{dF}{dV} = \frac{\frac{2q[u(\widehat{w}_1)-u(\widehat{w}_2)]}{u'(\widehat{w}_1)[p_1-\widehat{w}_1-q\widehat{\Pi}_1]}\phi + \frac{2(p_2-\widehat{w}_2)\overline{\Omega}_2}{\lambda\widehat{\Pi}_2^3 u'(\widehat{w}_2)}}{\left[\frac{[\frac{u'(\widehat{w}_1)^2}{-u''(\widehat{w}_1)}]\lambda\widehat{\Pi}_1[u(\widehat{w}_1)-u(\widehat{w}_2)]}{u'(\widehat{w}_1)[p_1-\widehat{w}_1-q\widehat{\Pi}_1]^2} + \frac{[\frac{u'(\widehat{w}_1)^2}{-u''(\widehat{w}_1)}]\lambda\widehat{\Pi}_1}{[p_1-\widehat{w}_1-q\widehat{\Pi}_1]} - \frac{[\frac{u'(\widehat{w}_2)^2}{-u''(\widehat{w}_2)}]\lambda\widehat{\Pi}_2}{[p_2-\widehat{w}_2-q\widehat{\Pi}_2]}\right]\phi + \frac{[\frac{u'(\widehat{w}_2)}{-u''(\widehat{w}_2)}]\overline{\Omega}_2}{\widehat{\Pi}_2[p_2-\widehat{w}_2-q\widehat{\Pi}_2]}} \tag{19}$$

where $q = \delta + \lambda(1-F)$ is the employee's exit rate from firm employment, and

$$\phi = \frac{\delta}{u(\widehat{w}_1) - u(\widehat{w}_2)}\left[\frac{(p_2 - \widehat{w}_2)\overline{\Omega}_2}{\lambda\delta\widehat{\Pi}_2^2} - 1\right]. \tag{20}$$

Although (19) is monstrous, note that along with Claim 1 it yields an autonomous differential equation system for $(\widehat{w}_1, \widehat{w}_2, \widehat{\Pi}_1, \widehat{\Pi}_2, F)$. Inspection of (19) establishes F' is singular at V_1^∞ which significantly complicates the analysis. We shall show below there are two possible types of equilibrium behavior. In one case the density of type 1 employees, $\phi(V)$, goes to zero as $V \to V_1^{\infty-}$ and F' is then continuous across V_1^∞. In the other case, ϕ does not go to zero in this limit and instead there is a mass $m_2 > 0$ of type 2 firms which post $V = V_1^\infty$.

Note the boundary with Region 3, $V = V_1^\infty$ is identified by (9); i.e. when

$$\delta V_1^\infty = u(w_1^\infty) + \lambda\int_{V_1^\infty}^{\overline{V}}[V - V_1^\infty]dF(V) \tag{21}$$

where $w_1^\infty = \max\{\overline{w}, p_1\}$.

Region 3 $[V_1^\infty, \overline{V}]$: Equilibrium implies offer density

$$\frac{dF}{dV} = \frac{-u''(\widehat{w}_2)}{u'(\widehat{w}_2)^2}\frac{2\delta\left[p_2 - \widehat{w}_2 - [\delta + \lambda(1-F)]\widehat{\Pi}_2\right]}{\overline{\Omega}_2}, \tag{22}$$

which (with Claim 1) determines the equilibrium dynamics $(\widehat{w}_2, \widehat{\Pi}_2, F)$ in Region 3.[2] The boundary $\overline{V}$ is identified where $F = 1$ (*i.e.* $F_2 = 1$).

A Market Equilibrium can be identified using backward induction through each of these regions. The differential equations above describe equilibrium within each region while at each boundary, the profit functions $\widehat{\Pi}_i$ must be continuous and $\widehat{w}_i$ consistent with (13). For a given candidate $\overline{V}$ and corresponding candidate $\overline{w}$, where $u(\overline{w}) = \delta\overline{V}$, Appendix C describes in detail how we to use the above differential equation systems in order to backward iterate through each of the three regions.[3] The iterative process stops when $G = U$, as $V = \underline{V}$ at this point. Thus for each $\overline{V}$ the algorithms described in Appendix C identify a candidate equilibrium distribution function F. But Lemma 1(a) further requires that $\underline{V} = V_u$; i.e. the lowest value contract yields the value of being unemployed. The candidate distribution function F therefore describes a Market Equilibrium if and only if

$$\delta\underline{V} = u(b) + \lambda \int_{\underline{V}}^{\overline{V}} [x - \underline{V}] dF(x). \tag{23}$$

The algorithm described in Appendix C uses a grid search over $\overline{V}$ so that the candidate F described above also satisfies (23).

An important complication, however, is that the differential equation describing F' is singular at V_1^∞ and F may contain a mass point there. Unfortunately this singularity implies we cannot provide a general existence proof. Nevertheless our numerical simulations always found an equilibrium, and sometimes multiple equilibria. We use numerical examples to demonstrate the richness of possible equilibrium behaviours.

4. NUMERICAL EXAMPLES

To illustrate we assume CRRA utility function $u(w) = w^{1-\sigma}/(1-\sigma)$ with $\sigma = 2.2$ (see Lentz (2005) for justification) and set $p_1 = 100$, $p_2 = 105$ and $b = 95$. Thus the surplus $(p_i - b)$ is twice as large in high productivity (type 2) firms than in low productivity (type 1) firms. Using a year as the reference unit of time, we set $\lambda = 1$

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[2] Note this solution for F' is consistent with F' given by (19) above but with $\phi = 0$ and $\widehat{\Pi}_2 = \sqrt{[p_2 - \widehat{w}_2]\frac{\overline{\Omega}_2}{\delta\lambda}}$ in Region 3 (see Appendix B).

[3] there are two algorithms depending on the equilibrium case (see below).

(an expected arrival rate of one outside offer per year) and $\delta = 0.04$ so that workers discount at 4% per annum. The implied equilibrium unemployment rate is 3.8%. Over a working lifetime, an employed worker expects to receive 25 outside offers. As unemployed workers receive offers at the same rate as employed worker, this structure suggests too long unemployment spells (expected duration of one year) and also too short expected working lifetimes ($1/\delta = 25$ years). Allowing different job arrival rates for employed and unemployed workers and introducing Poisson job destruction shocks (as in B/M) could be used to improve these fits but would needlessly complicate the analysis.

The singularity in (19) yields two types of equilibria. We begin with a Case 1 equilibrium.

Case 1 Equilibria: A Mass Point $m_2 > 0$ in F **at** V_1^∞.

Case 1 equilibria are defined as those Market Equilibria where the distribution of job offers, F, contains a mass point at $V = V_1^\infty$. For ease of exposition, we illustrate such an equilibrium using a numerical example; assume $\alpha_1 = \alpha_2 = 0.5$. Figure 2 describes the corresponding equilibrium baseline salary scales where, for greater clarity, we revert to describing wages paid as functions of tenure.

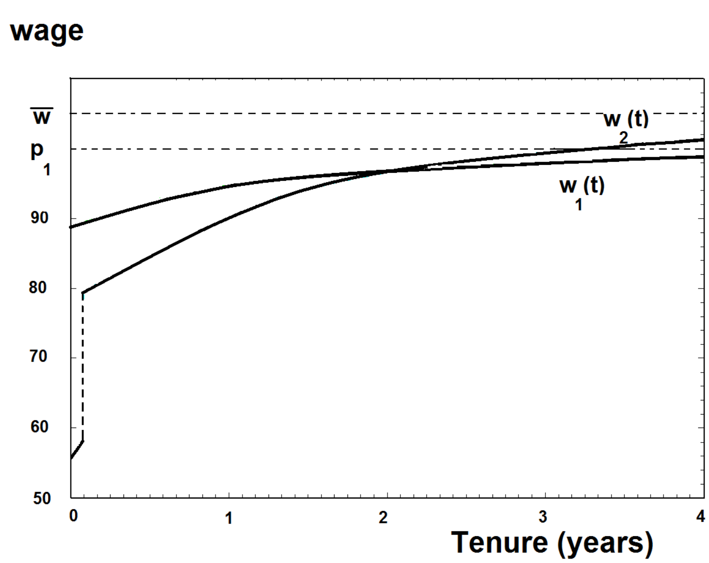

Figure 2: Equilibrium Baseline Salary Scales (case 1)

In this numerical example, 82% of all type 1 firms (i.e., mass $m_1 = 0.41$) offer $t_0 = 0$ on the type 1 baseline salary scale. This wage/tenure contract yields lifetime value $\underline{V} = V_u$ and corresponds to a starting wage $w_1(0) = 88.7$, rising quickly to $w = 96.9$ within 2 years tenure, and gradually asymptotes to marginal product $p_1 = 100$ thereafter. The starting wage $w_1(0)$ is significantly below b and reflects a foot-in-the-door effect - unemployed workers accept a low starting wage as an investment in higher future wages. All type 1 firms offer starting points $t_0 \leq T_1$ on the type 1 baseline salary scale, where $T_1 = 1.7$ years corresponds to starting wage $w_1(T_1) = 95.2$ and a contract which yields value $\overline{V}_1$. Workers with (arbitrarily) long tenures earn marginal product p_1 and enjoy value V_1^{∞}.

Starting point $t_0 = 0$ on the type 2 baseline salary scale yields $\overline{V}_1$, while starting point $t_0 = 1$ month yields value V_1^{∞}. Thus type 2 contracts which offer starting point $t_0 < 1$ month yield expected lifetime values $V_0 \in [\overline{V}_1, V_1^{\infty}]$, and so correspond to Region 2 in the previous section. The type 2 baseline salary scale is not continuous at 1 month. Solving (13) with mass point $m_2 = 0.28$ implies $\widehat{w}_1(V_1^{\infty -}) = 58.5$ and $\widehat{w}_1(V_1^{\infty +}) = 79.1$. Thus, the type 2 baseline salary scale implies an initial low wage spell which lasts for one month. During this early spell, $w_2(0) = 55.5$, rising to $w_2 = 58.5$ by the end of the month. After this "probationary" spell, the wage jumps to 79.1 and smoothly increases with tenure thereafter, asymptoting to $\overline{w} = 104.9$.

This example thus illustrates the following behavior as consistent with equilibrium: (i) Tenure effects are large only at short tenures and possibly include very short "probationary spells" where wages are initially low and increase by a discrete amount at the end of the spell;

(ii) A job quit from a type 1 firm to a type 2 firm might involve a pay-cut. A worker earning his/her marginal product at a type 1 firm ($w = 100$) will quit to any type 2 job offer with starting point $t_0 \geq 1$ month; i.e. a type 2 contract with starting salary no lower than 79.1. They accept such a low starting wage as wages along the type 2 baseline salary scale rise with tenure and attain a much higher level ($\overline{w} = 104.9$). Once employed at a type 2 firm, a worker never quits to a lower wage but will quit to a type 2 firm offering a higher wage. However, quit rates in type 2 firms quickly decrease with tenure as most type 2 contract offers are skewed around the bottom end of the type 2 baseline salary scale (56% of all type 2 firms offer $t_0 = 1$ month).

(iii) The first job of 50% of the workers is in a type 1 firm. It is not unreasonable to think of type 1 jobs as dead-end jobs - wages compensate for the value of being unemployed but offer relatively low wage promotion rates. Tenures at type 1 firms are correspondingly small; the arrival rate of a type 2 job offer is 0.5 (i.e., an expected wait of 2 years) whereupon the employee is likely to quit.

(iv) Turnover is not socially efficient. Type 1 employees with $V > \overline{V}_1$ (i.e. on salary point exceeding 1.7 years) will reject some type 2 job offers. For example a type 1 employee with arbitrarily long tenure (i.e., with $V \simeq V_1^{\infty}$) will reject a type 2 job offer which includes a probationary spell ($t_0 < 1$ month), even though the type 2 job is more productive.

The critical property of a Case 1 equilibrium is that ϕ, the density of type 1 employees, does not converge to zero as $V \to V_1^{\infty-}$. Inspection of (19) establishes that if $\phi \nrightarrow 0$ as $V \to V_1^{\infty-}$ then the limiting job offer density, denoted f^c, is

$$\lim_{V \to V_1^{\infty-}} F' = f^c = \frac{2}{\lambda}\left[\frac{-u''(p_1)}{u'(p_1)^2}\right][\delta + \lambda(1 - F(V_1^{\infty-})]^2.$$

A simple contradiction argument further establishes that if $\phi \nrightarrow 0$ as $V \to V_1^{\infty-}$, then F must contain a mass point at V_1^{∞}.[4] That mass point, m_2, is then tied down by the following boundary condition. Given an arbitrary choice for $m_2 > 0$ at V_1^{∞}, (13) determines $\widehat{w}_2$ at $V_1^{\infty-}$ and (20) then determines $\phi(V_1^{\infty-}) > 0$. The dynamics in Region 2 for $V < V_1^{\infty}$ are then uniquely determined and so determine $\phi(.)$ for all $V \in [\overline{V}_1, V_1^{\infty})$. But the additional boundary condition

$$\phi = \frac{\delta - (\delta + \lambda(1 - F))\overline{\Pi}_2/\widehat{\Pi}_2}{u'(\widehat{w}_1)[p_1 - \widehat{w}_1 - [\delta + \lambda(1 - F)]\widehat{\Pi}_1]} \text{ at } V = \overline{V}_1, \tag{24}$$

[4] In region 3, Appendix B establishes

$$\widehat{\Pi}_2^2 = [p_2 - \widehat{w}_2]\frac{\overline{\Omega}_2}{\delta\lambda}$$

for all $V \in (V_1^{\infty}, \overline{V}]$. If there is no mass point in F at V_1^{∞}, then $\widehat{w}_2$ is continuous across V_1^{∞}. But (20) then implies $\phi = 0$ at $V = V_1^{\infty-}$ which is the required contradiction.

which follows from Claim 2(ii), ties down this degree of freedom in m_2. The algorithm for a Case 1 equilibrium, described in Appendix C, thus embeds an additional grid search for equilibrium m_2 so that $\phi(.)$ satisfies (24).

Case 2 Equilibria: F' continuous at $\mathbf{V}_1^\infty$.

Case 2 equilibria are defined as those Market Equilibria where the density of job offers, F', is continuous across $V = V_1^\infty$. We again describe such equilibria using a numerical example: this time set $\alpha_2 = 10^{-4}$. Figure 3 describes the corresponding equilibrium baseline salary scales.

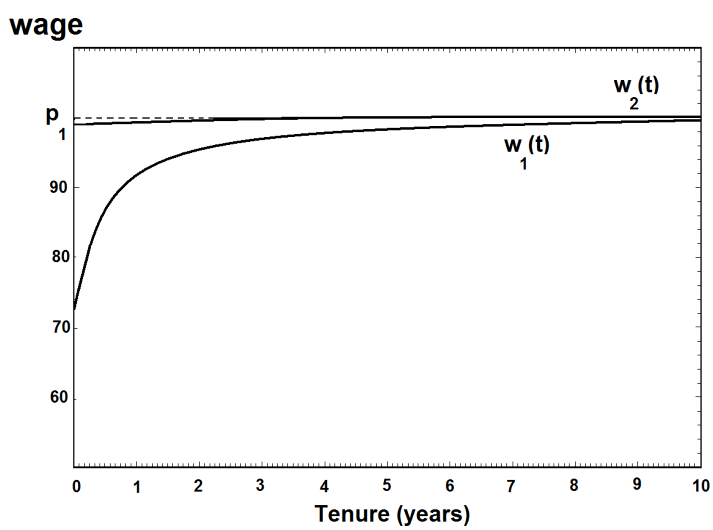

Figure 3: Equilibrium Baseline Salary Scales (case 2)

In this example 63% of type 1 firms (i.e. $m_1 = 0.63$) offer $t_0 = 0$ on the type 1 baseline salary scale. This contract yields value $\underline{V} = V_u$ with starting wage $w_1(0) = 72.4$ rising to wage $w_1 = 96.0$ after 2 years tenure. This wage then asymptotes to marginal product, $p_1 = 100$. The ceiling on type 1 starting offers, T_1, is extremely high, exceeding 20 years tenure in this case.

The type 2 baseline salary scale is continuous, has high starting value $w_2(0) = 99.96$ and limits to $\overline{w} = 100.03$. Thus in contrast to the previous case, this example demonstrates tenure effects may be negligible in high productivity firms. The intuition is most readily seen by first considering the single type case where $\alpha_1 = 1$.

If $\alpha_1 = 1$, a market equilibrium implies $\overline{w} = 99.96$. The type 1 baseline salary scale has the same starting wage $w_1(0) = 72.4$ but a slightly lower limiting wage $w_1^\infty = \overline{w} = 99.96$. Introducing a few type 2 firms, $\alpha_2 = 10^{-4}$, implies type two firms offer wages above 99.96 to attract (almost) all type 1 employees. The added competition across type 2 firms pulls $\overline{w}$, the highest wage paid in the market, above p_1. Type 1 firms respond to the added competition by raising wages with tenure until wages reach marginal product.

A Case 2 equilibrium requires that the type 1 employment density ϕ converges quickly to zero as $V \to V_1^{\infty-}$; specifically $\phi/(V_1^\infty - V) \to 0$ as $V \to V_1^{\infty-}$. (19) then implies the limiting density is the same density as implied by Region 3; i.e.

$$\lim_{V \to V_1^{\infty-}} F' = \frac{-u''(\widehat{w}_2)}{u'(\widehat{w}_2)^2} \frac{2\delta \left[p_2 - \widehat{w}_2 - [\delta + \lambda(1-F)]\widehat{\Pi}_2\right]}{\overline{\Omega}_2} \equiv \left[\frac{dF}{dV}\right]_{\text{Region3}}.$$

The equilibrium density of job offers F' is then continuous across V_1^∞. Iterating backwards from V_1^∞ finds that ϕ grows exponentially (or rather ϕ declines exponentially as $V \to V_1^\infty$). Appendix C describes how to identify such an equilibrium, noting that the solution must satisfy the boundary condition (24) for ϕ at $\overline{V}_1$.[5]

5. A Plethora of Equilibrium Configurations

We complete the discussion by describing Market Equilibria for a range of α_2 and so describe the set of possible Market Equilibrium configurations. We also establish multiple equilibria may occur. For each α_2, Table 1 reports the equilibrium case, the range of wages offered across each baseline salary scale and the implied mass points.

[5] The algebra suggests a third equilibrium might exist where $\phi = 0(V - V_1^\infty)$ and $\lim F'(V) \in [f, f^c]$. But $\phi = 0(V - V_1^\infty)$ and (37) requires $\lim F'(V) = 3f^c/8$. The difficulty is that this limiting density uniquely determines ϕ for V in a neighbourhood of V_1^∞. But that solution, with (19), does not (generically) yield $F'(V) \to 3f^c/8$. Our algorithms have not identified any such equilibria.

Table 1: Possible Equilibrium Configurations

Row		α_2	$\overline{w}$	$[\underline{w}_1, \mathbf{w}_1^\infty]$	$[\underline{w}_2, \mathbf{w}_2^\infty]$	m_1	m_2	*Case*
1		0	**99.959**	[**72.4**,$\overline{w}$]		**0.63**	-	–
2		10^{-5}	**99.983**	[**72.4**,$\overline{w}$]	[**99.6**,$\overline{w}$]	**0.63**	0	**2**
3		10^{-4}	**100.033**	[**72.4**, **100**]	[**98.9**,$\overline{w}$]	**0.63**	0	**2**
4		10^{-4}	**100.012**	[**72.4**, **100**]	[**98.9**,$\overline{w}$]	**0.63**	$\mathbf{2.7 \times 10^{-5}}$	**1**
5		0.1	**103.776**	[**78.5**, **100**]	[**69.5**,$\overline{w}$	**0.64**	**0.081**	**1**
6		0.5	**104.857**	[**88.7**, **100**]	[**55.5**,$\overline{w}$]	**0.41**	**0.28**	**1**
7		0.99	**104.944**	[88.6, 100]	[**65.2**,$\overline{w}$]	0.01	**0.06** (**0.48** at $\underline{V}$)	**1**
8		1	**104.942**		[**65.9**,$\overline{w}$]		(**0.52** at $\underline{V}$)	–

Row 1 ($\alpha_2 = 0$) describes the situation where there are only type 1 firms. Row 2 ($\alpha_2 = 10^{-5}$) describes an equilibrium where Region 3 is degenerate. This requires that α_2 is sufficiently small that $\overline{w}$ remains below p_1 and both baseline salary scales then converge to $\overline{w} < p_1$. Note that Lemma 1(c) implies only a Case 2 equilibrium can exist when $\overline{w} < p_1$. Of course α_2 this small (equal to 10^{-5}) makes little quantitative difference to aggregate outcomes relative to the single type case (row 1).

Row 3 ($\alpha_2 = 10^{-4}$) describes a Case 2 equilibrium when $\overline{w} > p_1$ and was described in detail above. Note that relative to row 2, the higher value of α_2 implies greater competition between type 2 firms which pulls $\overline{w}$ above p_1. This implies region 3 is no longer degenerate. For the same parameter values, a Case 1 equilibrium also exists - see row 4. Thus multiple equilibria are possible. As α_2 is so small, however, the change in contract posting behavior by type 2 firms makes little quantitative difference to the market outcome.

The subsequent rows describe what happens as we increase α_2, the proportion of type 2 firms. Somewhat surprisingly, it is the Case 2 equilibrium (F' continuous) which fails to survive. An increase in the number of type 2 firms increases the competition for type 1 employees in the intermediate Region 2. Type 1 firms respond by raising wages more quickly with tenure and equilibrium implies $\phi \nrightarrow 0$ as $V \to V_1^\infty$. In these latter examples, only Case 1 equilibria (interior mass points) could be found.

Row 6 ($\alpha_2 = 0.5$) was described in detail above, while row 7 ($\alpha_2 = 0.99$) illustrates what happens as α_2 becomes large. In this case Region 1 becomes degenerate (i.e. $\overline{V}_1 = \underline{V}$); all type 1 firms offer the type 1 baseline salary scale. A mass of type 2 firms, 0.48, also offer $V = \underline{V} = \underline{V}_2$; in their case the type 2 baseline salary scale. As $\alpha_2 \to 1$, the mass of type 2 firms offering $\underline{V}$ converges to 0.52 as described in the final row, while the mass point m_2 at V_1^∞ shrinks to zero. As $\phi \to 0$ in the overlap region, (19) implies the equilibrium density of job offers $F' \to f$ consistent with Region 3.

6. CONCLUSION

The model analyzed above leads to a rich pattern of behavior. Workers change jobs from time to time, firms raise wages with tenure, workers with greater experience tend to earn higher wages (via greater job shopping), workers with higher wages are less likely to change job, the greater a worker's tenure at a job the less likely a job change, and some workers who change jobs suffer a decline in wages. Obviously, certain simplification were used in the analysis. Perhaps the most important of these is that job destruction was ruled out and therefore movements from employment to unemployment do not occur. Allowing for job destruction would much complicate the analysis but is a doable task that we leave for the present.

There is a related (but significantly different) literature that attempts to explain why a worker's wage is positively related to both job tenure and general work experience; e.g. Topel (1991), Altonji and Shakotko (1992), Altonji and Williams (1997), Dustman and Meghir (2005), Kamborov and Manovskii (2005). The focus in this literature is on human capital accumulation; as a worker works at any given firm, both general human capital and firm specific human capital may be accumulated. Given a worker is rewarded according to both his/her firm specific human capital and general human capital, wages will increase with both firm tenure and general work experience. Determining by how much a worker's wage growth is due to tenure and how much to experience is the difficult task addressed. Above we have shown that even when there is no accumulation of human capital by workers (either firm specific or general), the desired patterns of behavior can be generated based only on a reasonable specification of market frictions. That is not to say that human capital accumulation is unimportant. Rather identifying empirically between training effects and pure tenure effects in a frictional labour market appears problematic. For example we have shown with no human capital accumulation, that short probationary spells can arise where new employees initially earn low wages, and on completion of that spell, wages jump by a discrete amount. Here such spells reflect pure rent extraction by the hiring firm. Such spells might also arise when a firm offers an initial training spell and the wage increase at the end of the spell reflects an increase in worker skills. An important direction for future research is to embed human capital accumulation into this search framework and so obtain identifying restrictions for empirical work.

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APPENDIX A

Proof of Theorem 1.

Given productivity p_i, an F satisfying $A2$ and any $V_0 \in [\underline{V}, V_i^\infty]$, a firm's optimal contracting problem is

$$\max_{w(.)} \int_0^\infty \psi(t|w(.))[p_i - w(t)]dt$$

subject to (a) $w(.) \geq 0$ and (b) $V(0|w(.)) = V_0$, where ψ is defined by (3). For ease of notation, let $w(.)$ denote the optimal wage contract and $\{V(.), \Pi(.)\}$ denote a worker's and a firm's continuation payoffs given this optimal contract.

The following extends the arguments given in B/C who describe the optimal contract when F is also restricted to being continuously differentiable and productivity $p > \overline{w}$. Throughout this proof, we suppose F only has a connected support (i.e., satisfies A2) and any p_i. Corresponding to that F and p_i is a $\overline{w}$ and $(w_i^\infty, V_i^\infty, \Pi_i^\infty)$ as defined in the Theorem. We first establish some preliminary results (Claims A1,A2).

Claim A1.

For any F satisfying A2, the optimal wage contract of a type i firm offering starting payoff $V_0 = V_i^\infty$ is $w(t) = w_i^\infty$ for all $t \geq 0$.

Proof of Claim A1

Suppose first $p_i \geq \overline{w}$, which implies $w_i^\infty = \overline{w}$ and $V_i^\infty = \overline{V}$. Consider then the wage contract $w(t) = \overline{w}$ for all t. The Bellman equation (2) implies $V(t|w(.)) = \overline{V}$ for all t, and so the worker never quits. As this contract also offers full insurance and $p_i > \overline{w}$ implies not quitting is jointly efficient, this contract is therefore jointly efficient. As this contract also extracts maximal employee rents - it offers expected lifetime payoff $V_0 = V_i^\infty$ as required - it describes the firm's optimal contract.

Consider instead $p_i < \overline{w}$ which implies $w_i^\infty = p_i$ and $V_i^\infty < \overline{V}$. In this case the wage contract $w(t) = p_i$ for all t implies the worker earns (marginal) product p_i and (2) implies $V(t|w(.)) = V_i^\infty$ for all t. Being paid marginal product implies the worker has efficient quit incentives - the worker will quit if and only if the value of an employment offer elsewhere exceeds the value of the current match. As this contract is jointly efficient and extracts maximal employee rents (given $V_0 = V_i^\infty$) it describes the firm's optimal contract. This completes the proof of Claim A1.

The next claim follows directly from B/C and so we omit the proof (see the proof of Claim A1 in B/C but with V_i^{∞} instead of $\overline{V}$).

Claim A2

Given $V_0 \in [\underline{V}, V_i^{\infty})$ optimality implies:

(a) $V_u \leq V(\tau) \leq V_i^{\infty}$ for all $\tau \geq 0$, and so V satisfies the Bellman equation (2) subject to the boundary condition that V is uniformly bounded.

(b) $0 \leq \Pi(\tau) \leq p_i/\delta$ for all $\tau \geq 0$.

Claim A2 establishes that the firm's optimal contract solves the programming problem

$$\max_{w \geq 0} \int_0^{\infty} \psi(t)[p_i - w(t)]dt \tag{25}$$

where

$$\dot{\psi} = -[\delta + \lambda(1 - F(V))]\psi \tag{26}$$

$$\dot{V} = \delta V - u(w) - \lambda \int_V^{\overline{V}} [1 - F(x)]dx, \tag{27}$$

with starting values

$$\psi(0) = 1; V(0) = V_0 \tag{28}$$

and V is uniformly bounded, satisfying

$$V_u \leq V(t) \leq V_i^{\infty} \text{ for all } t. \tag{29}$$

Following B/C, the following maximizes (25) but only subject to (26)-(28). The solution finds (29) is automatically satisfied; i.e. it is never a binding constraint.

Define the Hamiltonian

$$H = \psi[p_i - w] - x_{\psi}[\delta + \lambda(1 - F(V))]\psi + x_V[\delta V - u(w) - \lambda \int_V^{\overline{V}} [x - V]dF(x)] \tag{30}$$

where x_{ψ}, x_V denote the costate variables associated with the state variables ψ, V. The Maximum Principle implies the necessary conditions for a maximum are:

$$\begin{aligned} \text{either } w &= 0 \text{ and } \partial H/\partial w \leq 0 \Leftrightarrow x_V/\psi \geq 0 \\ \text{or } w &> 0 \text{ and } \psi + x_V u'(w) = 0 \Leftrightarrow x_V/\psi < 0 \text{ and } 1/u'(w) = -x_V/\psi. \end{aligned} \tag{31}$$

and

$$\dot{x}_\psi = -[p - w] + x_\psi[\delta + \lambda(1 - F)]. \tag{32}$$

Note, the standard approach also states a differential equation for x_V; i.e., $\dot{x}_V = -\partial H/\partial V$. We do not use this latter equation as H is not differentiable in V (as F is not differentiable). Instead no discounting implies the additional constraint

$$H = 0 \tag{33}$$

(**e.g.** p298, Leonard and Long (1992)) and this condition is sufficient to establish the Theorem.

Assumption $A1$ implies $w > 0$ along the optimal path, and so (31) implies $x_V = -\psi/u'(w)$.Thus, $H = 0$ requires

$$0 = [p_i - w] - x_\psi[\delta + \lambda(1 - F(V))] - \frac{[\delta V - u(w) - \lambda \int_V^{\overline{V}} [x - V] dF(x)]}{u'(w)}$$

As (32) is a linear differential equation, integration yields:

$$x_\psi(t) = \int_t^\infty e^{-\int_t^s [\delta + \lambda(1 - F(V(x))] dx} (p - w(s)) ds + A_0 e^{\int_0^t [\delta + \lambda(1 - F(V(x)))] dx}$$

where A_0 is the constant of integration. Note the first term is the firm's continuation profit $\Pi(t)$. Thus

$$x_\psi(t) = \Pi(t) + A_0 e^{\int_0^t [\delta + \lambda(1 - F(V(x)))] dx}$$

and the $H = 0$ condition implies

$$\begin{aligned} 0 \quad = \quad & [p_i - w] - \left[\Pi(t) + A_0 e^{\int_0^t [\delta + \lambda(1 - F(V(x)))] dx}\right] [\delta + \lambda(1 - F(V))] \\ & - \frac{[\delta V - u(w) - \lambda \int_V^{\overline{V}} [x - V] dF(x)]}{u'(w)} \end{aligned} \tag{34}$$

A contradiction argument now establishes $A_0 = 0$. As Π and V are uniformly bounded (Claim A2), $A_0 \neq 0$ implies the second term in (34) grows exponentially as $t \to \infty$. Thus, (34) requires $w \to 0$ in this limit. However, A1 implies $u(w)/u'(w) \to 0$ in this limit and so (34) must fail at large enough t, which is the required contradiction. Thus, $A_0 = 0$.

We have $x_\psi(t) = \Pi(t)$ and (32) now implies (6). Further, $A_0 = 0$ in the equation above and some rearranging yields (7). Using (7) to substitute out $u(w)$ in (27) and rearranging using (6) then yields (5).

All that is left is establishing the boundary conditions. Claim A2 implies Π is positive and (7) then implies w is increasing in tenure (see (11) in the text and the discussion around it)). Hence, V is increasing in tenure. As V is uniformly bounded, monotonicity implies V must converge to some limit value. Connected F implies $(w_i^\infty, V_i^\infty, \Pi_i^\infty)$ is this unique stationary point of this system and so must describe the optimal limiting contract.

Finally, note that V increasing implies $V \in [V_0, V_i^\infty]$ for all t, and so this solution also satisfies (29). This completes the proof of Theorem 1.

Proof of Lemma 1(c). $1 - G(V)$ describes the steady state number of workers employed whose current lifetime utility is at least V. The proof is by contradiction - suppose in equilibrium that $1 - G$ has a mass $\mu > 0$ at $\overline{V}$. Strictly positive profit requires that any firm which in equilibrium offers starting payoff $V_0 = \overline{V}$ must have productivity $p_i > \overline{w}$. Further, Claim A1 implies that any such firm enjoys steady state profit flow $\Omega_i^*(\overline{V}) = \lambda(1-\mu)[(p_i - \overline{w})/\delta]$. Consider instead a firm of that type but which instead offers $w = \overline{w} + \varepsilon$ for some $\varepsilon > 0$ and so obtains flow profit $\Omega = \lambda[(p_i - \overline{w} - \varepsilon)/\delta]$. As $\overline{w} < p_i$ then $\mu > 0$ implies $\Omega > \Omega_i^*(\overline{V})$ for ε small enough which contradicts the definition of a market equilibrium. As $1 - G$ cannot have a mass point at $\overline{V}$, then Claim A1 implies F cannot have a mass point at $\overline{V}$. This completes the proof of Lemma 1c.

Proof of Lemma 2.

In this proof only, we extend the notation in the text by rewriting $\widehat{w}_i(V)$ as $\widehat{w}(V, p_i)$. Thus $\widehat{w}(V, p)$ describes the wage paid in an optimal contract given value V and firm productivity p. Similarly we rewrite $\widehat{\Pi}_i(V)$ as $\widehat{\Pi}(V, p_i)$.

Thus, fix an i, a $V \in [V_u, V_i^\infty)$ and consider any firm with productivity $p \geq p_i$. Lemma 2(a) requires showing that $\partial \widehat{w}(V, p)/\partial p$ is strictly decreasing in p. Given the extended notation, (13) in Claim 1 implies

$$u(\widehat{w}(V,p))+u'(\widehat{w}(V,p))[p-\widehat{w}(V,p)-[\delta+\lambda(1-F)]\widehat{\Pi}(V,p)] = \delta V - \lambda\int_V^{\overline{V}}[1-F(x)]dx.$$

For V fixed, differentiating partially with respect to p yields

$$\frac{\partial\widehat{w}(V,p)}{\partial p} = \frac{u'(\widehat{w})}{-u''(\widehat{w})}\frac{1-[\delta+\lambda(1-F)]\frac{\partial\widehat{\Pi}(V,p)}{\partial p}}{p-\widehat{w}-[\delta+\lambda(1-F)]\widehat{\Pi}}.$$

To characterize $\partial\widehat{\Pi}(V,p)/\partial p$, note that at the contract optimum

$$\widehat{\Pi}(V,p) \equiv \Pi(\widehat{t},p) = \int_{\tau=\widehat{t}}^{\infty}\frac{\Psi(\tau|w^*)}{\Psi(\widehat{t}|w^*)}[p-w^*(\tau,p)]d\tau,$$

where $w^*(.)$ denotes the optimal wage tenure contract of a type p firm and $\Psi(\tau|w^*)/\Psi(\widehat{t}|w^*)$ is the probability the worker remains an employee at tenure τ given current tenure $\widehat{t}(V,p)$. Optimality of that contract and the Envelope Theorem now yield

$$\frac{\partial\widehat{\Pi}(V,p)}{\partial p} = \int_{\tau=\widehat{t}}^{\infty}\frac{\Psi(\tau|w^*)}{\Psi(\widehat{t}|w^*)}d\tau;$$

i.e., a marginal increase in p increases marginal profit at each future tenure should the worker survive. Assumption A2 (connected F), however, implies worker quit rates strictly decrease with tenure for $V < V_i^{\infty} \leq \overline{V}$. Thus at tenures $\tau > \widehat{t}$

$$\Psi(\tau|w^*) > e^{-[\delta+\lambda(1-F(V)][\tau-\widehat{t}]}\Psi(\widehat{t}|w^*)$$

and so

$$\frac{\partial\widehat{\Pi}(V,p)}{\partial p} > \int_{\tau=\widehat{t}}^{\infty}e^{-[\delta+\lambda(1-F(V)][\tau-\widehat{t}]}d\tau = \frac{1}{\delta+\lambda(1-F(V))}.$$

This condition and the above establishes $\partial\widehat{w}(V,p)/\partial p < 0$ as required.

As Claim 1 implies $\partial\widehat{\Pi}/\partial V = -1/u'(\widehat{w})$, Lemma 2(b) follows from part (a) and that $u(.)$ is strictly concave. This completes the proof of Lemma 2.

Proof of Claim 2.

Region 1: For $V < \overline{V}_1$, steady state turnover on baseline salary scale 1, over arbitrarily short period of time $\Delta > 0$, implies

$$\begin{aligned} U\lambda\Delta\alpha_1 F_1(V) &= [\widehat{N}_1(V) - \widehat{N}_1(V - [dV_1(\widehat{t}_1)/dt]\Delta)] \\ &\quad + \widehat{N}_1(V)[\delta + \lambda[1-\alpha_1 F_1(V)]]\Delta + o(\Delta). \end{aligned}$$

where the left hand side describes the inflow of type 1 workers with payoff less than V,and the right hand side describes outflow - the first term describes those whose tenure rises above $\widehat{t}_1(V)$, the second is the outflow due to exiting the labor market, or receiving a better outside offer. Taking the limit $\Delta \to 0$ implies the condition stated.

Region 2: For $V \in [\overline{V}_1, V_1^{\infty}]$, steady state turnover on baseline salary scale 1, over arbitrarily short period of time $\Delta > 0$, implies

$$\begin{aligned} U\lambda\alpha_1\Delta &= [\widehat{N}_1(V) - \widehat{N}_1(V - [dV_1(\widehat{t}_1)/dt]\Delta)] \\ &\quad + \widehat{N}_1(V)\delta\Delta + \int_{\underline{V}}^{V} d\widehat{N}_1(x)\lambda\Delta[\alpha_2(1 - F_2(x))] + o(\Delta). \end{aligned}$$

where the left hand side describes the inflow of workers onto baseline salary scale one with payoff less than V, and the right hand side describes the outflow due to increasing tenure, exiting the labor market or receiving a better outside offer from a type 2 firm. Rearranging and letting $\Delta \to 0$ implies the first condition stated.

Steady state turnover for all employed workers with payoff no greater than $V \in [\overline{V}_1, V_1^{\infty}]$ implies

$$\begin{aligned} U\lambda[\alpha_1 + \alpha_2 F_2(V)]\Delta &= [\widehat{N}_1(V) - \widehat{N}_1(V - [dV_1(\widehat{t}_1)/dt]\Delta)] \\ &\quad + [\widehat{N}_2(V) - \widehat{N}_2(V - [dV_2(\widehat{t}_2)/dt]\Delta)] \\ &\quad + [\widehat{N}_1(V) + \widehat{N}_2(V)][\delta + \lambda\alpha_2(1 - F_2(V))]\Delta \end{aligned}$$

where the left hand side describes the inflow, the right hand side describes the outflow due to tenure effects (both types), exiting the labor market and receiving an outside offer greater than V. Rearranging and letting $\Delta \to 0$ implies the second condition.

Region 3: For $V \in [V_1^{\infty}, \overline{V}]$, steady state turnover on baseline salary scale 2 implies

$$\begin{aligned} [\overline{N}_2 - \widehat{N}_2]\delta\Delta &= [\widehat{N}_2(V) - \widehat{N}_2(V - [dV_2(\widehat{t}_2)/dt]\Delta)] \\ &\quad + [U + \overline{N}_1 + \widehat{N}_2(V)]\lambda\Delta\alpha_2[1 - F_2(V)] + o(\Delta) \end{aligned}$$

where the left hand side describes the outflow of type 2 workers with payoff greater than V, and the right hand side describes the inflow - those whose tenure rises above $\widehat{t}_2(V)$,and those who receive an outside offer greater than V. Rearranging, noting that

$U + \overline{N}_1 + \overline{N}_2 = 1$, and letting $\Delta \to 0$ implies the condition stated. This completes the proof of Claim 2.

APPENDIX B - Identifying Equilibrium F'.

It is convenient to consider the regions in reverse order.

Region 3. $V \in [V_1^\infty, \overline{V}]$

As $dG/dV = d\widehat{N}_2/dV$ in this region, equilibrium requires solving

$$G\frac{d\widehat{\Pi}_2}{dV} + \frac{d\widehat{N}_2}{dV}\widehat{\Pi}_2 = 0 \text{ for all } V \in [V_1^\infty, \overline{V}],$$

with $\lambda G\widehat{\Pi}_2 = \overline{\Omega}_2$. Now Claim 1 implies $d\widehat{\Pi}_2/dV = -1/u'(\widehat{w}_2)$ and rearranging the above implies

$$\frac{d\widehat{N}_2}{dV} = \frac{\overline{\Omega}_2}{\lambda u'(\widehat{w}_2)\widehat{\Pi}_2^2}.$$

Noting $\overline{N}_2 - \widehat{N}_2 \equiv 1 - G$, Claim 2(iii) implies

$$\frac{d\widehat{N}_2}{dV} = \frac{\left[\delta - \frac{\overline{\Omega}_2}{\lambda\widehat{\Pi}_2}[\delta + \lambda(1-F)]\right]}{u'(\widehat{w}_2)[p_2 - \widehat{w}_2 - [\delta + \lambda(1-F)]\widehat{\Pi}_2]},$$

over this region. Equating these two expressions and simplifying yields

$$\widehat{\Pi}_2^2 = [p_2 - \widehat{w}_2]\frac{\overline{\Omega}_2}{\delta\lambda}.$$

As $\widehat{\Pi}_2$ is differentiable, differentiating wrt V implies

$$2\widehat{\Pi}_2\frac{d\widehat{\Pi}_2}{dV} = -\frac{d\widehat{w}_2}{dV}\frac{\overline{\Omega}_2}{\delta\lambda}.$$

Using claim 1 and (14) to substitute out $d\widehat{\Pi}_2/dV$ and $d\widehat{w}_2/dV$ then implies F' given by (22).

Region 2. $V \in [\overline{V}_1, V_1^\infty]$.

Equilibrium in this region requires

$$G\frac{d\widehat{\Pi}_2}{dV} + [\frac{d\widehat{N}_1}{dV} + \frac{d\widehat{N}_2}{dV}]\widehat{\Pi}_2 = 0 \text{ for all } V \in [\overline{V}_1, V_1^\infty],$$

and $\lambda G\widehat{\Pi}_2 = \overline{\Omega}_2$. As Claim 1 implies $d\widehat{\Pi}_2/dV = -1/u'(\widehat{w}_2)$, and letting $\phi = d\widehat{N}_1/dV$, the previous equation implies

$$\phi = \frac{\overline{\Omega}_2}{\lambda u'(\widehat{w}_2)\widehat{\Pi}_2^2} - \frac{d\widehat{N}_2}{dV}.$$

Noting that $\widehat{N}_1 + \widehat{N}_2 \equiv G - U$, $U = \delta/(\lambda+\delta)$ and $G = \overline{\Omega}_2/(\lambda\widehat{\Pi}_2)$, Claim 2(ii) implies

$$\frac{d\widehat{N}_2}{dV} = \frac{\delta - [\delta+\lambda(1-F)]\frac{\overline{\Omega}_2}{\lambda\widehat{\Pi}_2} - u'(\widehat{w}_1)[p_1 - \widehat{w}_1 - [\delta+\lambda(1-F)]\widehat{\Pi}_1]\phi}{u'(\widehat{w}_2)[p_2 - \widehat{w}_2 - [\delta+\lambda(1-F)]\widehat{\Pi}_2]}. \tag{35}$$

Now use this condition to substitute out $d\widehat{N}_2/dV$ in the previous equation. Simplifying further by using (7), which implies

$$u(\widehat{w}_1)+u'(\widehat{w}_1)[p_1-\widehat{w}_1-[\delta+\lambda(1-F)]\widehat{\Pi}_1] = u(\widehat{w}_2)+u'(\widehat{w}_2)[p_2-\widehat{w}_2-[\delta+\lambda(1-F)]\widehat{\Pi}_2],$$

then implies ϕ must satisfy:

$$\phi = \frac{\delta}{u(\widehat{w}_1) - u(\widehat{w}_2)}\left[\frac{(p_2-\widehat{w}_2)\overline{\Omega}_2}{\lambda\delta\widehat{\Pi}_2^2} - 1\right]$$

which establishes (20) in the text.

Equation (16) in Claim 2(ii) implies ϕ satisfies

$$\phi[u'(\widehat{w}_1)[p_1-\widehat{w}_1-[\delta+\lambda(1-F(V))]\widehat{\Pi}_1]] = \left[\alpha_1\lambda U - \delta\widehat{N}_1 - \alpha_2\lambda\int_{\underline{V}}^{V}[1-F_2(x)]d\widehat{N}_1(x)\right],$$

Differentiating wrt V and noting that (7) implies

$$\frac{d}{dV}\left[u'(\widehat{w}_i)[p_i - \widehat{w}_i - [\delta+\lambda(1-F(V))]\widehat{\Pi}_i\right] = \delta + \lambda(1-F) - u'(\widehat{w}_i)\frac{d\widehat{w}_i}{dV} \tag{36}$$

then yields

$$\frac{1}{\phi}\frac{d\phi}{dV} = \frac{u'(\widehat{w}_1)\frac{d\widehat{w}_1}{dV} - 2[\delta+\lambda(1-F)]}{u'(\widehat{w}_1)[p_1 - \widehat{w}_1 - [\delta+\lambda(1-F)]\widehat{\Pi}_1]}. \tag{37}$$

Finally, differentiate (20) with respect to V to obtain $d\phi/dV$. Then set this equal to $d\phi/dV$ (given by (37)). Using (7), Claim 1 and (14) to substitute out the resulting $d\widehat{w}_1/dV$, $d\widehat{w}_2/dV$, and $d\widehat{\Pi}_2/dV$ terms yields (19).

Region 1 $V \in [\underline{V}, \overline{V}_1]$**.**

The arguments used to establish F' in Region 3 also apply to Region 1. In particular, note that the solutions for F' have identical structures.

APPENDIX C - The Algorithms.

There are two algorithms, one for each type of equilibrium. We describe the Case 1 algorithm first.

Case 1 Algorithm: A Mass Point m_2 at V_1^∞.

Fix a $\overline{w} \in (p_1, p_2)$ and set $\overline{V} = u(\overline{w})/\delta$, $\overline{\Omega}_2 = \lambda(p_2 - \overline{w})/\delta$.[6] In Region 3, set initial values $(\widehat{w}_2, \widehat{\Pi}_2, F) = (\overline{w}, (p_2 - \overline{w})/\delta, 1)$ at $V = \overline{V}$,and backward iterate for $(\widehat{w}_2, \widehat{\Pi}_2, F)$ using the differential equations (14),(6), and (22). The iteration stops when:

$$\delta V = u(p_1) + \lambda \int_V^{\overline{V}} [1 - F(x)]dx$$

which identifies V_1^∞. We then move to Region 2. Let $(\widehat{w}_2^+, \widehat{\Pi}_2^+, F^+)$ denote the values of $(\widehat{w}_2, \widehat{\Pi}_2, F)$ thus obtained at V_1^∞.

Fix an $m_2 \in (0, F^+)$ and let $(\widehat{w}_2^-, \widehat{\Pi}_2^-, F^-)$ denote the values of $(\widehat{w}_2, \widehat{\Pi}_2, F)$ at $V_1^{\infty -}$.The mass point implies $F^- = F^+ - m_2$, $\widehat{\Pi}_2^- = \widehat{\Pi}_2^+$ as $\widehat{\Pi}_2$ is continuous and (13) implies $\widehat{w}_2^-$ solves:

$$u(\widehat{w}_2^-)+u'(\widehat{w}_2^-)[p_2-\widehat{w}_2^- -[\delta+\lambda(1-F^-)]\widehat{\Pi}_2^-] = u(\widehat{w}_2^+)+u'(\widehat{w}_2^+)[p_2-\widehat{w}_2^+ -[\delta+\lambda(1-F^+)]\widehat{\Pi}_2^+] \tag{38}$$

The implied value of $\phi > 0$ is given by (20) and must satisfy $\phi < G$. If $\phi > G$ then pick a smaller m_2.

Thus, for given choices $(\overline{w}, m_2)$ we have initial values $(\widehat{w}_2^-, \widehat{\Pi}_2^-, F^-)$ at V_1^∞ and also initial values $(\widehat{w}_1, \widehat{\Pi}_1) = (p_1, 0)$. The text defines the differential equation system $(\widehat{w}_1, \widehat{w}_2, \widehat{\Pi}_1, \widehat{\Pi}_2, F)$ over Region 2. The problem is that (19) for F and (14) for $\widehat{w}_1$ are both singular at V_1^∞. Examination of (19) with ϕ bounded away from zero implies

$$\lim_{V \to V_1^{\infty -}} F' = f^c = \frac{2}{\lambda} \left[\frac{-u''(p_1)}{u'(p_1)^2} \right] [\delta + \lambda(1 - F^-)]^2.$$

[6] there cannot be a case 1 equilibrium for $\overline{w} \leq p_1$.

Using l'Hopital's rule on (14) implies

$$\lim_{V\to V_1^{\infty-}} \frac{d\widehat{w}_1}{dV} = \lim_{V\to V_1^{\infty-}} \left[\frac{u'(\widehat{w}_1)}{-u''(\widehat{w}_1)}\right] \frac{\lambda F'(V)\frac{d\widehat{\Pi}_1}{dV}}{\left[-\frac{d\widehat{w}_1}{dV} - [\delta+\lambda(1-F)]\frac{d\widehat{\Pi}_1}{dV}\right]} \tag{39}$$

where Claim 1 implies $\frac{d\widehat{\Pi}_1}{dV} = -1/u'(\widehat{w}_1)$.This yields a quadratic equation for $\lim_{V\to V_1^{\infty-}} d\widehat{w}_1/dV$. As one root implies $\lim_{V\to V_1^{\infty-}} d\widehat{w}/dV < 0$, which is inconsistent with optimality. There is only one relevant solution

$$\lim_{V\to V_1^{\infty-}} \frac{d\widehat{w}_1}{dV} = \frac{[\delta+\lambda(1-F^-)]}{2u'(p_1)} + \sqrt{\left[\frac{[\delta+\lambda(1-F^-)]}{2u'(p_1)}\right]^2 + \left[\frac{\lambda f^c}{-u''(p_1)}\right]}.$$

Thus, in Region 2, we can backward iterate from $V = V_1^{\infty}$ with start values $(\widehat{w}_1, \widehat{w}_2, \widehat{\Pi}_1, \widehat{\Pi}_2, F) = (p_1, \widehat{w}_2^-, 0, \widehat{\Pi}_2^-, F^-)$ using the differential equations (14), (6),and (19) in the text, but for the first iteration use the above limiting solutions for $dF/dV, d\widehat{w}_1/dV$. The resulting system is numerically stable.

Associated with the iterated path in Region 2 is a path for $G = \overline{\Omega}_2/(\lambda\widehat{\Pi}_2)$ and for ϕ given by (20). The iteration stops as soon as either $\phi = G$ or $G = U$ and let $\overline{V}_1$ denote the corresponding value of V. If the iteration stops as $\phi = G$, then set α_1^0 equal to the current value of F. If instead the iteration stops because $G = U$, then set

$$\alpha_1^0 = \frac{\lambda+\delta}{\lambda\delta}\phi u'(\widehat{w}_1)[p_1 - \widehat{w}_1 - [\delta+\lambda(1-F)]\widehat{\Pi}_1].$$

We now move to Region 1 with start values $(\widehat{w}_1, \widehat{\Pi}_1, F)$ obtained from the end of Region 2 and set $\overline{\Omega}_1$ equal to the current value of $\lambda G\widehat{\Pi}_1$. If Region 2 ended as $\phi = G$ (and so $G > U$), we backward iterate $(\widehat{w}_1, \widehat{\Pi}_1, F)$ using (14), (6),and (18) until $G = U$. When this stopping point occurs, let $\underline{V}$ denote the corresponding value of V. If Region 2 ended because $G = U$ then Region 1 is degenerate and $\underline{V} = \overline{V}_1$.

Given the $F(.)$ obtained by the above iterative process, let b^0 denote the solution to

$$\delta\underline{V} = u(b^0) + \lambda\int_{\underline{V}}^{\overline{V}} [1 - F(x)]dx. \tag{40}$$

By construction, the above identifies a Market Equilibrium if $b = b^0$ and $\alpha_1 = \alpha_1^0$ (see footnote ??). For given parameters α_1 and b, the algorithm uses a grid search over $\overline{w}$ and m_2 so that $\alpha_1^0 = \alpha_1$ and $b^0 = b$. A computer program using Gauss is available from the authors.

Case 2 Algorithm: F' continuous across V_1^∞.

Fix a $\overline{w} \in (b, p_2)$ and set $\overline{V} = u(\overline{w})/\delta$, $\overline{\Omega}_2 = \lambda(p_2 - \overline{w})/\delta$. If $\overline{w} > p_1$, Region 3 is non-degenerate. In that case, set initial values $(\widehat{w}_2, \widehat{\Pi}_2, F) = (\overline{w}, (p_2 - \overline{w})/\delta, 1)$ and backward iterate the differential equations (14),(6),and (22) until

$$\delta V = u(p_1) + \lambda \int_V^{\overline{V}} [1 - F(x)] dx$$

which identifies V_1^∞. The current values $(\widehat{w}_2, \widehat{\Pi}_2, F)$ then provide the start values for $(\widehat{w}_2, \widehat{\Pi}_2, F)$ in Region 2. If instead $\overline{w} \leq p_1$,then Region 3 is degenerate (as $V_1^\infty = \overline{V}$) and Region 2 starts with values $(\widehat{w}_2, \widehat{\Pi}_2, F) = (\overline{w}, (p_2 - \overline{w})/\delta, 1)$.

In Region 2, at V_1^∞, $(\widehat{w}_2, \widehat{\Pi}_2, F)$ has the above start values and set $(\widehat{w}_1, \widehat{\Pi}_1) = (w_1^\infty, \Pi_1^\infty)$ as defined in Theorem 1. Again the problem is (19) for F and (14) for $\widehat{w}_1$ are both singular at V_1^∞. Unfortunately, the approach above, using the limiting analytic solutions for F' for the first iteration, does not yield a stable numerical system. Instead we integrate up analytically and so remove those singularities.

Appendix B shows, in Region 2, that the constant profit condition implies equation (20), i.e.,

$$\phi = \frac{\delta}{u(\widehat{w}_1) - u(\widehat{w}_2)} \left[\frac{(p_2 - \widehat{w}_2)\overline{\Omega}_2}{\lambda \delta \widehat{\Pi}_2^2} - 1 \right]$$

and ϕ evolves according to $d\phi/dV$ given by (37). Equilibrium F' was obtained by differentiating the equation above to get $d\phi/dV$ and setting it equal to $d\phi/dV$ in (37) and then solving for F'.

A a different approach is to integrate up $d\phi/dV$ in (37) and set that solution equal to ϕ above. Integrating up (37) yields [7]

$$\phi u'(\widehat{w}_1)[p_1 - \widehat{w}_1 - [\delta + \lambda(1-F)]\widehat{\Pi}_1] = \int_V^{V_1^\infty} [\delta + \lambda(1 - F(x))]\phi(x)dx. \qquad (41)$$

which, given ϕ in the previous equation, yields an integral equation determining F, rather than a differential equation describing F'.

Thus over Region 2, the Case 2 algorithm uses the differential equations

$$\frac{d\widehat{\Pi}_i}{dV} = -\frac{1}{u'(\widehat{w}_i)}$$

to determine the $\widehat{\Pi}_i$ (with start values described above), and level equations

$$u(\widehat{w}_i) + u'(\widehat{w}_i)[p_i - \widehat{w}_i - [\delta + \lambda(1-F)]\widehat{\Pi}_i] = \delta V - \lambda \int_V^{\overline{V}} [1 - F(x)]dx.$$

$$\phi = \frac{\delta}{u(\widehat{w}_1) - u(\widehat{w}_2)} \left[\frac{(p_2 - \widehat{w}_2)\overline{\Omega}_2}{\lambda\delta\widehat{\Pi}_2^2} - 1 \right]$$

$$u'(\widehat{w}_1)[p_1 - \widehat{w}_1 - [\delta + \lambda(1-F)]\widehat{\Pi}_1] = \int_V^{V_1^\infty} [\delta + \lambda(1 - F(x))]\frac{\phi(x)}{\phi(V)}dx,$$

which determine the $\widehat{w}_i, \phi$,and F respectively. Note, there are no singularities. Given an (extremely) small starting value for $\phi(V_1^\infty - h) = \overline{\phi} > 0$, where h is the step size, this system can be iterated backwards from V_1^∞ for $(\widehat{w}_1, \widehat{w}_2, \widehat{\Pi}_1, \widehat{\Pi}_2, F, \phi)$. The iteration stops as soon as either $\phi = G$, or $G = U$, and let $\overline{V}_1$ denote the corresponding value of V. If the iteration stops as $\phi = G$, then set α_1^0 equal to the current value of F. If instead the iteration stops as $G = U$, then set

$$\alpha_1^0 = \frac{\lambda + \delta}{\lambda\delta}\phi u'(\widehat{w}_1)[p_1 - \widehat{w}_1 - [\delta + \lambda(1-F)]\widehat{\Pi}_1].$$

[7] A simpler derivation notes that steady state turnover for type 1 employees implies

$$\alpha_1 \lambda U = \delta \overline{N}_1 + \lambda \int_{\underline{V}}^{V_1^\infty} \alpha_2[1 - F_2(x)]\phi(x)dx.$$

Using this in (16), noting also that $\alpha_2[1 - F_2(x)] = 1 - F$ for $V > \overline{V}_1$, yields (41) directly. Note this derivation does not require F differentiable.

We then move to Region 1 with start values $(\widehat{w}_1, \widehat{\Pi}_1, F)$ obtained from the end of Region 2. Given the $F(.)$ obtained, again let b^0 denote the solution to

$$\delta \underline{V} = u(b^0) + \lambda \int_{\underline{V}}^{\overline{V}} [1 - F(x)]dx.$$

By construction the above identifies a Market Equilibrium if $b = b^0$ and $\alpha_1 = \alpha_1^0$. For given parameters α_1, b, the algorithm uses a grid search over $\overline{w}$ and $\overline{\phi}$ so that $\alpha_1^0 = \alpha_1$ and $b^0 = b$. A computer program (using Gauss) is available from the authors.