ISSN 1755-5361

University of Essex

Department of Economics

Discussion Paper Series

No. 638 September 2007

Differentiated Networks: Equilibrium and Efficiency

Rossella Argenziano

Note : The Discussion Papers in this series are prepared by members of the Department of Economics, University of Essex, for private circulation to interested readers. They often represent preliminary reports on work in progress and should therefore be neither quoted nor referred to in published work without the written consent of the author.

Differentiated Networks: Equilibrium and Efficiency*

Rossella Argenziano†
University of Essex

September 7, 2007

Abstract

We consider a model of price competition in a duopoly with product differentiation and network effects. The value of a good for a consumer is the sum of a common and an idiosyncratic component. The first captures the vertical dimension of quality, the second captures horizontal differentiation. Each consumer privately observes his own value for each good, but cannot separate the common and the idiosyncratic component. Therefore, he has incomplete information about the value of the goods for the other consumers. After firms announce prices, consumers choose simultaneously which network to join, facing a coordination problem.

In the efficient allocation, both networks are active and the firm with the highest expected quality has the largest market share. To characterize the equilibrium allocation, we derive necessary and sufficient conditions for uniqueness of the equilibrium of the coordination game played by consumers for given prices. The equilibrium allocation differs from the efficient one for two reasons. First, the equilibrium allocation of consumers to the networks is too balanced, since consumers fail to internalize network externalities. Second, if access to the networks is priced by strategic firms, then the product with the highest expected quality is also the most expensive. This further reduces the asymmetry between market shares and therefore social welfare.

Keywords: network externalities, product differentiation, price competition, coordination games, global games.
Jel Classification: D43, D62, L14, L15.

*This paper is a revised version of chapter 1 of my PhD dissertation at Yale University. I am indebted to my advisors, Dirk Bergemann and Stephen Morris, and to Dino Gerardi and Ben Polak for their guidance. I would like to thank Simon Anderson, Luigi Balletta, Alessandro Bonatti, Don Brown, Giancarlo Corsetti, Chaim Fershtman, Neil Gandal, Bernardo Guimaraes, Barry Nalebuff, Dmitry Shapiro, Hyun Song Shin, Helen Weeds and seminar participants at Yale University, Harvard Business School, Bocconi University, Insead, the Sloan School of Management, the University of British Columbia, the University of Rochester, the University of Mannheim, the University of Essex, the W.P. Carey School of Business, the University of Western Ontario, the WISE workshop in Salerno, the conference "The Economics of Two-Sided markets" in Toulouse, the Econometric Society World Congress of 2005 and the 2006 conference of the EARIE for helpful comments and conversations and three anonymous referees for suggestions that greatly helped me to improve the paper. Financial support for this project was provided by Yale University in the form of a John F. Enders Award and a Robert M. Leylan Dissertation Fellowship.

†Department of Economics, University of Essex, Colchester, CO4 3SQ, U.K., rargenz@essex.ac.uk

1 Introduction

Many economic decisions, such as the purchase of a good by a population of consumers, or the adoption of a standard of production by a set of firms, exhibit a specific form of strategic complementarity known as network externality. The set of consumers buying a specific product, and the set of firms producing according to a given standard, constitute virtual networks: the externality arises when the payoff of each player who belongs to a network is increasing in the total size of the network itself.

This paper brings two contributions to the literature on network competition.[1] The first contribution is methodological. There is a problem with modelling the demand function for network goods, since the game played by consumers choosing which network to join, after prices have been announced, constitutes a coordination game, and typically these games have multiple equilibria. Therefore, in a basic model of Bertrand competition between identical networks, the demand for each good is not a well-defined function of prices. As a consequence, in order to analyze network competition, it is necessary to select one equilibrium for each of the coordination games that consumers play after observing different price announcements, e.g. one that is Pareto efficient,[2] or the one that is the most favorable to one firm, which benefits from a reputation advantage.[3]

Instead, we enrich the basic model by allowing for both horizontal and vertical differentiation, and for incomplete information about the quality of the goods. Rather than complicating the analysis, these more realistic assumptions allow us to derive a well-defined demand function for a large class of markets. In our set-up, consumers' network choices, for given prices, constitute a global game with correlated private values. Therefore, we can derive necessary and sufficient conditions that guarantee the existence of a unique equilibrium of this game and therefore yield a well-defined demand function. These conditions relate to the amount of horizontal differentiation and the quality of the information present on the market.

The second contribution is to address a classic issue in network industries, namely whether the presence of network effects gives rise to excessive or insufficient asymmetry of the market shares of rival firms. In the presence of network effects and product differentiation, characterizing the social optimum is a non-trivial problem: the aggregate surplus from the network effect is maximized if all players join the same network, while the surplus from the intrinsic utility from consumption of the good is maximized if every player buys the product he likes the most.

This trade-off is exemplified by the case of PC and Macintosh. If there was only one standard, every computer owner could run all the existing software and easily exchange files and know-how with anybody else. On the other hand, one of the main reasons why the two standards do coexist is that the two types of computers have different strengths that appeal to the needs of different consumers.[4]

Another example is the case of matrix programming languages. Matlab seems to be best suited to analyze time-series data, while Gauss seems to perform better when analyzing panel data.[5] At the same time, all programmers would benefit from the existence of a unique common language because they could exchange suggestions and pieces of code with a larger group of people.

In this paper, we investigate this trade-off between maximizing the aggregated network effect and letting consumers use their favorite product, and show that the market outcome is characterized by insufficient asymmetry of the market shares. This is due to the presence of two sources of inefficiency. The first is that consumers fail to internalize the network externality. The second is that, in equilibrium, the relative price of the high-quality product is too high.

In our model, we assume that two new, alternative network products are introduced and that consumers simultaneously choose which one to join. We consider two cases, one where the networks are "sponsored" and one where the networks are "unsponsored". In the network literature, this distinction refers to the cost an individual bears to join a network: if a network is sponsored, access to it is priced by a strategic firm. If it is unsponsored, access is available at marginal cost. While most telecommunication networks are typically sponsored, there are many cases of unsponsored networks as well. For example, by learning a new language, an individual can join a "communication network" that clearly exhibits network externalities. In this case, there is no centralized entity that owns the language and strategically sets a price for the access to the network. The "price" is simply the time spent studying the language.

We assume that the goods are both vertically and horizontally differentiated, but neither firms nor consumers can perfectly observe the vertical dimension of quality. More precisely, they observe a noisy public signal about the vertical quality of each good, and one of the two networks has a higher expected quality. Each consumer privately and perfectly observes his private value for each good but he cannot distinguish the common component (the objective quality of the good) from his idiosyncratic taste component. When choosing which network to join, each individual takes into account three elements: his private valuation for each good, the expected size of each network, and the cost he has to bear to join it.

Consider, for example, the introduction of two new, non-interconnected telecommunication networks, such as two softwares for videoconference. The assumption of noisy information about the vertical quality of each good captures the fact that the overall performance of such products critically depends on how well they interact with the complementary hardware and software, and this interaction cannot be perfectly tested before the product is introduced. The assumption of horizontal differentiation captures the idiosyncratic taste consumers might have for the more "recreational" features of these products, such as the graphic interface. Finally, the assumption that each consumer perfectly observes his private value for each good before making a purchase captures the fact that software developers typically make trial versions of their products available for free.

We address the question of how the equilibrium allocation of consumers across networks compares to the social optimum. We find that the optimal allocation of consumers across networks is asymmetric, with both networks having positive market shares and more than one half of the population joining the high-quality network. We then look at the allocation implemented by the market, distinguishing between the two cases of sponsored and unsponsored networks.

In the case of unsponsored networks, we identify a first source of inefficiency. Since consumers fail to internalize network effects, the equilibrium allocation is too balanced from a social point of view: the market share of the high-quality firm, though larger than one half, is still smaller than would be socially optimal. This inefficiency is exacerbated as the amount of horizontal differentiation vanishes. Looking at the case of sponsored networks, we find that strategic pricing makes the inefficiency even worse: the high-quality firm has an intrinsic advantage that is reflected in an equilibrium price higher than the one charged by the competitor, and this in turn reduces its market share even more.

A number of papers analyze price competition between two differentiated networks.

In models with network effects and pure vertical differentiation, such as the one presented by Farrell and Katz (1998), there are multiple sets of self-fulfilling expectations in the coordination game played by consumers. The same holds for most models where the goods are horizontally differentiated. For example, Griva and Vettas (2004) give a full characterization of the set of equilibria of a model with horizontal differentiation à la Hotelling, allowing also for vertical quality differences.

The closest model to ours is presented by De Palma and Leruth (1993). They allow only for horizontal differentiation and prove that a large degree of heterogeneity in consumers' tastes is necessary and sufficient for uniqueness of the equilibrium . This condition is similar to the condition we derive in our paper, for the limit case of our model where all the players perfectly observe the vertical dimension of quality and the latter is identical for the two goods.

Our discussion of the optimal allocation of consumers between networks is related to Farrell and Saloner (1986), who address the issue of whether complete standardization is efficient if consumers have heterogeneous preferences.[6] Our model differs from theirs in that in our model, where heterogeneity in consumers' tastes is modeled as a probit, neither the market equilibrium nor the social optimum can involve complete standardization. In our model, the relevant question is rather what is the optimal amount of asymmetry in the market shares of two networks, when there is enough product differentiation to guarantee that both are, and should be, active.

More recently, Mitchell and Skrzypacs (2005) have addressed the issue of the optimal amount of disparity in market shares in the long run, in a dynamic model of network competition with product differentiation. In the static version of their model, they find a result similar to ours: the market equilibrium yields more equal market shares than is socially desirable. The main difference between our analysis and the static version of their model is that, while they directly assume that the demand function is well-defined and downward sloping, we focus on the distribution of consumer preferences and on the information structure of the coordination game they play for given prices, in order to characterize and explain the necessary and sufficient conditions that guarantee that the demand function is well-defined and downward sloping.

Finally, Jullien (2007) analyzes network competition with perfect price-discrimination and finds a result analogous to the one we establish for the case of sponsored networks: in equilibrium, the largest network is too small. There are two main differences between our model and his. First, in his model the coordination game played by consumers after prices have been announced has multiple equilibria, and he assumes that consumers always select an equilibrium that maximizes the market share of a given "focal firm". We instead characterize the necessary and sufficient conditions for the existence of a unique equilibrium of the coordination game. Second, he assumes that firms announce prices sequentially (the focal firm first, then its competitor) and can perfectly price-discriminate, while we assume uniform pricing and simultaneous price announcements.

Our work is also connected to the literature on duopolistic price competition with differentiated goods. For a discussion of this literature, see Tirole (1988).

In our paper, we model consumers' choice as a global game with correlated private values. Global games, first analyzed by Carlsson and van Damme (1993), have been typically applied to economic situations where players have common values, such as currency crises and regime switches. Morris and Shin (2004) analyze a private-value global game where two players choose between two alternative actions and each player privately observes his own payoff type which is the sum of a common component and an idiosyncratic component. They derive a condition on the information structure that is necessary and sufficient for a unique equilibrium in that context. The solution of the model is identical in the case of a continuum of players choosing between two actions (Morris an Shin (2003)). In our model, we apply the Morris-Shin condition for uniqueness, and relate it to the degree of horizontal differentiation and the precision of the available information about the vertical dimension of quality.

The paper is organized as follows. In Section 2, we introduce the formal model. In Section 3, we characterize the socially optimal allocation of consumers to networks. In Section 4, we solve the consumer coordination game, characterize the equilibrium allocation for both the case of unsponsored and sponsored networks, and compare it to the efficient one. Section 5 concludes. All the proofs are relegated to the Appendix.

2 The Model

We assume that two indivisible network goods, a and b, become available to a population of consumers represented by a continuum of mass 1. The two goods are produced at the same, constant marginal cost c. If the networks are sponsored, each good is produced and sold by a firm who strategically chooses its price to maximize its profits:

$$\pi_j = \left(p^j - c\right) n^j$$

where p^j is the price charged by firm j (with $j = a, b$) and n^j is the number of units it sells. If the networks are unsponsored, there are no strategic firms and each good can be purchased at marginal cost.

Preferences exhibit network externalities: the utility that any consumer i derives from joining network j is increasing in n^j. More precisely, we assume:

$$U_i^j = x_i^j + n^j - p^j$$

where x_i^j represents the intrinsic value of good j for consumer i.

We further assume that the two goods are both vertically and horizontally differentiated. The intrinsic value of a good for a consumer is the sum of a common value component $\widetilde{\theta}^j$, representing the vertical dimension of quality of brand j, and an idiosyncratic component $\widetilde{\varepsilon}_i^j$, representing individual taste:

$$\widetilde{x}_i^j \equiv \widetilde{\theta}^j + \widetilde{\varepsilon}_i^j.$$

We assume that each $\widetilde{\theta}^j$ is a normal random variable:

$$\widetilde{\theta}^j \sim N\left(y^j, \frac{2}{\alpha}\right).$$

The expected value of $\widetilde{\theta}^j$, given by y^j, can be interpreted as a noisy public signal about $\widetilde{\theta}^j$.

We model consumer heterogeneity assuming that each individual's idiosyncratic taste component is normally distributed around zero:

$$\widetilde{\varepsilon}_i^j \sim N\left(0, \frac{2}{\beta}\right).$$

Finally, we assume that $\widetilde{\theta}^a$, $\widetilde{\theta}^b$ and each $\widetilde{\varepsilon}_i^j$ are independently distributed. All the above distributional assumptions are common knowledge.

Each consumer privately observes the vector

$$\mathbf{x}_i = \left(x_i^a, x_i^b\right) \in \mathbb{R}^2$$

which constitutes his type. We will denote a type profile for all consumers as

$$\mathbf{x} = \times_{i\in[0,1]}\mathbf{x}_i \in \left(\mathbb{R}^2\right)^{[0,1]}.$$

Notice that consumers have correlated private values: types are correlated, due to the presence of a common component, but each consumer's payoff does not depend directly on other consumers' types. We emphasize that in the general case of finite precision of the public signals, neither firms nor consumers can perfectly observe the vertical quality of each good. Therefore, a consumer can only observe his type but he cannot observe θ^j and ε_i^j separately. However, all the results in the paper still hold in the limit case where the precision of the public signals is infinite, i.e. if consumers can perfectly learn θ^j and ε_i^j by observing y^j.

We can now formally define actions, strategies and payoff functions for the players. If the networks are sponsored, then the two firms and the population of consumers play a game of incomplete information in two stages. In the first stage, firms announce prices $\left(p^a, p^b\right)$ simultaneously and non-cooperatively. An action for firm j is a price $p^j \in \mathbb{R}$. The strategy space for the firms coincides with their action space and a strategy profile for the firms is a vector $\mathbf{p} = \left(p^a, p^b\right) \in \mathbb{R}^2$.

In the second stage of the game, consumers learn their types, observe prices and simultaneously choose a network. We assume complete market coverage and exclusivity: each consumer buys exactly one unit of one good, thus joining either network a or network b. Formally, an action for consumer i is $r_i \in \{a, b\}$, and an action profile for all the consumers is $\mathbf{r} \in \{a, b\}^{[0,1]}$. A pure strategy for a consumer is

$$s_i : \mathbb{R}^2 \times \mathbb{R}^2 \to \{a, b\}$$

and a strategy profile for all consumers is $\mathbf{s} \equiv \times_{i \in [0,1]} s_i$. The size of network j when consumers play strategy profile $\mathbf{s}$ and firms play strategy profile $\mathbf{p}$ is $n^j \left(\mathbf{s}\left(\mathbf{x}, \mathbf{p}\right)\right)$.[7]

In the first stage, each firm maximizes its expected profits solving

$$\max_{p^j \in \mathbb{R}} \mathbb{E}_{\mathbf{x}} \left[\pi_j \left(n^j \left(\mathbf{s}\left(\mathbf{x}, \mathbf{p}\right)\right), \mathbf{p}\right)\right] = \max_{p^j \in \mathbb{R}} (p^j - c) \mathbb{E}_{\mathbf{x}} \left[n^j \left(\mathbf{s}\left(\mathbf{x}, \mathbf{p}\right)\right)\right].$$

In the second stage, each consumer maximizes his expected net surplus solving

$$\max_{j \in \{a,b\}} \mathbb{E}_{\mathbf{x}_{-i}} \left[U_i^j \left(\mathbf{x}_i, \left(\mathbf{s}\left(\mathbf{x}, \mathbf{p}\right)\right), \mathbf{p}\right) | \mathbf{x}_i\right] = \max_{j \in \{a,b\}} \left[x_i^j + \mathbb{E}_{\mathbf{x}_{-i}} \left[n^j \left(\mathbf{s}\left(\mathbf{x}, \mathbf{p}\right)\right) | \mathbf{x}_i\right] - p^j\right].$$

If networks are unsponsored, then both goods are priced at marginal cost and consumers play a static game of incomplete information equivalent to the second stage of the above game for the case $p^a = p^b = c$.

For notational convenience, we also define the following differences:

$$\begin{aligned} y &\equiv \frac{y^a - y^b}{2} \\ \widetilde{\theta} &\equiv \frac{\widetilde{\theta}^a - \widetilde{\theta}^b}{2} \sim N\left(y, \frac{1}{\alpha}\right) \\ \widetilde{\varepsilon}_i &\equiv \frac{\widetilde{\varepsilon}_i^a - \widetilde{\varepsilon}_i^b}{2} \sim N\left(0, \frac{1}{\beta}\right) \\ \widetilde{x}_i &\equiv \frac{\widetilde{x}_i^a - \widetilde{x}_i^b}{2} \sim N\left(y, \sigma^2\right) \end{aligned}$$

where

$$\sigma^2 = \sigma^2(\alpha, \beta) = \frac{1}{\alpha} + \frac{1}{\beta}.$$

The above variables can be easily interpreted. The random variable $\widetilde{\theta}$ is an index of vertical differentiation. For positive realizations of $\widetilde{\theta}$, good a has a higher objective quality than good b, and the opposite is true for negative values. The parameter y is a public signal about the difference in quality between the two goods. Without loss of generality, we will assume that $y > 0$ throughout the paper (i.e. we will assume that good a has a weakly higher expected quality than good b). The parameter α represents the precision of the public information about the difference in quality between the two goods. The random variable $\widetilde{\varepsilon}_i$ captures the idiosyncratic differences in taste among consumers and β, the precision of its distribution, is an index of the amount of heterogeneity among consumers.[8] More precisely, the smaller is β, the larger is the amount of horizontal differentiation between the two goods. The random variable $\widetilde{x}_i$ is a measure of consumer i's relative preference for good a, the good with the highest expected quality, with respect to good b. For positive realizations of $\widetilde{x}_i$, he prefers a to b. For negative realizations, his idiosyncratic preference for b is so strong that he prefers b to a. Notice that we use the notation $\mathbf{x}_i$ for the vector $\left(x_i^a, x_i^b\right)$ and the notation x_i for the difference $\frac{x_i^a - x_i^b}{2}$.

Finally, we define

$$p\left(p^a, p^b\right) \equiv \frac{p^a - p^b}{2}.$$

In what follows, we will use the simplified notation p to denote $p\left(p^a, p^b\right)$.

3 Efficient Allocation

In this section, we derive the ex-ante efficient allocation of consumers across networks. We choose to address the issue of efficiency from an ex-ante point of view because we believe that the right benchmark to compare to the allocation implemented by the market is the allocation that maximizes welfare given all, and only, the public information available on the market.[9] [10]

We therefore consider the problem faced by a benevolent social planner who has access to all the public information available on this market, and the power to choose ex-ante an allocation of consumers to the networks, i.e. an anonymous rule assigning each consumer i to either network a or network b according to his type x_i alone.

First, we need to define the welfare criterion. Given our assumption of quasi-linear utility functions, aggregate welfare equals consumer gross surplus minus total cost, since prices are a transfer from consumers to firms. Also, since we assumed that the two firms have the same marginal cost and that the total number of units sold in the market is constant, we can ignore costs as we solve the welfare maximization problem. Finally, the only choice variable in the welfare maximization problem is the allocation of consumers to the networks.

Let $(\mathcal{A}, \mathcal{B})$ be an allocation chosen ex-ante by the planner, and $\mathcal{A}_{\mathbf{x}}$ and $\mathcal{B}_{\mathbf{x}}$ denote the set of consumers who must join networks a and b, respectively, for a given realization of $\mathbf{x}$, when the allocation chosen is $(\mathcal{A}, \mathcal{B})$.[11] Also, let $n^a(\mathcal{A}_{\mathbf{x}}, \mathcal{B}_{\mathbf{x}})$ and $n^b(\mathcal{A}_{\mathbf{x}}, \mathcal{B}_{\mathbf{x}})$ denote the Lebesgue measure of the sets $\mathcal{A}_{\mathbf{x}}$ and $\mathcal{B}_{\mathbf{x}}$, respectively.

The welfare maximization problem is equivalent to the following problem:

$$\max_{\mathcal{A},\mathcal{B}} \mathbb{E}_{\mathbf{x}}\left[W(\mathcal{A}_{\mathbf{x}}, \mathcal{B}_{\mathbf{x}})\right] = \max_{\mathcal{A},\mathcal{B}} \mathbb{E}_{\mathbf{x}}\left[\int_{i\in\mathcal{A}_{\mathbf{x}}} \left(x_i^a + n^a(\mathcal{A}_{\mathbf{x}}, \mathcal{B}_{\mathbf{x}})\right) di + \int_{i\in\mathcal{B}_{\mathbf{x}}} \left(x_i^b + n^b(\mathcal{A}_{\mathbf{x}}, \mathcal{B}_{\mathbf{x}})\right) di\right] \tag{$*$}$$

$$s.t. \quad \mathcal{A}_{\mathbf{x}} \cup \mathcal{B}_{\mathbf{x}} = [0,1] \quad and \quad \mathcal{A}_{\mathbf{x}} \cap \mathcal{B}_{\mathbf{x}} = \varnothing \quad for\ every\ \mathbf{x}.$$

Next, we show that the optimal choice of the allocation $(\mathcal{A}, \mathcal{B})$ can be characterized as a threshold allocation. By this we mean that a benevolent social planner would maximize ex-ante social welfare by choosing a threshold t, and establishing the following rule: those consumers with a preference for a larger than t should join network a, and those with a preference for a smaller than t should join network b. Formally, we define a threshold allocation as follows:

Definition 1
A ***threshold allocation*** *is a pair* $(\mathcal{A}(t), \mathcal{B}(t))$ *such that* $\exists t \in \mathbb{R}$ *such that, for every realization of* $\mathbf{x}$, $\mathcal{A}_{\mathbf{x}}(t) = \{i \in [0,1] : x_i > t\}$ *and* $\mathcal{B}_{\mathbf{x}}(t) = \{i \in [0,1] : x_i \leq t\}$.

Lemma 1
The welfare maximizing allocation is a threshold allocation.

In what follows, we will denote by $(\mathcal{A}^*, \mathcal{B}^*)$ the solution to problem $(*)$ and by t^* the corresponding threshold .

The intuition for the result presented in Lemma 1 is the following. Consider an arbitrary rule to allocate consumers to the networks. If this is not a threshold allocation, then it must be such that there are some "relatively high" values of x_i for which it requires that consumers join b and some "relatively low" values of x_i for which it requires that consumers join a. One can show that the allocation obtained by switching the network assignment associated to some of these values of x_i, while being careful to keep the expected network sizes constant, yields a higher expected social welfare, hence the initial allocation could not be welfare maximizing.

In what follows, we will denote the welfare associated to a given threshold allocation $(\mathcal{A}(t), \mathcal{B}(t))$ as

$$\mathbb{E}_{\mathbf{x}}[W(t)] \equiv \mathbb{E}_{\mathbf{x}}[W(\mathcal{A}_{\mathbf{x}}(t), \mathcal{B}_{\mathbf{x}}(t))].$$

Given that the optimal allocation is associated to a threshold, the final step to solve the welfare maximization problem is to identify the optimal threshold t^*. First, notice that the welfare function can be rewritten as the sum of three components:

$$\mathbb{E}_{\mathbf{x}}[W(t)] = \mathbb{E}_{\mathbf{x}}\left[\left(\int_{i\in\mathcal{A}_{\mathbf{x}}(t)} \theta^a di + \int_{i\in\mathcal{B}_{\mathbf{x}}(t)} \theta^b di\right) + \right. \tag{1}$$

$$+\left(\int_{i\in\mathcal{A}_{\mathbf{x}}(t)} \varepsilon_i^a di + \int_{i\in\mathcal{B}_{\mathbf{x}}(t)} \varepsilon_i^b di\right) +$$

$$\left. + \left(\int_{i\in\mathcal{A}_{\mathbf{x}}(t)} n^a(\mathcal{A}_{\mathbf{x}}(t), \mathcal{B}_{\mathbf{x}}(t))\, di + \int_{i\in\mathcal{B}_{\mathbf{x}}(t)} n^b(\mathcal{A}_{\mathbf{x}}(t), \mathcal{B}_{\mathbf{x}}(t))\, di\right)\right].$$

The first component measures the aggregate surplus consumers derive from the vertical quality of the goods, the second measures the aggregate surplus derived from idiosyncratic taste and finally the third component measures the aggregate surplus derived from network effects. By evaluating expression 1 for a given realization of $\left(\widetilde{\theta}^a, \widetilde{\theta}^b\right)$ and then taking the expectation, the welfare function can be finally be re-written as

$$\begin{aligned} \mathbb{E}_{\mathbf{x}}[W(t)] &= \mathbb{E}_{\theta^a,\theta^b}\left\{\theta^a - 2\theta\Phi\left((t-\theta)\sqrt{\beta}\right) + \right. \\ &\quad + \sqrt{\frac{2}{\pi\beta}} e^{-\frac{(t-\theta)^2\beta}{2}} + \\ &\quad \left. + 2\left[\Phi\left((t-\theta)\sqrt{\beta}\right)\right]^2 - 2\Phi\left((t-\theta)\sqrt{\beta}\right) + 1\right\} \end{aligned} \tag{2}$$

where $\Phi(\cdot)$ denotes the cdf of a standard normal random variable.[12]

Before we characterize the optimal threshold, it is worth to analyze in larger detail the three components of the welfare function.

For a given realization of $\left(\widetilde{\theta}^a, \widetilde{\theta}^b\right)$, the first component,

$$\theta^a - 2\theta\Phi\left((t-\theta)\sqrt{\beta}\right), \tag{3}$$

measures the aggregate surplus derived from the vertical quality of the goods. Expression (3) can be easily interpreted. If $t = -\infty$ it reduces to θ^a : all consumers are assigned to network a and a measure 1 of individuals enjoy quality θ^a. For a finite t, if we denote by $F(\cdot)$ the cdf of x_i, then only $1 - F(t)$ consumers are assigned to a, while the remaining $F(t)$ consumers are assigned to b and enjoy quality θ^b. Substituting to $F(t)$ its expression and using the variable θ defined in section 2, we get (3). If the realized quality difference θ is positive, it easy to see that (3) is maximized by $t = -\infty$, i.e. by an allocation where all consumers join the high quality network (see Figure 1[13]).

The second welfare component for a given realization of $\left(\widetilde{\theta}^a, \widetilde{\theta}^b\right)$,

$$\sqrt{\frac{2}{\pi\beta}} e^{-\frac{(t-\theta)^2\beta}{2}}, \tag{4}$$

measures the portion of aggregate surplus due to consumers' idiosyncratic taste for the network to which they are allocated. Expression 4 is maximized at $t = \theta$ (see Figure 2). Intuitively, this idiosyncratic surplus component is maximized when each consumer is assigned to the network for which he has an idiosyncratic preference. If θ were perfectly observable, this could be achieved by choosing $t = \theta$, so that every consumer with a positive ε_i would join a and the others would join b.

Finally, for given $\left(\widetilde{\theta}^a, \widetilde{\theta}^b\right)$, the last welfare component,

$$2\left[\Phi\left((t-\theta)\sqrt{\beta}\right)\right]^2 - 2\Phi\left((t-\theta)\sqrt{\beta}\right) + 1 \tag{5}$$

measures the aggregated network effect. Expression (5) can be easily interpreted. The total network effect is given by the sum of the squared market shares (all consumers in a network receive a positive effect measured by the size of the network itself). Denoting again by $F(\cdot)$ the cdf of x_i, for a given threshold t market shares are $n^a = 1 - F(t)$ and $n^b = F(t)$. Substituting to $F(t)$ its expression we get (5).

This component of consumer surplus is maximized by $t \in \{-\infty, +\infty\}$, since for any of those values all consumers join the same network and therefore each of them enjoys the largest possible network effect (see Figure 3). Given this decomposition, it appears that there is a trade-off between maximizing the first and the third component of welfare by assigning all consumers to the same network, and maximizing the second component by splitting consumers equally. Proposition 1 characterizes the socially optimal allocation.

Proposition 1 (Ex-Ante Efficient Allocation)
Social welfare is maximized by an asymmetric threshold allocation such that

$$-1 < t^* < 0 < y.$$

As one would intuitively expect, $t^* < y$: the optimal allocation is such that the network with the largest expected quality has the largest expected market share.

Nonetheless, the presence of enough heterogeneity in consumers' taste results in an interior optimum ($-\infty < t^*$).[14] Next, notice that, in the absence of network externalities, the optimal threshold would simply be $t^* = 0$: each consumer should consume the good he privately prefers. Because there are network effects, the optimal threshold is instead smaller than zero: all the consumers with a positive x_i, who therefore have a private preference for a, and who constitute more than one half of the population, and some of those with a small preference for b, should join the large network a. The minority of consumers with a strong preference for b should form another, much smaller network. The intuition for this result is the following: suppose the threshold were zero, then network a would be larger than network b. For this reason, moving the marginal consumer to network a would increase the utility of a large measure of consumers, through the network effect, and decrease the utility of a smaller number of consumers. Also, notice that the lower bound on t^* is equal to minus one because we implicitly normalized the maximum network externality (the externality derived from being in a network of measure one) to one. If we re-scaled the network effect by writing the utility function as $U_i^j = x_i^j + kn^j - p^j$, with $k \in \mathbb{R}_+$, then the lower bound on t^* would be $-k$.

We conclude this section presenting a simple comparative statics result about the effect on t^* of a marginal increase in either α, the precision of the common value quality component θ, or β, the precision of the idiosyncratic component ε_i.

Corollary 1 (Comparative Statics of t^*)
The socially optimal threshold t^* is decreasing in both α and β.

To get an intuition for this result, it is helpful to consider the value of t^* for given values of α and β, keeping in mind that it represents the (negative) intrinsic preference for good a of the marginal consumer allocated by the planner to network b. In t^*, the marginal benefit, in terms of increased aggregate network effects, that the planner could achieve by moving the threshold further to the left and increasing the asymmetry, is exactly balanced by the loss of intrinsic utility that would then be imposed on the marginal consumer with preference t^*, who would be re-allocated to a instead of b.

An increase in either α or β results in a decrease in σ^2, the unconditional variance of the consumers' private values. This change in the shape of the distribution implies that, for the initial t^*, a smaller set of consumers are allocated to network b and a larger set of consumers are allocated to network a. In turn, this stronger asymmetry increases the marginal network effect gain from moving the threshold to the left of the initial t^* and makes it optimal to do so. In other words, the increased marginal benefit from the network effects makes it socially optimal to impose a stronger sacrifice, in terms of intrinsic utility, on the marginal consumer.

4 The Equilibrium Allocation

In this section, we characterize the allocations that prevail in equilibrium in the market, for the case of unsponsored and sponsored networks, in order to compare them to the efficient one that we characterized in Section 3. First, we will solve the coordination game consumers play when they choose which network to join, for given prices. Then, we will consider separately the case where prices are simply equal to the marginal cost and the case where they are set by strategic firms.

4.1 Consumer Coordination

The next result characterizes the equilibrium of the coordination game played by consumers for given price difference p and given expected quality difference y. This coordination game constitutes a global game with correlated private values, such as the one analyzed by Morris and Shin (2004). We apply their methodology to characterize the equilibrium and identify the necessary and sufficient condition for its uniqueness. Then, we illustrate how this condition relates to the heterogeneity of consumers' preferences and to the quality of the public information available on the market.

Suppose consumers observe a quadruple $\left\{p^a, p^b, y^a, y^b\right\}$. Consider a type $\widehat{\mathbf{x}}_i$ of consumer i with the following property: if a consumer has type $\widehat{\mathbf{x}}_i$, and he believes that every consumer with a preference for a larger than his own will join a, and every consumer with a preference for a smaller than his own will choose b, then he is indifferent between the two networks. Formally, $\widehat{\mathbf{x}}_i\left(p,y\right)$ is a solution to:

$$x_i^a + \Pr\left[\left(x_{i'} > x_i\right) \middle| \, \mathbf{x_i}\right] - p^a = x_i^b + \Pr\left[\left(x_{i'} \leq x_i\right) \middle| \, \mathbf{x_i}\right] - p^b. \tag{6}$$

Let $t\left(p,y\right)$ be the intrinsic preference for a of a consumer of type $\widehat{\mathbf{x}}_i\left(p,y\right)$. That is:

$$t\left(p,y\right) \equiv \frac{\widehat{x}_i^a\left(p,y\right) - \widehat{x}_i^b\left(p,y\right)}{2} .$$

By rearranging (6), it can be shown that $t\left(p,y\right)$ is implicitly defined by the following equation:

$$t - \Phi\left[\left(t-y\right)z\right] + \frac{1}{2} - p = 0 \tag{7}$$

where

$$z = z(\alpha,\beta) \equiv \sqrt{\frac{\alpha^2\beta}{\left(\alpha+\beta\right)\left(\alpha+2\beta\right)}} . \tag{8}$$

The following proposition holds:

Proposition 2 (Equilibrium of the Coordination Game)
If $z \leqslant \sqrt{2\pi}$, the coordination game played by consumers after any price announcement $\left(p^a, p^b\right)$ has

the following unique Nash-equilibrium:

$$s_i^*(\mathbf{x},\mathbf{p}) = \begin{cases} a \text{ if } x_i > t(p,y), \\ \\ b \text{ if } x_i \leq t(p,y), \end{cases} \quad \forall i \in [0,1].$$

where $t(p,y)$ *satisfies*

$$t(p,y) - \Phi\left[\left(t(p,y) - y\right) z\right] + \frac{1}{2} - p = 0.$$

The threshold $t(p,y)$ *is strictly increasing in* p *and strictly decreasing in* y*.*

The equilibrium described by Proposition 2 is a symmetric equilibrium in switching strategies around the threshold $t(p,y)$: consumers with a preference for a larger than $t(p,y)$ join a and the others join b. Therefore, the equilibrium allocation is a threshold allocation. This will prove particularly useful as we compare the market allocation to the efficient one.

The two available pieces of public information, namely the public signal y about the relative quality of the two goods and the difference p between the two prices, affect the equilibrium threshold in a very intuitive way. For a given quality signal, if a becomes more expensive with respect to b, the threshold moves to the right, i.e. a loses some consumers to b. For given prices, instead, if a's expected quality advantage y increases, the threshold moves to the left, i.e. a gains some consumers.

Next, we illustrate the intuition behind the uniqueness condition mentioned in the statement of Proposition 2. It can be shown that, if the game has only one equilibrium in the class of equilibria in switching strategies, then the game has no other equilibrium of any type.[15] This implies that the key condition for uniqueness is that there exists only one value of the threshold t that solves (7). In turn, (7) has a unique solution $t(p,y)$ if the second term is (almost) invariant with respect to t. That term represents the strategic uncertainty of a player who observes a private signal exactly equal to the threshold t. More precisely, it represents his expectation of the proportion of consumers who will choose b because their private value x_i is smaller than t. Therefore, uniqueness is guaranteed if strategic uncertainty is not very sensitive to the value of t.

This low level of sensitivity is achieved in the parameter region defined by the inequality

$$z \leqslant \sqrt{2\pi}.$$

Following Morris and Shin (2004) and Ui (2006), z can be rewritten as

$$z = \sqrt{\frac{\alpha^2\beta}{(\alpha+\beta)(\alpha+2\beta)}} = \sqrt{\frac{\alpha\beta}{\alpha+\beta}\left(\frac{1-\frac{\beta}{\alpha+\beta}}{1+\frac{\beta}{\alpha+\beta}}\right)} = \sqrt{\frac{1}{\sigma^2}\left(\frac{1-\rho}{1+\rho}\right)} \tag{9}$$

where σ^2 is the unconditional variance of the consumers' private values and

$$\rho \equiv \frac{\beta}{\alpha + \beta}$$

is their correlation coefficient. From this reparametrization, it is easy to see that the uniqueness condition is satisfied if either the unconditional variance or the correlation is sufficiently large.

The intuition for this result is the following.[16] When consumers' private values are strongly correlated, strategic uncertainty is not very sensitive to a player's type because no matter what specific realization of $\widetilde{x}_i$ he observed, any consumer i believes that it is about as likely for any consumer i' to observe $x_{i'} > x_i$ or $x_{i'} < x_i$.

In the case of high unconditional variance of consumers' private values, the density approaches the uniform density. Therefore, also in this case strategic uncertainty is not very sensitive to a player's type: whatever the realization of $\widetilde{x}_i$ he observed, consumer i believes that approximately one half of the remaining consumers observed values $x_{i'} > x_i$, and the other half observed $x_{i'} < x_i$.

In the parameter space (α, β) the uniqueness condition is satisfied either if β is sufficiently large, for any α, or if α is large and β is smaller than some upper bound that approaches 2π as α goes to infinity. In the first case, uniqueness is guaranteed by a high correlation of consumers' private values. In the second case, correlation is small, but the low β guarantees that the unconditional variance is sufficiently large to guarantee that $z \leqslant \sqrt{2\pi}$.

In terms of market features, the first case corresponds to markets where the amount of horizontal differentiation is sufficiently small, with respect to the noise of the public information about the vertical quality of the goods, and the second case to markets with rather precise public information and a large amount of horizontal differentiation.[17]

Throughout the rest of the paper, we will assume that the condition for uniqueness, $z \leqslant \sqrt{2\pi}$, is satisfied, focusing our attention on those network markets where the demand function is naturally well-defined.

Before we consider the two cases of sponsored and unsponsored networks in more detail, we present one more result about the issue of consumer coordination.

Proposition 3 (Ex-Post Impact of Public Information)
Let $z \leqslant \sqrt{2\pi}$. For any pair (p, y) and for a given realization of θ, in equilibrium the networks realized market shares are

$$n^a = 1 - n^b = \Pr\left[x_i > t(p, y)\right] = 1 - \Phi\left[\left(t\left(p, y\right) - \theta\right)\sqrt{\beta}\right]. \tag{10}$$

Each firm's ex-post market share is strictly increasing in its expected quality.

This result has a natural interpretation. For a given θ, the market share of each network is given by the proportion of consumers with a value of x_i above or below the equilibrium threshold. After

θ has been drawn, the distribution of x_i is uniquely determined and clearly it does not depend on y, the expected value of θ. Therefore, the fact that a network's ex-post market share is increasing in its ex-ante expected quality has to be due to the fact that the equilibrium threshold $t(p, y)$ is a function of y. In turn, the result that $t(p, y)$ is affected by y might appear counterintuitive, since consumers play the coordination game that determines $t(p, y)$ after each of them has perfectly observed his own private value x_i. Since this is a model with private values, one might expect that, after observing his own x_i, each consumer i would discard the noisy information contained in its expected value y. Still, consumer i cares not only about his own x_i but also about the distribution of the private values of other consumers. This is true because of the presence of network effects: the signals observed by other consumers will affect their choices, and i needs to take such choices into account because of the strategic complementarity. This is where y becomes relevant. The prior distribution of each $x_{i'}$ is centered around y. When consumer i observes x_i, and updates his belief about the distribution of $x_{i'}$, whatever the value of x_i he observed, the posterior expectation of $x_{i'}$ is increasing in y. Since in equilibrium consumers with a high $x_{i'}$ choose a, the higher y, the higher a's expected market share from the point of view of consumer i, the higher his own convenience in choosing a.

4.2 Unsponsored Networks

We now consider the case where the two networks are unsponsored, in order to highlight the possible coordination failure arising on these markets when we abstract from the issue of strategic pricing. Without sponsors, consumers can join any network by paying a price equal to the marginal cost:

$$p^a = p^b = c, \qquad \text{and} \qquad p = 0.$$

The next proposition describes the equilibrium allocation and compares it to the efficient one.

Proposition 4 (Inefficiency with Unsponsored Networks)
With unsponsored networks, the equilibrium allocation is a threshold allocation with threshold

$$t^u \equiv t(0, y) \in (-0.5, 0).$$

Moreover,

$$t^u > t^* \quad \text{and} \quad \mathbb{E}_{\mathbf{x}}[W(t^u)] < \mathbb{E}_{\mathbf{x}}[W(t^*)].$$

The allocation implemented by the market if the networks are unsponsored shares some qualitative features with the efficient allocation: both are threshold allocations and in both cases, since the threshold is smaller than y, the network with higher expected quality has an expected market share larger than one half. Moreover, in both cases the threshold is negative: network a includes not only all consumers with a positive x_i, but also some consumers with a moderate private preference for b.

Nonetheless, the market does not implement the welfare maximizing allocation. In particular, the market allocation is more balanced than the efficient one. The source of this market failure is the presence of network effects that consumers fail to internalize. To get an intuition for this result, it is useful to revise the decomposition of the welfare function that we presented in Section 3. For any realized level of quality of the two goods, social welfare is the sum of the gross surplus derived from the objective quality of the goods, gross surplus derived from idiosyncratic taste and, finally, gross surplus derived from network effects. The efficient threshold t^* is therefore the result of the compromise among different forces: from a social point of view, horizontal differentiation makes a symmetric allocation more desirable while vertical differentiation and the presence of network externalities make asymmetry more desirable. Since individual consumers do not internalize the network externality, society implements an allocation that is more symmetric than the efficient one.

In order to discuss how this inefficiency is affected by the precision α of the public signal y, and by the amount of horizontal differentiation, we present the following corollary.

Corollary 2 (Comparative Statics of t^u)
The equilibrium threshold for the case of unsponsored networks, t^u, is decreasing in α. Moreover,

$$\text{for } \beta < \frac{\alpha}{\sqrt{2}}, \qquad \frac{\partial t^u}{\partial \beta} < 0$$

$$\text{for } \beta > \frac{\alpha}{\sqrt{2}}, \qquad \frac{\partial t^u}{\partial \beta} > 0,$$

and

$$\lim_{\beta \to +\infty} t^u = 0.$$

To capture the intuition for this result, it is useful to remember that, for given values of α and β, t^u measures the intrinsic preference for good a of a consumer who, observing that both goods are available at the same price, and assuming that all consumers with $x_i \leq t^u$ consume good b and all consumers with $x_i > t^u$ consume good a, is indifferent between joining network a or network b. In particular, since $t^u < 0$, this "threshold- consumer" has a strict intrinsic preference for good b, which is exactly compensated by the fact that, conditional on his observation t^u and the equilibrium strategies, he expects network a to be larger than network b. More precisely, using the reparametrization presented in equation (9), his expectation of the difference between the size of the two networks can be written as

$$\mathbb{E}_{\mathbf{x}_{-i}}\left[n^a \,|x_i = t^u\right] - \mathbb{E}_{\mathbf{x}_{-i}}\left[n^b \,|x_i = t^u\right] = 1 - 2\mathbb{E}_{\mathbf{x}_{-i}}\left[n^b \,|x_i = t^u\right] = 1 - 2\Phi\left[(t^u - y)\sqrt{\frac{1}{\sigma^2}\left(\frac{1-\rho}{1+\rho}\right)}\right]. \tag{11}$$

Let us now examine the impact of a marginal increase of the precision α of the public signal y on this expectation. Both the unconditional variance σ^2 and the correlation ρ decrease. As a consequence, the value of expression (11) increases: a consumer with preference for a equal to the initial value of t^u now expects a stronger asymmetry in the size of the two networks. This implies that he cannot be indifferent anymore, but has a strict preference for joining network a. Hence, the new indifferent consumer has to be someone with a stronger intrinsic preference for good b. In other words, the new threshold has to be to the left of the initial one.

By Corollaries 1 and 2, both the socially optimal threshold t^* and the equilibrium threshold t^u are decreasing in α, so we cannot identify a monotonicity in α of the inefficiency illustrated in Proposition 4.

Next, consider the impact of a marginal increase of the precision β of the idiosyncratic component ε_i on expression (11). As β increases, the unconditional variance σ^2 decreases, while the correlation ρ increases. It can be shown that the net effect on expression (11) depends on the value of $\frac{\beta}{\alpha}$. In particular, for small values of $\frac{\beta}{\alpha}$, the expected difference in market shares from the point of view of the initially indifferent consumer increases, hence this consumer cannot be indifferent anymore but instead strictly prefers to join network a, and the new equilibrium threshold has to be to the left of the initial one. For large values of $\frac{\beta}{\alpha}$, the opposite happens.

The last result in Corollary 2 is that for a given α, as β goes to infinity the correlation ρ tends to one and the equilibrium threshold t^u goes to zero. The intuition for this result is that as the amount of heterogeneity among consumers goes to zero, each consumer thinks of himself as "typical", hence expects that if consumers follow the equilibrium strategies, and his own private value constitutes the threshold, the two network sizes will be approximately equal. Given this inference, the only consumer who can be indifferent between joining a or b is one with intrinsic preference for a over b equal to zero.

In the light of Corollary 1, this result bears important consequences in terms of social welfare. For very small levels of horizontal differentiation, i.e. as β approaches infinity, the inefficiency of the market allocation in markets with unsponsored networks becomes extreme: t^u approaches its maximum value, zero, while t^* approaches its minimum value (for a given α).

4.3 Sponsored Networks

After identifying a source of inefficiency arising on network markets, we now ask the question of whether strategic pricing mitigates or aggravates this inefficiency. More precisely, we will now assume that the two networks are sponsored and that therefore access to each of them is priced by a strategic, profit-maximizing firm. Then, we will derive the equilibrium of the two-stage price competition game and compare the allocation of consumers induced in equilibrium to both the ex-ante efficient one and the one implemented by the market in the absence of strategic pricing.

The next Lemma describes the demand functions for the two goods.

Lemma 2 (Expected Demand Functions)
Let $z \leqslant \sqrt{2\pi}$. The expected demand function for each network is well defined for any price pair $\left(p^a, p^b\right)$ and it is given by

$$\mathbb{E}_{\mathbf{x}}\left[n^a\right] = 1 - \mathbb{E}_{\mathbf{x}}\left[n^b\right] = 1 - \Phi\left[\frac{t\left(p,y\right) - y}{\sigma}\right]$$

Moreover, for given prices each firm's expected market share is strictly increasing in the expected quality of its product.

The assumption $z \leqslant \sqrt{2\pi}$ guarantees that there is a unique equilibrium of the coordination game played by consumers in the second stage, therefore the demand function is well defined.

It can be interesting to compare the comparative statics results contained in Lemma 2 and Proposition 3. Proposition 3 looks at the market shares for a given realization of θ: the expected quality of the products affects ex-post market shares because it affects the equilibrium threshold $t\left(p,y\right)$. Lemma 2 instead, considers the ex-ante market shares. In this case, there are two mechanisms through which the larger a network's expected quality, the larger its expected market share: one is the change in $t\left(p,y\right)$, the other is the shift in the distribution of x_i (which is centered around y).

We now have the tools to characterize the pure-strategy subgame-perfect equilibrium of the price competition game. Substituting the expected demand functions into the profit functions, we can write the two firms' optimization problem as

$$\begin{aligned}
\max_{p^a \in \mathbb{R}} \mathbb{E}_{\mathbf{x}}\left[\pi_a\right] &= \left(p^a - c\right)\left[1 - \Phi\left[\frac{t\left(p,y\right) - y}{\sigma}\right]\right] \\
\max_{p^b \in \mathbb{R}} \mathbb{E}_{\mathbf{x}}\left[\pi_b\right] &= \left(p^b - c\right)\Phi\left[\frac{t\left(p,y\right) - y}{\sigma}\right].
\end{aligned}$$

Let p^s denote the value of p associated to the firms' equilibrium strategies and t^s denote $t\left(p^s, y\right)$. The following proposition holds:

Proposition 5 (Strategic Pricing Inefficiency)
If a pure strategy SPNE of the price competition game exists,[18] *then*

$$p^s \in \left(0, y\right) \quad \text{and} \quad t^s \in \left(t^u, y\right).$$

Moreover,

$$t^* < t^u < t^s$$

and

$$\mathbb{E}_{\mathbf{x}}\left[W\left(t^{s}\right)\right] < \mathbb{E}_{\mathbf{x}}\left[W\left(t^{u}\right)\right] < \mathbb{E}_{\mathbf{x}}\left[W\left(t^{*}\right)\right].$$

The main qualitative features of the equilibrium of the price competition game are consistent with the standard results in duopolistic models of price competition with vertical differentiation:[19] in equilibrium, the firm selling the best product charges the highest price ($p^s > 0$) but the difference in quality more than compensates the difference in prices ($y > p^s$), so that it also attracts a fraction of consumers larger than one half.[20]

What is relevant for efficiency, though, is *by how much* this market share exceeds one half. We have already shown that in the absence of strategic pricing the market share of firm a is too small, from a social point of view. According to Proposition 5, a's market share is even smaller if the two networks are sponsored, and this equilibrium provides even less welfare than the equilibrium with unsponsored networks. The reason why strategic pricing reduces welfare is quite straightforward: with vertical differentiation, the firm selling the best product has a natural advantage that is reflected in a higher equilibrium price. In turn, the fact that a is more expensive than b, ceteris paribus, shifts some consumers from a to b.

We have therefore identified two separate sources of inefficiency on network markets: the first is the presence of network externalities that consumers do not internalize, which determines a market allocation that is too balanced from a social point of view. The second, which arises only if networks are sponsored, is the presence of strategic pricing, that further reduces the asymmetry between network sizes.

In section 4.2, we have shown that the first type of inefficiency is exacerbated as the amount of horizontal differentiation vanishes. An analogous result cannot be established for the second type of inefficiency. It can be easily shown that the comparative statics results stated in Corollary 2 hold not only for t^u, but more generally for $t(p, y)$, for any (p, y) such that $y > 0$ and $p < y$. What is ambiguous though, is the impact of a marginal increase in either α or β on the equilibrium price difference p^s. Therefore, the total impact on t^s, which is the sum of a direct effect and an indirect effect through p^s, is ambiguous as well.

5 Conclusions

We analyzed a model of duopolistic competition between two vertically and horizontally differentiated networks. We characterized the necessary and sufficient condition that guarantee that the coordination game played by the consumers for given prices has a unique equilibrium, and that therefore the demand function for each of the products is well-defined. Uniqueness is guaranteed either by a large amount of horizontal differentiation, or by a sufficient amount of noise in the public information available about the vertical quality of the goods.

We also addressed the issue of efficiency of network markets. We found that social welfare is maximized by an asymmetric allocation such that the network with the highest expected quality has the largest market share. We then compared this allocation with the ones implemented by the market if the networks are unsponsored or sponsored, respectively. Two sources of inefficiency emerged from the analysis: the failure to internalize network externalities, which is exacerbated in the case of vanishing horizontal differentiation, and the presence of sponsors that strategically set prices for the products. Both these facts induce the market to implement an allocation of consumers that is too symmetric.

We analyzed efficiency from an ex-ante point of view. One possible alternative is to analyze efficiency from an ex-post point of view, assuming that a benevolent social planner can observe the realized distribution of private values in the population. It can be shown that also in that case the efficient allocation is a threshold allocation such that both networks are active and one of them has a larger market share than the other. The main difference with the ex-ante case is that the ex-post optimal threshold depends on the realization of θ. Also, in the ex-post efficient allocation the network with the largest market share is the one with the highest *realized* quality, while in the ex-ante case it is the one with the highest *expected* quality.

Comparing the equilibrium allocation to the ex-post efficient allocation, the main result is that strategic pricing can increase welfare if the firm with the highest expected quality has the lowest realized quality, and decrease welfare in the opposite case. The intuition for this result is the following. If the public signal is in favor of, say, firm a, in equilibrium it will charge a price higher than the competitor and ex-post this will negatively affect its market share (everything else equal). Therefore, if ex-post a is the best firm, strategic pricing shifts some consumers towards the worst firm, while if a is the worst firm, then strategic pricing shifts some consumers towards the best firm.

In our model, we assumed that consumer heterogeneity is unbounded while the utility derived from the network effect is bounded. The result that the ex-ante efficient threshold is an interior optimum holds under more general assumptions. For symmetric, bounded network effects, if the distribution of x_i is continuous and symmetric around the mean, then it is sufficient that the upper bound of the support of x_i is weakly larger than the individual surplus from being in a network of size one.

We also assumed complete market coverage. This assumption allowed us to abstract from the possible deadweight loss associated to strategic pricing and to focus only on the inefficiency arising from the difference in equilibrium prices.

Finally, we assumed that firms have access only to the public information available about the vertical quality of the products. A natural extension of the model would be to allow each firm to observe a private signal about the vertical quality of its product. This would add the issue of whether firms can use prices to signal their quality.

We leave for future research an extension of this model to a dynamic environment, where consumers choose sequentially and firms can adjust prices over time.

A Appendix

A.1 Proofs

Proof of Lemma 1. We will prove by contradiction that any allocation $(\mathcal{A}', \mathcal{B}')$ that is not a threshold allocation cannot maximize the ex-ante welfare function.

Since there is a continuum of consumers, for almost every realization $\mathbf{x}$ the distribution of types x_i in the population is the same as the distribution of each individual x_i. Then, by construction of $(\mathcal{A}', \mathcal{B}')$, there exists at least a pair $(\mathcal{S}, \mathcal{T})$ such that:

- for every $\mathbf{x}$, $\mathcal{S}_\mathbf{x} \subset \mathcal{A}'_\mathbf{x}$, $\mathcal{T}_\mathbf{x} \subset \mathcal{B}'_\mathbf{x}$, and $x_i < x_{i'}$ for every (i, i') such that $i \in \mathcal{S}_\mathbf{x}$ and $i' \in \mathcal{T}_\mathbf{x}$,
- for almost every $\mathbf{x}$, $\int_{i \in \mathcal{S}_\mathbf{x}} di = \int_{i \in \mathcal{T}_\mathbf{x}} di$.

Consider now a different allocation $(\mathcal{A}'', \mathcal{B}'')$ such that $\mathcal{A}'' = (\mathcal{A}'/\mathcal{S}) \cup \mathcal{T}$ and $\mathcal{B}'' = (\mathcal{B}'/\mathcal{T}) \cup \mathcal{S}$.

We will show that $(\mathcal{A}'', \mathcal{B}'')$ yields a larger expected welfare than $(\mathcal{A}', \mathcal{B}')$.

First, notice that since there is a continuum of consumers, the sets $\mathcal{A}'_\mathbf{x}$, $\mathcal{B}'_\mathbf{x}$, $\mathcal{S}_\mathbf{x}$, $\mathcal{T}_\mathbf{x}$, $\mathcal{A}''_\mathbf{x}$, $\mathcal{B}''_\mathbf{x}$ are identical for almost every realization of $\mathbf{x}$, up to a permutation of the $i's$. In other words, for each if these sets it holds that its measure is the same for almost every $\mathbf{x}$, but the identities of the consumers that belong to it can change.

The total change in ex-ante welfare when moving from allocation $(\mathcal{A}', \mathcal{B}')$ to allocation $(\mathcal{A}'', \mathcal{B}'')$ is the following:

$$\mathbb{E}_\mathbf{x}\left[W\left(\mathcal{A}''_\mathbf{x}, \mathcal{B}''_\mathbf{x}\right)\right] - \mathbb{E}_\mathbf{x}\left[W\left(\mathcal{A}'_\mathbf{x}, \mathcal{B}'_\mathbf{x}\right)\right] =$$

$$= \mathbb{E}_\mathbf{x}\left[\int_{i \in \mathcal{A}''_\mathbf{x}} \left(n^a\left(\mathcal{A}''_\mathbf{x}, \mathcal{B}''_\mathbf{x}\right) + x_i^a\right) di + \int_{i \in \mathcal{B}''_\mathbf{x}} \left(n^b\left(\mathcal{A}''_\mathbf{x}, \mathcal{B}''_\mathbf{x}\right) + x_i^b\right) di + \right.$$

$$\left. - \int_{i \in \mathcal{A}'_\mathbf{x}} \left(n^a\left(\mathcal{A}'_\mathbf{x}, \mathcal{B}'_\mathbf{x}\right) + x_i^a\right) di - \int_{i \in \mathcal{B}'_\mathbf{x}} \left(n^b\left(\mathcal{A}'_\mathbf{x}, \mathcal{B}'_\mathbf{x}\right) + x_i^b\right) di\right] =$$

$$\begin{aligned}
= \quad & \mathbb{E}_\mathbf{x}\left[\int_{i \in (\mathcal{A}'_\mathbf{x}/\mathcal{S}_\mathbf{x})} n^a\left(\mathcal{A}''_\mathbf{x}, \mathcal{B}''_\mathbf{x}\right) di + \int_{i \in \mathcal{T}_\mathbf{x}} n^a\left(\mathcal{A}''_\mathbf{x}, \mathcal{B}''_\mathbf{x}\right) di\right. \\
& + \int_{i \in (\mathcal{B}'_\mathbf{x}/\mathcal{T}_\mathbf{x})} n^b\left(\mathcal{A}'_\mathbf{x}, \mathcal{B}'_\mathbf{x}\right) di + \int_{i \in \mathcal{S}_\mathbf{x}} n^b\left(\mathcal{A}'_\mathbf{x}, \mathcal{B}'_\mathbf{x}\right) di + \\
& - \int_{i \in (\mathcal{A}'_\mathbf{x}/\mathcal{S}_\mathbf{x})} n^a\left(\mathcal{A}''_\mathbf{x}, \mathcal{B}''_\mathbf{x}\right) di - \int_{i \in \mathcal{S}_\mathbf{x}} n^a\left(\mathcal{A}''_\mathbf{x}, \mathcal{B}''_\mathbf{x}\right) di + \\
& - \int_{i \in (\mathcal{B}'_\mathbf{x}/\mathcal{T}_\mathbf{x})} n^b\left(\mathcal{A}'_\mathbf{x}, \mathcal{B}'_\mathbf{x}\right) di - \int_{i \in \mathcal{T}_\mathbf{x}} n^b\left(\mathcal{A}'_\mathbf{x}, \mathcal{B}'_\mathbf{x}\right) di + \\
& + \int_{i \in (\mathcal{A}'_\mathbf{x}/\mathcal{S}_\mathbf{x})} x_i^a di + \int_{i \in \mathcal{T}_\mathbf{x}} x_i^a di + \int_{i \in (\mathcal{B}'_\mathbf{x}/\mathcal{T}_\mathbf{x})} x_i^b di + \int_{i \in \mathcal{S}_\mathbf{x}} x_i^b di +
\end{aligned}$$

$$-\int_{i\in(\mathcal{A}'_{\mathbf{x}}/\mathcal{S}_{\mathbf{x}})} x_i^a di - \int_{i\in\mathcal{S}_{\mathbf{x}}} x_i^a di - \int_{i\in(\mathcal{B}'_{\mathbf{x}}/\mathcal{T}_{\mathbf{x}})} x_i^b di - \int_{i\in\mathcal{T}_{\mathbf{x}}} x_i^b di \Bigg] =$$

$$= \mathbb{E}_{\mathbf{x}}\left[\int_{i\in\cup\mathcal{T}_{\mathbf{x}}} x_i^a di + \int_{i\in\cup\mathcal{S}_{\mathbf{x}}} x_i^b di - \int_{i\in\cup\mathcal{S}_{\mathbf{x}}} x_i^a di - \int_{i\in\mathcal{T}_{\mathbf{x}}} x_i^b di\right] =$$

$$= -\mathbb{E}_{\mathbf{x}}\left[\int_{i\in\mathcal{S}_{\mathbf{x}}} 2x_i di - \int_{i\in\mathcal{T}_{\mathbf{x}}} 2x_i di\right] > 0$$

where the last inequality holds because by construction $x_i < x_{i'}$ for every (i, i') such that $i \in \mathcal{S}_{\mathbf{x}}$ and $i' \in \mathcal{T}_{\mathbf{x}}$. This proves that allocation $(\mathcal{A}'', \mathcal{B}'')$ yields a large expected welfare than allocation $(\mathcal{A}', \mathcal{B}')$. ■

Proof of Proposition 1. The proof of Proposition 1 is divided into 4 claims. In Claim 1 we derive expression (2) for the welfare function. In Claim 2 we prove that the welfare function has a global maximum and the maximum is attained at a point $t \in (-1, 1)$. In Claim 3 we prove that the optimum cannot be in the interval $t \in [0, y]$. For the case $y \geq 1$, this completes the proof. Claim 4 considers the case where $y < 1$ and shows that it cannot be the case that $t^* \in (y, 1)$. This concludes the proof.

Claim 1: Expression (2) represents the ex-ante welfare function associated to a generic threshold allocation.

For a given realization of $\left(\widetilde{\theta}^a, \widetilde{\theta}^b\right)$, take a generic threshold allocation $(\mathcal{A}(t), \mathcal{B}(t))$. Let $I(\cdot)$ denote the indicator function. The level of welfare associated to $(\mathcal{A}(t), \mathcal{B}(t))$ is

$$\mathbb{E}_{\mathbf{x}}\left[W(t) \middle| \theta^a, \theta^b\right] =$$

$$\begin{aligned}
= \quad & \mathbb{E}_{\mathbf{x}}\Bigg[\int_{-\infty}^{t-\theta} \theta^b f(\varepsilon_i)\, d\varepsilon_i + \int_{t-\theta}^{+\infty} \theta^a f(\varepsilon_i)\, d\varepsilon_i + \\
& + \int_{-\infty}^{t-\theta} (1 - n^a(\mathcal{A}(t), \mathcal{B}(t))) f(\varepsilon_i)\, d\varepsilon_i + \int_{t-\theta}^{+\infty} n^a(\mathcal{A}(t), \mathcal{B}(t)) f(\varepsilon_i)\, d\varepsilon_i + \\
& + \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \varepsilon_i^a I\left(\frac{\varepsilon_i^a - \varepsilon_i^b}{2} \geqslant t - \theta\right) f(\varepsilon_i^a) f\left(\varepsilon_i^b\right) d\varepsilon_i^a d\varepsilon_i^b + \\
& + \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \varepsilon_i^b I\left(\frac{\varepsilon_i^a - \varepsilon_i^b}{2} \leqslant t - \theta\right) f(\varepsilon_i^a) f\left(\varepsilon_i^b\right) d\varepsilon_i^a d\varepsilon_i^b\Bigg] =
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}_{\mathbf{x}}\left[\theta^b \int_{-\infty}^{t-\theta} f(\varepsilon_i)\, d\varepsilon_i + \theta^a \int_{t-\theta}^{+\infty} f(\varepsilon_i)\, d\varepsilon_i + \right.\\
&\quad + \left(1 - n^a\left(\mathcal{A}(t), \mathcal{B}(t)\right)\right) \int_{-\infty}^{t-\theta} f(\varepsilon_i)\, d\varepsilon_i + n^a\left(\mathcal{A}(t), \mathcal{B}(t)\right) \int_{t-\theta}^{+\infty} f(\varepsilon_i)\, d\varepsilon_i + \\
&\quad + \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \varepsilon_i^a I\left(\frac{\varepsilon_i^a - \varepsilon_i^b}{2} \geqslant t - \theta\right) f(\varepsilon_i^a)\, f\left(\varepsilon_i^b\right) d\varepsilon_i^a d\varepsilon_i^b + \\
&\quad \left. + \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \varepsilon_i^b I\left(\frac{\varepsilon_i^a - \varepsilon_i^b}{2} \leqslant t - \theta\right) f(\varepsilon_i^a)\, f\left(\varepsilon_i^b\right) d\varepsilon_i^a d\varepsilon_i^b\right] =
\end{aligned}
$$

$$
\begin{aligned}
&= \mathbb{E}_{\mathbf{x}}\left[\theta^b \Phi\left((t-\theta)\sqrt{\beta}\right) + \theta^a\left[1 - \Phi\left((t-\theta)\sqrt{\beta}\right)\right] + \right.\\
&\quad + \left(1 - n^a\left(\mathcal{A}(t), \mathcal{B}(t)\right)\right)\Phi\left((t-\theta)\sqrt{\beta}\right) + n^a\left(\mathcal{A}(t), \mathcal{B}(t)\right)\left[1 - \Phi\left((t-\theta)\sqrt{\beta}\right)\right] + \\
&\quad + \int_{-\infty}^{+\infty}\left[\int_{\varepsilon_i^b + 2(t-\theta)}^{+\infty} \varepsilon_i^a f(\varepsilon_i^a)\, d\varepsilon_i^a\right] f\left(\varepsilon_i^b\right) d\varepsilon_i^b + \\
&\quad \left. + \int_{-\infty}^{+\infty}\left[\int_{\varepsilon_i^a - 2(t-\theta)}^{+\infty} \varepsilon_i^b f\left(\varepsilon_i^b\right) d\varepsilon_i^b\right] f(\varepsilon_i^a)\, d\varepsilon_i^a\right] =
\end{aligned}
$$

$$
\begin{aligned}
&= \mathbb{E}_{\mathbf{x}}\left[\theta^b \Phi\left((t-\theta)\sqrt{\beta}\right) + \theta^a\left[1 - \Phi\left((t-\theta)\sqrt{\beta}\right)\right] + \right.\\
&\quad + \left[\Phi\left((t-\theta)\sqrt{\beta}\right)\right]^2 + \left[1 - \Phi\left((t-\theta)\sqrt{\beta}\right)\right]^2 + \\
&\quad + \int_{-\infty}^{+\infty}\left[\sqrt{\frac{2}{\beta}}\phi\left(\left(\varepsilon_i^b + 2(t-\theta)\right)\sqrt{\frac{\beta}{2}}\right)\right] f\left(\varepsilon_i^b\right) d\varepsilon_i^b + \\
&\quad \left. + \int_{-\infty}^{+\infty}\left[\sqrt{\frac{2}{\beta}}\phi\left(\left(\varepsilon_i^a - 2(t-\theta)\right)\sqrt{\frac{\beta}{2}}\right)\right] f(\varepsilon_i^a)\, d\varepsilon_i^a\right] =
\end{aligned}
$$

$$
\begin{aligned}
&= \Phi\left((t-\theta)\sqrt{\beta}\right)\left[\theta^b - \theta^a\right] + \theta^a + \\
&\quad + 2\left[\Phi\left((t-\theta)\sqrt{\beta}\right)\right]^2 - 2\Phi\left((t-\theta)\sqrt{\beta}\right) + 1 + \\
&\quad + \sqrt{\frac{2}{\pi\beta}} e^{-\frac{(t-\theta)^2\beta}{2}}. \qquad (12)
\end{aligned}
$$

Therefore, the expected welfare as a function of a generic threshold t is:

$$
\begin{aligned}
\mathbb{E}_{\mathbf{x}}\left[W(t)\right] &= \mathbb{E}_{\theta^a,\theta^b}\left\{\theta^a - 2\theta\Phi\left((t-\theta)\sqrt{\beta}\right) + \right.\\
&\quad + \sqrt{\frac{2}{\pi\beta}} e^{-\frac{(t-\theta)^2\beta}{2}} + \\
&\quad \left. + 2\left[\Phi\left((t-\theta)\sqrt{\beta}\right)\right]^2 - 2\Phi\left((t-\theta)\sqrt{\beta}\right) + 1\right\}.
\end{aligned}
$$

Claim 2: The welfare function has a global maximum and the maximum is attained at a point $t \in (-1, 1)$.

Given Claim 1, the optimal threshold t^* solves the following problem:

$$\begin{aligned}\max_t \mathbb{E}_{\mathbf{x}}[W(t)] &= \max_t \mathbb{E}_{\theta^a,\theta^b}\left\{\left[\theta^a - 2\theta\Phi\left((t-\theta)\sqrt{\beta}\right)\right] + \right.\\ &+ \left[\sqrt{\frac{2}{\pi\beta}} e^{-\frac{(t-\theta)^2\beta}{2}}\right] + \\ &\left. + \left[2\left[\Phi\left((t-\theta)\sqrt{\beta}\right)\right]^2 - 2\Phi\left((t-\theta)\sqrt{\beta}\right) + 1\right]\right\}.\end{aligned}$$

The welfare function is continuous and differentiable in t, and by Leibnitz' rule, its first derivative can be written as:

$$\frac{\partial}{\partial t}\mathbb{E}_{\mathbf{x}}[W(t)] =$$

$$= \mathbb{E}_{\theta^a,\theta^b}\left[\left(2\Phi\left((t-\theta)\sqrt{\beta}\right) - 1 - \theta - t + \theta\right)\left(2\sqrt{\beta}\phi\left((t-\theta)\sqrt{\beta}\right)\right)\right] =$$

$$= \mathbb{E}_{\theta^a,\theta^b}\left[\left(2\Phi\left((t-\theta)\sqrt{\beta}\right) - 1 - t\right)\left(2\sqrt{\beta}\phi\left((t-\theta)\sqrt{\beta}\right)\right)\right].$$

Notice that

$$2\sqrt{\beta}\phi\left((t-\theta)\sqrt{\beta}\right) > 0 \qquad \forall\theta.$$

This implies that, since for every $t \leq -1$

$$2\Phi\left((t-\theta)\sqrt{\beta}\right) - 1 - t > 0 \qquad \forall\theta$$

then it is true that for every $t \leq -1$

$$\mathbb{E}_{\theta^a,\theta^b}\left[\left(2\Phi\left((t-\theta)\sqrt{\beta}\right) - 1 - t\right)\left(2\sqrt{\beta}\phi\left((t-\theta)\sqrt{\beta}\right)\right)\right] > 0.$$

So the derivative is always positive for any $t \leq -1$.

Analogously, since for every $t \geq 1$ it is true that

$$2\Phi\left((t-\theta)\sqrt{\beta}\right) - 1 - t < 0 \qquad \forall\theta,$$

then for every $t \geq 1$ it is also true that

$$\mathbb{E}_{\theta^a,\theta^b}\left[\left(2\Phi\left((t-\theta)\sqrt{\beta}\right) - 1 - t\right)\left(2\sqrt{\beta}\phi\left((t-\theta)\sqrt{\beta}\right)\right)\right] < 0,$$

i.e. the derivative is always negative for any $t \geq 1$.

If we restrict the domain of the welfare function to any set $[t_1, t_2]$ such that $(-1, 1) \subset [t_1, t_2]$, the function has a global maximum $t^* \in [t_1, t_2]$ by Weierstrass theorem. If then we extend the domain to $\mathbb{R}$, it is true that t^* is still the global maximum because $\mathbb{E}_{\mathbf{x}}[W(t)] < \mathbb{E}_{\mathbf{x}}[W(-1)]$ $\forall t < t_1$ and $\mathbb{E}_{\mathbf{x}}[W(t)] < \mathbb{E}_{\mathbf{x}}[W(1)]$ $\forall t > t_2$.

By a similar argument, it has to be the case that $t^* \in (-1,1) \subset [t_1, t_2]$ because $\mathbb{E}_{\mathbf{x}}[W(t)] < \mathbb{E}_{\mathbf{x}}[W(-1)]$ $\forall t$ such that $t_1 < t \leq -1$ and $\mathbb{E}_{\mathbf{x}}[W(t)] < \mathbb{E}_{\mathbf{x}}[W(1)]$ $\forall t$ such that $1 \leq t < t_2$.

Claim 3 The optimum cannot be in the interval $t \in [0, y]$.

We will prove this claim by showing that welfare is strictly decreasing $\forall t \in [0, y]$.

By the definition of a derivative,

$$\frac{\partial}{\partial t}\mathbb{E}_{\mathbf{x}}[W(t)] = \lim_{\delta \to 0} \frac{\mathbb{E}_{\mathbf{x}}[W(t+\delta)] - \mathbb{E}_{\mathbf{x}}[W(t)]}{\delta} = \lim_{\delta \to 0^+} \frac{\mathbb{E}_{\mathbf{x}}[W(t+\delta)] - \mathbb{E}_{\mathbf{x}}[W(t)]}{\delta}.$$

The welfare function $\mathbb{E}_{\mathbf{x}}[W(t)]$ can be rewritten as:

$$\mathbb{E}_{\mathbf{x}}[W(t)] =$$

$$= \mathbb{E}_{\mathbf{x}}\left[\int_{i \in \mathcal{A}_{\mathbf{x}}(t)} n^a(\mathcal{A}_{\mathbf{x}}(t), \mathcal{B}_{\mathbf{x}}(t))\, di + \int_{i \in \mathcal{A}_{\mathbf{x}}(t)} x_i^a di + \int_{i \in \mathcal{B}_{\mathbf{x}}(t)} n^b(\mathcal{A}_{\mathbf{x}}(t), \mathcal{B}_{\mathbf{x}}(t))\, di + \int_{i \in \mathcal{B}_{\mathbf{x}}(t)} x_i^b di\right] =$$

$$= \mathbb{E}_{\mathbf{x}}\left[\left(\int_{i \in \mathcal{A}_{\mathbf{x}}(t)} di\right)^2 + \left(\int_{i \in \mathcal{B}_{\mathbf{x}}(t)} di\right)^2 + \int_{i \in \mathcal{A}_{\mathbf{x}}(t)} x_i^a di + \int_{i \in \mathcal{B}_{\mathbf{x}}(t)} x_i^b di\right].$$

Hence, for any positive δ :

$$\mathbb{E}_{\mathbf{x}}[W(t+\delta)] - \mathbb{E}_{\mathbf{x}}[W(t)] =$$

$$= \mathbb{E}_{\mathbf{x}}\left[\left(\int_{i \in \mathcal{A}_{\mathbf{x}}(t+\delta)} di\right)^2 - \left(\int_{i \in \mathcal{A}_{\mathbf{x}}(t)} di\right)^2 + \left(\int_{i \in \mathcal{B}_{\mathbf{x}}(t+\delta)} di\right)^2 - \left(\int_{i \in \mathcal{B}_{\mathbf{x}}(t)} di\right)^2 + \right.$$

$$\left. + \int_{i \in \mathcal{A}_{\mathbf{x}}(t+\delta)} x_i^a di - \int_{i \in \mathcal{A}_{\mathbf{x}}(t)} x_i^a di + \int_{i \in \mathcal{B}_{\mathbf{x}}(t+\delta)} x_i^b di - \int_{i \in \mathcal{B}_{\mathbf{x}}(t)} x_i^b di\right] =$$

$$= \mathbb{E}_{\mathbf{x}}\left[\left(\int_{i \in \mathcal{A}_{\mathbf{x}}(t+\delta)} di\right)^2 - \left(\int_{i \in \mathcal{A}_{\mathbf{x}}(t)} di\right)^2 + \left(\int_{i \in \mathcal{B}_{\mathbf{x}}(t+\delta)} di\right)^2 - \left(\int_{i \in \mathcal{B}_{\mathbf{x}}(t)} di\right)^2 + \right.$$

$$\left. + \int_{i \in \mathcal{A}_{\mathbf{x}}(t+\delta)/\mathcal{A}_{\mathbf{x}}(t)} x_i^a di + \int_{i \in \mathcal{B}_{\mathbf{x}}(t+\delta)/\mathcal{B}_{\mathbf{x}}(t)} x_i^b di\right] =$$

$$= \mathbb{E}_{\mathbf{x}}\left[\left(\int_{i \in \mathcal{A}_{\mathbf{x}}(t+\delta)} di\right)^2 - \left(\int_{i \in \mathcal{A}_{\mathbf{x}}(t)} di\right)^2 + \left(\int_{i \in \mathcal{B}_{\mathbf{x}}(t+\delta)} di\right)^2 - \left(\int_{i \in \mathcal{B}_{\mathbf{x}}(t)} di\right)^2 + \right.$$

$$\left. + \int_{i \in \mathcal{B}_{\mathbf{x}}(t+\delta)/\mathcal{B}_{\mathbf{x}}(t)} \left(x_i^b - x_i^a\right) di\right] =$$

$$= \mathbb{E}_{\mathbf{x}}\left[\left(\int_{i\in\mathcal{A}_{\mathbf{x}}(t+\delta)} di\right)^2 - \left(\int_{i\in\mathcal{A}_{\mathbf{x}}(t)} di\right)^2 + \right.$$

$$\left. + \left(\int_{i\in\mathcal{B}_{\mathbf{x}}(t+\delta)} di\right)^2 - \left(\int_{i\in\mathcal{B}_{\mathbf{x}}(t)} di\right)^2 - \int_{i\in\mathcal{B}_{\mathbf{x}}(t+\delta)/\mathcal{B}_{\mathbf{x}}(t)} 2x_i di\right]. \qquad (13)$$

where the third inequality follows from the definition of x_i.

Since there is a continuum of consumers, and we assume that law of large numbers holds, each of the sets $\mathcal{A}_{\mathbf{x}}(t+\delta)$, $\mathcal{A}_{\mathbf{x}}(t)$, $\mathcal{B}_{\mathbf{x}}(t+\delta)$, $\mathcal{B}_{\mathbf{x}}(t)$ is identical for almost every realization of $\mathbf{x}$, up to a permutation of the $i's$, hence the expression can be rewritten as

$$[1 - F(t+\delta)]^2 - [1 - F(t)]^2 + [F(t+\delta)]^2 - [F(t)]^2 - \int_t^{t+\delta} 2x_i f(x_i)\, dx_i \qquad (14)$$

where $F(\cdot)$ denotes the cdf of the random variable x_i.

Let

$$H_1(\delta) \equiv [1 - F(t+\delta)]^2 - [1 - F(t)]^2 + [F(t+\delta)]^2 - [F(t)]^2 - 2(t+\delta)[F(t+\delta) - F(t)].$$

Since

$$-\int_t^{t+\delta} 2x_i f(x_i)\, dx_i > -2(t+\delta)\int_t^{t+\delta} f(x_i)\, dx_i = -2(t+\delta)[F(t+\delta) - F(t)] \qquad \forall\delta > 0,$$

then

$$H_1(\delta) < \mathbb{E}_{\mathbf{x}}[W(t+\delta)] - \mathbb{E}_{\mathbf{x}}[W(t)] \qquad \forall\delta > 0,$$

which implies

$$\frac{H_1(\delta)}{\delta} < \frac{\mathbb{E}_{\mathbf{x}}[W(t+\delta)] - \mathbb{E}_{\mathbf{x}}[W(t)]}{\delta} \qquad \forall\delta > 0.$$

Similarly, let

$$H_2(\delta) \equiv [1 - F(t+\delta)]^2 - [1 - F(t)]^2 + [F(t+\delta)]^2 - [F(t)]^2 - 2t[F(t+\delta) - F(t)].$$

Since

$$-\int_t^{t+\delta} 2x_i f(x_i)\, dx_i < -2t\int_t^{t+\delta} f(x_i)\, dx_i = -2t[F(t+\delta) - F(t)] \qquad \forall\delta > 0,$$

then

$$\mathbb{E}_{\mathbf{x}}[W(t+\delta)] - \mathbb{E}_{\mathbf{x}}[W(t)] < H_2(\delta) \qquad \forall\delta > 0,$$

which implies

$$\frac{\mathbb{E}_{\mathbf{x}}\left[W\left(t+\delta\right)\right]-\mathbb{E}_{\mathbf{x}}\left[W\left(t\right)\right]}{\delta}<\frac{H_2\left(\delta\right)}{\delta}\qquad\forall\delta>0.$$

Hence, it holds that

$$\frac{H_1\left(\delta\right)}{\delta}<\frac{\mathbb{E}_{\mathbf{x}}\left[W\left(t+\delta\right)\right]-\mathbb{E}_{\mathbf{x}}\left[W\left(t\right)\right]}{\delta}<\frac{H_2\left(\delta\right)}{\delta}\qquad\forall\delta>0.$$

Next, notice that

$$\lim_{\delta\to0^+}\frac{H_1\left(\delta\right)}{\delta}=\lim_{\delta\to0^+}\frac{\left[1-F\left(t+\delta\right)\right]^2-\left[1-F\left(t\right)\right]^2+\left[F\left(t+\delta\right)\right]^2-\left[F\left(t\right)\right]^2-2\left(t+\delta\right)\left[F\left(t+\delta\right)-F\left(t\right)\right]}{\delta}\tag{15}$$

is equal, by l'Hôpital's rule, to

$$\begin{aligned}&\lim_{\delta\to0^+}\left\{f\left(t+\delta\right)\left[2F\left(t+\delta\right)-2+2F\left(t+\delta\right)-2\left(t+\delta\right)\right]-2\left[F\left(t+\delta\right)-F\left(t\right)\right]\right\}=\\&=\lim_{\delta\to0^+}\left\{2f\left(t+\delta\right)\left[2F\left(t+\delta\right)-1-\left(t+\delta\right)\right]-2\left[F\left(t+\delta\right)-F\left(t\right)\right]\right\}\\&=2f\left(t\right)\left\{2F\left(t\right)-1-t\right\}.\end{aligned}$$

Similarly,

$$\lim_{\delta\to0^+}\frac{H_2\left(\delta\right)}{\delta}=\lim_{\delta\to0^+}\frac{\left[1-F\left(t+\delta\right)\right]^2-\left[1-F\left(t\right)\right]^2+\left[F\left(t+\delta\right)\right]^2-\left[F\left(t\right)\right]^2-2t\left[F\left(t+\delta\right)-F\left(t\right)\right]}{\delta}\tag{16}$$

is equal, by l'Hôpital's rule, to

$$\begin{aligned}&\lim_{\delta\to0^+}\left\{f\left(t+\delta\right)\left[2F\left(t+\delta\right)-2+2F\left(t+\delta\right)-2t\right]\right\}=\\&=\lim_{\delta\to0^+}\left\{2f\left(t+\delta\right)\left[2F\left(t+\delta\right)-1-t\right]\right\}\\&=2f\left(t\right)\left\{2F\left(t\right)-1-t\right\}.\end{aligned}$$

Since

$$\frac{H_1\left(\delta\right)}{\delta}<\frac{\mathbb{E}_{\mathbf{x}}\left[W\left(t+\delta\right)\right]-\mathbb{E}_{\mathbf{x}}\left[W\left(t\right)\right]}{\delta}<\frac{H_2\left(\delta\right)}{\delta}\qquad\forall\delta>0$$

and

$$\lim_{\delta\to0^+}\frac{H_1\left(\delta\right)}{\delta}=\lim_{\delta\to0^+}\frac{H_2\left(\delta\right)}{\delta}=2f\left(t\right)\left\{2F\left(t\right)-1-t\right\}$$

we can conclude that

$$\frac{\partial}{\partial t}\mathbb{E}_{\mathbf{x}}\left[W\left(t\right)\right]=\lim_{\delta\to0^+}\frac{\mathbb{E}_{\mathbf{x}}\left[W\left(t+\delta\right)\right]-\mathbb{E}_{\mathbf{x}}\left[W\left(t\right)\right]}{\delta}=2f\left(t\right)\left\{2F\left(t\right)-1-t\right\}.\tag{17}$$

Moreover, since $y > 0$ implies that

$$2f(t)\{2F(t) - 1 - t\} < 0 \qquad \forall t \in [0, y],$$

we can conclude that

$$\frac{\partial}{\partial t}\mathbb{E}_{\mathbf{x}}[W(t)] < 0 \qquad \forall t \in [0, y]$$

and that therefore the welfare maximizing threshold cannot be in the interval $t \in [0, y]$.

Claim 4. It cannot be the case that $t^* \in (y, 1)$.

We will prove Claim 4 by proving that for any $t > y$ there exists a $t' < y$ such that $\mathbb{E}_{\mathbf{x}}[W(t')] > \mathbb{E}_{\mathbf{x}}[W(t)]$.

Let $t \in (y, 1)$ and take $t' < y$ such that

$$\left|y - t'\right| = |y - t| \equiv \Delta.$$

It holds that

$$\begin{aligned}
&\mathbb{E}_{\mathbf{x}}\left[W\left(t'\right)\right] - \mathbb{E}_{\mathbf{x}}[W(t)] = \\
&= \mathbb{E}_{\theta^a,\theta^b}\left[\theta^a - 2\theta\Phi\left(\left(t' - \theta\right)\sqrt{\beta}\right)\right] - \mathbb{E}_{\theta^a,\theta^b}\left[\theta^a - 2\theta\Phi\left((t - \theta)\sqrt{\beta}\right)\right] + \\
&+\mathbb{E}_{\theta^a,\theta^b}\left[\sqrt{\frac{2}{\pi\beta}}e^{-\frac{(t'-\theta)^2\beta}{2}}\right] - \mathbb{E}_{\theta^a,\theta^b}\left[\sqrt{\frac{2}{\pi\beta}}e^{-\frac{(t-\theta)^2\beta}{2}}\right] + \\
&+\mathbb{E}_{\theta^a,\theta^b}\left[\left[2\left[\Phi\left(\left(t' - \theta\right)\sqrt{\beta}\right)\right]^2 - 2\Phi\left(\left(t' - \theta\right)\sqrt{\beta}\right) + 1\right]\right] + \\
&-\mathbb{E}_{\theta^a,\theta^b}\left[\left[2\left[\Phi\left((t - \theta)\sqrt{\beta}\right)\right]^2 - 2\Phi\left((t - \theta)\sqrt{\beta}\right) + 1\right]\right] = \\
&= \mathbb{E}_{\theta}\left[-2\theta\Phi\left(\left(t' - \theta\right)\sqrt{\beta}\right)\right] - \mathbb{E}_{\theta}\left[-2\theta\Phi\left((t - \theta)\sqrt{\beta}\right)\right].
\end{aligned}$$

The last equality holds because both

$$\mathbb{E}_{\theta^a,\theta^b}\left[\sqrt{\frac{2}{\pi\beta}}e^{-\frac{(t-\theta)^2\beta}{2}}\right]$$

and

$$\begin{aligned}
&\mathbb{E}_{\theta^a,\theta^b}\left[\left[2\left[\Phi\left((t - \theta)\sqrt{\beta}\right)\right]^2 - 2\Phi\left((t - \theta)\sqrt{\beta}\right) + 1\right]\right] = \\
&= \mathbb{E}_{\theta^a,\theta^b}\left[\left[\Phi\left((t - \theta)\sqrt{\beta}\right)\right]^2 + \left[1 - \Phi\left((t - \theta)\sqrt{\beta}\right)\right]^2\right]
\end{aligned}$$

are symmetric with respect to y.

To establish the sign of

$$= \mathbb{E}_\theta \left[-2\theta \Phi \left(\left(t' - \theta \right) \sqrt{\beta} \right) \right] - \mathbb{E}_\theta \left[-2\theta \Phi \left(\left(t - \theta \right) \sqrt{\beta} \right) \right]$$

rewrite the expression as

$$\int_{-\infty}^{+\infty} 2\theta \left[\Phi \left(\left(y + \Delta - \theta \right) \sqrt{\beta} \right) - \Phi \left(\left(y - \Delta - \theta \right) \sqrt{\beta} \right) \right] g\left(\theta\right) d\theta$$

where $g(\theta)$ denotes the pdf of the random variable θ.

Notice that

$$\int_{-\infty}^{+\infty} 2\theta \left[\Phi \left(\left(y + \Delta - \theta \right) \sqrt{\beta} \right) - \Phi \left(\left(y - \Delta - \theta \right) \sqrt{\beta} \right) \right] g\left(\theta\right) d\theta =$$

$$= \int_{-\infty}^{0} 2\theta \left[\Phi \left(\left(y + \Delta - \theta \right) \sqrt{\beta} \right) - \Phi \left(\left(y - \Delta - \theta \right) \sqrt{\beta} \right) \right] g\left(\theta\right) d\theta +$$

$$+ \int_{0}^{2y} 2\theta \left[\Phi \left(\left(y + \Delta - \theta \right) \sqrt{\beta} \right) - \Phi \left(\left(y - \Delta - \theta \right) \sqrt{\beta} \right) \right] g\left(\theta\right) d\theta +$$

$$+ \int_{2y}^{+\infty} 2\theta \left[\Phi \left(\left(y + \Delta - \theta \right) \sqrt{\beta} \right) - \Phi \left(\left(y - \Delta - \theta \right) \sqrt{\beta} \right) \right] g\left(\theta\right) d\theta.$$

It holds that

$$\int_{0}^{2y} 2\theta \left[\Phi \left(\left(y + \Delta - \theta \right) \sqrt{\beta} \right) - \Phi \left(\left(y - \Delta - \theta \right) \sqrt{\beta} \right) \right] g\left(\theta\right) d\theta > 0$$

since the integrand is nonnegative $\forall \theta \in [0, 2y]$ and strictly positive $\forall \theta \in (0, 2y]$.

Moreover,

$$\int_{-\infty}^{0} 2\theta \left[\Phi \left(\left(y + \Delta - \theta \right) \sqrt{\beta} \right) - \Phi \left(\left(y - \Delta - \theta \right) \sqrt{\beta} \right) \right] g\left(\theta\right) d\theta \leq 0$$

because the integrand is nonpositive $\forall \theta \in (-\infty, 0]$ and

$$\int_{2y}^{+\infty} 2\theta \left[\Phi \left(\left(y + \Delta - \theta \right) \sqrt{\beta} \right) - \Phi \left(\left(y - \Delta - \theta \right) \sqrt{\beta} \right) \right] g\left(\theta\right) d\theta \geq 0$$

because the integrand is nonnegative $\forall \theta \in [2y, +\infty)$.

Next, we show that

$$\int_{-\infty}^{0} 2\theta \left[\Phi \left(\left(y + \Delta - \theta \right) \sqrt{\beta} \right) - \Phi \left(\left(y - \Delta - \theta \right) \sqrt{\beta} \right) \right] g\left(\theta\right) d\theta +$$

$$+ \int_{2y}^{+\infty} 2\theta \left[\Phi \left(\left(y + \Delta - \theta \right) \sqrt{\beta} \right) - \Phi \left(\left(y - \Delta - \theta \right) \sqrt{\beta} \right) \right] g\left(\theta\right) d\theta > 0.$$

Take any $\theta \in [2y, +\infty)$ and denote $\theta - 2y = d > 0$.

For any such θ, there exists $\theta' \in (-\infty, 0]$ such that $\theta' = -d$. By symmetry of the distribution of θ, $g(\theta') = g(\theta)$.
Moreover,

$$2\theta'\left[\Phi\left(\left(y+\Delta-\theta'\right)\sqrt{\beta}\right)-\Phi\left(\left(y-\Delta-\theta'\right)\sqrt{\beta}\right)\right]+$$
$$+2\theta\left[\Phi\left(\left(y+\Delta-\theta\right)\sqrt{\beta}\right)-\Phi\left(\left(y-\Delta-\theta\right)\sqrt{\beta}\right)\right]=$$
$$=-2d\left[\Phi\left(\left(y+d+\Delta\right)\sqrt{\beta}\right)-\Phi\left(\left(y+d-\Delta\right)\sqrt{\beta}\right)\right]+$$
$$+\left(4y+2d\right)\left[\Phi\left(\left(-y-d+\Delta\right)\sqrt{\beta}\right)-\Phi\left(\left(-y-d-\Delta\right)\sqrt{\beta}\right)\right]=$$
$$=4y\left[\Phi\left(\left(-y-d+\Delta\right)\sqrt{\beta}\right)-\Phi\left(\left(-y-d-\Delta\right)\sqrt{\beta}\right)\right]>0.$$

Since $\forall\theta \in [2y, +\infty)$ there exists $\theta' \in (-\infty, 0]$ such that

$$2\theta\left[\Phi\left(\left(y+\Delta-\theta\right)\sqrt{\beta}\right)-\Phi\left(\left(y-\Delta-\theta\right)\sqrt{\beta}\right)\right]g(\theta)+$$
$$+2\theta'\left[\Phi\left(\left(y+\Delta-\theta'\right)\sqrt{\beta}\right)-\Phi\left(\left(y-\Delta-\theta'\right)\sqrt{\beta}\right)\right]g(\theta')>0,$$

then

$$\int_{-\infty}^{0}2\theta\left[\Phi\left(\left(y+\Delta-\theta\right)\sqrt{\beta}\right)-\Phi\left(\left(y-\Delta-\theta\right)\sqrt{\beta}\right)\right]g(\theta)\,d\theta+$$
$$+\int_{0}^{2y}2\theta\left[\Phi\left(\left(y+\Delta-\theta\right)\sqrt{\beta}\right)-\Phi\left(\left(y-\Delta-\theta\right)\sqrt{\beta}\right)\right]g(\theta)\,d\theta\geq 0$$

and combining this inequality with

$$+\int_{0}^{2y}2\theta\left[\Phi\left(\left(y+\Delta-\theta\right)\sqrt{\beta}\right)-\Phi\left(\left(y-\Delta-\theta\right)\sqrt{\beta}\right)\right]g(\theta)\,d\theta>0$$

we get

$$\mathbb{E}_{\theta}\left[-2\theta\Phi\left(\left(t'-\theta\right)\sqrt{\beta}\right)\right]-\mathbb{E}_{\theta}\left[-2\theta\Phi\left(\left(t-\theta\right)\sqrt{\beta}\right)\right]>0.$$

This implies that if $0 < y \leq 1$ it cannot be the case that the welfare function attains its global maximum for $t \in (y, 1]$.
We conclude that both if $y \in (0, 1]$ and if $y > 1$ it has to be the case that the global maximizer of the welfare function, t^*, satisfies

$$-1 < t^* < 0 < y.$$

■

Proof of Corollary 1. From the proof of Proposition 1, we know that t^* solves

$$\frac{\partial}{\partial t}\mathbb{E}_{\mathbf{x}}\left[W(t)\right]\Big/_{t=t^*}=2f(t^*)\left\{2F(t^*)-1-t^*\right\}=2\phi\left[\frac{t^*-y}{\sigma}\right]\left\{2\Phi\left[\frac{t^*-y}{\sigma}\right]-1-t^*\right\}=0. \quad (18)$$

where

$$\sigma \equiv \sqrt{\frac{1}{\alpha}+\frac{1}{\beta}}$$

is the unconditional standard error of the consumers' private values.

Using the chain rule

$$\frac{\partial t^*}{\partial \alpha}=\frac{\partial t^*}{\partial \sigma}\frac{\partial \sigma}{\partial \alpha}, \text{ and } \frac{\partial t^*}{\partial \beta}=\frac{\partial t^*}{\partial \sigma}\frac{\partial \sigma}{\partial \beta}.$$

First, notice that

$$\frac{\partial \sigma}{\partial \alpha}=\frac{1}{2\sqrt{\frac{1}{\alpha}+\frac{1}{\beta}}}\left(-\frac{1}{\alpha^2}\right)<0$$

and

$$\frac{\partial \sigma}{\partial \beta}=\frac{1}{2\sqrt{\frac{1}{\alpha}+\frac{1}{\beta}}}\left(-\frac{1}{\beta^2}\right)<0.$$

Since t^* has to satisfy equation (18), it follows from the implicit function theorem that

$$\frac{\partial t^*}{\partial \sigma}=-\frac{\frac{\partial\left\{2\phi\left[\frac{t^*-y}{\sigma}\right]\left\{2\Phi\left[\frac{t^*-y}{\sigma}\right]-1-t^*\right\}\right\}}{\partial \sigma}}{\frac{\partial\left\{2\phi\left[\frac{t^*-y}{\sigma}\right]\left\{2\Phi\left[\frac{t^*-y}{\sigma}\right]-1-t^*\right\}\right\}}{\partial t^*}}. \tag{19}$$

The denominator of expression (19) is negative, since t^* is a maximum. Next, we show that the numerator is positive.

$$\frac{\partial\left\{2\phi\left[\frac{t^*-y}{\sigma}\right]\left\{2\Phi\left[\frac{t^*-y}{\sigma}\right]-1-t^*\right\}\right\}}{\partial \sigma}=$$

$$=\left\{2\Phi\left[\frac{t^*-y}{\sigma}\right]-1-t^*\right\}2\phi\left[\frac{t^*-y}{\sigma}\right]\frac{(t^*-y)^2}{\sigma^3}-4\left\{\phi\left[\frac{t^*-y}{\sigma}\right]\right\}^2\frac{(t^*-y)}{\sigma^2}=$$

$$=-4\left\{\phi\left[\frac{t^*-y}{\sigma}\right]\right\}^2\frac{(t^*-y)}{\sigma^2}>0$$

where the second equality holds because the first term is null by condition (18), and the inequality holds because by Proposition 1, $t^*-y<0$.

We can conclude that

$$\frac{\partial t^*}{\partial \sigma}>0.$$

and that therefore

$$\frac{\partial t^*}{\partial \alpha}=\frac{\partial t^*}{\partial \sigma}\frac{\partial \sigma}{\partial \alpha}<0 and \frac{\partial t^*}{\partial \beta}=\frac{\partial t^*}{\partial \sigma}\frac{\partial \sigma}{\partial \beta}<0.$$

■

Proof of Proposition 2. The characterization of the equilibrium and derivation of the

uniqueness condition follow directly from the Appendix of Morris and Shin (2004).
The evaluate the sign of $\frac{\partial t(p,y)}{\partial p}$,and $\frac{\partial t(p,y)}{\partial y}$, notice that since $t(p,y)$ has to satisfy expression (7), then:

$$\begin{aligned}\frac{\partial t(p,y)}{\partial p} &= -\frac{\frac{\partial\left[t-\Phi[(t-y)z]+\frac{1}{2}-p\right]}{\partial p}}{\frac{\partial\left[t-\Phi[(t-y)z]+\frac{1}{2}-p\right]}{\partial t}} = -\frac{-1}{1-\phi\left[(t-y)z\right]z} \\ \frac{\partial t(p,y)}{\partial y} &= -\frac{\frac{\partial\left[t-\Phi[(t-y)z]+\frac{1}{2}-p\right]}{\partial y}}{\frac{\partial\left[t-\Phi[(t-y)z]+\frac{1}{2}-p\right]}{\partial t}} = -\frac{\phi\left[(t-y)z\right]z}{1-\phi\left[(t-y)z\right]z}.\end{aligned}$$

When the uniqueness condition is satisfied, the denominators of the two expressions above are strictly positive, therefore

$$\frac{\partial t(p,y)}{\partial p} > 0 \text{ and } \frac{\partial t(p,y)}{\partial y} < 0.$$

■

Proof of Proposition 3. If consumers play the equilibrium strategy profile, then all the consumers who observe $x_i > t(p,y)$ and who therefore have an idiosyncratic taste component $\varepsilon_i > t(p,y) - \theta$, buy a and everyone else buys b.
For the second part of the claim, note that

$$\begin{aligned}\frac{\partial n^a(\cdot)}{\partial y^a} &= \frac{\partial n^a}{\partial y}\frac{1}{2} = \left(\frac{1}{2}\right)\left\{-\phi\left[(t(p,y)-\theta)\sqrt{\beta}\right]\sqrt{\beta}\left(\frac{\partial t}{\partial y}\right)\right\} = \\ &= \left(\frac{1}{2}\right)\left\{-\phi\left[(t(p,y)-\theta)\sqrt{\beta}\right]\sqrt{\beta}\left(-\frac{\phi\left[(t-y)z\right]z}{1-\phi\left[(t-y)z\right]z}\right)\right\} > 0\end{aligned}$$

and

$$\begin{aligned}\frac{\partial n^b(\cdot)}{\partial y^b} &= \frac{\partial n^b}{\partial y}\left(-\frac{1}{2}\right) = \left(-\frac{1}{2}\right)\left\{\phi\left[(t(p,y)-\theta)\sqrt{\beta}\right]\sqrt{\beta}\left(\frac{\partial t}{\partial y}\right)\right\} = \\ &= \left(-\frac{1}{2}\right)\left\{\phi\left[(t(p,y)-\theta)\sqrt{\beta}\right]\sqrt{\beta}\left(-\frac{\phi\left[(t-y)z\right]z}{1-\phi\left[(t-y)z\right]z}\right)\right\} > 0.\end{aligned}$$

■

Proof of Proposition 4. For the first statement, notice that the equilibrium threshold for the case of unsponsored networks, $t^u \equiv t(0,y)$, satisfies

$$t^u - \Phi\left[(t^u - y)z\right] + \frac{1}{2} = 0. \tag{20}$$

Under the uniqueness condition, the lhs of (20) is monotonically increasing in t and t^u is well-defined. Since $\Phi[\cdot] \in [0,1]$, for any $t \geq 0$ it holds that the lhs of (20) is strictly negative if $t \leq -\frac{1}{2}$. Therefore, it has to be the case that $t^u > -\frac{1}{2}$. Moreover, for $y = 0$ the solution to (20) is $t^u = 0$, and since $t(p,y)$ is strictly decreasing in y, it has to be the case that $t^u \in \left(-\frac{1}{2}, 0\right)$.

For the second and the third statement, we know from Proposition 1 that $t^* \in (-1, 0)$. Next, we will prove that the welfare function is strictly decreasing in the interval $t \in [t^u, 0]$, which in turn implies that $t^* \in (-1, t^u)$ and that t^u cannot be a point where the function attains its global maximum.

Let $t = t^u$. From the proof of Proposition 1, we know:

$$\frac{\partial}{\partial t}\mathbb{E}_{\mathbf{x}}\left[W(t)\right]\Big/_{t=t^u} = 2f(t^u)\left\{2F(t^u) - 1 - t^u\right\}$$

We will show that

$$2f(t^u)\left\{2F(t^u) - 1 - t^u\right\} \tag{21}$$

is strictly negative.

First, notice that (21) has the same sign as $2F(t^u) - 1 - t^u$, since $f(t^u) > 0$.

From the definition of t^u

$$t^u = \Phi\left[(t^u - y)z\right] - \frac{1}{2}.$$

By assumption, $F(t^u) = \Phi\left[\frac{t^u - y}{\sigma}\right]$ where

$$\sigma^2 \equiv \frac{\alpha + \beta}{\alpha\beta}$$

is the unconditional variance of the consumers' private values.

Therefore, we need to check the sign of

$$2\Phi\left[\frac{t^u - y}{\sigma}\right] - 1 - \Phi\left[(t^u - y)z\right] + \frac{1}{2} = \Phi\left[\frac{t^u - y}{\sigma}\right] - \Phi\left[(t^u - y)z\right] + \Phi\left[\frac{t^u - y}{\sigma}\right] - \frac{1}{2}.$$

Since $\frac{1}{\sigma} > z$ and $t^u - y < 0$, $\Phi\left[\frac{t^u-y}{\sigma}\right] - \Phi\left[(t^u - y)z\right] < 0$ and $\Phi\left[\frac{t^u-y}{\sigma}\right] < \frac{1}{2}$, then the derivative of the welfare function evaluated at t^u, which is equal to (21), is strictly negative.

Finally, let $t \in (t^u, 0]$. We will prove that in this interval $\frac{\partial}{\partial t}\mathbb{E}_{\mathbf{x}}\left[W(t)\right] < 0$ as well. From the proof of Proposition 1,

$$\frac{\partial}{\partial t}\mathbb{E}_{\mathbf{x}}\left[W(t)\right] = 2f(t)\left\{2F(t) - 1 - t\right\}$$

Since $2f(t)$ is always positive, the last expression has the same sign as $2F(t) - 1 - t$. The function $2F(t) - 1 - t$ is the difference of the two strictly increasing functions $r(t) = 2F(t)$ and $s(t) = 1 + t$. This difference is positive for $t < -1$, negative for $t = t^u$ and negative in $t = 0$ (since $F(\cdot)$ is the cdf of x_i which is distributed normally around $y > 0$).

Since the slope of $s(t)$ is constant and the slope of $r(t)$ is increasing in the interval $t \in (-1, 0]$, it cannot be the case that their difference is nonnegative in any point $t \in (t^u, 0)$. Therefore, it holds that $2F(t) - 1 - t < 0$ in the interval $t \in (t^u, 0]$, which in turn implies that the welfare function is strictly decreasing in the interval $t \in [t^u, 0]$.

We can conclude that $t^* \in (-1, t^u)$. ■

Proof of Corollary 2. Using the chain rule,

$$\frac{\partial t^u}{\partial \alpha} = \frac{\partial t^u}{\partial z}\frac{\partial z}{\partial \alpha}$$

and

$$\frac{\partial t^u}{\partial \beta} = \frac{\partial t^u}{\partial z}\frac{\partial z}{\partial \beta}.$$

From Proposition 2, t^u has to satisfy equation (20). It follows from the implicit function theorem that

$$\frac{\partial t^u}{\partial z} = -\frac{\frac{\partial\left[t^u - \Phi[(t^u - y)z] + \frac{1}{2}\right]}{\partial z}}{\frac{\partial\left[t^u - \Phi[(t^u - y)z] + \frac{1}{2}\right]}{\partial t^u}} = -\frac{-\phi\left[(t-y)\,z\right](t^u - y)}{1 - \phi\left[(t-y)\,z\right]z}.$$

Under the uniqueness condition $z \leqslant \sqrt{2\pi}$, the denominator is positive. Also, from Proposition 4 $t^u - y < 0$, hence the numerator is positive as well, and we can conclude that $\frac{\partial t^u}{\partial z} < 0$.

Finally, since

$$\frac{\partial z}{\partial \alpha} = \frac{1}{2}\sqrt{\frac{(\alpha+\beta)(\alpha+2\beta)}{\alpha^2\beta}} \cdot \frac{\alpha\beta\left(3\alpha\beta + 4\beta^2\right)}{(\alpha+\beta)^2(\alpha+2\beta)^2} > 0,$$

we can conclude that

$$\frac{\partial t^u}{\partial \alpha} = \frac{\partial t^u}{\partial z}\frac{\partial z}{\partial \alpha} < 0.$$

Similarly, since

$$\frac{\partial z}{\partial \beta} = \frac{1}{2}\sqrt{\frac{(\alpha+\beta)(\alpha+2\beta)}{\alpha^2\beta}} \cdot \frac{\alpha^2\left(\alpha^2 - 2\beta^2\right)}{(\alpha+\beta)^2(\alpha+2\beta)^2},$$

we can conclude that

$$\text{for } \beta < \frac{\alpha}{\sqrt{2}},\ \frac{\partial z}{\partial \beta} > 0 \text{ and } \frac{\partial t^u}{\partial \beta} = \frac{\partial t^u}{\partial z}\frac{\partial z}{\partial \beta} < 0,$$

$$\text{for } \beta > \frac{\alpha}{\sqrt{2}},\ \frac{\partial z}{\partial \beta} < 0 \text{ and } \frac{\partial t^u}{\partial \beta} = \frac{\partial t^u}{\partial z}\frac{\partial z}{\partial \beta} > 0.$$

Finally, we consider the limit cases. It holds that

$$\lim_{\beta \to +\infty} z = 0$$

which implies that as β goes to infinity, equation(20) which defines t^u converges to

$$t^u = 0.$$

■

Proof of Lemma 2. To calculate the expected market shares, it is sufficient to take the expectation over θ of the market shares expressed in Proposition 3. For the second part of the claim, note that

$$\begin{aligned}
\frac{\partial \mathbb{E}_{\mathbf{x}}\left[n^{a}\right]}{\partial y^{a}} &= \frac{\partial \mathbb{E}_{\mathbf{x}}\left[n^{a}\right]}{\partial y}\frac{1}{2}=\left(\frac{1}{2}\right)\left\{-\phi\left[\frac{t\left(p,y\right)-y}{\sigma}\right]\frac{1}{\sigma}\left(\frac{\partial t}{\partial y}-1\right)\right\}= \\
&= \left(\frac{1}{2}\right)\left\{-\phi\left[\frac{t\left(p,y\right)-y}{\sigma}\right]\frac{1}{\sigma}\left(-\frac{\phi\left[\left(t-y\right)z\right]z}{1-\phi\left[\left(t-y\right)z\right]z}-1\right)\right\}>0
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \mathbb{E}_{\mathbf{x}}\left[n^{b}\right]}{\partial y^{b}} &= \frac{\partial \mathbb{E}_{\mathbf{x}}\left[n^{b}\right]}{\partial y}\left(-\frac{1}{2}\right)=\left(-\frac{1}{2}\right)\left\{\phi\left[\frac{t\left(p,y\right)-y}{\sigma}\right]\frac{1}{\sigma}\left(\frac{\partial t}{\partial y}-1\right)\right\}= \\
&= \left(-\frac{1}{2}\right)\left\{\phi\left[\frac{t\left(p,y\right)-y}{\sigma}\right]\frac{1}{\sigma}\left(-\frac{\phi\left[\left(t-y\right)z\right]z}{1-\phi\left[\left(t-y\right)z\right]z}-1\right)\right\}>0.
\end{aligned}$$

■

Proof of Proposition 5. First, we prove that if a pure strategy SPNE exists, then it has to be the case that $p^s \in (0, y)$.
The first order conditions of the firms optimization problem are

$$\begin{aligned}
\frac{\partial \mathbb{E}_{\mathbf{x}}\left[\pi_{a}\right]}{\partial p^{a}} &= 1-\Phi\left[\frac{t\left(p,y\right)-y}{\sigma}\right]-\left(p^{a}-c\right)\phi\left[\frac{t\left(p,y\right)-y}{\sigma}\right]\frac{\partial t}{\partial p}\frac{1}{2\sigma}=0 \\
\frac{\partial \mathbb{E}_{\mathbf{x}}\left[\pi_{b}\right]}{\partial p^{b}} &= \Phi\left[\frac{t\left(p,y\right)-y}{\sigma}\right]-\left(p^{b}-c\right)\phi\left[\frac{t\left(p,y\right)-y}{\sigma}\right]\frac{\partial t}{\partial p}\frac{1}{2\sigma}=0.
\end{aligned}$$

Both must be satisfied in equilibrium. Therefore, it has to be the case that $\left(p^a, p^b\right)$ and $p^s = \frac{p^a - p^b}{2}$ satisfy:

$$1-2\Phi\left[\frac{t\left(p,y\right)-y}{\sigma}\right]=\left(p^{a}-p^{b}\right)\phi\left[\frac{t\left(p,y\right)-y}{\sigma}\right]\frac{\partial t}{\partial p}\frac{1}{2\sigma}. \tag{22}$$

First, note that (22) can't be satisfied if $p^s < 0$ because that would imply that the lhs is positive and the rhs is negative. Also, it cannot be the case that $p^s = 0$ in equilibrium because in that case the lhs would be positive and the rhs would be equal to zero. Moreover, it cannot be that $p^s = y$ in equilibrium because in that case the lhs would be equal to zero and the right hand side would be positive. Finally, it cannot be that $p^s > y$ in equilibrium because in that case the lhs would be negative and the rhs would be positive. Therefore, if a pure strategy SPNE exists it has to be such that $p^s \in (0, y)$. Notice that both the lhs and the rhs of 22 are continuous functions of p, therefore $G\left(p^s\right) \equiv lhs - rhs$ is continuous as well. Moreover, $G(p)$ is positive at $p = 0$ and is negative at $p = y$, therefore it must have at least one zero in the interval $p \in (0, y)$.
Next, we prove that $t^s \in (t^u, y)$.
From Proposition 2, $t\left(p, y\right)$ is strictly increasing in p and since $p^s > 0$ then it has to be the case that $t^u \equiv t\left(0, y\right) < t\left(p^s, y\right) \equiv t^s$.

From the definition of $t(p,y)$, it follows that $t(y,y) = y$. Since $t(p,y)$ is strictly increasing in p and $p^s < y$, then it has to be the case that $t^s < y$.
Finally, we need to show that $\mathbb{E}_{\mathbf{x}}[W(t^u)] > \mathbb{E}_{\mathbf{x}}[W(t^s)]$. This result holds because, as we have shown in the proof of Proposition 4, $\mathbb{E}_{\mathbf{x}}[W(t)]$ is strictly decreasing in the interval $t \in [t^u, y)$ and $t^s \in (t^u, y)$. ■

A.2 Discussion of the Existence of a Pure Strategy Equilibrium in Prices

We could not establish existence of a price equilibrium in pure strategies for general values of the parameters α and β. Caplin and Nalebuff (1991) provide sufficient conditions for the existence of an equilibrium in pure strategies for discrete choice models of differentiated product markets. Without network externalities, our model reduces to an asymmetric probit which satisfies the assumptions in Caplin and Nalebuff (1991). In the presence of network effects, instead, we cannot apply their techniques to prove quasi-concavity of the profit functions. Therefore, although we proved that the first order conditions of the firms maximization problems intersect at least once, this is not sufficient to guarantee that their intersection constitutes an equilibrium. Nonetheless, we have examined the shape of the profit functions for a grid of values of the parameters (α, β) in the relevant range $z(\alpha, \beta) \leqslant \sqrt{2\pi}$, and in all cases the profit function was strictly unimodal and the intersection of the first order conditions identified a pure strategy equilibrium.

B Figures

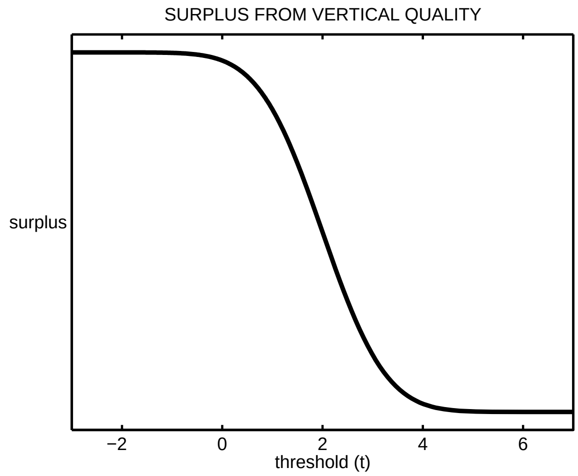

Figure 1: Aggregate surplus derived from the vertical quality of the goods for the case $\theta = 2$.

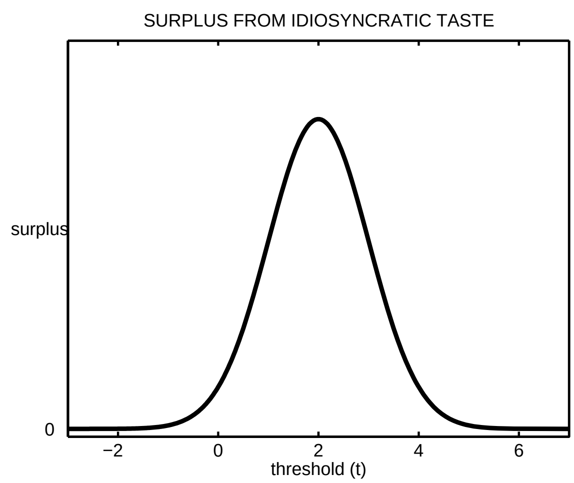

Figure 2: Aggregate surplus derived from idiosyncratic taste for the case $\theta = 2$.

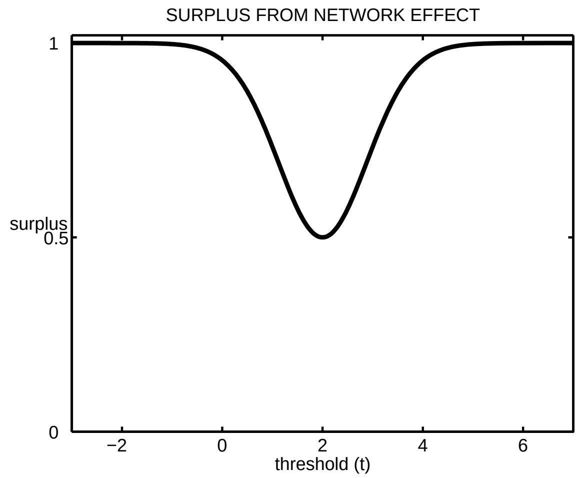

Figure 3: Aggregate surplus derived from the network effect for the case $\theta = 2$.

Notes

[1]For a review of this literature, see Farrell and Klemperer (2004).

[2]See, e.g., Baake and Boom (2001).

[3]See Jullien (2007).

[4]This differentiation is both vertical and horizontal. On its website, Apple lists the top ten reasons to switch to a Mac. Among them, "reason number 2" is that a Mac doesn't crash (clearly a claim of higher vertical quality) while "reason number 3" is the fact that Mac offers the best technology to store and play digital music (which is a feature that different consumers might value differently, depending on the specific use they make of their computers).

[5]See Rust (1993).

[6]Chou and Shy (1990) and Church and Gandal (1992) address the same issue in models with indirect network effects.

[7]More precisely, $n^j(\mathbf{s}(\mathbf{x},\mathbf{p}))$ is defined as the Lebesgue measure of the set consumers joining network j. We leave $n^j(\mathbf{s}(\mathbf{x},\mathbf{p}))$ undefined for the case where the latter set is not measurable. In equilibrium, the set will be measurable.

[8]Throughout the paper, we will assume that the law of large number holds for a continuum of independent variables, i.e. we will assume that the distribution of the idiosyncratic component in the population is the same as the distribution of any individual ε_i.(see Judd 1985).

[9]For the concept of ex-ante efficiency, see Holmström and Myerson (1983).

[10]For a discussion of the results using ex-post efficiency, see the Conclusions.

[11]Since there is a continuum of consumers, and we are assuming that the law of large numbers holds, for almost every realization $\mathbf{x}$ the distribution of types in the population is the same as the distribution of each individual x_i. Then, for a given allocation $(\mathcal{A},\mathcal{B})$, the sets $(\mathcal{A}_\mathbf{x},\mathcal{B}_\mathbf{x})$ are identical for almost every realization of $\mathbf{x}$, up to a permutation of the $i's$. In other words, the measures of these sets are the same, but the identities of the consumers that belong to each of them can change. In what follows, we will restrict attention to allocations $(\mathcal{A},\mathcal{B})$ such that $\mathcal{A}_\mathbf{x}$ and $\mathcal{B}_\mathbf{x}$ are Lebesgue measurable for almost every realization of $\mathbf{x}$.

[12]The derivation of 2 is included in the proof of Proposition 1.

[13]The main purpose of Figure 1 and Figure 2 is to describe the qualitative features of the components of consumer surplus associated with vertical and horizontal quality for a given positive θ that we fix equal to two. For this illustrative purpose, we abstract from the absolute values each of these functions takes for specific values of the parameters (α,β) and specific realizations of (θ^a,θ^b). Therefore, we do not report the scale of the vertical axis.

[14]For a discussion of this result, see the Conclusions.

[15]More precisely, in that case the strategy profile that constitutes the only equilibrium in switching strategies is also the unique rationalizable strategy profile of the game (see Morris and Shin (2004)).

[16]For an analogous discussion of the uniqueness condition in this class of games, see Morris and Shin (2004), section

2.3.

[17] In their (1993) paper, De Palma and Leruth show that in a duopolistic network market with horizontally differentiated products the demand function is well-defined if and only if the amount of horizontal differentiation, as measured by the variance in consumers' idiosyncratic taste for the goods, is sufficiently large. Their setting corresponds to the special case of our model in which the common quality component of the goods is perfectly observed ($\alpha = +\infty$), and there is no vertical differentiation ($\theta = y = 0$). Our results are consistent with those in De Palma and Leruth (1994), since in this special case our uniqueness condition is satisfied iff $\beta \leq 2\pi$, i.e. iff there is a sufficient amount of horizontal differentiation.

[18] For a discussion of the existence of a pure strategy equilibrium see the Appendix.

[19] See, e.g., Shaked and Sutton (1982) for a model of pure vertical differentiation and Anderson and de Palma (2001) for a model with both horizontal differentiation and quality differences.

[20] Note that in this model, the firm that has an advantage is not literally the firm "selling the best product" but is the firm that is expected to sell the best product.

References

[1] Anderson, S.P. and de Palma, A.(2001): "Product Diversity in Asymmetric Oligopoly: Is the Quality of Consumer Goods too Low?", *Journal of Industrial Organization*, 49, 113-135.

[2] Apple Computer: "Top Ten Reasons to Switch", *http://www.apple.com/switch/whyswitch*

[3] Baake, P. and Boom, A. (2001): "Vertical Product Differentiation, Network Externalities , and Compatibility Decisions", *International Journal of Industrial Organization*, 19, 267-284.

[4] Caplin, A. and Nalebuff, B. (1991): "Aggregation and Imperfect Competition: on the Existence of Equilibrium", *Econometrica,* 59, 25-59.

[5] Carlsson, H., and E. van Damme (1993): "Global Games and Equilibrium Selection", *Econometrica*, **61**: 989-1018.

[6] Chou, C. and Shy, O. (1990): "Network Effects without Network Externalities", *International Journal of Industrial Organization*, 8, 259-270.

[7] Church, J. and Gandal, N. (1992): "Network Effects without Network Externalities", *Journal of Industrial Organization*, 40, 85-104.

[8] De Palma, A and Leruth, L. (1993): "Equilibrium in Competing Networks with Differentiated Products", *Transportation Science,* 27, 73-80.

[9] Farrell, J. and Katz, M. (1998): "The Effects of Antitrust and Intellectual Property Law on Compatibility and Innovation," *Antitrust Bulletin*, Fall/Winter 1998, 609-650.

[10] Farrell, J. and Klemperer, P. (2004): "Coordination and Lock-In: Competition with Switching Costs and Network Effects", *mimeo,* University of California, Berkeley and Oxford University.

[11] Farrell, J. and Saloner, G. (1986): "Standardization and Variety", *Economics Letters,* 20, 71-74.

[12] Griva, K and Vettas, N. (2004): "Price Competition in a Differentiated Products Duopoly Under Network Effects",*CEPR Discussion Paper 4574*.

[13] Holmström, B and Myerson, R.B. (1983): "Efficient and Durable Decision Rules with Incomplete Information", *Econometrica,* 51, 1799-1819.

[14] Judd, K. (1985), "The Law of Large Numbers with a Continuum of IID Random Variables", *Journal of Economic Theory*, **35**: 19–25.

[15] Jullien, B. (2007): "Price Skewness and Competition in Multi-Sided Markets: How to Divide and Conquer," *mimeo.*

[16] Mitchell, M. and Skrzypacs, A. (2005): "Network externalities and Long-Run Market Shares", *Economic Theory*, forthcoming.

[17] Morris, S. and Shin, H.S. (2003): "Global Games: Theory and Applications", in *Advances in Economics and Econometrics (Proceedings of the Eighth World Congress of the Econometric Society)*, edited by M. Dewatripont, L. Hansen and S. Turnovsky. Cambridge, England: Cambridge University Press (2003).

[18] Morris, S. and Shin, H.S. (2004): "Heterogeneity and Uniqueness in Interaction Games" *forthcoming in The Economy as an Evolving Complex System III* (L. Blume and S. Durlauf, Eds). Santa Fe Institute Studies in the Sciences of Complexity. New York: Oxford University Press.

[19] Rust, J (1993): "Gauss and Matlab: a Comparison", *Journal of Applied Econometrics*, 8, 307-324.

[20] Shaked, A. and Sutton, J. (1982): "Relaxing Price Competition through Product Differentiation", *Review of Economic Studies*, 49, 3-13.

[21] Tirole, J. (1988): *"The Theory of Industrial Organization"*, MIT Press, Cambridge, MA.

[22] Ui, T. (2006): "Correlated Quantal Responses and Equilibrium Selection", *Games and Economic Behavior*, 57, 361-369.