# Commitment and Observability in an Economic Environment*

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**Abstract**

Bagwell (1995) argues that commitment in undermined by the slightest imperfectness in observation. Guth, Ritzberger & Kirchsteiger (1998) question this assertion: for any finite leader-follower game, with arbitrary many players in each role and generic payoffs, they show that there always exists a subgame perfect equilibrium outcome that is *accessible,* i.e. it can be approximated by the outcome of a mixed equilibrium of the game with imperfect observation.We show that accessibility fails in a class of games played in economic environments, where the payoffs to commitment actions depend upon prices set by other agents, prices being chosen from a continuum. Accessibility requires either that commitment is not required or that the price setting agents have no monopoly power. Our result follows from a generalized indifference principle which mixed strategies must satisfy in such economic environments.

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# 1 Introduction

A key insight in Thomas Schelling's classic book, *The Strategy of Conflict,* is that the ability to commit oneself often confers a strategic advantage. While Schelling emphasized the value of commitment in military and social situations, his insight has since been explored and formalized in diverse fields of economics including industrial organization, international trade and political economy — indeed, it is perhaps the most persistent idea in the explosion of applications of game theory to economics since the 1980s. The foundations of this literature have been questioned in a provocative paper by Bagwell [1], who argues that the value of commitment in undermined by the slightest amount of imperfect observation. Bagwell considers a leader-follower model, where the leader's chosen action (or commitment) is observed noisily by the follower. He shows that the pure strategy equilibria of the noisy observation game coincide with the pure strategy equilibria of the simultaneous move game, where the follower has *no* observation of the leader's action. Bagwell interprets this result as saying that the slightest amount of noise completely undermines the leader's ability to commit.

Bagwell's claim, and his focus on pure strategy equilibria, have been questioned — pure strategy equilibria need not always exist in the noisy game, e.g. if the simultaneous move game fails to have a pure strategy equilibrium. van Damme and Hurkens [4] analyze the class of games with one leader and one follower and generic payoffs. Such games have a unique backward induction (or Stackelberg) outcome if observation is perfect. They show that the Stackelberg outcome is always *accessible* – there exists a mixed strategy equilibrium of the game with imperfect observation, whose outcome converges to the Stackelberg outcome, as the noise in observation vanishes.[1]

[1]The noisy observation game will, in general, have multiple equilibria;van Damme and Hurkens use equilibrium selection theory to argue that the mixed equilibrium supporting the Stackelberg outcome is more likely to be played than any other equilibria. On the other hand, Oecshsler and Schlag [10] use evolutionary dynamics to select the pure strategy non-Stackelberg equilibrium. Huck and Muller [6] present experimental evidence showing that

The accessibility result has been shown to be very general by Guth et. al. [5], who consider finite leader-follower games with an arbitrary number of leaders and arbitrary number of followers. If payoffs are generic, there always exists an accessible subgame perfect equilibrium outcome of the game with perfect observation. [2] The proof of this proposition is based on a fundamental property of generic extensive form games – the existence of a stable sets and an essential component. This suggests that existence of accessible outcomes is likely to be obtained for a very large class of games with generic payoffs. These results offer some comfort to the vast body of applied theory, which analyzes multi-stage games. In particular, if a game has a unique subgame perfect equilibrium outcome, this will be accessible in generic scenarios. This offers an intellectual justification for the fact that applied theory continues to analyze models of commitment without reference to Bagwell's claims.[3]

This paper argues that in a wide range of *economic* environments, accessibility may fail, so that the outcome of the game with perfect observation cannot be approximated by equilibrium outcomes under imperfect observability. By economic environments, we mean situations where incentives to commit are influenced by the prices chosen by other agents.[4] One specific class of games where our results apply is leader-follower games that are played in a contracting environment. A second example that we consider is the interaction between a strategic buyer, who seeks to commit to certain decisions today in order to influence her terms of trade tomorrow, and the sellers who quote prices to her. Our main finding is that an accessible outcome fails to

the outcome is close to Stakelberg when the noise is small.

[2]Not all subgame perfect equilibrium outcomes are accessible, even with generic payoffs — see the example in [5]. A similar result is obtained in finitely repeated games with imperfect private monitoring — Bhaskar and van Damme [3] show that efficient equilibrium outcomes under perfect monitoring may not be accessible with imperfect private monitoring.

[3]Since mixed equilibria can be purified by adding private payoff shocks (see Maggi [7]), commitment power can be restored even in pure strategy equilibrium.

[4]Since prices maybe chosen from a continuum, the results of Guth et. al. do not apply in our context.

exist under very general conditions. The failure of accessibility arises since mixed strategy equilibria in these economic environments have to satisfy a *generalized indifference principle* – not only must the player randomizing between two actions be indifferent between these actions, those quoting prices to him must also be so indifferent. So the contribution of this paper is to re-instate the Bagwell critique, for a specific class of games.

The layout of the remainder of the paper is as follows. Section 2 sets out a simple entry deterrence example, which illustrates our basic point of the failure of accessibility. Section 3 characterizes the equilibrium outcomes of leader follower games in a contracting environment, when there is perfect observation of the leaders' actions. Section 4 has our main results, when there is noisy observation in leader follower games. Section 5 presents an example, of a strategic buyer, that illustrates the more general economic contexts where our point applies. The final section concludes.

## 2 An Entry Deterrence Example

The basic point of our paper is made most simply by considering the following example of an entry-deterrence game, in Fig. 1. The leader is the incumbent firm can choose whether or not to incur a costly investment. The entrant observes whether investment is made ($I$) not ($N$), and decides whether to stay out or enter.

$I$, the act of investment on the part of the incumbent, requires the purchase of equipment. We shall assume initially that the investment good is purchased in a competitive market at cost price $c$, i.e. $p = c$. $v > c$, so that the backward induction outcome has the incumbent choosing $I$ while the entrant stays out. This simple game illustrates the value of commitment — the incumbent invests only in order to deter entry.

Bagwell [1] has argued that the slightest amount of imperfect observation undermines the commitment power of the incumbent. Suppose that when the incumbent chooses $I$, the entrant observes the signal $i$ with probability

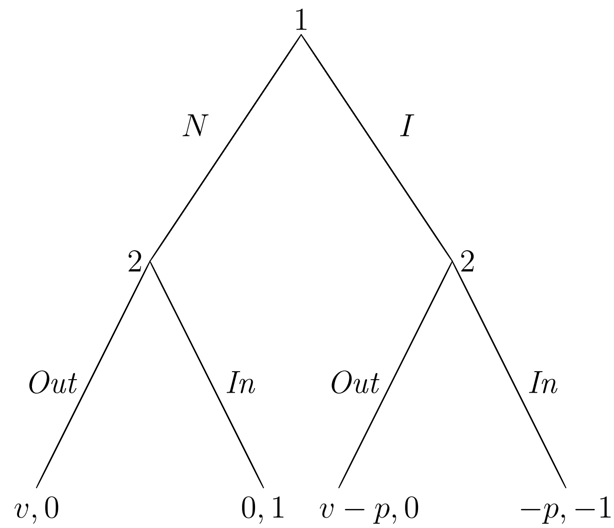


Figure 1: The Entry Deterrence Game

$1-\varepsilon$, and the signal $n$ with probability $\varepsilon$. If the incumbent chooses $N$, the entrant observes $n$ with probability $1-\eta$ and $i$ with probability $\eta$. Then there cannot be a pure strategy equilibrium where the incumbent chooses $I$ with probability one. The only pure strategy equilibrium is where the incumbent does not invest, and the entrant enters irrespective of his signal.

However, there exists a mixed strategy equilibrium where the incumbent invests with probability $\theta$ close to one ($\theta$ solves $\frac{\theta\varepsilon}{\theta\varepsilon+(1-\theta)(1-\eta)} = \frac{1}{2}$, so that the entrant believes that investment has taken place with probability one-half on observing $n$). The entrant stays out if he observes the signal $i$, and if he observes $n$, he enters with probability $\gamma$. The incumbent's payoff from investing equals

$$U(I,\rho) = (v - \gamma\varepsilon) - p. \tag{1}$$

Whereas his payoff from not investing is

$$U(N,\rho) = [(1-\gamma)(1-\eta) + \eta]\, v. \tag{2}$$

Equating these payoffs yields $\gamma = \frac{p}{v(1-\eta-\varepsilon)}$. Note that

$$\theta = \frac{1-\eta}{1-\eta-\varepsilon}, \tag{3}$$

so that the probability of investment converges to 1 as $\varepsilon \to 0$. Since the entrant stays out whenever he observes $i$, the outcome ($I$,OUT) occurs with a probability that tends to one as $\varepsilon \to 0$. This is an example of the result in van Damme and Hurkens [4], that in any game with one leader and one follower and generic payoffs, the subgame perfect equilibrium outcome is always accessible.

We now consider the implications of this game being played in a contracting environment. Let us assume that there is a monopoly supplier of equipment, whose cost of production is $c$. The game in the contracting environment, $\tilde{\Gamma}$, is as follows. The supplier chooses a price, $p$, that is quoted to the incumbent. This is observed by the incumbent alone, who then chooses his action. The entrant observes the incumbent's action and chooses his own action. This game has a unique perfect Bayesian equilibrium, as follows. The supplier chooses $p = v$. The incumbent chooses $I$ if and only if $p \leq v$, and chooses $N$ otherwise. The entrant stays out if he observes $I$ and enters if he observes $N$. Note that in this equilibrium, the supplier earns a profit of $v - c$, and the action profile ($I$,OUT) is played with probability one – we call this the *outcome* of the equilibrium. Call the outcome of this equilibrium the Stackelberg outcome — the incumbent invests, buying equipment at price $v$, and the entrant stays out. Observe that the action profile played in this equilibrium coincides with that played when investment good is purchased on a competitive market; however, the monopoly supplier earns profits that equal his marginal contribution to the incumbent's payoff at this equilibrium.

Now let us assume that the investment decision is observed only imperfectly. Recall that the prices quoted by the incumbent's suppliers are not observed by the entrant. We claim that there is no equilibrium with an outcome that is close to the Stackelberg outcome. An equilibrium outcome of the game with noisy observation consists of an (expected) profit for the supplier and a probability distribution over player action profiles, i.e. it is an element of Euclidean space. So more precisely, if we let $\varepsilon \to 0$, there does not exist a sequence of equilibria of the associated games $\tilde{\Gamma}(\varepsilon)$, the outcome of which converges to the Stackelberg outcome in the Euclidean metric. Let $\varepsilon > 0$ and assume that there is an equilibrium where the incumbent chooses $I$ with high probability, and where the supplier chooses any price $p > c$. Note first in such an equilibrium, the incumbent cannot invest with probability one, for the same reason as in the original Bagwell argument. For in this case, the entrant will stay out regardless of the signal, and the incumbent would therefore prefer not to invest. So let us assume that the outcome is such that incumbent plays both $I$ and $N$ with positive probability. Let the total probability that $N$ is played be $\alpha$. Consider first the case where the supplier does not randomize, and let $p > c$ be the supplier's price chosen, so that his payoff is

$$(p - c)(1 - \alpha). \tag{4}$$

We now show that $\alpha > 0$ implies that $p$ cannot be optimal for supplier $A$, for he can do better by choosing $p' < p$, where $p'$ is arbitrarily close to $p$. If he chooses any price less than $p$, then we claim that the incumbent will choose to invest with probability one. To verify this, inspect the expressions for (1) and (2) — since the two are equal at $p$, it follows that the former will be strictly greater when $p' < p$. It follows that the supplier can earn $(p' - c)$ for any $p' < p$, which will exceed $(p - c)(1 - \alpha)$ for $p$ sufficiently close to $p$. Hence there cannot exist an equilibrium with outcome close to the Stackelberg outcome, i.e. with $p$ close to $v$,and where the incumbent invests with probability close to one.

It remains to consider the case where the supplier randomizes across prices. Suppose that he chooses $p, p'$ with positive probability, where $p > p'$. At $p$, the incumbent must invest with positive probability, since otherwise the supplier's payoff is zero. The argument of the previous paragraph implies that at $p'$, the incumbent invests with probability one. But then the supplier can choose $p''$ such that $p > p'' > p'$, and the incumbent will still invest with probability one, implying that $p'$ cannot be optimal. We have therefore shown that for any $\varepsilon > 0$, there does not exist an equilibrium where investment occurs with positive probability, and where the supplier chooses a price which is greater than $c$. Since supplier profits equal $v - c$ in the Stackelberg outcome with perfect observability, this outcome is not accessible, since it cannot be approximated in the game with imperfect observation.

This example does not rely upon the supplier having complete monopoly power over the supply of the investment good, it only requires that there is *some* monopoly power. Suppose that there is a second supplier for the same good, whose cost of production where $v > c' > c$. The equilibrium outcome in the game without noise would have $p = c' - c$, so that the efficient supplier makes positive profits. [5] However, in the game with noisy observation where investment takes place, the incumbent must randomize, so positive profits for the supplier are inconsistent with equilibrium.

The above argument makes clear that in any equilibrium where the incumbent randomizes between $I$ and $N$, the supplier must also be indifferent between these actions, i.e. the supplier's price must equal marginal cost $c$. There does exist such an equilibrium — the supplier quotes $p = c$, and the incumbent randomizes between $I$ and $N$ at this price.[6] In such an equilibrium, the incumbent's payoff is approximately $v - c$ when the noise is small, which is strictly greater than his payoff under perfect observability, 0. In this equilibrium, the incumbent appropriates the supplier profits, which he cannot do under perfect observability. In other words, if incumbent retains his commitment power under imperfect observation, he also enhances his power vis-a-vis his supplier, and captures all the surplus.

[5] This is true in any cautious equilibrium, where the inefficient supplier does not choose price below $c'$.

[6] The precise continuation strategies for the strategies of the incumbent and entrant are as given in equations [1]-[3], with $p = c$. If the supplier deviates and chooses $p > c$, the incumbent chooses $N$ for sure.

## 2.1 A game where the leader has no incentive to deviate

The entry deterrence game is a classic instance of a game where commitment is important – the incumbent firm has no incentive to invest except in order to deter entry. However, the failure of accessibility also applies when the leader has no incentive to deviate from his subgame perfect equilibrium action in the underlying game. Our second example, in Fig. 2, illustrates this. Assume that $x < 2$ and $x \neq 1$, so that the underlying game has a unique subgame perfect equilibrium.

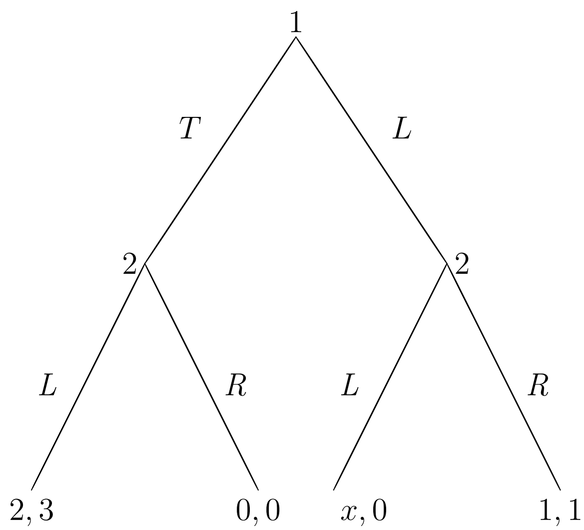


Figure 2: Leader has no incentive to deviate, $x < 2$.

Consider the game of Fig. 2. The subgame perfect equilibrium outcome is $(T, L)$. Since this is a Nash equilibrium of the simultaneous move game, it is also a pure strategy equilibrium outcome of the game with noisy observation. Now consider the game played in a contracting environment where player 1, the leader, requires to contract with a monopoly supplier, supplier $T$, in order to play $T$. None of the other actions of either player need contracting with a supplier. With perfect observation of the leader's action, the price charged by this supplier, $p(T)$, must equal 1, the difference between the leader's payoff at $(T, L)$ and $(B, R)$. However, with imperfect observation, the outcome where $(T, L)$ and supplier $T$ makes a profit of 1 is not accessible. In any equilibrium where $T$ is played with probability one, we must have $p(T) = 2 - x$, since the follower will play $L$ irrespective of the signal that he observes. On the other hand, if the leader randomizes, $p(T) = 0$ for the same reasons as in the entry deterrence example. Thus we have a failure of accessibility in this example as well, as long as $x \neq 1$.

The basic characteristic of this example is that the leader's action is *relevant* to the follower, in the sense that the follower's best response varies depending on whether $T$ or $B$ is played). This condition, in conjunction with supplier monopoly power, is sufficient to ensure failure of accessibility.

## 2.2 Robustness: Discrete Prices

Why is there a failure of accessibility? Our example, of a game played in a contracting environment are clearly not generic extensive form game, in the sense of Guth et. al. [5]— the game is not finite, since prices are chosen from a continuum. Assume now that prices must be chosen from a discrete grid. More precisely, each supplier must choose a price $p_i$ from the set $\{\frac{k}{m} : k = 0$ or $k \in N\}$, where $m \in N$ indexes the fineness of the grid. The noisy game $\tilde{\Gamma}$ can now be parameterized by the pair $(\varepsilon, m)$ where $\varepsilon$ is the noise in observation and $m$ is the grid size. We shall demonstrate two results. First, for any given level of noise parameterized by $\varepsilon$, there exists a grid size $m(\varepsilon)$ such that the limit perfect equilibrium outcomes in the sequence of games $\tilde{\Gamma}(\varepsilon, m)$ $(\varepsilon > 0, m \geq m(\varepsilon))$ as $\varepsilon \to 0$ and $m \to \infty$ are disjoint from the limit perfect equilibrium outcomes in the sequence of noiseless games $\tilde{\Gamma}(0, m)$ as $m \to \infty$. However, if we fix the grid size at some $\bar{m}$, this discontinuity does not appear, i.e. there exists a perfect equilibrium outcome of $\tilde{\Gamma}(0, \bar{m})$ which is a limit of a sequence of perfect equilibrium outcomes of the games $\tilde{\Gamma}(\varepsilon, \bar{m})$ as $\varepsilon \to 0$. This latter result is not implied by the results of Guth et. al. [5] since homogeneous good Bertrand competition with discrete prices is not a generic game — there are ties in payoffs at the terminal nodes of the game tree, so long as both suppliers must choose from the same grid of prices.[7]

It will be expositionally convenient to assume that the payoff $v$ in Fig. 1 is an irrational number. Given the price grid $m$, let $p^*(m)$ be the largest price that is less than $v$, and let the grid be sufficiently fine that $p^*(m) > c$. Let us now characterize perfect Bayesian equilibrium outcomes in the game where the incumbent's action is perfectly observed. Clearly, in any equilibrium outcome, the incumbent will invest with probability one and entrant will stay out with probability one. If the supplier chooses the price $p^*(m)$, then he sells with probability one, since $v > p^*(m)$. Thus the unique perfect Bayesian equilibrium has the supplier making profits $p^*(m) - c$, with ($I$,OUT) being played with probability one.

Now let us consider the situation where commitment is imperfectly observed. Fix an equilibrium of the game $\tilde{\Gamma}(\varepsilon, m)$, where $I$ is played by the incumbent with positive probability, and let $\hat{p}(m)$ be the largest price, that is chosen by the supplier in this equilibrium. Clearly, at $\hat{p}(m)$ the incumbent must buy with positive probability, since otherwise $\hat{p}$ will not be chosen by the supplier. Since it is optimal for the incumbent to buy at $\hat{p}$, it must be strictly optimal to buy at any price strictly below $\hat{p}$. Hence the supplier can ensure the payoff of $\hat{p}(m) - \frac{1}{m}$ by choosing the price $\hat{p}(m) - \frac{1}{m}$. If $\alpha$ is the probability that the incumbent buys at price $\hat{p}$, we must have that

$$\alpha \hat{p}(m) \geq \hat{p}(m) - \frac{1}{m}. \tag{5}$$

Let $\beta$ be the probability that the supplier's price equals $\hat{p}$. Hence the total probability that the incumbent does not invest equals

$$\beta(1 - \alpha) \leq 1 - \alpha \leq \frac{1}{\hat{p}(m)m}. \tag{6}$$

If the grid of prices is sufficiently fine, then $m$ is large and the right hand side above will be close to zero unless $\hat{p}(m)$ is also small. However, the total

[7] For example, one can show that such a game will have a continuum of perfect equilibrium outcomes as long as the grid of prices is sufficiently fine.

probability that the incumbent does not invest must equal $1 - \theta$, where $\theta$ is given by 3. Therefore,

$$\hat{p}(m) \leq \frac{1 - \eta - \varepsilon}{\varepsilon m}. \tag{7}$$

We have therefore proved the following:

1. For any $\varepsilon > 0$, there exists $m^*(\varepsilon)$ such that if we consider the sequence of games $\tilde{\Gamma}(\varepsilon, m(\varepsilon))$ where $\varepsilon \to 0$ and $m(\varepsilon) \geq m^*(\varepsilon)$, the Stackelberg outcome is not accessible.

2. If we fix $m$, and consider the sequence $\tilde{\Gamma}(\varepsilon, m)$ where $\varepsilon \to 0$, the Stackelberg outcome is accessible.

To interpret this statement, consider an example where $c = 0$, and $v$ = £100 + $\xi$, where $\xi$ is a tiny irrational number. The supplier makes a profit of £100 when commitment is perfectly observed. In the game with noisy observation, suppose that the probability of the incumbent does not invest $(1 - \theta)$ must equal $\varepsilon$ in order to make the entrant indifferent between entering and staying out when she observes $n$. If the price grid is in pennies, then the supplier will have no incentive to reduce price below *£*100 only if $(1 - \varepsilon)100 \geq 99.99$, i.e. $\varepsilon$ must be smaller than 0.0001. In other words, if the profits that the supplier makes are large relative to the minimum unit of account, the noise must be very small indeed.

The remainder of the paper sets out how the insight contained in these examples generalize. Our first task, in section 3, is to provide a characterization of equilibrium outcomes in leader-follower games played in a contracting environment, where the followers' have perfect observation of leaders' actions. In section 4, we will consider how noisy observation results in a failure of accessibility quite generally.

# 3 Perfect observation

We consider a leader-follower game played in a contracting environment, with perfect observation of the leaders' commitment. This follows the set up in Bhaskar [2], although the exposition here is self-contained. We will use the term player for someone who plays the game in question, and the term supplier to denote someone with whom a player may need to contract with in order to be able to adopt some strategy in the game. Among the players, we will distinguish between the set of leaders, $L = \{1, 2, ..., m\}$ and the set of followers, $F = \{m+1, ..., n\}$. $I = L \cup F$ is the set of players, and each player $i$ has a finite action set $A_i$, whose generic element will also be denoted by $a_i^j$ or $a_i$. Let $A = \times_{i \in I} A_i$ be the set of action profiles, and let $g_i : A \to R$ be the *gross payoff* of player $i$. These gross payoffs at the profile $a = (a_i)_{i \in I}$ will in general differ from the usual (net) payoffs of a player since she may have to contract with a supplier in order to be able to play the action $a_i$. Let $\bar{A}_i \subset A_i$ be the set of actions for which the player needs a supplier. For any player $i$ and any action $a_i^j \in \bar{A}_i$ , let $\Sigma_{ij} = \{ij_1, ..., ij_{m_{ij}}\}$ denote the set of competing suppliers – the player needs to contract with exactly one of these suppliers in order to take action $a_i^j$. The $h-$th supplier, $ij_h$, has a cost of supply, $c_{ijh}$. Let $ij_1$ index the efficient supplier for action $ij$, that is the one with the lowest cost. We will also use the notation $\phi(ij)$ to denote the efficient supplier for action $ij$. $\Sigma_i = \cup_{j \in \bar{A}_i} \Sigma_j$ denotes the set of suppliers for player $i$. Let $p_{ijh}$ denote the price which is charged by supplier $ij_h$ for enabling the action $a_i^j$, and let $p_{ij} = (p_{ijh})_{ijh \in \Sigma_{ij}}$, and $p_i = (p_{ij})_{j \in \bar{A}_i}$. If $a_i^j \notin \bar{A}_i$ we set the price of this action, $p_{ij}$, to zero. The net payoff at the profile $a = (a_i^j, a_{-i})$ where $a_{-i}$ is the vector of actions of players $h \neq i$, and player $i$ contracts with supplier $ijh$, is given by

$$u_i(a, p_i) = g_i(a_i^j, a_{-i}) - p_{ijh}. \tag{8}$$

The payoff to supplier $ijh$ is given by $p_{ijh} - c_{ijh}$, where $c_{ijh}$ is the cost of producing the input required for this action. If the player does not play action $a_i^j$, the payoff to any supplier of this action is zero. Let us normalize prices and gross payoffs by measuring them net of the minimum cost of supply (equal to $c_{ij1}$ for any action $ij$), so that a zero price corresponds to pricing at minimum cost. Henceforth, the gross payoff $g_i(a_i, a_{-i})$ will denote the payoff when player pays the minimum cost of action $a_i$. We extend, in the usual way, the gross payoff function $g_i$ to correlated action profiles: $g_i(\alpha)$ is the payoff to player $i$ when $\alpha \in \Delta(A)$ is the vector of correlated actions played.

The *leader-follower game with private contracts*, $\Gamma$, is as follows:

1. Each supplier in $\Sigma_L = \cup_{i \in L} \Sigma_i$ quotes a price for each input that he supplies.

2. Each leader $i \in L$ observes the price vector $p_i$ (but not the prices quoted to other players), and chooses an action.

3. Each supplier in $\Sigma_F = \cup_{i \in F} \Sigma_i$ observes the action profile $a_L$ chosen by the leaders and chooses her price.

4. Each follower $i \in F$ observes $a_L$ and her own price vector $p_i$, and followers simultaneously choose actions.

We make the following assumptions regarding the game $\Gamma$.

Assumption **A1.** (No Complementary Inputs): For any player $i$ and any action $a_i^j$, no more than one supplier is required.

Assumption **A2.** For every player $i$ there exists an action $a_i^0$ such that no input is required to play this action.

This assumption ensures that the minimum payoff that any player in $I$ can receive is bounded and given by $\min_{a_{-i}} g_i(a_i^0, a_{-i})$.

Assumption **A3.** A supplier supplies at most one player, i.e. $\Sigma_i$ and $\Sigma_j$ are disjoint if $i \neq j$.

This assumption plays an essential role in our analysis of private contracts, since it ensures that the beliefs of player $i$ regarding the actions chosen by other players do not vary with the prices that player $i$ is quoted by her supplier.

Assumption **A4.** A supplier supplies at most one action of any player, i.e. $\forall i$, $\Sigma_{ij}$ and $\Sigma_{ik}$ are disjoint if $j \neq k$.

Define $\tilde{c}_{ij} = \min_{ijh \in \Sigma_{ijh}, h \neq 1}\{c_{ijh}\}$. $\tilde{c}_{ij} = \infty$ if $\phi(ij)$ is a monopoly supplier.

Assumption **A5.** For any player $i$ and action $a_i^j \in \bar{A}_i \subset A_i$, $\tilde{c}_{ij} > 0$ .

A5 states that efficient supplier has some monopoly power, and is made for expositional convenience. It is without essential loss of generality – if there is more than one efficient supplier for an action, then in any equilibrium where this action is played, the price must equal zero, and this is equivalent to assuming that this action does not belong to $\bar{A}_i$.

We will restrict attention to perfect Bayesian equilibria of the game Γ. Beliefs of the players are given by Bayes rule along the equilibrium path. If a supplier deviates, then assumption A3 implies that this does not affect the beliefs of a player to whom this price applies, i.e. her beliefs about the actions of other players are unaffected. In addition, we want to rule out "unreasonable equilibria", where inactive suppliers (i.e. those who do not make a sale) choose strictly negative prices. Such equilibria are sometimes called *cautious,* and can be ruled out by considerations of trembling hand perfection.[8] So henceforth, by "equilibrium" we mean a cautious perfect Bayesian equilibrium where all supplier prices are non-negative.

A pure strategy for a supplier $ijh$ (for action $a_i^j$) is a price $p_{ijh} \in R_+$. A mixed strategy is a probability measure $\pi_{ijh}$ on $R_+$. A strategy for leader $i$ is a map $\sigma_i : R^{|\Sigma_i|} \to \Delta(A_i)$. A pure strategy for supplier $ijh$, who supplies a follower's action $a_i^j$, is a map $\rho_{ijh} : A_L \to R_+$, while a mixed strategy $\tilde{\rho}_{ijh}$ specifies a probability measure on $R_+$ for every $a_L \in A_L$. A strategy for follower $j$ is a map $\sigma_j : R^{|\Sigma_j|} \times A_L \to \Delta(A_j)$. A strategy profile $\sigma$ is a collection $((\pi_{ijh})_{ijh \in \Sigma_L}, (\sigma_i)_{i \in L}, (\tilde{\rho}_{ijh})_{ijh \in \Sigma_F}, (\sigma_j)_{j \in F})$.

In usual terminology, the outcome of a strategy profile $\sigma$ is the induced

[8] Simon and Stinchcombe [11] set out refinements for continuum normal form games, but there are no universally accepted refinements for continuum extensive form games. However, we can discretize the price space, and consider trembling hand perfect equilibria of the discretized game. We may restrict attention to equilibria of the continuum game which are limit points of a sequence of trembling hand equilibria of discrete games as the grid of prices becomes increasingly finer. It is easy to see that any equilibrium with negative prices will not be a limit of such trembling hand perfect equilibria.

distribution over the terminal nodes of the game tree. In price setting games, the set of equilibrium outcomes usually contains considerable redundancy. For example, in the case of Bertrand competition between three firms with differing unit costs, the price set by the highest cost firm is irrelevant, and can therefore be chosen arbitrarily. It will be therefore be more useful to focus on a refinement of the set of outcomes. The *action outcome* associated with a strategy profile is the induced distribution over the set of player-action profiles, $A$. The *supplier payoffs* associated with $\sigma$ are simply the payoffs to the suppliers under this profile. For the purposes of this paper, the *outcome* of a strategy profile is defined as the pair consisting of the action outcome and the supplier payoffs, and is an element of Euclidean space.

Our results will relate the equilibrium action outcomes in $\tilde{\Gamma}$, the leader follower game in a contracting environment to those in a standard leader-follower game. This is the leader-follower game $\Gamma$ where all supplier prices are exogenously fixed at zero (i.e. all inputs are supplied at cost) and players net payoffs equal their gross payoffs. A strategy for a leader in $\Gamma$ is a mixed action $\alpha_i \in \Delta(A_i)$, while a follower's strategy is a map $\beta_j : A_L \to \Delta(A_j)$. Let $E^{\Gamma}$ denote the set of subgame perfect equilibria of $\Gamma$. The *outcome* of a strategy profile $(\alpha_L, \beta_F)$ is the induced distribution over the elements of $A$. Let $\Omega^{\Gamma} \subset \Delta(A)$ denote the set of subgame perfect equilibrium outcomes of the game $\Gamma$. Given $a_L \in A_L$, let $E^{\Gamma(a_L)}$ denote the set of Nash equilibria in the subgame that results following the play of $a_L$. Let $\alpha_L = (\alpha_i)_{i\in L}, \beta_F = (\beta_i)_{i\in F}$, and let $(\alpha_L, \beta_F) \in E^{\Gamma}$. Given an vector $a = (a_i)_{i=1}^n$, we use the notation $a\backslash a_i'$ to denote the vector that results when the $i$-th component $a_i$ is replaced by $a_i'$.Given $a_L \in A_L$, if follower $i$'s continuation $\beta_i(a_L)$ is a pure action, we define follower $i$'s deviation loss $\delta_i(a_L, \beta_F)$ as :

$$\delta_i(a_L, \beta_F) = g_i(a_L, \beta_F(a_L)) - \max_{a_i \neq \beta_i(a_L)} g_i(a_L, \beta_F(a_L)\backslash a_i). \tag{9}$$

That is, $\delta_i(\alpha_L, \beta_F)$ is follower $i$'s loss from choosing his next best action rather than the recommendation of the equilibrium. [9] If $\beta_i(a_L)$ is a mixed,

[9] In the game $\tilde{\Gamma}$, if a supplier is required for taking action $\beta_i(a_L)$, $\delta_i(a_L, \beta_F)$ can be

$\delta_i(\alpha_L, \beta_F) = 0$.

Consider now a leader, $i \in L$. If the leader plays a pure action $\hat{a}_i$ as part of the equilibrium $(\alpha_L, \beta_F)$, we define his deviation loss $\delta_i(\alpha_L, \beta_F)$ as :

$$\delta_i(\alpha_L, \beta_F) = g_i(\alpha_L, \beta_F(a_L)) - \max_{a_i \neq \hat{a}_i} g_i(a_L \backslash a_i, \beta_F(a_L \backslash a_i)). \tag{10}$$

If the leader plays a mixed action, $\delta_i(\alpha_L, \beta_F) = 0$.

Let $\Omega^{\tilde{\Gamma}}$ denote the set of equilibrium action outcomes of the game $\tilde{\Gamma}$, i.e. the actions played by the leader and follower in a cautious perfect Bayesian equilibrium of the game $\tilde{\Gamma}$.

**Proposition 1** $\Omega^{\tilde{\Gamma}} = \Omega^{\Gamma}$ *i.e. the cautious perfect Bayesian equilibrium action outcomes of* $\tilde{\Gamma}$ *coincide with the subgame perfect equilibrium outcomes of* $\Gamma$. *In any equilibrium of* $\tilde{\Gamma}$ *with action outcome* $(a_L, \beta_F(a_L))$, *an active supplier for player* $i$ *earns his marginal contribution* $\min\{\delta_i(a_L, \beta_F(a_L)), \tilde{c}_{ij}\}$.

**Proof:** While this proposition may be proved more succinctly by an application of theorem 1 in Bhaskar [2] and an induction argument, we present the complete argument here for the reader's convenience. We show first the correspondence between equilibrium action outcomes in the two games in the second stage, when the followers choose their actions. Let $a_L$ be an arbitrary action profile chosen by the leaders. In the game $\tilde{\Gamma}$, the continuation game that follows the choice of $a_L$ is a game of imperfect information, since the prices chosen by the suppliers in stage 1 are not observed by either the followers or their suppliers. Nevertheless, these prices are payoff irrelevant, and the only payoff relevant variable, $(a_L)$, is commonly observed by all followers and their suppliers. In the game $\Gamma$, let $E(a_L)$ denote the set of Nash equilibria in the subgame following the play of $a_L$, and let $\beta(a_L) \in E(a_L)$. We show first that in the game $\tilde{\Gamma}$, there exists an equilibrium where, following the play of $a_L$ in stage one, $\beta(a_L)$ is played by the followers and every supplier of a follower earns his marginal contribution. If follower $i$ plays a pure action at

thought of as the marginal contribution of this supplier to the follower's payoff at this subgame perfect equilibrium.

$\beta(a_L)$, let the most efficient supplier for this action choose a price equal to his marginal contribution, $\delta_i(a_L, \beta(a_L))$, while all other suppliers in $\Sigma_i$ choose a price of zero. If follow $i$ randomizes at $\beta(a_L)$, then let every supplier in $\Sigma_i$ choose a price of zero. Let each follower $i$ play $\beta_i(a_L)$ if no active supplier deviates, and play any best response action otherwise.

We now show that in any equilibrium of $\tilde{\Gamma}$, the action outcome following the choice of $a_L$ by the leaders is an element of $E(a_L)$. Fix an equilibrium of $\tilde{\Gamma}$, and consider the decisions of follower $i$ and his suppliers. From the point of view of these agents, $a_L$ is given, and the strategies of the suppliers of other followers and the strategies of the followers, induce a mixed action profile, $\alpha_{-i} \in \times_{j\in F, j\neq i}\Delta(A_j)$. Furthermore, by assumption A3, the player $i$'s beliefs about the profile played by other followers do not change with the prices charged by $i$'s suppliers. So our results follow from standard results on Bertrand competition between asymmetric sellers and a single buyer. Let $a_i^j \in \arg\max_{a_i} g_i(a_i, \alpha_{-i}, a_L)$, and let $a_i^k \in \arg\max_{a_i \neq a_i^j} g_i(a_i, \alpha_{-i}, a_L)$ be the next best action. Consider first the case where $g_i(a_i^j, \alpha_{-i}, a_L) > g_i(a_i^k, \alpha_{-i}, a_F)$. In this case, we show that $a_i^j$ must be played with probability one, and supplier $\phi(a_i^j)$ must earn a payoff $\delta_i(a_i^j, \alpha_{-i}, a_F)$ in any cautious equilibrium. By choosing a price $\delta_i(a_i^j, \alpha_{-i}, a_F) - \varepsilon$, $\phi(a_i^j)$ can ensure a sale with probability one, since the prices of other suppliers are non-negative, and thus his payoff must be no less than $\delta_i(a_i^j, \alpha_{-i}, a_F)$ . If his payoff is strictly greater than $\delta_i(a_i^j, \alpha_{-i}, a_F)$, then the support of $\phi(a_i^j)$ 's mixed strategy must consist of prices strictly greater than $\delta_i(a_i^j, \alpha_{-i}, a_F)$, and $\phi(a_i^k)$ can also earn positive profits. Thus in any mixed strategy equilibrium suppliers $\phi(a_i^j)$ and $\phi(a_i^k)$ must both earn positive profits. By assumption A2, the prices of the two sellers must also be bounded, by $[g_i(a_i^j, \alpha_{-i}, a_F) - g_i(a_0, \alpha_{-i}, a_F)]$ and $[g_i(a_i^k, \alpha_{-i}, a_F) - g_i(a_0, \alpha_{-i}, a_F)]$ respectively (recall that $a_0$ is the action that does not require any supplier). We now show that at least one seller's mixed strategy has in its support a price that earns zero profits, contradicting our earlier result that each seller earns positive profits. Let $x(\phi(a_i^j))$ (resp. $x(\phi(a_i^k)))$ denote the supremum of prices that lie in the support of $\phi(a_i^j)$'s (resp. $\phi(a_i^k)$ mixed strategy. If $x(\phi(a_i^j)) > x(\phi(a_i^k)) + \delta_i(a_i^j, \alpha_{-i}, a_L)$, then $\phi(a_i^j)$ chooses a price which makes zero profits, while if the inequality is reversed, this is the case for $\phi(a_i^k)$. If both expressions are equal, then both sellers must choose a price which makes zero profits, unless each player's mixed strategy has an atom at the supremum. But in this case, a player can do strictly better by choosing a price $\varepsilon$ below the supremum, where $\varepsilon$ is sufficiently small. We conclude therefore that if $a_i^j$ is the unique maximizer element of $\arg\max_{a_i} g_i(a_i, \alpha_{-i}, a_L)$, then it must be played with probability one, and $\phi(a_i^j)$ earns his marginal contribution. Similar arguments establish that if $\arg\max_{a_i} g_i(a_i, \alpha_{-i}, a_L)$ has several elements, then the player's mixed strategy can only assign probability to one of these, and the prices must equal zero. We have therefore established that following the play of $a_L$ in stage 1, each player must assign positive probability only to actions that maximize his gross payoffs, given any the induced beliefs over the actions of other players. Thus the action outcome of must be an element of $E(a_L)$, and supplier payoffs are as in the statement of the theorem.

We now proceed to the first stage of the game $\tilde{\Gamma}$. Given any action profile $a_L$ chosen by the leaders, gross payoffs to any leader $i$ are given by $g_i(a_L, \beta(a_L))$ where $\beta(a_L) \in E^{\Gamma(a_L)}$. Thus any equilibrium strategy profile of the followers defines a strategic form game for the leaders played in a contracting environment, and applying the theorem again, the actions chosen by the leaders must constitute a Nash equilibrium. Finally, the payoff loss to any leader from choosing his best deviant action or an alternative supplier is $\min\{\delta_i(a_L, \beta_F(a_L)), \tilde{c}_{ij}\}$, which specifies the payoff to each active supplier to a leader. ■

The above proposition establishes the following: given any subgame perfect equilibrium of the game $\Gamma$, there is associated a unique outcome $\omega$ of the game $\tilde{\Gamma}$, since the supplier payoffs are uniquely defined by their marginal contributions.

**Remark 1** *Since the proof is based on induction over two stages, the propo-*

*sition can be generalized as follows. Consider an arbitrary finite multistage game* $\Gamma$ *with observed actions, with finitely many actions, players and stages. Now embed* $\Gamma$ *in a contracting environment, resulting in the game* $\tilde{\Gamma}$*, where in each stage, the suppliers involved choose prices after having observed the actions played in previous stages. The equilibrium distributions over players actions in the games* $\tilde{\Gamma}$ *and* $\Gamma$ *coincide.*

This proposition has the following corollary. Consider a game $\Gamma$ with one leader and one follower. For generic payoffs, the game $\Gamma$ has a unique backwards induction equilibrium in pure strategies, that we denote by $(a_1^*, \beta_2^*)$, where $a_1^*$ is the leader's action, and $\beta_2^*$ is the follower's best response . The action profile played in this equilibrium is denoted by $(a_1^*, a_2^*)$, where $a_2^* = \beta^*(a_1^*)$. Now for generic payoffs, $\delta_1(a_1^*, \beta_2^*)$, the deviation loss suffered by the leader at the unique equilibrium, is strictly positive, as is that of the follower. We have therefore that:

**Corollary 2** *Any one-leader one-follower game* $\tilde{\Gamma}$ *with generic payoffs has a unique equilibrium action outcome. If a supplier is required for the leader (or follower) to take his equilibrium action, this supplier makes strictly positive profits.*

# 4 The noisy leader- follower game

We now assume that the leaders' action profile is observed with some noise: given that $a_L^k \in A_L$ is chosen by the leader, nature chooses signal $a_L^h \in A_L$ with probability $\lambda_{hk}$. $\lambda$ is a stochastic matrix defined on $A_L \times A_L$. Let $\Lambda$ be the set of possible signal structures, and $Int(\Lambda)$ be the set of signal structures with full support, i.e. the set of $\lambda$ such that $\lambda_{hk} > 0 \; \forall \; hk$. Let $\lambda_0$ denote the identity matrix – this corresponds to perfect observation. The game $\tilde{\Gamma}(\lambda)$ is defined as follows:

1. Each supplier in $\Sigma_L$ quotes a price for each input that he supplies.
2. Each leader $i \in L$ observes the price vector $p_i$, and chooses an action.

3. Given the action profile chosen by the leaders, nature chooses a signal in $A_L$, according to the stochastic matrix $\lambda$. The signal is observed by agents in $\Sigma_F \cup F$.

4. Each supplier in $\Sigma_F$ quotes a price for each input that he supplies.

5. Each follower $i$ observes the price vector $p_i$ and followers simultaneously choose actions.

The gross payoffs to players depend only on action profile realized, and not upon the signal. As before, net payoffs to players are equal to gross payoffs minus the prices paid.

Note that all agents have exactly the same strategy sets in the games $\tilde{\Gamma}$ and $\tilde{\Gamma}(\lambda)$ – only the payoffs associated with strategy profiles differ in the two games. We restrict attention to cautious perfect Bayesian equilibria of the game $\tilde{\Gamma}(\lambda)$. Fix an equilibrium of the game $\tilde{\Gamma}(\lambda)$. The action outcome of an equilibrium is the element of $\Delta(A)$ induced by the equilibrium. The outcome of an equilibrium is the pair consisting of the action outcome and the profile of supplier payoffs. Let $\Xi(\lambda)$ denote the set of equilibrium outcomes of $\tilde{\Gamma}(\lambda)$, and let $\Xi(0)$ denote the set of equilibrium outcomes of $\tilde{\Gamma}$. Expected supplier payoffs are real numbers, while the player action outcomes are probability distributions over a finite set. Thus $\Xi(\lambda) \subset R^{|\Sigma_L \cup \Sigma_F|} \times \Delta(A)$, a subset of Euclidean space. Since the set of outcomes is a subset of Euclidean space, we may use the usual norm in order to define convergence. We say that a sequence $\omega_n \rightarrow \omega$ if this convergence is in the usual topology.

**Definition 3** $\omega \in \Xi(0)$ *is* accessibl*e if* $\exists$ *countable sequences* $(\lambda, \tilde{\Gamma}(\lambda), \omega(\lambda))$, $\omega(\lambda) \in \Xi(\lambda)$ *with* $\lambda \in Int(\Lambda), \lambda \rightarrow \lambda_0$ *such that* $\tilde{\Gamma}(\lambda) \rightarrow \tilde{\Gamma}(\lambda_0)$ *and* $\omega(\lambda) \rightarrow \omega$.

Let us consider the class of games $\Gamma$ with a subgame perfect equilibrium $(a_L^*, \beta_F)$, where $a_L^* \in A_L$ is a pure action profile. We assume :

**Assumption A6:** The subgame following the play of $a_L^*$ has a unique equilibrium.

Let us denote the outcome of this equilibrium by $(a_L^*, \alpha_F^*)$, where $\alpha_F^* = \beta_F(a_L^*)$. Notice that if $\Gamma$ has one leader and one follower, then generically $\Gamma$ has a unique subgame perfect equilibrium, and therefore satisfies A6.

We now set out the conditions under which accessibility fails.

**Definition 4** *Leader $i \in L$ has an incentive to deviate at $(a_L^*, \alpha_F^*)$ if*

$$\max_{a_i} g_i(a_L^* \backslash a_i, \alpha_F^*) > g_i(a_L^*, \alpha_F^*).$$

This definition is straightforward – a leader has an incentive to deviate at a subgame perfect equilibrium if he can increase his payoff, given the choices of other leaders and given that followers do not respond to this deviation. In standard leader-follower games outside a contracting environment, the question of commitment is only relevant if the leader has an incentive to deviate. Our entry deterrence example belongs to this class. However, our negative results on accessibility apply to a larger class equilibria, in a larger class of games.

**Definition 5** *Leader $i$'s action is relevant at $(a_L^*, \beta_F)$ if*

$$\max_{a_i \neq a_i^*} g_i(a_L^* \backslash a_i, \alpha_F^*) \neq \max_{a_i \neq a_i^*} [g_i(a_L^* \backslash a_i, \beta_F(a_L^* \backslash a_i)].$$

Let $\hat{a}_i \in \arg\max_{a_i \neq a_1^*} g_i(a_L^* \backslash a_i, \beta_F(a_L^* \backslash a))$. If $\beta(a_L^* \backslash \hat{a}_i)$ differs from $\alpha_F^*$, then leader $i$'s action will be relevant (for the followers) at $(a_L^*, \beta_F)$, provided that payoffs are generic. In other words, any game where some follower's best response depends upon whether leader $i$ chooses his commitment action $a_i^*$ or deviates (optimally) from this, leader $i$'s action will be relevant. Clearly, in any game where the leader has an incentive to deviate, his action will be relevant, but the converse is not true.

To clarify our definitions, let us consider a one leader-one follower game, with a unique subgame perfect equilibrium, $(a_1^*, \beta_2)$. Let $\hat{a}_1 \in \arg\max_{a_1 \neq a_1^*} g_1(a_1, a_2^*)$, where $a_2^* = \beta_2(a_1^*)$. If $\beta_2(\hat{a}_1) \neq a_2^*$, then the leader's action will be relevant, provided that payoffs are generic. The game in Fig. 2 (p. 9) provides an illustration: in the (generically unique) subgame perfect equilibrium, the leader has an incentive to deviate only if $x > 2$; however, as long as $x \neq 1$, the leader's action is always relevant.

Let $\gamma_i(a_L^*, \alpha_F^*) = g_i(a_L^*, \alpha_F^*) - \max_{a_i \neq a_i^*} g_i(a_L^* \backslash a_i, \alpha_F^*)$ denote the deviation loss of the leader in the simultaneous move game, where the follower has no observation of the leader's action. If $\gamma_i(a_L^*, \alpha_F^*) \neq \delta_i(a_L^*, \beta_F)$, then leader $i'$s action is relevant.

**Theorem 6** *Let $\Gamma$ be a leader follower game with subgame perfect equilibrium $(a_L^*, \beta_F)$, with $\alpha_F^* = \beta(a_L^*)$, that satisfies A6. Let $\tilde{\Gamma}$ be the associated game in a contracting environment, and let $\omega^*$ denote the equilibrium outcome of $\tilde{\Gamma}$ that is associated with $(a_L^*, \beta_F)$. If either a) leader i's action is relevant at $(a_L^*, \beta_F)$ for some $i \in L$, and there is a monopoly supplier for action $a_i^*$, or b) some leader $j \in L$ has an incentive to deviate at $(a_L^*, \alpha_F^*)$ and needs to contract with a supplier to take action $a_j^*$, then the outcome $\omega^*$ is not accessible.*

**Proof.** From proposition 1, we know that in the game $\tilde{\Gamma}$, there exists an equilibrium with $p(a_i^*) > 0$, with the leaders choosing $a_L^*$ and the followers chooses $\alpha_F^*$ on observing $a_L^*$. Let $\omega^*$ denote the outcome of this equilibrium. Consider the noisy game $\tilde{\Gamma}(\lambda)$ where $\lambda \in Int(\Lambda)$ is sufficiently close to $\lambda_0$ and suppose that we have an equilibrium with an outcome $\omega(\lambda)$ that is sufficiently close to $\omega^*$ where the probability that $a_L^*$ is played by the leaders is at least $1 - \varepsilon$ and for any action $a_i$ played by leader $i$, the probability that the signal $(a_L^*/a_i)$ is observed is at least $1 - \varepsilon$. Suppose that leader $i$ plays $a_i^*$ for sure in this equilibrium. Then the followers must believe that he has chosen $a_i^*$ irrespective of the signal that is observed. Assumption A6 implies that if $\varepsilon$ is sufficiently close to zero, the followers must play $\alpha^*$ after every signal $(a_L^*/a_i)$.

Suppose b), so that leader $i$ has an incentive to deviate. In this case, since the followers play $\alpha^*$ after every signal $(a_L^*/a_i)$, there cannot be an equilibrium where $i$ plays $a_i^*$ for sure. Consider next an equilibrium where the leader randomizes between $a_i^*$ and some other action. Since $a_i^*$ is played with probability less than one, the equilibrium price $p(a_i^*)$ cannot be strictly positive, for if this was the case, supplier $a_i^*$ has a profitable deviation – by reducing his price by any $\eta > 0$, he can ensure a sale for sure. Thus in any such equilibrium, $p(a_i^*) = 0$, so that the outcome cannot be close to $\omega^*$.

Suppose a) so that in the noiseless game $p(a_i^*) = \delta_i(a_L^*, \beta_F)$. If the leader chooses $a_i^*$ for sure in the noisy game, the followers play $\alpha^*$after every signal $(a_L^*/a_i)$. So if $\gamma_i(a_L^*, \alpha_F^*) < \delta_i(a_L^*, \beta_F)$ and $p(a_i^*) > \gamma_i(a_L^*, \alpha_F^*)$, it is optimal for the leader to deviate and choose the action $a_1 \in \arg\max_{a_1 \neq a_1^*} g_1(a_1, a_2^*)$, so that the payoff of supplier $a_i^*$ cannot be close to $\delta_i(a_L^*, \beta_F)$. Conversely, if $\gamma_i(a_L^*, \alpha_F^*) > \delta_i(a_L^*, \beta_F)$ supplier $a_i^*$ can increase his price to $\gamma_i(a_L^*, \alpha_F^*)$ and it will still be optimal for the leader to buy. Thus we cannot have an equilibrium where $a_i^*$ is played with probability one and where supplier $p(a_i^*)$ is close to $\delta_i(a_L^*, \beta_F)$. Once again, if $i$ randomizes, $p(a_i^*) = 0$, so that the outcome cannot be close to $\omega^*$. ■

Having shown that we cannot, in general, approximate the equilibrium outcomes of the noiseless game $\tilde{\Gamma}$ when we allow for noise, one may ask a more limited question. Is it possible to approximate equilibrium *action profiles* taken by the players in the game, even if one does not approximate the supplier payoffs? Let us consider games $\Gamma$ with one leader, player 1, and one follower, player 2. We assume that the gross payoffs in the game satisfy the following genericity assumption:

**A7:** For any $a, a' \in A, a \neq a', g_1(a) \neq g_1(a')$ and $g_2(a) \neq g_2(a')$.

A7 implies that the game $\Gamma$ has a unique subgame perfect equilibrium, the outcome of which we denote by $a^* = (a_1^*, a_2^*)$. By proposition 1, in the associated game $\tilde{\Gamma}$ played in a contracting environment, $\Omega^{\tilde{\Gamma}}$ consists of a singleton set, $\{a^*\}$.

**Theorem 7** *Let $\Gamma$ be a one-leader one follower game that satisfies A7, with unique subgame perfect equilibrium outcome $a^*$. For any countable sequence $\lambda \to \lambda_0, \lambda \in Int(\Lambda)$ and associated sequence of noisy games $\tilde{\Gamma}(\lambda)$, there exists a sequence $\tilde{\alpha}(\lambda) \in \Omega(\tilde{\Gamma}(\lambda))$ such that $\tilde{\alpha}(\lambda) \to a^*$.*

**Proof.** We consider two separate cases, depending upon whether the leader has an incentive to deviate or not. Suppose that the leader has no incentive to deviate, so that $a_1^* \in \arg\max_{a_1} g_1(a_1, a_2^*)$. In the noisy game, let supplier $a_1^*$ choose a price equal to $\min\{\gamma_1(a_1^*, a_2^*), \tilde{c}_1^*\}$, where $\tilde{c}_1^*$ denotes the cost of the next most efficient supplier for $a_1^*$ ( if no supplier is needed for $a_1^*$, set this price equal to zero), and let suppliers for other actions choose a price of zero. Let the leader choose $a_1^*$, and let the suppliers of the follower and the follower choose the continuation strategies in $\tilde{\Gamma}$ (the noiseless) that follow the play of $a_1^*$, regardless of the signal that is observed – these are clearly optimal given that $a_1^*$ is played with probability one by the leader, and given that $\lambda \in Int(\Lambda)$. Given the follower's behavior, it is optimal for the leader to play $a_1^*$, since $\max_{a_1 \neq a_1^*} g_1(a_1, a_2^*) = g_1(a_1^*, a_2^*) - \gamma_1(a_1^*, a_2^*) \leq g_1(a_1^*, a_2^*) - p_1(a_1^*)$. Thus the outcome $a^*$ is an equilibrium outcome of the noisy game.

Suppose now that the leader has an incentive to deviate at $a^*$. From van Damme and Hurkens [4], we know that if A7 is satisfied, there exists a sequence $\lambda \to \lambda_0, \lambda \in Int(\Lambda)$ , with an associated sequence of leader-follower games $\Gamma(\lambda)$, such that in each of these games there exists an equilibrium $(\alpha_1(\lambda), \beta_2(\lambda))$, where the outcomes of this sequence of equilibria converge to $a^*$. For any $\lambda$ in this sequence, we shall construct an equilibrium $\sigma(\lambda)$ in $\tilde{\Gamma}(\lambda)$, the noisy game played in a contracting environment, with the property that $\sigma(\lambda)$ induces the same behavior by the players as $(\alpha_1(\lambda), \beta_2(\lambda))$. Consider $\tilde{\Gamma}(\lambda)$ and assume that for any signal $a_1$ observed by the followers and their suppliers, $\hat{p}_2(a_1)$ and $\sigma_2(a_1,\ \hat{p}_2(a_1))$ is such that $\sigma_2(a_1,\ \hat{p}_2(a_1)) = \beta_2(a_1, \lambda)$. Assume further that every supplier of the leader chooses a price of zero. Since $\alpha_1(\lambda)$ is optimal for the leader in $\Gamma(\lambda)$, it is also optimal for the leader to play $\alpha_1(\lambda)$ in $\tilde{\Gamma}(\lambda)$ since gross payoffs are the same in the two games. Furthermore, given that the leader has an incentive to deviate at $a^*$, $\alpha_1(\lambda)$ does not assign probability one to $a_1^*$, i.e. the leader is randomizing between two or more actions. Thus it is optimal for every seller to choose a price of zero, since any active seller who increases his price will fail to sell with probability one.

It remains to verify our assumption that for any signal $a_1$ observed by the followers and their suppliers, $\hat{p}_2(a_1)$ and $\sigma_2(a_1,\ \hat{p}_2(a_1))$ is such that $\sigma_2(a_1,$

$\hat{p}_2(a_1)) = \beta_2(a_1, \lambda)$. This follows from proposition 1, since given any beliefs over the $A_1$, the continuation game is a (one player) game in a contracting environment. Proposition 1 implies that the equilibrium distributions over player 2's actions coincide in the continuation game played in a contracting environment and the continuation game without contracting being required, where player 2's payoffs equal his gross payoffs. Thus if $\beta_2(a_1, \lambda)$ is optimal in the latter, there exist optimal supplier prices $\hat{p}_2(a_1)$ and $\sigma_2(a_1, \hat{p}_2(a_1))$ such that $\sigma_2(a_1, \hat{p}_2(a_1)) = \beta_2(a_1, \lambda)$. ■

One interpretation of theorem 7 is that it shows accessibility of equilibrium action profiles, even if supplier payoffs cannot be approximated. Thus it might be argued that in games played in a contracting environment, imperfect observation has distributional consequences, but has no implications for the actions that are taken. In our view this is not an appropriate interpretation: the payoffs to suppliers will have incentive effects and will therefore affect outcomes in a broader sense.

# 5 Application: Commitment by a strategic buyer

We consider an example, and demonstrate the failure of accessibility in a price setting context. There there is one buyer, indexed $C$, and two sellers, $A$ and $B$. The buyer has a consumption opportunity for one unit of the product, for each of two periods, $t = 1, 2$. The valuation of the buyer for the product is 1 in each period . Each of these buyers has one unit of the product, and values this at zero. In each period $t$, each seller with positive inventory simultaneously quotes the price for a single unit, $p_i^t$, and the buyer makes a choice to buy from one or none of the sellers. Let $d^t \in \{A, B, \emptyset\}$ denote the buyer's purchase decision, where $\emptyset$ denotes the choice of not buying. We shall assume that the prices chosen by the sellers are private, i.e. they are not observed by the other seller. However, we may distinguish two distinct information structures, public transactions and private transaction. In the case of public transactions, the buyer's decision at $t = 1, d^1$, is commonly observed by both the sellers. In the case of private transactions, the buyer's decision is only observed directly by the party with whom the transaction occurs. As before, our focus is on cautious perfect Bayesian equilibria.

With public transactions, the buyer has substantial commitment power. Suppose that the buyer chooses not to buy at $t = 1$. This implies that there is Bertrand competition in the final period, and hence the buyer gets the product at price 0. It follows, that in any equilibrium, the buyer's utility is at least 1. Indeed, the equilibrium payoff set consists of the convex hull of the points $(0, 0, 1), (1, 0, 1)$ and $(0, 1, 1)$, where the three components represent the payoffs of $A, B$ and $C$ respectively. Any payoff in this set, say $(\lambda_A, \lambda_B, 1)$ can be sustained by the following strategies: at $t = 1$, both sellers choose prices equal to zero. The buyer buys from seller $i$ with probability $\lambda_j, j \neq i$, as long as neither seller has chosen a price greater than zero. If either buyer has chosen a price greater than zero, the buyer chooses not to buy. In period 2, there is either Bertrand competition at price zero (if both sellers have inventory) or one buyer is a monopolist and chooses price equal to one.

Now let us consider the case where transactions are private. That is, if the buyer chooses $A$, then $A$ observes this, but $B$ is only sure that the buyer has not chosen $B$, i.e. he knows that $d^1 \in \{A, \emptyset\}$. We may allow for the possibility that in this case $B$ observes a private signal that is informative about the buyer's decision; however, we shall assume that all signals have positive probability under both decisions.

We show first that there exists an equilibrium where the buyer gets utility 0, while the two sellers each get utility 1. Equilibrium strategies are as follows (if prices are greater than 1, the buyer never buys):

At $t = 1$ :

$p_A^1 = p_B^1 = 1$.

$d^1 = A$ if $p_A^1 = p_B^1 \leq 1$. $d^1 = i$ if $p_i^1 < p_j^1$ and $p_i^1 \leq 1$.

At $t = 2$ :

$p_i^2 = 1$ if $d^1 \neq i$,for $i = A, B$.

$d^2 = A$ if $p_A^2 = p_B^2 \leq 1$. $d^2 = i$ if $p_i^2 < p_j^2$ and $p_i^2 \leq 1$.

This equilibrium has the outcome where the buyer buys from $A$ at price 1 at $t = 1$ and from $B$ at price 1 at $t = 2$. Since each seller makes his maximal feasible profit, clearly neither has any incentive to deviate along the equilibrium path. So consider deviations by the buyer at $t = 1$. If the buyer deviates to $d^1 = \emptyset$, then seller $A$ knows that there has been a deviation, but seller $B$ does not know that there has been a deviation. Hence $B$ continues with his equilibrium strategy, and prices at 1 at $t = 2$. Seller $A$ does not know whether the buyer has deviated to $\emptyset$ or $B$; however, irrespective of his beliefs, he knows that he can ensure that the buyer purchases with probability one as long as he prices strictly below one, and the tie breaking rule embodied in the buyer's continuation strategy implies this is also the case if $p_A^2 = 1$, regardless of the form of the buyer's deviation. Hence it is optimal for $A$ to price at 1, and the buyer's deviation is unprofitable. Similarly, it is easy to verify that deviating by buying from $B$ at $t = 1$ is unprofitable.

Our main result is the following proposition, showing that the buyer loses his commitment power in every equilibrium when there is imperfect observation.

**Proposition 8** *If transactions are private, the payoff* $(1, 1, 0)$ *is the unique equilibrium payoff.*

**Proof.** Consider first an equilibrium where the buyer buys with probability one at $t = 1$. Fix any such equilibrium where $d^1 = j$ with positive probability along the equilibrium path, and assume that seller $i$ has chosen his equilibrium price $p_i^1$; then $d^1 \neq i \Rightarrow i$ believes that $d^1 = j$ for any signal that he receives. Hence $i$ will choose the price 1 at $t = 2$ if the buyer does not buy from him at $t = 1$. We show that this implies that $p_j^1 = 1$. If this is not the case, and $p_j^1 < 1$, then $j$ can increase his payoff by choosing $p' \in (p_j^1, 1)$. If the buyer's equilibrium response to this deviation is to choose $d^1 = i$, then $j$ will be a monopolist at $t = 2$, and hence this deviation is beneficial for $j$.

Suppose that the buyer's equilibrium response to $j$'s deviation is to choose $d^1 = \emptyset$. We have established that $d^1 \neq i \Rightarrow i$ believes that $d^1 = j$ for any signal that he receives, and hence $i$ believes that he is a monopolist at $t = 2$, and will choose price 1. $j$ can therefore ensure that the buyer buys from him at $t = 2$ by choosing any price $p_j^2 < 1$, and thus he has a profitable deviation. We conclude that in any equilibrium where the buyer buys with probability one at $t = 1$, he pays a price of 1, and he also buys with probability one at $t = 2$, also at a price of 1.

Consider next a candidate equilibrium where the buyer fails to buy with probability one at $t = 1$. Hence the price of both firms at $t = 2$ equals zero. Suppose now that $A$ offers a price $p_A^1 < 1$. The buyer will certainly buy, since this gives him positive utility and does not affect his continuation value, since seller $B$ cannot observe this deviation. Hence there cannot be an equilibrium where the buyer fails to buy with probability one at $t = 1$.

Finally, we consider the class of candidate equilibria where the buyer randomizes between buying and not buying at $t = 1$. Consider first an equilibrium where $d^1 = \emptyset$ with probability $\theta$ and $d^1 = A$ with probability $1 - \theta$, and where $A$'s price at $t = 1$ is $p_A^1$. Write $V_i^2(d^1 = x)$ for the expected continuation value of agent $i$ $(i \in (A, B, C))$ conditional on the buyer's decision $d^1 = x$ $(x \in \{A, B, \emptyset\})$. Since the buyer must be indifferent between buying and not buying, we must have

$$1 - p_A^1 = V_C^2(d^1 = \emptyset) - V_C^2(d^1 = A). \tag{11}$$

Furthermore, if $A$ charges any price less than $p_A^1$, the buyer will strictly prefer to buy. Hence $A$ must also be indifferent between making a sale in period two at price $p_A^1$ and making a sale at $t = 1$ in competition with seller $B$, i.e.

$$p_A^1 = V_A^2(d = \emptyset). \tag{12}$$

Adding these expressions we obtain

$$V_C^2(d^1 = \emptyset) + V_A^2(d^1 = \emptyset) - V_C^2(d^1 = A) = 1. \tag{13}$$

However, since the total available value at $t = 2$ is 1, this implies that $V_C^2(d^1 = A) = 0$ (and also $V_B^2(d^1 = \emptyset) = 0$). However $V_C^2(d^1 = A) = 0$ implies $p_B^2 = 1$.

However, $p_B^2 = 1$ is inconsistent with an equilibrium where $d^1 = \emptyset$ with probability $\theta > 0$; in the event that $d^1 = \emptyset$, $A$ can ensure himself of a payoff arbitrarily close to 1 in period 2 by choosing a price $1 - \varepsilon$. Hence equilibrium requires that $A$ also choose a price of 1 at $t = 2$, and that the buyer buys from $A$ when $p_A = p_B = 1$. However, this implies that $p_B^2 = 1$ is not optimal for $B$, since he does better by choosing a price slightly below 1 thereby making a sale for sure. Thus we cannot have such an equilibrium where the buyer randomizes between $d^1 = \emptyset$ and $d^1 = A$.

Finally we consider an equilibrium where the buyer randomizes between $d^1 = \emptyset$, $d^1 = A$ and $d^1 = B$. In this case, in addition to the above expressions, one similarly also obtains

$$V_C^2(d^1 = \emptyset) + V_B^2(d^1 = \emptyset) - V_C^2(d^1 = B) = 1, \tag{14}$$

which implies that $V_C^2(d^1 = \emptyset) = 1$, so that at least one seller's price must be zero at $t = 2$ if the buyer does not buy from this seller. However we also have $V_C^2(d^1 = A) = 0$ and $V_C^2(d^1 = B) = 0$, which is inconsistent with this, and hence we cannot have an equilibrium where the buyer randomizes between all three decisions. ■

This application shows a sharp discontinuity between imperfect monitoring of transactions, and perfect monitoring. Whereas the buyer must get a payoff of at least one when his transaction (i.e. his commitment) is perfectly observed, he must get zero when his transaction is imperfectly observed. That is, he loses his commitment power entirely. The key to this example is that the pricing decisions of the sellers influence the terms on which the buyer makes his commitment. In any equilibrium where the buyer randomizes between buying and not buying, the sellers must also be indifferent between his actions and this turns out to be impossible.

# 6 Conclusion

We have already discussed the literature on imperfectly observed commitment in the introduction to this paper. A related class of games, introduced by Várdy [12], is leader-follower games with costly observation, where the follower must pay a small cost in the event that she chooses to observe the leader's action. Várdy shows that the Stackelberg outcome of a generic finite one leader - one follower game, where the follower *automatically* observes the leader's action, can be approximated by the mixed equilibrium outcome of the game with costly observation (this requires a modification of the definition of "outcome", since the action sets in two classes of game differ).[10] In a recent paper, Morgan and Várdy [9] analyze a class of costly observation games with continuum action sets, and show that the subgame perfect equilibrium outcome when the follower automatically observes the leader's action cannot be approximated in the game with costly observation. For the leader to retain commitment power, she must randomize; however, if the follower's payoff function is strict concave, then he plays pure at every information set. If the leader's payoff function satisfies some strict concavity assumptions, then it has a unique maximizer, and randomization becomes impossible. This is an interesting and important point, and follows from considerations that are quite distinct from those in the present paper. However, this is not a failure of accessibility result strictly speaking, since the action sets in the limit game, where the follower automatically observes the leader's action, are different from those in the costly observation game, where

[10] Morgan and Várdy [8] conduct an experimental investigation of this type of game. For small observation costs, the leader's payoff is close to that of the Stackelberg outcome; however, the randomization probabilties do not vary with the observation cost as predicted by theory.

the follower has to make a choice.[11]

To conclude, our basic results follow from the fact that mixed strategies played in an economic environment must satisfy a stringent generalized indifference principle. Since mixed equilibria are required in order to approximate Stackelberg outcomes, outcomes with perfect observation may well be very different from those under imperfect observation. In consequence, Bagwell's point, that one should be cautious in focusing upon commitment effects under perfect observation, appears to be valid when we consider games played in economic environment. It is possible that many economic applications may fall into this category, since payoffs to agents who choose actions are affected by the prices set by other agents.

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