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Non-abelian T-duality, Ramond fields and coset geometries

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Abstract: We extend previous work on non-abelian T-duality in the presence of Ramond fluxes to cases in which the duality group acts with isotropy such as in backgrounds containing coset spaces. In the process we generate new supergravity solutions related to D-brane configurations and to standard supergravity compactifications.

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1 Introduction

The idea of extending abelian T-duality [1, 2] to non-abelian isometry groups has a long history that both can be given a path integral formulation. However, there are a number of notable differences that clearly distinguish the two cases. Unlike the abelian case, when the isometries are non-commuting, they are no longer present in the T-dual background and the transformation is non-invertible in a path integral approach. Additionally, in general, one cannot establish non-abelian duality as an exact equivalence between partition functions. Nonetheless, such a transformation can still have powerful applications as a solution generating technique in supergravity. Also, in the examples that have been constructed, even if the original non-abelian group G is compact, the associate variables of the T-dual background are non-compact. The last remark, together with some earlier observation in [6] and technical advancements in dealing with backgrounds lacking manifest isometries [9], led recently to an improvement of our understanding. In particular, it was realized that

Background	Coset	Group Dualised
$AdS_3 \times \mathbf{S^3} \times T^4$	SO(4)/SO(3)	SO(4)
$AdS_5 \times \mathbf{S^5}$	SO(6)/SO(5)	SO(6)
$AdS_4 \times \mathbf{CP^2} \times S^2$	SU(3)/(SU(2) × U(1))	SU(3)
$AdS_4 \times \mathbf{S^2} \times \mathbf{S^2} \times \mathbf{S^2}$	$(\mathrm{SU}(2)/\mathrm{SO}(2))^3$	$\mathrm{SU}(2)^3$
$AdS_4 \times \mathbf{CP^3}$	SU(4)/(SU(3) × U(1))	SU(4)

Table 1. Examples studied; the relevant coset manifold shown in bold.

non-abelian T-duality in pure NS backgrounds can be thought of as describing infinitely large spin sectors of a parent theory [10]. When in the latter's theory σ-model the target space coordinates undergo a stretching or contraction one obtains the T-dual σ-model we are interested in.

In some sense, the situation is similar to fermionic T-duality [11] which provided an explanation of the dual superconformal symmetry of $\mathcal{N} = 4$ SYM when applied to $AdS_5 \times S^5$, which also is not an exact symmetry. This development motivates a reconsideration of non-abelian T-duality, in the context of geometries supported by Ramond (RR) fluxes. In [12], non-abelian T-duality was considered for target spaces which included some group manifold, G, as a subspace and whose curvature was supported by RR fluxes. These theories possess a $G_L \times G_R$ isometry group and it was shown how to implement the non-abelian duality with respect to the G_L symmetry. These situations can naturally occur in the near horizon geometries of D-brane configurations. An example is the case of $AdS_3 \times S^3 \times T^4$; here a dualisation with respect to an $\mathrm{SU}(2)_L$ symmetry of the S^3 results in a solution of massive IIA supergravity. Performing a similar dualisation on an $\mathrm{SU}(2) \subset \mathrm{SO}(6)$ isometry for the case of $AdS_5 \times S^5$ gave rise to a solution whose M-theory lift captures generic features of the geometries proposed in [13] (for similar geometries constructed in type-IIA see [14]) as gravity duals to $\mathcal{N} = 2$ gauge theories.

The formulation of non-abelian T-duality in the presence of Ramond fluxes in [12] overcame certain technical difficulties. To appreciate it, recall that in the abelian case the unique dimensional reduction to nine dimensions of the type-II supergravities provided for the transformation rules [15]. However, in non-abelian cases an approach along these lines seems more demanding and hasn't been explored so far. Following this work, it is natural to ask whether the situation can be generalized further to include the case where the isometry is realized via a coset manifold. For instance, one may consider, as we indeed do in a particular example, the dualization of the entire SO(6) isometry that acts on the five-sphere within $AdS_5 \times S^5$. This is a rather non-trivial extension at both the technical and conceptual levels.

The aim of this paper is to address exactly this situation and to provide a whole class of new examples of non-abelian T-dual backgrounds by considering target spaces containing coset manifolds. More precisely, for target spaces containing a coset G/H manifold we will perform a duality with respect to the full G isometry group and demonstrate how the Ramond fluxes transform under the duality. We illustrate this by providing several examples of dualisation in interesting supergravity backgrounds detailed in table 1.

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Initial Background	Initial RR-Fields	T-Dual RR-Fields
$AdS_3 \times \mathbf{S^3} \times T^4$	F_3	$F_1\,,\ F_5$
$AdS_5 \times \mathbf{S^5}$	F_5	F_2
$AdS_4 \times \mathbf{CP^2} \times S^2$	$F_2\,,\ F_4$	$F_2\,,\ F_4$
$AdS_4 \times \mathbf{S^2} \times \mathbf{S^2} \times \mathbf{S^2}$	$F_2\,,\ F_4$	F_3
$AdS_4 \times \mathbf{CP^3}$	$F_2\,,\ F_4$	F_3

Table 2. Initial and T-dual backgrounds with the corresponding Ramond fluxes indicated.

Unlike the case of group manifolds, the G isometry group typically acts on the coset G/H with isotropy and it is this feature that introduces some technical challenges. This is very evident in the Buscher procedure in which the $\dim(G)$ isometry group is gauged; one will have $\dim(G)$ Lagrange multipliers enforcing a flat connection. Among all these variables $\dim(G/H)$ will become the T-dual coordinates and the remainder will be gauge fixed. We will exploit the fact that the dual geometry can be parametrised by H invariant combinations of the Lagrange multipliers to address this issue and to provide simplified geometries produced by dualisation. Expanding the techniques of [12] we are able to construct the full Ramond fluxes required to support these geometries as supergravity solutions which we summarise in table 2. A general feature is that the chirality of the dual theory changes when $\dim(G)$ is odd and is preserved when this is even. One may also see that in all of the dual backgrounds there is no NS two-form, something attributable to the fact that the coset spaces are symmetric and the group we dualized with is the maximal symmetry group (of the corresponding factor in bold in table 2).

The structure of the rest of this paper is as follows: In section 2 we review the general strategy of T-duality in the presence of Ramond fields and then in section 3 we show how this may be applied to the coset geometries in general. In section 4 we then present the explicit examples studied. Due to its additional complexity we leave the case of $AdS_4 \times CP^3$ as an appendix A to the main article. We have also included appendix B with useful information on the geometry and Killing vectors of group and coset spaces, appendix C with the action of the spinor-Lorentz-Lie derivative on the Killing vectors of the $AdS_5 \times S^5$ space and appendix D on the Killing vectors of S^5 as a coset space and the proof of a useful identity.

2 General strategy

Given a supergravity background, in order to perform the non-abelian T-duality transformation we first allocate the group of isometries with respect to which we will perform the transformation. Next we derive the T-duals of the NS fields which on their own form a closed set. This can be done using, for instance, path integral methods following Buscher's treatment of abelian T-duality [1, 2] adapted appropriately for non-abelian isometries [3]. Alternatively, we may achieve the same result by applying a canonical transformation in the phase space of the two-dimensional σ-model [7, 8, 16]. Neither of the above procedures is fully adequate to compute the transformation rules for the Ramond flux fields. In [12] we developed a general procedure that solved this problem which is based on the

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construction of a Lorentz transformation matrix Λ relating the frames naturally defined by the transformations of the left and right world sheet derivatives under T-duality.

This Lorentz transformation induces an action on spinors [17] given by a matrix Ω obtained by requiring that

$$\Omega^{-1}\Gamma^i\Omega = \Lambda^i{}_j\Gamma^j \ . \tag{2.1}$$

To include RR-fields into the discussion we combine them into a bi-spinor according to the type-II supergravity to which they belong. Specifically, we have that

$$\text{IIB}: \qquad P = \frac{e^\Phi}{2}\sum_{n=0}^{4}\frac{1}{(2n+1)!}\ \not{F}_{2n+1} \tag{2.2}$$

and

$$\text{(massive) IIA}: \qquad P = \frac{e^\Phi}{2}\sum_{n=0}^{5}\frac{1}{(2n)!}\ \not{F}_{2n}\,, \tag{2.3}$$

where we used the standard notation $\not{F}_p = \Gamma^{\mu_1\cdots\mu_p}F_{\mu_1\cdots\mu_p}$. In the definition of P we have used the democratic formulation of type-II supergravities [18] wherein all forms up to order ten appear on equal footing. In this formulation and for Minkowski signature spacetimes the conditions

$$F_{2n} = (-1)^n \star F_{10-2n}\,, \qquad F_{2n+1} = (-1)^n \star F_{9-2n}\,, \tag{2.4}$$

should be imposed so that one remains with the right degrees of freedom. However, in checking our solutions to supergravity we shall, in general, work with the standard formulations of type-II supergravities in which no higher forms than five appear.

The Ramond fluxes then transform according to

$$\hat{P} = P\Omega^{-1}\,, \tag{2.5}$$

where we have denoted by a hat the bi-spinor obtained after the duality. In some sense, this relation asserts that, demanding independence of Physics on the frame choice leads to a tranformation of the flux fields within the two-member family of type-II supergravity. The details of the matrix Ω corresponding to cases of non-abelian T-duality have to be worked out in the various cases of interest. We recall for comparison that for the case of abelian T-duality this is simply given as $\Omega = \Gamma_{11}\Gamma_1$ [17], where the 1 labels the isometry direction and Γ_{11} the product of all Gamma matrices. In the abelian case we go from IIA to IIB and vice-versa. However, in non-abelian cases we might change or stay within the same chirality theory [12].

3 Non-abelian T-duals in coset spaces

In [12] it was shown that the Lorentz rotation that acts on spinors can be calculated from the transformation rules of the world sheet derivatives. These rules are easily obtained in the canonical approach to T-duality. We now want to understand the same construction for the coset space σ-models.

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3.1 Review of T-duals in group spaces

We first recap the results of [12] which we generalize slightly to incorporate a wider class of σ-models on group manifolds than just the Principal Chiral Model (PCM). Consider an element g in a group G. We construct the components of the left invariant Maurer-Cartan forms as $L^a_\mu = -i\ \mathrm{Tr}(t^a g^{-1}\partial_\mu g)$, where the representation matrices t^a obey the corresponding Lie algebra with structure constants $f^{ab}{}_c$. The most general σ-model, that is invariant under the global symmetry $g \to g_0 g$, with $g_0 \in G$, is of the form

$$S = \frac{1}{2}\int d^2\sigma\, E_{ab} L^a_+ L^b_- \,, \qquad L^a_\pm = L^a_\mu \partial_\pm X^\mu \,, \tag{3.1}$$

where E is a dim(G) square invertible constant matrix (actually E may depend on other coordinates that have only a spectator rôle in the whole discussion, although this will not be needed for our purposes). For the case where E is proportional to just the Cartan metric, taken to be the identity matrix in this paper, this σ-model is just the PCM on G. However in what follows it will be important to us that one can still perform a duality for a general matrix E.

The non-abelian T-dual σ-model to eq. (3.1) with respect to the full G symmetry is constructed by following the standard Buscher-like approach by introducing gauge fields and a Lagrange multiplier term. Alternatively, we may employ a canonical transformation in phase space. With either method the result is

$$\tilde{S} = \frac{1}{2}\int d^2\sigma\, (M^{-1})^{ab}\partial_+ v_a \partial_- v_b \,, \tag{3.2}$$

in which

$$M_{ab} = E_{ab} + f_{ab} \,, \qquad f_{ab} = f_{ab}{}^c v_c \ . \tag{3.3}$$

There is also a dilaton induced as a quantum effect given by

$$\Phi = -\frac{1}{2}\ln\det M \ . \tag{3.4}$$

The canonical transformation relating these models is entirely encoded in the transformation of the world sheet derivatives

$$L^a_+ = (M^{-1})^{ba}\partial_+ v_b \,, \qquad L^a_- = -(M^{-1})^{ab}\partial_- v_b \ . \tag{3.5}$$

As an immediate consequence of the identity

$$\frac{1}{2}(M^{-1} + M^{-T}) = M^{-T}\eta M^{-1} = M^{-1}\eta M^{-T} \,, \tag{3.6}$$

in which η denotes the symmetric part of E, both M^{-1} and M^{-T} occurring in eq. (3.5) define frame fields for the metric of the dual σ-model eq. (3.2). These two frames are related by a Lorentz transformation

$$\Lambda = -\kappa M^{-T} M \kappa^{-1} = -\kappa^{-T} M M^{-T}\kappa^T \,, \tag{3.7}$$

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where the matrix κ is such that the constant matrix $\eta = \kappa^T\kappa$. Given this form of the Lorentz transformation we may explicitly solve eq. (2.1) to find the corresponding spinorial representation Ω. We first expand M around minus the identity by treating as small parameters the coordinates v_a as well as the antisymmetric part of the matrix E which we will denote by S. After determining the infinitesimal transformation and subsequent exponentiation we find that

$$\Omega = e^{\frac{1}{2}\tilde{f}_{ab}\Gamma^{ab}} \prod_{i=1}^{\dim(G)} (\Gamma_{11}\Gamma_i)\,, \qquad \tilde{f} = \kappa^{-T}(S+f)\kappa^{-1} = -\tilde{f}^T\,. \tag{3.8}$$

The reason that we may obtain the result by an exponentiation of the infinitesimal form is that the matrices Γ_{ab} close into an $so(\dim(G))$ algebra. From the above expression it is clear that if the duality group is even then we stay in the same type-II supergravity, whereas if it is odd then we flip from (massive) type-IIA supergravity to type-IIB and vice versa.

Whilst generically the σ-model eq. (3.2) has no isometries it is possible for particular forms of the matrix E to obtain residual symmetries. These correspond to extra isometries of the original σ-model eq. (3.1) that commute with the symmetry that we used to perform the non-abelian T-duality. Of course, the matrix Ω in eq. (3.8) should respect this symmetry. For example, in the case of $E = \mathbf{1}$ the original σ-model in eq. (3.1) enjoys a global $G_L \times G_R$ isometry which will lead to a residual G_R symmetry in the dual theory. This is indeed the case in the examples worked out in [12] in which a non-abelian dual of S^3 is performed with respect to $\mathrm{SU}(2)_L$ of the total isometry group $\mathrm{SO}(4) \simeq \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$; the $\mathrm{SU}(2)_R$ symmetry is manifestly preserved in the dual background.

3.2 Non-abelian T-duals in coset spaces via reduction

To extend the discussion for σ-models corresponding to coset G/H spaces we split for notational purposes the index $a = (i, \alpha)$, where the indices i and α belong to the subgroup $H \in G$ and the corresponding coset G/H, respectively. The σ-model is

$$S = \frac{1}{2}\int d^2\sigma\, (E_0)_{\alpha\beta} L^\alpha_\mu L^\beta_\nu \partial_+ X^\mu \partial_- X^\nu\,, \tag{3.9}$$

so that it has the same form as that for group spaces in eq. (3.1). The restriction of the matrix E in eq. (3.1) to coset space requires that E_0 is G-invariant which severely restricts its form. In most cases of interest this will be taken to be proportional to the Killing metric. The key point that enables one to obtain the explicit form eq. (3.2) of the non-abelian T-dual for the case of group manifolds relied on the fact that the symmetry acts with no isotropy. In technical terms that means that, in the Buscher-like approach, one can gauge fix the group element g to unity, so that the dual σ-model contains only the Lagrange multipliers. For coset models this is not possible and one has to gauge fix some of the Lagrange multipliers as well, in which the group acts with isotropy, i.e. as $\delta v^a = f_{bc}{}^a \epsilon^b v^c$. Hence there exist fixed points of this transformation.

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For our purposes it is convenient to proceed by using a reduction method introduced in [19].[1] The reduction procedure is taken as follows: Consider a matrix E of the form

$$E = \text{diag}\left(E_0, \lambda\, \mathbf{1}_{\dim(H)}\right) , \tag{3.10}$$

where E_0 is a $\dim(G/H)$ square invertible constant matrix and λ is a parameter. Then the dual models eq. (3.1) and eq. (3.2) are perfectly consistent and have $\dim(G)$ target spaces. In the limit $\lambda \to 0$ the Maurer-Cartan forms associated with the subgroup in eq. (3.1) drop out. Then, we are left with the σ-model for the coset space G/H eq. (3.9) and eq. (3.2) represents its dual. For the whole procedure to be consistent one has to ensure that the corresponding target spaces are reduced to $\dim(G/H)$. It can be shown that this is ensured if E_0 is indeed G-invariant [19]. The above remarks imply that we may fix $\dim(H)$ among the v_a's and denote the remaining ones by x_α. Alternatively, we may think of the x_α's as the H-subgroup invariants one can form using the $\dim(G)$ variables parameterizing $g \in G$. This is completely analogous and in fact inspired by a treatment of the gauge fixing procedure in gauged WZW models in [20].

To find out the transformation rules of the world sheet derivatives we define the $\dim(G/H)$ square matrices $N_\pm$ from the relations

$$\begin{aligned} L_+^\alpha &= (M^{-1})^{b\alpha}\partial_+ v_b = N_+^{\alpha\beta}\partial_+ x_\beta \,, \\ L_-^\alpha &= -(M^{-1})^{\alpha b}\partial_- v_b = N_-^{\alpha\beta}\partial_- x_\beta \,, \end{aligned} \tag{3.11}$$

where we have taken the $\lambda \to 0$ limit. Then the Lorentz transformation is given by

$$\Lambda = \kappa_0 N_+ N_-^{-1} \kappa_0^{-1} \,, \tag{3.12}$$

where κ_0 is the restriction of the frame matrix κ to the coset obeying $E_0 = \kappa_0^T \kappa_0$. It should be possible to obtain Ω, to be used in eq. (2.5), by appropriately taking the $\lambda \to 0$ limit in eq. (3.8). In that respect, whether or not one changes or stays in the same type-II theory depends entirely on $\dim(G)$ and not on $\dim(G/H)$.

4 Examples

We present below several examples from D-brane configurations in string theory and from some standard compactifications in type-II supergravity.

4.1 Non-abelian T-dual in the D1-D5 near horizon

As a first example we consider the $AdS_3 \times S^3 \times T^4$ geometry that arises as the near horizon limit of the D1-D5 brane system. The type-IIB supergravity background consists of a metric

$$ds^2 = ds^2(\text{AdS}_3) + ds^2(\text{S}^3) + ds^2(\text{T}^4) \,, \tag{4.1}$$

[1] This was actually considered in the more general context of σ-models related by Poisson-Lie T-duality, of which non-abelian duality is just a particular case.

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where the normalization is such that $R_{\mu\nu} = \mp 2g_{\mu\nu}$ for the AdS_3 and S^3 factors, respectively, supported by the Ramond flux

$$F_3 = 2\left(\mathrm{Vol}(AdS_3) + \mathrm{Vol}(S^3)\right) , \tag{4.2}$$

whereas the dilaton $\Phi = 0$. To construct the bi-spinor of fluxes we need the Hodge-dual of the above three-form

$$F_7 = -(\star F_3) = 2\left(\mathrm{Vol}(S^3) + \mathrm{Vol}(AdS_3)\right) \wedge \mathrm{Vol}(T^4) . \tag{4.3}$$

Note that we have completely absorbed all constant factors by appropriate rescalings. The presence of S^3 indicates a global SO(4) with respect to which we will perform the non-abelian transformation. For comparison, we recall that the non-abelian T-dual with respect to the $\mathrm{SU}(2)_L$ subgroup of SO(4) was constructed in [12]. However, in that case, unlike here, the group's action is without isotropy.

To proceed we need to determine the matrix M in eq. (3.2). Let's recall that we may construct the SO(N) algebra by first defining matrices t_{ab} with $a = 1, 2, \ldots, N$, with

$$(t_{ab})_{cd} = \delta_{ac}\delta_{bd} . \tag{4.4}$$

Then

$$J_{ab} = t_{ab} - t_{ba} , \tag{4.5}$$

obey the SO(N) algebra. An SO($N-1$) subalgebra is generated by the matrices J_{ij} with $i = 2, 3, \ldots, N$, whereas the coset SO(N)/SO($N-1$) currents are given by J_{1i}.

For the case at hand, $N = 4$, we define

$$\begin{aligned} S_a &= J_{1,a+1} , \qquad a = 1,2,3 , \\ S_{a+3} &= J_{2,a+2} , \qquad a = 1,2 , \\ S_6 &= J_{34} . \end{aligned} \tag{4.6}$$

In this arrangement the elements S_a with $a = 4, 5, 6$ obey an SO(3) subalgebra. We organize the structure constants by computing

$$[S_a, S_b] = f_{ab}{}^c S_c . \tag{4.7}$$

According to the previous discussion we now need to gauge fix three of the six v_a. For this simple case one could, of course, do this just by inspection. However, for more complicated cases this is not such an easy thing to do. To this end we employ some group theoretical reasoning developed in the context of gauged WZW models in [20]. Under $\mathrm{SO}(4) \to \mathrm{SO}(3)$ the adjoint decomposes $\mathbf{6} \to \mathbf{3} \oplus \mathbf{3}$. If we label the first triplet as X and the second as Y, we have explicitly

$$X = (v_1 + v_6, v_5 - v_2, v_3 + v_4) , \qquad Y = (v_1 - v_6, -v_5 - v_2, v_3 - v_4) . \tag{4.8}$$

There are three independent invariants under SO(3) that one can construct from these triplets given by

$$t_1 = X^2 , \qquad t_2 = X \cdot Y , \qquad t_3 = Y^2 . \tag{4.9}$$

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To fix the residual SO(3) one imposes some constraints $F_i(v) = 0$, with $i = 1, 2, 3$. Clearly a valid gauge fixing choice cannot eliminate these invariants. In other words, after gauge fixing there must remain three parameters in one-to-one correspondence with these invariants. We now make the following gauge choice $v_1 = v_2 = v_6 = 0$, and rename the remaining coordinates

$$(x_1, x_2, x_3) = (v_3, v_4, v_5)\,, \tag{4.10}$$

such that the invariants are given by

$$t_1 = (x_1 + x_2)^2 + x_3^2\,, \quad t_2 = x_1^2 - x_2^2 - x_3^2\,, \quad t_3 = (x_1 - x_2)^2 + x_3^2\,. \tag{4.11}$$

To construct the dual we now need the matrix $M = E + f$, which in the $\lambda \to 0$ coset limit is given by

$$M = \begin{pmatrix} 1 & -v_4 & -v_5 & v_2 & v_3 & 0 \\ v_4 & 1 & -v_6 & -v_1 & 0 & v_3 \\ v_5 & v_6 & 1 & 0 & -v_1 & -v_2 \\ -v_2 & v_1 & 0 & 0 & -v_6 & v_5 \\ -v_3 & 0 & v_1 & v_6 & 0 & -v_4 \\ 0 & -v_3 & v_2 & -v_5 & v_4 & 0 \end{pmatrix}. \tag{4.12}$$

Applying the gauge fixing we find the matrices $N_\pm$ appearing in the canonical transformation of the derivatives eq. (3.11) as

$$N_+ = \frac{1}{x_1 x_3}\begin{pmatrix} 0 & x_2 & x_3 \\ 0 & x_2^2 - x_1^2 & x_2 x_3 \\ x_1 x_3 & x_2 x_3 & x_3^2 \end{pmatrix}, \quad N_- = \frac{1}{x_1 x_3}\begin{pmatrix} 0 & x_2 & x_3 \\ 0 & x_1^2 - x_2^2 & -x_2 x_3 \\ -x_1 x_3 & -x_2 x_3 & -x_3^2 \end{pmatrix}. \tag{4.13}$$

These define two frames for the dual geometry whose metric is explicitly given by

$$\begin{aligned} ds^2 &= dx_1^2 + 2\frac{dx_1(x_2 dx_2 + x_3 dx_3)}{x_1} + \frac{dx_2^2\left[x_1^4 - 2{x_1}^2 x_2^2 + x_2^2\left(x_2^2 + x_3^2 + 1\right)\right]}{x_1^2 x_3^2} \\ &\quad + 2\frac{dx_2 dx_3 x_2\left(-x_1^2 + x_2^2 + x_3^2 + 1\right)}{x_1^2 x_3} + \frac{{dx_3}^2\left({x_2}^2 + {x_3}^2 + 1\right)}{{x_1}^2}\,, \end{aligned} \tag{4.14}$$

plus of course the terms $ds^2(\mathrm{AdS}_3) + ds^2(\mathrm{T}^4)$. The NS two-form vanishes and the dilaton computed from eq. (3.4) is

$$\Phi = -\ln(x_1 x_3)\,. \tag{4.15}$$

The Lorentz transformation relating the frames is found using eq. (3.12) with $\kappa_0 = \mathbf{1}$. It reads

$$\Lambda = \mathrm{diag}(1, -1, -1)\,. \tag{4.16}$$

Hence the corresponding transformation for the spinors is

$$\Omega = -\Gamma_2 \Gamma_3\,, \tag{4.17}$$

as if we had two successive abelian T-dualities. The reason for this is that the lack of isometries in the T-dual background prevents Ω from having some non-trivial structure.[2] Then we compute the RR forms

$$\begin{aligned} F_1 &= 2x_1x_3e_1 = 2(x_2dx_2 + x_3dx_3) \\ F_5 &= (1+\star)(F_1 \wedge \mathrm{Vol}(T^4)) \,, \end{aligned} \tag{4.18}$$

supplemented by an F_9 obeying $\star F_9 = F_1$ as it should.

The metric eq. (4.14) is quite complicated. It turns out that it considerably simplifies if we use the invariants eq. (4.9) as coordinates for the dual geometry. After some manipulations we find that the natural one-forms associated with N_+ can be expressed quite simply as

$$\begin{aligned} e^1 &= \frac{1}{8x_1x_3}\left(dt_1 - 2dt_2 + dt_3\right) \,, \\ e^2 &= \frac{1}{4x_1x_3}\left[(x_2 - x_1)dt_1 + (x_2 + x_1)dt_3\right] \,, \\ e^3 &= \frac{1}{4x_1}\left(dt_1 + dt_3\right) \,, \end{aligned} \tag{4.19}$$

where for the time being we leave these x_α's as implicit functions of the new coordinates (they can be explicitly obtained by inverting eq. (4.11)). For the metric we find

$$\begin{aligned} ds^2 = \frac{1/16}{t_1t_3 - t_2^2} &\Big[(-2dt_2 + dt_3)^2 + 4t_1dt_3^2 + 2dt_1(-2dt_2 + (1-4t_2)dt_3) \\ &+ (1+4t_3)dt_1^2\Big] + ds^2(AdS_3) + ds^2(T_4) \,. \end{aligned} \tag{4.20}$$

The dilaton and the fluxes are

$$\Phi = -\ln\left[\frac{1}{2}\sqrt{t_1t_3 - t_2^2}\right] \tag{4.21}$$

and

$$F_1 = \frac{1}{4}(dt_1 - 2dt_2 + dt_3) \,, \qquad F_5 = (1+*)(F_1 \wedge \mathrm{Vol}(T^4)) \,. \tag{4.22}$$

Note that the geometry is singular at $x_1x_3 = 0$. This is due to the fact that the duality group acts with isotropy on the Lagrange multipliers. In addition, we have verified that the supergravity equations of motion are indeed satisfied by the T-dual background. Similar comments apply to the other examples below.

4.1.1 (No) supersymmetry of the dual

The dual background given by eq. (4.14), eq. (4.15) and eq. (4.18) does not preserve any supersymmetry. This can easily be seen from the dilatino variation (non-democratic form)

$$\delta\lambda = \left(\partial\!\!\!/\phi + ie^\phi F\!\!\!/_1\right)\epsilon - \frac{1}{2}\left(H\!\!\!\!/ + ie^\phi F\!\!\!/_3\right)\epsilon^* \,, \tag{4.23}$$

[2] In that respect one can check out Ω in eq. (3.10) of [12]. In that case there is a residual rotational symmetry after the T-duality is performed, so that the matrix Ω could have this symmetry.

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in which $\epsilon = \epsilon_1 + i\epsilon_2$ for two Majorana-Weyl supersymmetry parameters of same chirality. For the geometry above, in which we have vanishing three NS form, this simply reduces to an equation of the form

$$(a\Gamma^3 + b\Gamma^4 + c\Gamma^5)\epsilon = 0\,, \tag{4.24}$$

where a, b, c have some coordinate dependence. By squaring one can see that this implies $(a^2 + b^2 + c^2)\epsilon = 0$ and hence the only solution is the trivial one $\epsilon = 0$.

This conclusion agrees with our expectation from the spinor-Lorentz-Lie derivative (Kosmann derivative) [21, 22]. It was shown in [12] that for the Killing spinor of $AdS_3 \times S^3 \times T^4$ to be invariant under the $\mathrm{SU}(2)_L$ Killing vectors

$$(\mathcal{P}_- \otimes \mathbf{1}_{32})\varepsilon = 0\,, \tag{4.25}$$

where we used the doublet $\varepsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}$ and introduced projectors $\mathcal{P}_\pm = \frac{1}{2}\left(\mathbf{1}_2 \pm \sigma_1\right)$. The fact that the projector for the left action is $\mathcal{P}_-$ can be traced to the $\mathrm{SU}(2)_L$ invariant 1-forms (and corresponding dual vector fields) which obey the Maurer-Cartan equations

$$dL^a = \frac{1}{2} f_{bc}{}^a L^b \wedge L^c\,. \tag{4.26}$$

If we were instead to consider the $\mathrm{SU}(2)_R$ action, since the right invariant forms obey

$$dR^a = -\frac{1}{2} f_{bc}{}^a R^b \wedge R^c\,, \tag{4.27}$$

we would find a projector condition

$$(\mathcal{P}_+ \otimes \mathbf{1}_{32})\varepsilon = 0\ . \tag{4.28}$$

It is clear that the only spinor that can be invariant under both the left and right actions is the trivial zero spinor.

4.2 Non-abelian T-dual in the D3 near horizon

Our second example concerns the type-IIB supergravity solution describing the near horizon limit of the D3-brane background. It consists of a metric

$$ds^2 = ds^2(\mathrm{AdS}_5) + ds^2(\mathrm{S}^5)\,, \tag{4.29}$$

normalized such that $R_{\mu\nu} = \mp 4 g_{\mu\nu}$ for the AdS_5 and S^5 factors, respectively, supported by the self-dual Ramond flux

$$F_5 = 4\left(\mathrm{Vol}(AdS_5) - \mathrm{Vol}(S^5)\right)\ . \tag{4.30}$$

As before the dilaton $\Phi = 0$ and we note that we have completely absorbed all constant factors by appropriate rescalings. The presence of S^5 indicates a global SO(6) with respect to which we will perform the non-abelian transformation.

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We construct the SO(6) algebra as in eq. (4.5) with $N = 6$ and we define

$$\begin{aligned} S_a &= J_{1,a+1}\,, & a &= 1,\ldots,5\,,\\ S_{a+5} &= J_{2,a+2}\,, & a &= 1,\ldots,4\,,\\ S_{a+9} &= J_{3,a+3}\,, & a &= 1,2,3\,,\\ S_{a+12} &= J_{4,a+4}\,, & a &= 1,2\,,\\ S_{15} &= J_{56}\,. \end{aligned} \tag{4.31}$$

In this arrangement the elements S_a with $a = 6, 7, \ldots, 15$ obey an SO(5) subalgebra. We organize the structure constants by computing eq. (4.7).

In order to gauge fix we find it convenient to form the five invariants of the antisymmetric matrix rep. **15** of SO(6) under the SO(5) subgroup. According to eq. (4.31) this splits into a vector and the antisymmetric rep., i.e. $\mathbf{15} \to \mathbf{5} \oplus \mathbf{10}$. These are explicitly constructed as

$$(V_i) = (v_1, v_2, \ldots, v_5)\,, \quad (A_{ij}) = \begin{pmatrix} 0 & v_6 & v_7 & v_8 & v_9 \\ -v_6 & 0 & v_{10} & v_{11} & v_{12} \\ -v_7 & -v_{10} & 0 & v_{13} & v_{14} \\ -v_8 & v_{11} & -v_{13} & 0 & v_{15} \\ -v_9 & -v_{12} & -v_{14} & -v_{15} & 0 \end{pmatrix}\,. \tag{4.32}$$

The invariants are[3]

$$\begin{aligned} t_1 &= V^2\,, \qquad t_2 = -\frac{1}{2}\mathrm{Tr}(A^2)\,, \qquad t_3 = \frac{1}{8}\epsilon^{ijklm}A_{ij}A_{kl}V_m\,,\\ t_4 &= -\frac{1}{4}\mathrm{Tr}(A^4) + \frac{1}{8}[\mathrm{Tr}(A^2)]^2\,, \qquad t_5 = -(A^2)_{ij}V_iV_j\,, \end{aligned} \tag{4.33}$$

where the various numerical factors have been introduced for later convenience. The gauge fixing of ten parameters among the fifteen v_a's should be such that the remaining five have a one to one correspondence to the above invariants. The transformation of the Lagrange multipliers is given by

$$\delta v^a = f_{bc}{}^a \epsilon^b v^c \quad \Longrightarrow \quad \delta v^i = f_{jk}{}^i \epsilon^j v^k\,, \quad \delta v^\alpha = f_{j\beta}{}^\alpha \epsilon^j v^\beta\,, \tag{4.34}$$

where we note that the infinitesimal parameters belong to the subgroup. One may explicitly check that eq. (4.33) indeed remain invariants. We choose to keep non-zero the variables v_a with $a = 5, 8, 10, 13, 15$. Hence in our notation

$$(x_1, x_2, x_3, x_4, x_5) = (v_5, v_8, v_{10}, v_{13}, v_{15})\,. \tag{4.35}$$

In terms of our variables $x_\alpha, \alpha = 1, 2, \ldots, 5$, the invariants become

$$\begin{aligned} t_1 &= x_1^2\,, \qquad t_2 = x_2^2 + x_3^2 + x_4^2 + x_5^2\,, \qquad t_3 = x_1x_2x_3\,,\\ t_4 &= x_3^2(x_2^2 + x_5^2)\,, \qquad t_5 = x_1^2x_5^2\,. \end{aligned} \tag{4.36}$$

[3]The characteristic polynomial satisfied by the matrix A, according to the Cayley-Hamilton theorem, is of degree 5. However, because of antisymmetry of A, we have that $(A^5)_{ij}V_iV_j = 0$. Hence, the next available invariant $(A^4)_{ij}V_iV_j$ is not an independent one.

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Then the matrix M significantly simplifies. The matrices that define the frames are computed using eq. (3.11). They turn out to be

$$N_+ = \frac{1}{x_1}\begin{pmatrix} 0 & \frac{x_1^2-x_2^2}{x_5} & \frac{x_2(x_1^2-x_3^2)}{x_3x_5} & -\frac{x_2x_4}{x_5} & -x_2 \\ 0 & 0 & \frac{x_3^2-x_2^2-x_5^2}{x_4x_5} & \frac{x_3}{x_5} & 0 \\ 0 & -\frac{x_2x_4}{x_5} & \frac{x_1^2(x_3^2-x_2^2)+x_3^2(x_2^2-x_3^2-x_4^2+x_5^2)}{x_3x_4x_5} & \frac{x_1^2-x_3^2-x_4^2}{x_5} & -x_4 \\ 0 & -\frac{x_2}{x_5} & -\frac{x_3}{x_5} & -\frac{x_4}{x_5} & -1 \\ x_1 & x_2 & x_3 & x_4 & x_5 \end{pmatrix}, \tag{4.37}$$

as well as a similar expression for N_- in such a way that the Lorentz transformation eq. (3.12) (we use that $\kappa_0 = \mathbf{1}$) is

$$\Lambda = \mathrm{diag}(-1, 1, -1, 1, -1) \ . \tag{4.38}$$

The metric is obtained using either of the above frames. The NS two-form turns out to be zero and the dilaton is

$$\Phi = -\ln(x_1^2 x_3 x_4 x_5^2) \ . \tag{4.39}$$

The corresponding transformation for the spinors is (we omit an overall sign)

$$\Omega = \Gamma_{11}\Gamma_1\Gamma_3\Gamma_5 \, , \tag{4.40}$$

leading to the RR form

$$F_2 = 4x_1^2 x_3 x_4 x_5^2 \; e_2 \wedge e_4 \, , \tag{4.41}$$

together with an F_8 obeying $\star F_8 = -F_2$ as it should.

As before we may express the background in terms of the invariants in eq. (4.33). After some manipulations we find that the natural one-forms associated with N_+ can be expressed quite simply as

$$\begin{aligned} e^1 &= \frac{1}{2x_1^2 x_3 x_5}\left(2t_1 dt_3 - t_3 dt_1 - t_3 dt_2\right) , \\ e^2 &= \frac{1}{2x_1^3 x_3 x_4 x_5^3}\left[(t_1 t_4 - t_3^2)dt_2 - t_5 dt_4\right] , \\ e^3 &= \frac{1}{2x_1^3 x_3 x_4 x_5^3}\Big[t_4 t_5 dt_1 + (t_2 t_3^2 + t_1^2 t_4 - t_1(t_3^2 + t_2 t_4) + t_4 t_5)dt_2 \\ &\qquad -2t_3 t_5 dt_3 - t_3^2 dt_4 + t_1 t_4 dt_4 + t_3^2 dt_5 - t_1 t_4 dt_5)\Big] , \\ e^4 &= -\frac{1}{2x_1 x_5} dt_2 \, , \\ e^5 &= \frac{1}{2x_1}(dt_1 + dt_2) \ . \end{aligned} \tag{4.42}$$

where the x_α's are implicit functions of the new coordinates. The dilaton and flux are

$$\Phi = -\frac{1}{2}\ln\left[(t_3^2 + t_2 t_5 - t_1 t_4)(t_1 t_4 - t_3^2) - t_4 t_5^2\right] \ . \tag{4.43}$$

and

$$F_2 = -dt_2 \wedge dt_4 \, , \tag{4.44}$$

which is a manifestly exact form.

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The involved form of the solution suggests that supersymmetry is broken — this is indeed the case as can be established from a consideration of the dilatino and gravitino supersymmetry variations of type-IIA. As detailed in appendix C one reaches the same conclusion by demanding that the spinor-Lie derivative of the Killing spinors of $AdS_5 \times S^5$ vanishes for the SO(6) killing vectors generating the isometry.

4.3 Non-abelian T-dual of $AdS_4 \times CP_2 \times S^2$

There is a class of solutions of eleven-dimensional supergravity labeled as $M(m,n)$, where m and n are integers, which were constructed in [23] (for a review see [24]) and are U(1) bundles over $CP^2 \times S^2$. By dimensionally reducing one obtains a type-IIA supergravity solution. The metric is

$$ds^2 = ds^2(\mathrm{AdS}_4) + ds^2(\mathrm{CP}^2) + ds^2(\mathrm{S}^2)\,, \tag{4.45}$$

where we have normalized in such a way that $R_{\mu\nu} = -2g_{\mu\nu}$, $\Lambda_4 g_{\mu\nu}$ and $\Lambda_2 g_{\mu\nu}$ for AdS_4, CP^2 and S^2, respectively. The geometry is supported by a two-form flux written as a linear combination of the Kähler forms on CP^2 and S^2

$$F_2 = \frac{2}{3}\Lambda_4\, mJ_{\mathrm{CP}^2} + \Lambda_2\, nJ_{\mathrm{S}^2}\,, \tag{4.46}$$

with

$$J_{\mathrm{CP}^2} = e^1 \wedge e^2 + e^3 \wedge e^4\,, \qquad J_{\mathrm{S}^2} = \mathrm{Vol}(\mathrm{S}^2) = e^5 \wedge e^6\ . \tag{4.47}$$

In addition, there is a four-form flux

$$F_4 = A_{m,n}\mathrm{Vol}(AdS_4)\ . \tag{4.48}$$

Consistency with the equations of motion requires that

$$\Lambda_2 = \frac{4}{1+2x}\,, \qquad \Lambda_4 = \frac{4x}{1+2x}\,, \tag{4.49}$$

and

$$A^2_{m,n} = 16\frac{8m^2x^3 - 9n^2(1+x)}{9(x-1)(1+2x)^2}\,, \tag{4.50}$$

and that there is a constant dilaton

$$e^{-2\Phi} = \frac{2}{9}\frac{9n^2 - 4m^2x^2}{(1-x)(1+2x)}\ . \tag{4.51}$$

The parameter x is determined from a cubic equation

$$\frac{m^2}{n^2} = \frac{9}{4}\frac{2x-1}{x^2(3-2x)}\,, \tag{4.52}$$

which has only one real root in the interval $x \in [\frac{1}{2}, \frac{3}{2}]$. For $x = 1$ one easily sees that consistency requires that $2m = 3n$. In this particular case the eleven-dimensional solution has either $N = 2$ or $N = 0$ supersymmetry, but the dimensionally reduced type-IIA solution in which we are interested has no supersymmetry whatsoever, so it is highly unexpected that the T-dual geometry will be supersymmetric. Note that when $2m = 3n$ is satisfied then there is no singularity when $x = 1$.

For our purposes we need the higher forms

$$\begin{aligned} F_6 &= -(\star F_4) = A_{m,n}\, e^1 \wedge e^2 \wedge e^3 \wedge e^4 \wedge e^5 \wedge e^6\,, \\ F_8 &= \star F_2 = \frac{4}{1+2x}\mathrm{Vol}(\mathrm{AdS}_4) \wedge \left(\frac{2}{3} m x J_{\mathrm{CP}^2} \wedge J_{\mathrm{S}^2} + n \mathrm{Vol}(\mathrm{CP}^2)\right)\,. \end{aligned} \tag{4.53}$$

The presence of CP^2 indicates a global SU(3) with respect to which we will perform the non-abelian transformation. We will use as a basis the standard Gell-Mann matrices λ_a, $a = 1, 2, \ldots, 8$. To conform with our conventions we relabel $\lambda_1, \lambda_2, \lambda_3$ and λ_8, the generators of the subgroups SU(2) and U(1) respectively, as S_5, S_6, S_7, S_8, and $\lambda_4, \lambda_5, \lambda_6, \lambda_7$ as S_1, S_2, S_3, S_4.

Now if we consider the symmetry subgroup SU(2) $\times$ U(1), we would like to gauge fix by setting four of the Lagrange multipliers to zero. A suitable gauge fixing choice may be discerned by constructing the four invariants of the **8** representation of SU(3) under the SU(2) $\times$ U(1) subgroup. Under SU(2) $\times$ U(1), the **8** splits as $\mathbf{8} \to \mathbf{3} \oplus \mathbf{2} \oplus \bar{\mathbf{2}} \oplus \mathbf{1}$. These may be represented explicitly in terms of the eight Lagrange multipliers as

$$D = v_8\,, \qquad V^i = (v_1 - i v_2, v_3 - i v_4)\,, \qquad \bar{V}_i = (V^i)^* \tag{4.54}$$

and

$$A = \begin{pmatrix} v_7 & v_5 - i v_6 \\ v_5 + i v_6 & -v_7 \end{pmatrix}\,. \tag{4.55}$$

Then by ensuring that we gauge fix so that the remaining four Lagrange multipliers are in one to one correspondence with the independent invariants

$$t_1 = D\,, \qquad t_2 = \frac{1}{2}\mathrm{Tr}(A^2)\,, \qquad t_3 = V^i \bar{V}_i\,, \qquad t_4 = \frac{1}{2}\bar{V}_i A^i{}_j V^j, \tag{4.56}$$

we will determine a suitable gauge fixing choice. For the current case we adopt $v_2 = v_4 = v_6 = v_7 = 0$. This removes any residual freedom in SU(2) $\times$ U(1). We henceforth relabel

$$(x_1, x_2, x_3, x_4) = (v_1, v_3, v_5, v_8). \tag{4.57}$$

For this gauge choice the invariants are

$$t_1 = x_4\,, \qquad t_2 = x_3^2\,, \qquad t_3 = x_1^2 + x_2^2\,, \qquad t_4 = x_1 x_2 x_3\,. \tag{4.58}$$

The matrices defining the frames may then be read off from the earlier prescription with N_+ taking the form

$$N_+ = \begin{pmatrix} 1 & 0 & \frac{x_1(x_2^2 - x_3^2) + \sqrt{3} x_2 x_3 x_4}{x_3(x_2^2 - x_1^2)} & \frac{x_2(x_2^2 + x_1^2 - 2x_3^2) + 2\sqrt{3} x_1 x_3 x_4}{2\sqrt{3} x_3 (x_1^2 - x_2^2)} \\ 0 & 0 & \frac{x_2}{x_2^2 - x_1^2} & \frac{x_1}{\sqrt{3}(x_1^2 - x_2^2)} \\ 0 & 1 & \frac{x_2(x_1^2 - x_3^2) + \sqrt{3} x_1 x_3 x_4}{x_3(x_1^2 - x_2^2)} & \frac{x_1(x_1^2 + x_2^2 - 2x_3^2) + 2\sqrt{3} x_2 x_3 x_4}{2\sqrt{3} x_3 (x_2^2 - x_1^2)} \\ 0 & 0 & \frac{x_1}{x_1^2 - x_2^2} & \frac{x_2}{\sqrt{3}(x_2^2 - x_1^2)} \end{pmatrix}\,. \tag{4.59}$$

The corresponding Lorentz transformation is

$$\Lambda = \mathrm{diag}(-1, 1 - 1, 1)\,, \tag{4.60}$$

from which one may identify the transformation for the spinors (again omitting an overall sign) as

$$\Omega = \Gamma_1\Gamma_3. \tag{4.61}$$

The NS two-form is zero and the dilaton turns out to be

$$\Phi = \Phi_0 - \ln\left(2\sqrt{3}x_3(x_1^2 - x_2^2)\right)\,, \tag{4.62}$$

where Φ_0 denotes the constant original dilaton in eq. (4.51). The RR fluxes supporting the transformed geometry become

$$\begin{aligned} F_2 &= -\frac{4}{\sqrt{3}}\,\Lambda_4 m\; x_3(x_1^2 - x_2^2)(e^2\wedge e^3 + e^1\wedge e^4)\,,\\ F_4 &= 2\sqrt{3}\; x_3(x_1^2 - x_2^2)(\Lambda_2 n\; e^1\wedge e^3 + A_{m,n}\; e^2\wedge e^4)\wedge \mathrm{Vol}(\mathrm{S}^2)\,. \end{aligned} \tag{4.63}$$

In addition we obtain an F_6 and an F_8, obeying eq. (2.4).

4.4 Non-abelian T-dual of $AdS_4 \times S^2 \times S^2 \times S^2$

A class of solutions of eleven-dimensional supergravity labeled as $O(n_1, n_2, n_3)$, where the n_i's are integers, was constructed in [25] (for a review see [24]) and are U(1) bundles over $S^2 \times S^2 \times S^2$. By dimensionally reducing one obtains a type-IIA supergravity solution. The metric is

$$ds^2 = ds^2(\mathrm{AdS}_4) + \sum_{i=1}^{3} ds^2(\mathrm{S}^2_{\mathrm{i}})\,, \tag{4.64}$$

where we have normalized in such a way that $R_{\mu\nu} = -2g_{\mu\nu}$ and $\Lambda_i g_{\mu\nu}$ for AdS_4 and each of the S^2's, respectively. The geometry is supported by the two-form flux

$$F_2 = \Lambda_1 n_1\; e^1\wedge e^2 + \Lambda_2 n_2\; e^3\wedge e^4 + \Lambda_3 n_3\; e^5\wedge e^6\,. \tag{4.65}$$

In addition, there is a four-form flux which for consistency assumes the form

$$F_4 = A_{n_1,n_2,n_3}\;\mathrm{Vol}(AdS_4)\,, \qquad A_{n_1,n_2,n_3} = \sqrt{3}(\Lambda_1^2 n_1^2 + \Lambda_2^2 n_2^2 + \Lambda_3^2 n_3^2)^{1/2}\,. \tag{4.66}$$

Further, consistency with the equations of motion requires that

$$\Lambda_1 + \Lambda_2 + \Lambda_3 = 4 \tag{4.67}$$

and

$$\frac{n_1^2}{n_2^2} = \frac{\Lambda_2^2}{\Lambda_1^2}\frac{\Lambda_1 - 1}{\Lambda_2 - 1}\,, \qquad \text{and cyclic in } 1, 2, 3\,, \tag{4.68}$$

and that there is a constant dilaton

$$e^{-\Phi} = \frac{1}{\sqrt{6}}A_{n_1,n_2,n_3}\,. \tag{4.69}$$

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We need the higher forms

$$\begin{aligned} F_6 &= -(\star F_4) = A_{n_1,n_2,n_3}\ e^1\wedge e^2\wedge e^3\wedge e^4\wedge e^5\wedge e^6\,, \\ F_8 &= \star F_2 = \mathrm{Vol}(\mathrm{AdS}_4)\wedge\big(\Lambda_1 n_1\ e^3\wedge e^4\wedge e^5\wedge e^6 \\ &\quad + \Lambda_2 n_2\ e^1\wedge e^2\wedge e^5\wedge e^6 + \Lambda_3 n_3\ e^1\wedge e^2\wedge e^3\wedge e^4\big)\ . \end{aligned} \tag{4.70}$$

We will perform a non-abelian T-duality transformation with respect to the SU(2) symmetry of each one of the S^2 factors. Let's concentrate on just one of them with metric normalized so that $R_{ij} = g_{ij}$. We can gauge fix as $v_1 = 0$ and define

$$(v_2, v_3) = (\rho, z)\ . \tag{4.71}$$

The matrices defining the frames are

$$N_+ = \begin{pmatrix} 0 & -\frac{1}{\rho} \\ 1 & \frac{z}{\rho} \end{pmatrix}\,, \qquad N_- = \begin{pmatrix} 0 & -\frac{1}{\rho} \\ -1 & -\frac{z}{\rho} \end{pmatrix}\,, \tag{4.72}$$

related by the Lorentz transformation

$$\Lambda = \mathrm{diag}(1, -1)\ . \tag{4.73}$$

The metric is

$$ds^2(\mathrm{S}^2_{\mathrm{d}}) = \frac{dz^2}{\rho^2} + \left(d\rho + \frac{z}{\rho}dz\right)^2\,, \tag{4.74}$$

whereas the corresponding would be dilaton factor is $\Phi = -\ln\rho$ and the NS two-form is zero.

Taking the above into account we find that the non-abelian dual has metric

$$ds^2 = ds^2(\mathrm{AdS}_4) + \sum_{i=1}^{3}\Lambda_i^{-1} ds^2(\mathrm{S}^2_{\mathrm{d,i}})\,, \tag{4.75}$$

where the i-factor contains (ρ_i, z_i). The dilaton is

$$e^{-\Phi} = \frac{1}{\sqrt{6}} A_{n_1,n_2,n_3}\ \rho_1\rho_2\rho_3\ . \tag{4.76}$$

To find the non-abelian space requires (we omit again an overall sign)

$$\Omega = \Gamma_{11}\Gamma_2\Gamma_4\Gamma_6\ . \tag{4.77}$$

Hence we obtain

$$\begin{aligned} F_3 &= \rho_1\rho_2\rho_3\big(\Lambda_1 n_1\ e^1\wedge e^4\wedge e^6 + \Lambda_2 n_2\ e^2\wedge e^3\wedge e^6 + \Lambda_3 n_3\ e^2\wedge e^4\wedge e^5 \\ &\quad + A_{n_1,n_2,n_3}\ e^1\wedge e^3\wedge e^5\big) \end{aligned} \tag{4.78}$$

and an F_7 obeying $\star F_3 = -F_7$. Note that from the the original isometry only the permutation symmetry remains and there is no supersymmetry.

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5 Concluding remarks

In the present paper we have established the rules for performing non-abelian T-duality transformations in cases where the isometry group acts with isotropy and the supergravity backgrounds have non-trivial Ramond flux fields. In particular, we have concentrated on coset spaces that frequently appear in important classical supergravity solutions.

We presented examples starting from D-brane configurations, namely the D1-D5 and the D3 near horizon brane systems, and also from various supergravity compactifications on spheres and CP-spaces. In a similar way to other non-isotropic cases in [12] it is possible to stay in the same type-II theory or change chirality from type-IIA to type-IIB and vice versa, depending solely on the dimension of the duality group, and irrespectively of the details of the background.

Due to the isotropy there are fixed points of the isometry group acting on the dual variables. These give rise to singularities in the T-dual backgrounds we have constructed. In addition, as in previous examples, the T-dual backgrounds correspond to non-compact manifolds even though the duality groups are compact. It would be interesting to investigate possible relations to the near horizon limits of brane configurations. Then, the singularities could be related to the locations of the branes in the transverse space. Another avenue open to investigation is the possibility that our T-dual backgrounds represent effective theories for describing high spin sectors of some parent theories as it was shown for pure NS backgrounds in [10]. If true this will have further implications within the AdS/CFT correspondence.

Based on our examples, non-abelian T-duality generically breaks all isometries and supersymmetry when it is performed with respect to the maximal symmetry group. A further interesting question is to understand whether and how the original symmetries may be recovered as hidden non-local symmetries in the dual background.

Finally, it would be interesting to derive the same T-duality rules by dimensional reduction on appropriate manifolds in a similar fashion to the abelian case in [15]. For this to be possible one needs to establish relations between compactifications of type-II supergravity to lower dimensions as well as between their massive deformations. Besides an alternative proof of the non-abelian T-duality rules in the presence of non-trivial RR fluxes, this would also provide a deeper understanding of the involved supergravity theories.

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A Non-abelian T-dual of $AdS_4 \times CP_3$

In this appendix we will examine the type-IIA supergravity solution with metric

$$ds^2 = ds^2(\mathrm{AdS}_4) + ds^2(\mathrm{CP}^3)\,, \tag{A.1}$$

normalized such that $R_{\mu\nu} = -12 g_{\mu\nu}$ for the AdS_4 and $R_{\mu\nu} = 8 g_{\mu\nu}$ for the CP^3 factors, respectively. It is supported by the Ramond fluxes

$$F_2 = \pm 2J\,, \qquad F_4 = 6\mathrm{Vol}(AdS_4)\,, \tag{A.2}$$

where J is the Kähler form with components obeying $(J^2)_{\mu\nu} = g_{\mu\nu}$ (for the CP^3 metric indices only). The dilaton is $\Phi = 0$ and as before we note that we have completely absorbed all constant factors by appropriate rescalings.

The presence of CP^3 indicates a global SU(4) with respect to which we will perform the non-abelian transformation. The higher forms are

$$\begin{aligned} F_6 &= -(\star F_4) = 6\ \mathrm{Vol}(\mathrm{CP}^2) = 6\ e^1 \wedge e^2 \wedge e^3 \wedge e^4 \wedge e^5 \wedge e^6\,, \\ F_8 &= \star F_2 = \pm 2\mathrm{Vol}(\mathrm{AdS}_4) \wedge \big(e^3 \wedge e^4 \wedge e^5 \wedge e^6 \\ &\qquad + e^1 \wedge e^2 \wedge e^5 \wedge e^6 + e^1 \wedge e^2 \wedge e^3 \wedge e^4\big)\ . \end{aligned} \tag{A.3}$$

We will denote the generators of the SU(4) algebra by S_a, $a = 1, 2, \ldots, 15$ and we will choose the following anti-hermitian basis [26]

$$\begin{aligned} S_1 &= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \\ S_3 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \qquad S_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\ S_5 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{pmatrix}, \qquad S_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \end{aligned} \tag{A.4}$$

and

$$S_7 = -\frac{i}{\sqrt{6}}\mathrm{diag}(1, 1, 1, -3)\,, \qquad S_{7+i} = \begin{pmatrix} \lambda_i & 0 \\ 0 & 0 \end{pmatrix}, \quad i = 1, 2, \ldots, 8\,, \tag{A.5}$$

where λ_i are the Gell-Mann matrices for SU(3). In this basis the Kähler form is [26]

$$J = e^1 \wedge e^2 + e^3 \wedge e^4 + e^5 \wedge e^6\ . \tag{A.6}$$

Again we organize the structure constants by computing eq. (4.7). Under the $SU(3) \times U(1) \subset SU(4)$ the $\mathbf{15} \to \mathbf{8_0} \oplus \mathbf{3_+} \oplus \bar{\mathbf{3}}_- \oplus \mathbf{1_0}$. These can be represented in doubled line notation as $A_i{}^j$, V^i, $\bar{V}_i$ and D where explicitly in terms of the 15 v_a's we have

$$D = v_7\,, \qquad V^i = (v_1 - iv_2, v_3 - iv_4, v_5 - iv_6)\,, \qquad \bar{V}_i = (V^i)^* \tag{A.7}$$

and

$$(A^i{}_j) = \begin{pmatrix} v_{10} + \frac{v_{15}}{\sqrt{3}} & v_8 - iv_9 & v_{11} - iv_{12} \\ v_8 + iv_9 & \frac{v_{15}}{\sqrt{3}} - v_{10} & v_{13} - iv_{14} \\ v_{11} + iv_{12} & v_{13} + iv_{14} & -\frac{2v_{15}}{\sqrt{3}} \end{pmatrix}. \tag{A.8}$$

There are two classes of charge invariant operators that can be built by forming contractions; "glueballs" of the form $\mathrm{Tr}(A^n)$ and "mesons" of the form $V^i(A^n)_i{}^j\bar{V}_j$. However, trace relations similar to those mentioned in the main text, ensure that these are not all independent and a suitable basis is given by

$$\begin{aligned} t_1 &= V^i\bar{V}_i\,, \qquad & t_2 &= \bar{V}_iA^i{}_jV^j\,, \qquad & t_3 &= \bar{V}_i(A^2)^i{}_jV^j - \frac{1}{2}V^i\bar{V}_i\mathrm{Tr}(A^2) \\ t_4 &= \frac{1}{2}\mathrm{Tr}(A^2)\,, \qquad & t_5 &= \frac{1}{3}\mathrm{Tr}(A^3), \qquad & t_6 &= 2\sqrt{\frac{2}{3}}D\;. \end{aligned} \tag{A.9}$$

As already pointed out in the main text, the gauge fixing of nine parameters among the fifteen should be in one to one correspondence with the above invariants. We choose to keep non-zero the variables v_a with $a = 1, 6, 7, 8, 10, 12$. Adopting the notation

$$(x_1, x_2, x_3, x_4, x_5, x_6) = (v_1, v_6, v_7, v_8, v_{10}, v_{12})\,, \tag{A.10}$$

the invariants are

$$\begin{aligned} t_1 &= x_1^2 + x_2^2\,, \qquad t_2 = x_1\left(x_1x_5 - 2x_2x_6\right)\,, \\ t_3 &= -x_2\left(x_2(x_4^2 + x_5^2) + 2x_1x_5x_6\right)\,, \\ t_4 &= x_4^2 + x_5^2 + x_6^2\,, \qquad t_5 = x_5x_6^2\,, \qquad t_6 = 2\sqrt{\frac{2}{3}}x_3\;. \end{aligned} \tag{A.11}$$

One could now compute the matrices which define frames by following the procedure prescribed in eq. (3.11), however this entails the onerous task of inverting a large matrix. An alternative approach described in [19, 27], is to start directly with the generating functional of the canonical transformation between dual σ-models, apply the above gauge fixing and then calculate the remaining transformations. Finally one sets to zero the components of momenta in the direction of the subgroup since these drop out of the σ-model eq. (3.9) in the coset limit. In this way one may calculate the following explicit, albeit extremely complicated, expressions for the frames of the dual σ-model

$$
\begin{aligned}
2\Delta e^1 &= x_1 s_1 dt_1 + x_2^2 x_6 dt_2 + x_1 x_2 dt_3 + (x_1 s_1 + t_6 x_2^2 x_6) dt_4 + x_2 s_3 dt_5 \\
&\quad + \frac{1}{4}\left(s_2 s_3 - x_1^3 x_2 x_5 - 2x_2^4 x_6\right) dt_6\,, \\
2\Delta e^2 &= x_2^2 x_6 dt_4 + x_1 x_2 dt_5 + \frac{1}{4} x_1 s_2 dt_6\,, \\
2x_4 \Delta e^3 &= -x_1^{-1}\left(s_5 x_2 x_5 x_6 + s_6 s_7\right) dt_1 + s_6 dt_2 - s_5 dt_3 + \left(s_6(t_6 - x_5) + x_4^2 x_2^2 x_6\right) dt_4 \\
&\quad + (s_6 - s_5 t_6) dt_5 + \frac{1}{4}\left(\left(s_4 x_6 - s_7 s_2\right) t_6 + s_8\right) dt_6\,, \\
2x_4 \Delta e^4 &= s_6 dt_4 - s_5 dt_5 + \frac{1}{4}\left(s_5 x_6^2 - 2 s_6 x_5 + (x_1^2 + 2x_2^2) x_4^2 x_6\right) dt_6\,, \\
2\Delta e^5 &= x_1 x_2 x_6 dt_4 + x_1^2 dt_5 - \frac{s_4}{4} dt_6\,, \\
2\Delta e^6 &= -x_2^2 x_6^2 dt_1 - x_1 x_2 x_6 dt_2 - x_1^2 dt_3 - x_2 x_6 s_3 dt_4 - x_1 s_3 dt_5 \\
&\quad + \frac{1}{4}\left(t_6 s_4 + x_1^4 x_5 + 2 x_1 x_2^3 x_6 - x_6 s_2\right) dt_6\,,
\end{aligned}
\tag{A.12}
$$

in which we have defined

$$
\begin{aligned}
s_1 &= t_4 x_2 + x_1 x_5 x_6\,, \qquad s_2 = 2t_4 x_2 + 3x_1 x_5 x_6\,, \qquad s_3 = t_6 x_1 + x_2 x_6\,, \\
s_4 &= t_4 x_1^2 - 3x_2^2 x_6^2\,, \qquad s_5 = x_1 x_2 x_5 + (x_1^2 - x_2^2) x_6\,, \qquad s_6 = x_5 s_5 + x_1 x_2 x_4^2\,, \\
s_7 &= x_1 x_5 - x_2 x_6\,, \qquad \Delta = x_1^2 s_1 - x_2^3 x_6^2\,, \\
s_8 &= 2t_4(s_6 - t_1 x_1 x_2) + x_1^3 x_2 x_5^2 - (x_1^4 + 3x_1^2 x_2^2 - 2x_2^4) x_5 x_6 + (4x_1 x_2^3 - 3 s_5 x_5) x_6^2\,.
\end{aligned}
$$

The dual background has a dilaton given by

$$\Phi = -\ln(4\sqrt{2} x_4 \Delta) \tag{A.13}$$

and zero NS two-form field.[4] Whilst this background as presented is clearly very complicated one might hope that some more sophisticated group theoretic arguments could be brought to bear in order that it can be understood better.

The Lorentz transformation relating left and right movers is given by

$$\Lambda = \mathrm{diag}(-1, 1, -1, 1, 1, -1)\,, \tag{A.14}$$

which has the spinorial representation

$$\Omega = \Gamma_{11}\Gamma_1\Gamma_3\Gamma_6\,. \tag{A.15}$$

Therefore we conclude that the dual geometry is supported by the following flux

$$F_3 = \pm 8\sqrt{2} x_4 \Delta (3e^2 \wedge e^4 \wedge e^5 + e^1 \wedge e^3 \wedge e^5 - e^2 \wedge e^3 \wedge e^6 - e^1 \wedge e^4 \wedge e^6)\,. \tag{A.16}$$

There is also an $F_7 = -(\star F_3)$ as it should.

[4] Other gauge fixing choices may result in a non-zero NS two-form. However, these will be pure gauge with vanishing field strength.

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B Geometry and Killing vectors in group and coset spaces

For the reader's convenience we recapitulate some relevant results for our purposes concerning the geometry of groups and coset manifolds. Further details may be found in [29, 30]. Following our notation in the main text, let t_a be generators for G of which t_i correspond to the subgroup $H \subset G$ and t_α are the remaining coset generators. We assume that the generators are normalised such that $\mathrm{Tr}(t_a t_b) = \delta_{ab}$. An element $g \in G$, parameterized appropriately by $\dim(G)$ variables X^μ, can be used to define the G-algebra valued left-invariant and right-invariant one-forms $L = -ig^{-1}dg = L^a t_a$ and $R = -ig^{-1}dgg^{-1} = R^a t_a$, with components related by $R^a = D^{ab}L^b$, where

$$D_{ab}(g) = \mathrm{Tr}(g^{-1}t_a g t_b) \ . \tag{B.1}$$

This matrix is defined by the adjoint action of g and obeys $D_{ab}(g^{-1}) = D_{ba}(g)$. The metric in group space is

$$g_{\mu\nu} = L^a_\mu L^a_\nu = R^a_\mu R^a_\nu \ . \tag{B.2}$$

This metric has a $G_L \times G_R$ group of invariance. The Killing vectors for these left and right transformations are

$$K^{\mathrm{L}}_a = R^\mu_a \partial_\mu \,, \qquad K^{\mathrm{R}}_a = -L^\mu_a \partial_\mu \ . \tag{B.3}$$

They obey two commuting Lie-algebras for G as well as a completeness and a derivative relation

$$\sum_{a=1}^{\dim(G)} K^\mu_a K^\nu_a = g^{\mu\nu} \,, \quad \nabla_\mu K^a_\nu = -\frac{1}{2} f_{bc}{}^a K^b_\mu K^c_\nu \,, \tag{B.4}$$

for either set of Killing vectors, separately. Also $\partial_\mu D_{ab} = L^c_\mu D_{ad} f_{cb}{}^d$ proves useful in various algebraic manipulations.

Turning to coset spaces, an element of the coset G/H is given by a representative $\hat{g} \in G$ parameterized by $\dim(G/H)$ local coordinates x^μ, for example $\hat{g} = \exp(it_\alpha \delta^\alpha_\mu x^\mu)$. The left-invariant one-forms with coset indices L^α define a frame for the coset $e_\mu{}^\alpha = L^\alpha_\mu$, with inverse $e^\mu{}_\alpha$, such that the metric on the coset is given by

$$g_{\mu\nu} = e_\mu{}^\alpha e_\nu{}^\beta \delta_{\alpha\beta} \tag{B.5}$$

and has only a G_L invariance group. The corresponding Killing vectors are

$$K_a = D_{a\alpha}(\hat{g}) e^\mu_\alpha \partial_\mu \,, \tag{B.6}$$

which obey the relations eq. (B.4) with μ and ν parameterizing the coset manifold. It turns out that K_a can be obtained from the Killing vectors K^L_a defined in eq. (B.3) for group spaces as follows: Parameterizing a general group element in G as $g = \hat{g}h$, where $h \in H$, and denoting the variables parameterizing h by y^i, arranged such that $y^i = 0$ corresponds to $h = \mathbf{I}$, K_a can be obtained from K^L_a by "gauge fixing", i.e. by setting $y^i = 0$ and ignoring the corresponding derivatives ∂_i.

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C Spinor derivative on $AdS_5 \times S^5$ Killing vectors

The Killing spinors of the $AdS_5 \times S^5$ geometry in type-IIB supergravity obey the differential equation

$$0 = (\mathbf{1}_2 \otimes D_\mu)\epsilon + \frac{1}{8 \cdot 5!}(i\sigma_2 \otimes \not{F}_5 \Gamma_\mu)\epsilon \,, \tag{C.1}$$

or in terms of the complex Weyl spinor $\varepsilon = \epsilon_1 + i\epsilon_2$,

$$0 = D_\mu \varepsilon - \frac{i}{8 \cdot 5!}\not{F}_5 \Gamma_\mu \varepsilon \,. \tag{C.2}$$

One may choose a basis of Gamma-matrices (see the appendix of [28])

$$\Gamma_a = \sigma_1 \otimes \gamma_a \otimes 1 \,, \qquad \Gamma_i = \sigma_2 \otimes 1 \otimes \gamma_i \,, \tag{C.3}$$

for the AdS and sphere directions respectively, such that a chiral spinor may be decomposed as

$$\varepsilon = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \epsilon_{\mathrm{AdS}} \otimes \eta \,. \tag{C.4}$$

The components of eq. (C.2) in the directions of the sphere (with unit radius) become

$$D_\alpha \eta = \frac{i}{2}\gamma_\alpha \eta \,, \qquad \alpha = 5 \dots 9 \,, \tag{C.5}$$

(in which η has four complex but otherwise unconstrained components) whilst in the direction of AdS

$$D_a \epsilon_{\mathrm{AdS}} = \frac{1}{2}\gamma_a \epsilon_{\mathrm{AdS}}, \qquad a = 0, 1, \dots, 4 \,. \tag{C.6}$$

Then for the Killing vectors generating the SO(6) isometry we may act with the spinor-Lorentz-Lie derivative as

$$\begin{aligned} \mathcal{L}_{K^a}\eta &= K^{a\alpha} D_\alpha \eta - \frac{1}{4}\nabla_\alpha K^a_\beta \gamma^{\alpha\beta}\eta \\ &= \frac{i}{2}K^a_\alpha \gamma^\alpha \eta - \frac{1}{4}\nabla_\alpha K^a_\beta \gamma^{\alpha\beta}\eta \\ &= \frac{i}{2}K^a_\alpha \gamma^\alpha \eta - \frac{1}{4}f_{bc}{}^a K^b_\alpha K^c_\beta \gamma^{\alpha\beta}\eta \,. \end{aligned} \tag{C.7}$$

In the second line we have used the Killing spinor property and in the third the property obeyed by the Killing vectors in eq. (B.4). Next we contract the above equation with K^a_α and make use of the completeness relation on the Killing vectors in eq. (B.4) to obtain that

$$\sum_{a=1}^{\dim(G)} K^a_\alpha \mathcal{L}_{K^a}\eta = \frac{i}{2}\gamma_\alpha \eta - \frac{1}{4}f_{abc}K^a_\alpha K^b_\beta K^c_\gamma \gamma^{\beta\gamma}\eta \,. \tag{C.8}$$

However, as shown in eq. (D.4) below, it turns out that $f_{abc}K^a_\alpha K^b_\beta K^c_\gamma = 0$, $\forall\ \alpha, \beta$ and γ. Hence for a Killing spinor to be invariant under the SO(6) action it is necessary that

$$\sum_{a=1}^{\dim(G)} K^a_\alpha \mathcal{L}_{K^a}\eta = \frac{i}{2}\gamma_\alpha \eta = 0 \,, \tag{C.9}$$

to which the only solution is $\eta = 0$. Hence, we conclude that the non-abelian T-dual of the $AdS_5 \times S^5$ background of type-IIB does not preserve any supersymmetry.

D S^5 as a coset and its SO(6) Killing vectors

The five-sphere of half-unit radius can be defined by five stereographic coordinates $z^\alpha = y^\alpha(1/2 - y^6)^{-1}$ where $\vec{y}$ defines the embedding in $\mathbf{R}^6$. The isomorphism between the sphere and the coset SO(6)/SO(5) is given by identifying with a point $\vec{z}$ an SO(6) element that maps the north pole to that point, modulo the SO(5) stability group which leaves the north pole fixed. Corresponding to the point $\vec{z}$ one may take as the group element (see, for instance, the contribution of Van Nieuwenhuizen in [29])

$$\hat{g}(\vec{z}) = (1+z^2)^{-1} \left(\begin{array}{c|c} \delta^{\alpha\beta}(1+z^2) - 2z^\alpha z^\beta & 2z^\beta \\ \hline -2z^\alpha & 1-z^2 \end{array} \right) , \tag{D.1}$$

in a basis in which the subgroup generators act on the top left block and the coset acts on the remaining directions. With $(J_{a,b})_{cd} = \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}$ the coset generators are then given by $t_\alpha = J_{6,\alpha}$, $\alpha = 1, 2, \ldots, 5$ and the subgroup generators are all the rest. Following the steps described above one recovers the metric

$$ds^2 = \frac{4dz^\alpha dz^\alpha}{(1+z^2)^2} , \qquad z^2 = z^\alpha z^\alpha \tag{D.2}$$

and finds the Killing vectors to be

$$\begin{aligned} K_a &= z^{a+1}\partial_1 - z^1\partial_{a+1} , \qquad a = 1,2,3,4 , \\ K_{a+4} &= z^{a+2}\partial_2 - z^2\partial_{a+2} , \qquad a = 1,2,3 , \\ K_{a+7} &= z^{a+3}\partial_3 - z^3\partial_{a+3} , \qquad a = 1,2 , \\ K_{10} &= z^5\partial_4 - z^4\partial_5 , \\ K_{a+10} &= z^a z\cdot\partial + \frac{1-z^2}{2}\partial_a , \quad a = 1,2,\ldots,5 . \end{aligned} \tag{D.3}$$

A tedious direct calculation verifies that $f_{abc}K^a_\alpha K^b_\beta K^c_\gamma = 0$, $\forall \alpha, \beta$ and γ, where the summation acts on the entire set of SO(6) algebra indices. Equivalently the three-form

$$f_{abc}K^a_\alpha K^b_\beta K^c_\gamma dz^\alpha \wedge dz^\beta \wedge dz^\gamma = 0 . \tag{D.4}$$

It would be interesting to know to what extent this vanishing relation is valid for other cosets as well.

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