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A Characterization of Height-Based Extensions of Principal Filtral Opportunity Rankings

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Abstract: A parameterized characterization of height-based total extensions of Principal filtral opportunity rankings is provided and shown to include as a special case a version of the well-known Pattanaik-Xu characterization of the cardinality-based ranking.

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1 Introduction

A set-inclusion filtral preorder on a finite set X of basic alternatives/opportunities amounts to the set-inclusion partial order as augmented with a *minimal opportunity threshold* which is induced by an order-filter (to be defined below): under the threshold, opportunity sets are indifferent to each other and to the null opportunity set, while over the threshold *the set-inclusion partial order is simply replicated.* Therefore, the behaviour of a set-inclusion filtral preorder (henceforth SIFP) over the threshold is arguably *non-controversial.* On the other hand, since the threshold can be chosen in many different ways, SIFPs -unlike e.g. the cardinality preorder- also accommodate a non-negligible *diversity* of judgments concerning the most appropriate ranking of opportunity sets. Thus, SIFPs can be regarded as a format for opportunity rankings that, building upon a common and essentially 'objective' basis, gives some scope to a modicum of diversity in judgments. Furthermore, SIFPs are amenable to nice aggregation methods including majority voting (see Vannucci(1999)). However, when it comes to the assessment of inequality among opportunity profiles according to a majorization preorder, the very fact that a SIFP is in general *non-total* is undoubtedly a rather fastidious inconvenience. Two basic strategies may be devised to escape the foregoing difficulty while sticking to the notion of a majorization preorder, namely i) reformulating (and generalizing) the majorization construct in order to adapt it to the general case of arbitrary non-total preorders, or ii) extending SIFPs to total preorders in a suitably 'natural' manner.

Strategy i) is a quite radical move. Indeed, it can be shown that it demands a special tactic to cope with pairs of *non-isomorphic* lattices of order filters: thus, it runs deep to the very foundations of the majorization construct. By comparison, strategy ii) is much more conservative. One way to pursue strategy ii) is implicitly proposed and explored in Savaglio and Vannucci (2001). It consists in relying on the *height* function of a SIFP in order to extend the latter to a total preorder, the resulting 'higher than' relation. Now, the height of an element x counts the size of the longest strictly ascending chain having x as its maximum. Does this notion qualify as a 'natural' extension of the underlying SIFP? In the general case the answer is admittedly bound to be disputable. Indeed, the main problem here is that-generally speaking- SIFPs admit maximal strictly ascending chains of different size having the same minimum and the same maximum, i.e.SIFPs do not satisfy the so called Jordan-Dedekind chain condition, hence are not graded and therefore do not have any rank function for their elements (see e.g. Barbut and Monjardet (1970) for a general review of the foregoing notions as defined below in the text). Thus, reliance on height functions, which provide an instance of rank functions in the graded case, but are well-defined anyway, is a second-best choice of sorts. However, one may select a suitably defined well-behaved class of SIFPs.

In the present paper, I focus on *principal SIFPs,* namely on those SIPFs whose threshold-inducing order filter is *principal* or *latticial* i.e. closed under meet or set-intersection. The reasons for doing that are the following. First, it is shown below that principal SIFPs are graded and their height functions amount to rank functions. This implies that in the principal case heights provide a much more reliable numerical scale to rank the elements of a SIFP than they do in the general case. Therefore, height-based extensions are arguably more 'natural' and strongly grounded for principal SIFPs than they are for general SIFPs. Second, Savaglio and Vannucci (2001) prove that principal SIFPs do support an opportunity-profile counterpart to the classic characterization theorems of the majorization preorder on real sequences due to Hardy, Littlewood and Pólya (1952).

Then, a simple characterization of height-based extensions of *principal* SIFPs is provided. Our characterization relies on conditions that use the relevant filter as a fixed parameter. Indeed, it is quite clear that *over the filtral threshold* height-based extensions of principal SIFPs behave –essentially– as the cardinality-based preorder. Therefore, one should expect that a suitable reformulation of standard characterizations of the cardinality-based preorder should also work for height-based extensions of SIFPs. As a matter of fact, we show that SIFPs can be indeed characterized by a suitably adapted version of the axiom set employed by Pattanaik and Xu (1990) to obtain their well-known characterization of the cardinality-based preorder. We also show that in our setting a version of the Pattanaik-Xu characterization mentioned above is essentially recaptured as a special case which arises from a particular choice of the relevant filtral parameter.

2 Model and results

Let $(Y, \succcurlyeq)$ be a preposet (i.e. $\succcurlyeq$ is a reflexive and transitive binary relation on set Y); we shall denote by $([Y]_\sim, [\succcurlyeq]_\sim)$ its quotient poset w.r.t. the symmetric component $\sim$ of $\succcurlyeq$, namely the antisymmetric preposet on the set of $\sim$-equivalence classes as defined by the rule $[x]_\sim \ [\succcurlyeq]_\sim [y]_\sim$ iff $x \succcurlyeq y$.An antichain of $(Y, \succcurlyeq)$ is a set $Z \subseteq Y$ such that for any $z_1, z_2 \in Z$ if $z_1 \neq z_2$ then z_1 and z_2 are not $\succcurlyeq$-comparable. For any antichain Z of a finite non-empty preposet $(Y, \succcurlyeq)$, an order filter of $(Y, \succcurlyeq)$ with basis Z is the minimal set $F = F(Z) \subseteq Y$ such that $Y \supseteq Z$ and for any y, z, if $y \in F$ and $z \succ y$ then $z \in F$.

Thus, whenever Y is finite, an order filter F of $(Y, \succcurlyeq)$ is uniquely defined by a finite set $Z = Z_F = \{z_1, ..., z_l\} \subseteq Y$ such that, $F = \{y \in Y$:there exists $i \in \{1, ..., l\}$ such that either $y = z_i$ or $y \succ z_i\}$: Z is also denoted as the *basis* of F. In particular, if Z is a singleton i.e. $l = 1$ then is said to be *principal.* It should be remarked that if $(Y, \succcurlyeq)$ is a *lattice* (namely, $\succcurlyeq$ is antisymmetric and for any $x, y \in X$, the pair $\{x, y\}$ has both a least upper bound w.r.t. $\succcurlyeq$, and a greatest lower bound w.r.t. $\succcurlyeq$, denoted by $x \vee y$ and $x \wedge y$, respectively) then a principal order filter is also *latticial* or $\wedge$-*closed* namely $x \wedge y \in F$ whenever both $x \in F$ and $y \in F$.[It can also be shown that the converse holds for a finite lattice].

Let now consider a poset $(X, \geq)$. A chain of $(X, \geq)$ is a subset $X' \subseteq X$ which is totally ordered by $\geq$; by definition, the *length* of chain X' is $l(X') = |X'| - 1$.A chain X' of $(X, \geq)$ is *maximal* if there is no chain U of $(X, \geq)$ such that $X' \subset U$.If $(X,\geq)$ has a minimum $\bot$, one may define its *height function* $h_{\geq}: X \rightarrow \mathbb{Z}^{+} \cup \{\infty\}$by declaring the height $h_{\geq}(x)$ of any $x \in X$ to be the lowest upper bound of the set of the lenghts of *all (maximal)*chains $X'$of $(X,\geq)$ having x as their maximum. A poset $(X,\geq)$ is said to satisfy the *Jordan-Dedekind chain condition* if for any $x,y \in X$ and any pair of maximal chains X',X'' of $(X,\geq)$ having x as their common minimum and y as their common maximum, $l(X')$ =$l(X'')$ i.e. equivalently $|X'| = |X''|$. Furthermore, a poset $(X,\geq)$ is *graded* if it admits a *rank function* i.e. an integer-valued function $r: X \rightarrow \mathbb{Z}$ such that for any $x,y \in Y$: i) if $x \succ y$ then $r(x) > r(y)$ and ii) $r(x) = r(y)+1$ whenever x *covers* y i.e. $x > y$ and $\{z \in Y : x > z > y\} = \emptyset$. A general preposet $(Y,\succcurlyeq)$ will be said to satisfy the *Jordan-Dedekind chain condition* if its quotient poset $([Y]_{\sim},[\succcurlyeq]_{\sim})$ does, and to be *graded* if $([Y]_{\sim},[\succcurlyeq]_{\sim})$ is a graded poset.

We are now ready to turn to inclusion-filtral preorders. We shall confine ouselves to a *finite* set X.For any order filter F of $(\wp(X),\supseteq)$ the F-generated *set-inclusion filtral preorder (SIFP)* is the binary relation $\succcurlyeq_F$on $\wp(X)$ defined as follows: for any $A,B \in \wp(X)$ $A \succcurlyeq_F B$ if and only if $A \supseteq B$ or $B \notin F$.

Let F be an order filter of $(\wp(X),\supseteq)$ and $\succcurlyeq_F$ the filtral preorder induced by F. Then, the $\succcurlyeq_F$*-induced height function* $h_F: \wp(X) \rightarrow \mathbb{Z}_{+}$ is defined as follows: for any $A \subseteq X$

$$h_F(A) = \max\left\{ \begin{array}{c} \#\mathcal{C} : \mathcal{C} \text{ is a } \succcurlyeq_F \text{-chain, such that} \\ A \in \mathcal{C} \text{ and } A \succ_F B \text{ for any } B \in \mathcal{C} \setminus \{A\} \end{array} \right\}.$$

The *height-based (total) extension* of $\succcurlyeq_F$is the total preorder $\succcurlyeq_{h_F}$defined as follows: for any $A,B \subseteq X$, $A \succcurlyeq_{h_F} B$ if and only if $h_F(A) \geq h_F(B)$.

As mentioned in the Introduction, the main aim of the present paper is to provide a characterization of the height-based total preorder $\succcurlyeq_{h_F}$when the relevant order filter F is *principal*.Indeed, it turns out that in the latter case the SIFP $(\wp(X),\succcurlyeq_F)$ is *graded,* hence the height function h_F is a well-defined *rank function* which provides an unambiguous criterion to assess the comparative 'status' of opportunity sets according to $\succcurlyeq_F$. This claim is made precise by the following

Proposition 1 *Let F be a* principal *order filter of* $(\wp(X),\supseteq)$*. Then, the F-generated SIFP*$(\wp(X),\succcurlyeq_F)$ *is a* graded *preposet.*

Proof. It is a well-known fact that a finite set is graded if and only if it satisfies the Jordan-Dedekind condition as defined above :see e.g. Barbut and Monjardet (1970)). Thus it suffices to show that $(\wp(X),\succcurlyeq_F)$ does indeed satisfy the latter condition. Indeed, suppose it does not. Then there exist $A,B \subseteq X$ and maximal chains $\mathbf{C} = \{[C_0]_{\sim_F},..,[C_k]_{\sim_F}\}, \mathbf{C}'= \{[C'_0]_{\sim_F},..,[C'_{k'}]_{\sim_F}\}$ of

$([\wp(X)]_{\sim_F},[\succcurlyeq_F]_{\sim_F})$ with $C_i \succ_F C_{i*1}, i=0,.,k-1, C'_j \succ_F C'_{j*1}, j=0,..,k'-1$, $[C_0]_{\sim_F} = [C'_0]_{\sim_F} = [A]_{\sim_F}, [C_k]_{\sim_F} = [C'_{k'}]_{\sim_F} = [B]_{\sim_F}$, and such that $|\mathbf{C}| \neq |\mathbf{C}'|$ .Now, if $\{A,B\} \cap F = \emptyset$ then $|\mathbf{C}| = |\mathbf{C}'| = 1$ hence either $\{A,B\} \cap F = \{A\}$ or $\{A,B\} \subseteq F$.If in fact $B \notin F$ then by definition of $\succcurlyeq_F$and maximality of chains $\mathbf{C},\mathbf{C}'$ it must be the case that for any $i \in \{1,..k-1\}, j \in \{1,..,k'-1\}$,there exist $x_i \in X \setminus C_i, y_j \in X \setminus C'_j$ such that $C_{i+1} = C_i \cup \{x_i\}, C_{j+1} = C_j \cup \{x_j\}$, and both C_1 and C_1'belong to the basis of F.But then, since F is principal, $C_1 = C_1'$.It follows that $k-1 = |\mathbf{C} \setminus \{C_0, C_1\}| = |\mathbf{C}' \setminus \{C_0', C_1'\}| = k'-1$ whence $k = k'$ i.e. $|\mathbf{C}| = |\mathbf{C}'|$, a contradiction. Finally, if $B \in F$ then again, by definition of $\succcurlyeq_F$and maximality of chains $\mathbf{C}, \mathbf{C}'$ it must be the case that for any $i \in \{0, ..k-1\}, j \in \{0, .., k'-1\}$,there exist $x_i \in X \setminus C_i, y_j \in X \setminus C_j'$ such that $C_{i+1} = C_i \cup \{x_i\}, C_{j+1} = C_j \cup \{x_j\}$ whence by the same argument presented above $k = k'$, a contradiction. ■

Remark 1 *Of course, a general SIFP need not be graded. To check this fact, consider the following elementary example: let be* $X = \{x, y, z\}$, *and* F *the order filter of* $(\wp(X), \supseteq)$ *having* $\{\{x\}, \{y, z\}\}$ *as its basis, and* $(\wp(X), \succcurlyeq_F)$ *the resulting SIFP. Then consider* $\mathbf{C} = \{[X]_{\sim_F}, [\{x, y\}]_{\sim_F}, [\{x\}]_{\sim_F}, [\emptyset]_{\sim_F}\}$ *and* $\mathbf{C}' = \{[X]_{\sim_F}, [\{y, z\}]_{\sim_F}, [\emptyset]_{\sim_F}\}$. *Now* $\mathbf{C}$ *and* $\mathbf{C}'$*are two maximal chains of* $([\wp(X)]_{\sim_F}, [\succcurlyeq_F]_{\sim_F})$ *with different size (and length), having* $[X]_{\sim_F}$*as their common maximum and* $[\emptyset]_{\sim_F}$*as their common minimum. Thus,* $(\wp(X), \succcurlyeq_F)$ *does not satisfy the Jordan-Dedekind chain condition and as a consequence-being finite- is* not *graded.*

Let us now proceed to the announced characterization of $\succcurlyeq_{h_F}$. In order to do this, a few more definitions are needed.

Let $F \in \mathcal{F}((\wp(X), \cup, \cap))$, $F \neq \emptyset$ be any latticial filter of the (finite) lattice $(\wp(X), \cup, \cap)$ i.e. equivalently a principal order filter of the (finite) poset $(\wp(X), \supseteq)$.Then, the following F-parameterized properties of a binary relational system $(\wp(X), \succcurlyeq)$ can be defined:

F-Restricted Indifference between Singletons (F-RIS): $(\wp(X), \succcurlyeq)$ satisfies *F-RIS* if for all $A \in F$ and $x, y \in X \setminus A$, $A \cup \{x\} \sim A \cup \{y\}$.

F-Restricted Strict Monotonicity (F-RSM): $(\wp(X), \succcurlyeq)$ satisfies *F-RIS* if for all $A \in F$ and $x, y \in X$ such that $x \neq y$, $y \notin A$ entails $A \cup \{x, y\} \succ A \cup \{x\}$.

F-Restricted Independence (F-RIND): $(\wp(X), \succcurlyeq)$ satisfies *F-RIS* if for all $A, B \in F$ and $x \in X, x \notin A \cup B$ and $A \succcurlyeq B$ if and only if $A \cup \{x\} \succcurlyeq B \cup \{x\}$.

F-Threshold Effect (F-TE): $A \succ B \sim \emptyset$ for all $A, B \subseteq X$ such that $\emptyset \neq A \in F$ and $B \in \wp(X) \setminus F$.

It turns out that, in general, the foregoing properties are not mutually independent. Indeed, we have the following

Proposition 2 *Let* F *be a principal filter of* $(\wp(X), \supseteq)$ *and* $(\wp(X), \succcurlyeq)$ *a preordered set which satisfies both* F-RIS *and* F-RSM. *Then* $(\wp(X), \succcurlyeq)$ *satisfies* F-RIND *as well.*

Proof. Let us assume that $A, B \in F, x \in X \setminus (A \cup B)$ .Since, by definition of F, there exists $Y \subseteq X$ such that $F = \{C \subseteq X : C \supseteq Y\}$, it follows that there exist non-negative integers h, k and $\{a_1, .., a_h\} \subseteq X \setminus Y, \{b_1, .., b_k\} \subseteq X \setminus Y$ such that $A = Y \cup \{a_1, .., a_h\}, B = Y \cup \{b_1, .., b_k\}$.Now, assume $A \succcurlyeq B$. If $h < k$ then $A \sim Y \cup \{b_1, .., b_h\}$ by a repeated application of *F-RIS*. Therefore, by a repeated application of *F-RSM*, $B \succ Y \cup \{b_1, .., b_h\} \sim A$, whence, by transitivity of $\succcurlyeq$, $B \succ A$, a contradiction. Let us then assume without loss of generality that $h \geq k$.Thus -by a repeated application of *F-RIS* to $Y \cup \{x\}$- $Y \cup \{a_1, .., a_k, x\} \sim Y \cup \{b_1, .., b_k, x\}$.If $h = k$, then $A \cup \{x\} \sim B \cup \{x\}$ follows immediately. Otherwise, $A \cup \{x\} = Y \cup \{a_1, .., a_h, x\} \succ Y \cup \{b_1, .., b_k\}$ follows by a repeated application of *F-RSM*, and by transitivity of $\succcurlyeq$ .Conversely, let us assume that $A \cup \{x\} \succcurlyeq B \cup \{x\}$.If $A = Y \cup \{a_1, .., a_h\} \succcurlyeq B = Y \cup \{b_1, .., b_k\}$ does *not* hold, then it must be the case that $h < k$. But then, it follows by a repeated application of *F-RIS* as applied to $Y \cup \{x\} \in F$ that $Y \cup \{b_1, .., b_h, x\} \sim Y \cup \{a_1, .., a_h, x\} = A \cup \{x\}$.Thus, by a repeated application of *F-RSM* and by transitivity of $\succcurlyeq$, it also follows $B \cup \{x\} \succ A \cup \{x\}$, a contradiction. Hence $A \succcurlyeq B$, and *F-RIND* holds. ■

Theorem 3 *Let F be a principal filter of $(\wp(X), \supseteq)$ and $(\wp(X), \succcurlyeq)$ a preordered set. Then,$(\wp(X), \succcurlyeq)$is the* height-based extension $(\wp(X), \succcurlyeq_{h_F})$*of the set-inclusion principal filtral preorder $(\succcurlyeq_F)$ if and only if $(\wp(X), \succcurlyeq)$ satisfies F-RIS,F-RSM and F-TE.*

Proof. It is straightforward to check that $(\wp(X), \succcurlyeq_{h_F})$ is in fact a (total) preorder that satisfy F-RIS, F-SM, F-RIND, and F-TE. Indeed, let $F = \{A \subseteq X : A \supseteq Y\}$ where $Y \subseteq X$. If $A \in F$, and $x, y \in X \setminus F$ then by definition $h_F(A \cup \{x\}) = h_F(A \cup \{y\}) = \#(A \setminus Y) + 1$ whence $A \cup \{x\} \sim_F A \cup \{y\}$.Moreover, $A \in F$,$x, y \in X$ and $y \notin A$ clearly entails $h_F(A \cup \{x, y\}) = \#(A \setminus Y) + 3$ and $h_F(A \cup \{x\}) = \#(A \setminus Y) + 2$ if $x \notin A$, while $h_F(A \cup \{x, y\}) = \#(A \setminus Y) + 2$ and $h_F(A \cup \{x\}) = \#(A \setminus Y) + 1$ if $x \in A$: in any case, by definition, $A \cup \{x, y\} \succ_{h_F} A \cup \{x\}$. Finally, for all $A \in \wp(X) \setminus F$, and $B \in F, h_F(A) = 0 = h_F(\emptyset)$ while $h_F(B) \geq 1$ i.e. $B \succ_{h_F} A \sim_{h_F} \emptyset$.

Conversely, let $(\wp(X), \succcurlyeq)$ be a preordered set. To begin with, we define an auxiliary function $l_F : \wp(X) \to \mathbb{N}$ as follows: for any $A \subseteq X$

$l_F(A) = \max\{\#(A \setminus Z) : Z \in F, Z \subseteq A\} = \#(A \setminus Y)$, if $A \supseteq Y$ and

$l_F(A) = -1$ otherwise

(i.e. $l_F(A) = h_F(A) - 1$: l_F is the so-called length function of $(\wp(X), \succcurlyeq_F)$).

Next, we show that if $(\wp(X), \succcurlyeq)$ satisfies *F-RIS and F-RIND* then for any $A, B \in F$:

$l_F(A) = l_F(B)$ entails $A \sim B$ (or equivalently $h_F(A) = h_F(B)$ entails $A \sim B$).

We proceed by induction on $l_F(A)$.The case $l_F(A) = l_F(B) = 0$ is trivial in that it entails -by definition- $A = Y = B$ whence $A \sim B$.

Let us now suppose by inductive hypothesis that for any nonnegative integer m not larger than $n, l_F(A) = l_F(B) = m$ entails $A \sim B$.

Then, take a pair $C, D \subseteq X$ such that $l_F(C) = l_F(D) = n + 1$.If $C = D$ there is nothing to prove. If $C \neq D$ then there exist $x, y \in X$ and $A, B \subseteq X$ such that $A \cap B \supseteq Y, \#(A \setminus Y) = \#(B \setminus Y) = n, x \notin B, y \notin A$ and $C = A \cup \{x\}, D = B \cup \{y\}$.It follows that $\{x, y\} \cap Y = \emptyset$ hence, by definition, $l_F(A) = l_F(C) - 1 = l_F(D) - 1 = l_F(B) = n$,which entails $A \sim B$, by inductive hypothesis. Moreover, if $x \in A$ then $C = A$ a contradiction since $l_F(A) \neq l_F(C)$, thus indeed $x \notin A \cup B$, and $y \notin A \cup B$ by a similar argument. Therefore, $A \cup \{x\} \sim B \cup \{x\}$(and $A \cup \{y\} \sim B \cup \{y\}$) by *F-RIND.* Moreover, $A \cup \{x\} \sim A \cup \{y\}$(and $B \cup \{y\} \sim B \cup \{x\}$) by *F-RIS.* As a result, $A \cup \{x\} \sim B \cup \{y\}$ i.e. $C \sim B$, by transitivity of $\sim$, and the inductive thesis follows.

Now, take any pair $A, B \subseteq X$ such that $h_F(A) > h_F(B)$ i.e. equivalently $l_F(A) > l_F(B)$. Two cases should be distinguished, namely: i) $A \supseteq Y$ and $B \not\supseteq Y$; ii) $A \cap B \supseteq Y$. If case i) obtains, then, by definition of $F, A \in F$ and $B \notin F$ hence $A \succ B$ by *F-TE.* Under case ii) both $A \in F$ and $B \in F$, and there exist $A' \subseteq X \setminus Y, B' \subseteq X \setminus Y$ such that $A = Y \cup A'$, $B = Y \cup B'$ and $\#A' > \#B'$. Then, there also exists $A'' \subset A'$ such that $\#A'' = \#B'$.We also posit $\#(A' \setminus A'') = k$ and $A' = A'' \cup \{x_1, .., x_k\}$.Therefore, $l_F(Y \cup A'') = l_F(Y \cup B') = l_F(B)$ whence $(Y \cup A'') \sim B$ by the first part of this proof. Since $Y \cup A'' \in F$, $Y \cup A'' \cup \{x_1\} \succ Y \cup A''$, by *F-RSM.* By a repeated application of a similar argument - and by transitivity of $\succ$ - we can eventually establish that $A \succ Y \cup A'' \sim B$ whence $A \succ B$.

Thus, we have just shown that for any $A, B \subseteq X$: $h_F(A) = h_F(B)$ entails $A \sim B$ and $h_F(A) > h_F(B)$ entails $A \succ B$, i.e. $(\succcurlyeq_{h_F}) \subseteq (\succcurlyeq)$. Hence, in particular, $\succcurlyeq$ is a *total* preorder. But notice that if there exist $A, B \subseteq X$ such that $A \succcurlyeq B$ and *not* $A \succcurlyeq_{h_F} B$, then -since $\succcurlyeq_{h_F}$is also a *total* preorder by definition- it must be the case that $B \succcurlyeq_{h_F} A$ hence $h_F(B) \geq h_F(A)$.Moreover, *not* $A \succcurlyeq_{h_F} B$ entails $h_F(B) > h_F(A)$, whence $B \succ A$, a contradiction.

It follows that $(\succcurlyeq) \subseteq (\succcurlyeq_{h_F})$ as well, so that $(\succcurlyeq) = (\succcurlyeq_{h_F})$. ■

The foregoing characterization is tight. To see this, consider the following list of examples.

Example 1 *Take a principal order filter F of $(\wp(X), \supseteq)$ and the corresponding set-inclusion filtral preordered set $(\wp(X), \succcurlyeq_F)$ defined as follows: for any $A, B \subseteq X, A \succcurlyeq_F B$ if and only if [$A \supseteq B$ or $B \notin F$](see Vannucci(1999)). It is easily checked that $\succcurlyeq_F$is indeed a preorder, and satisfies* F-RSM *and* F-TE. *Moreover, let $A \in F, B \in F$ and $x \in X \setminus (A \cup B)$. Thus, $A \succcurlyeq_F B$ entails $A \supseteq B$ whence $A \cup \{x\} \supseteq B \cup \{x\}$ which in turn entails $A \cup \{x\} \succcurlyeq_F B \cup \{x\}$. Conversely, since obviously $\{A \cup \{x\}, B \cup \{x\}\} \subseteq F$, $A \cup \{x\} \succcurlyeq_F B \cup \{x\}$ entails $A \cup \{x\} \supseteq B \cup \{x\}$.Then $A \supseteq B$ as well, hence by definition $A \succcurlyeq_F B$.It follows that $\succcurlyeq_F$also satisfies* F-RIND. *However, for any $A \in F$ and $x, y \in A$ such that $x \neq y$, $A \cup \{x\}$ and $A \cup \{y\}$ are* not *$\succcurlyeq_F$-comparable, hence* F-RIS *fails.*

Example 2 *Let us consider again a principal order filter F of $(\wp(X), \supseteq)$, and the binary relational system $(\wp(X), \succcurlyeq_F^{\#})$ defined as follows: for any $A, B \subseteq X$, $A \succcurlyeq_F^{\#} B$ if and only if [either ($B \in F, A \in F$ and $\#B \geq \#A$) or $B \notin F$]. Notice that $\succcurlyeq_F^{\#}$is indeed a* preorder*: to check this, first observe that reflexivity of $\succcurlyeq_F^{\#}$follows trivially from the definition, and assume that $A \succcurlyeq_F^{\#} B$ and $B \succcurlyeq_F^{\#} C$. The following mutually exclusive and exhaustive cases should be distinguished: i) $\#B \geq \#A$, $\#C \geq \#B$ and $\{A, B, C\} \subseteq F$: in this case $\#C \geq \#A$ hence*

$A \succcurlyeq_F^{\#} C$ by the first clause; ii) $\#B \geq \#A, \{A, B\} \subseteq F$ and $C \notin F$: in this case $A \succcurlyeq_F^{\#} C$ by the second clause; iii) $B \notin F$ and $C \notin F$:here again $A \succcurlyeq_F^{\#} C$ follows immediately from the second clause. Thus, $\succcurlyeq_F^{\#}$is transitive. Also, if $A \in F, x \notin A$ and $y \notin A$, then clearly $\{A \cup \{x\}, A \cup \{y\}\} \subseteq F$ and $\#(A \cup \{x\}) = \#(A \cup \{y\})$ whence $A \cup \{x\} \sim_F^{\#} A \cup \{y\}$,i.e. F-RIS *is satisfied. Similarly, if $A \in F, B \in F, x \in X \setminus (A \cup B)$ and $A \succcurlyeq_F^{\#} B$ then $\#B \geq \#A$ and $\{A \cup \{x\}, B \cup \{x\}\} \subseteq F$.Thus, $\#(B \cup \{x\}) \geq \#(A \cup \{x\})$ whence, by definition, $A \cup \{x\} \succcurlyeq_F^{\#} B \cup \{x\}$. Conversely, if $A \in F, B \in F, x \in X \setminus (A \cup B)$ and $\#(A \cup \{x\}) \succcurlyeq_F^{\#} (B \cup \{x\})$ then by definition $\#(B \cup \{x\}) \geq \#(A \cup \{x\})$: it follows that $\#B \geq \#A$ as well hence by definition $A \succcurlyeq_F^{\#} B$.Thus,* F-RIND *is also satisfied. Finally,* F-TE *follows immediately from the definition. However,* F-RSM *is definitely* not *satisfied by $\succcurlyeq_F^{\#}$: indeed, if $A \in F, x \in X \setminus A$, $y \in X \setminus A$ and $x \neq y$ then, by definition, $A \cup \{x\} \succ_F^{\#} A \cup \{x, y\}$.*

Example 3 *Fix a principal order filter F of $(\wp(X), \supseteq)$ and take the binary relational system $(\wp(X), \succcurlyeq_{F^\partial}^{\#})$ defined as follows: for any $A, B \subseteq X$, $A \succcurlyeq_{F^\partial}^{\#} B$ if and only if [either $A \notin F$ or $A \in F, B \in F$ and $\#A \geq \#B$]. Obviously, $\succcurlyeq_{F^\partial}^{\#}$fails to satisfy* F-TE. *On the other hand, $\succcurlyeq_{F^\partial}^{\#}$is a* preorder. *To check this, observe that reflexivity follows immediately from the definition. As for transitivity, if $A \succcurlyeq_{F^\partial}^{\#} B$ and $B \succcurlyeq_{F^\partial}^{\#} C$ then the following two mutually exclusive and exhaustive cases are to be distinguished: i) $A \notin F$, and ii) $A \in F, B \in F, C \in F, \#A \geq \#B$ and $\#B \geq \#C$. In both cases, $A \succcurlyeq_{F^\partial}^{\#} C$ follows immediately from the definition. Furthermore,* F-RIS *and* F-RSM *of $\succcurlyeq_{F^\partial}^{\#}$are also easily seen to follow trivially from the definition.*

It should be emphasized here that the axioms used by Pattanaik and Xu (1990) in their well-known, and seminal, characterization of the cardinality-based preorder– namely Indifference between Singletons, Strict Monotonicity, and Independence– are implied by the corresponding axioms in our list when the reference filter F is taken to be the trivial or maximum filter $\wp(X)$. Moreover, it is immediately seen that for $F = \wp(X)$ the fourth axiom of our list i.e. *F-Threshold Effect,* which has no counterpart in the Pattanaik-Xu list, is in fact trivially satisfied when restricted to the original Pattanaik-Xu domain which only includes *non-empty* opportunity sets.

3 Concluding remarks

As mentioned in the Introduction the characterization of height-based extensions of *principal* filtral opportunity preorders provided in the present paper does *not* extend to the general case of *arbitrary* filtral opportunity preorders. This is due to the fact that when an order filter is not principal, the height function of the corresponding SIFP may exhibit a highly irregular behaviour.

Therefore, the height-based extension of a SIFP does not mimic the behaviour of the cardinality-based preorder over the filtral threshold. A simple example may help clarify this point.

Example 4 *Let* $X = \{x_1, .., x_7\}$, $Z = \{\{x_1, x_2\}, \{x_3, x_4, x_5, x_6\}, \{x_7\}\}$, *and* $F = F(Z)$ *(notice that* Z *is indeed an antichain of* $(\wp(X), \supseteq)$ *). Then, consider the height-based extension* $(\wp(X), \succcurlyeq_{h_F})$ *of the* F*-induced SIFP* $(\wp(X), \succcurlyeq_F)$, *and take* $A = \{x_3, x_4, x_5, x_6\}$.*Clearly,* $A \in F$. *However,* $h_F(A \cup \{x_1\}) = 2$ *while* $h_F(A \cup \{x_7\}) = 5$, *hence* $A \cup \{x_7\} \succ_{h_F} A \cup \{x_1\}$ *and* F-RIS *fails.*

By contrast, our characterization is in fact amenable to a simple generalization in another direction. Indeed, a counterpart to Theorem 2 for *arbitrary* (finite) lattices of sets is readily available provided that the axioms are suitably reformulated by replacing *join-irreducibles for singletons/atoms.* The details of this extension, however, will not be pursued here.

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