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The Price-Level Computation Method

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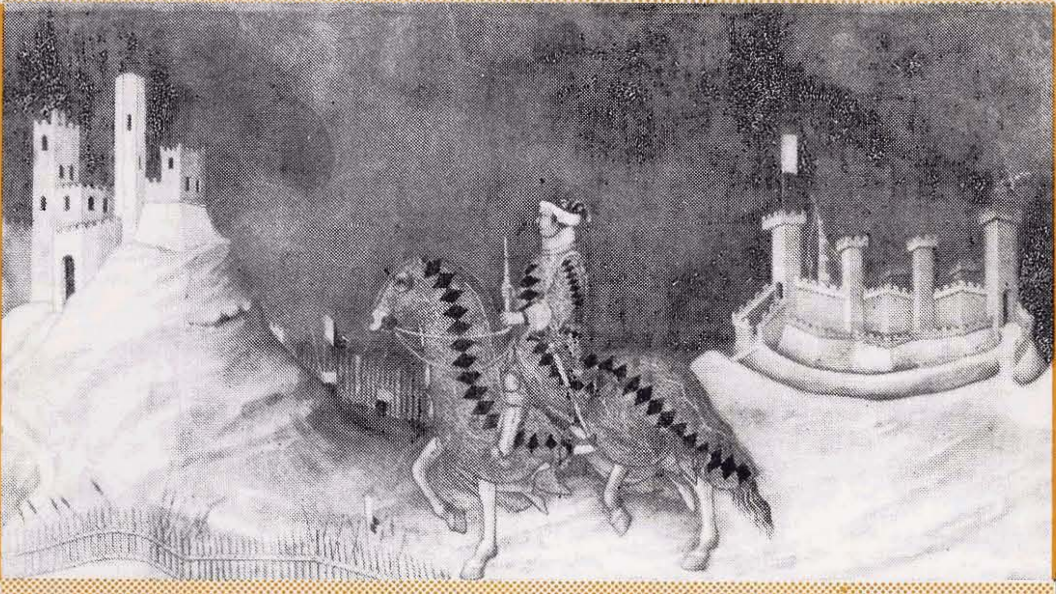

Abstract - It has been submitted that, for the very large number of different traditional type formulae to determine price indices associated with a pair of periods, which are joined with the longstanding question of which one to choose, they should all be abandoned. For the method proposed instead, price levels associated with periods are first all computed together, subject to a consistency of the data, and then price indices that are as taken together true are determined from their ratios. An approximation method can apply in the case of inconsistency. Here is an account of the mathematics of the method.

Keywords: inflation, index-number problem, non-parametric, price index, price level, revealed preference.

Jel Classification: C43, E31.

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1 Introduction

Prices change and an individual who enjoys a consumption that provides a certain standard of living at a certain money cost would like to know how much it will cost to maintain the same standard at the new prices.

Reference may be made to this first paragraph for the basis of the price-index idea.

The Price Index issued from the Statistical Office is a number that tells how to deal with the question, the index being the multiplier of old expenditure to determine the new.

The question of how to produce such a number is known as *The Index-Number Problem*. To proceed about it there are primitive points to be added. Let P_{st} denote the price index from period s to period t.

For a first point, the number must apply equally well to everyone experiencing the price change, whatever their standard of living. Hence an expenditure M_s in period *s, at whatever level*, must be replaced by

$$M_r = P_{rs} M_s$$

in period r to maintain the same standard of living. This point seems not to be explicitly represented among Irving Fisher's well-known "Tests", but the next points are, though we are not now considering applications to actual formulae, as usual, but rather to the basic idea of a price-index itself.

For the *Identity Test*, there is the statement

$$P_{tt} = 1,$$

that is, "when one year is compared with itself, the index shows 'no change'." Most formulae go along with this.

For the next, if the price change is reversed, so the new prices becoming the old and vice-versa, then the price index, the ratio that turns old expenditure into new, is replaced by the reciprocal. That is,

$$P_{ts} = (P_{st})^{-1}$$

which is the *Time Reversal Test.* Fisher's "ideal index" is just about the only formula that satisfies this. No wonder it is "ideal"

This thinking somehow seems to be as if the price index was derived as a ratio of *price-levels*, expressing *purchasing power of money* for obtaining a standard of living by purchase of consumption.

For a distinction and the language for it: *price level* has reference to a single period, while *price index* has reference to two, and is in principle the ratio of new level to old, so it is the multiplier of old expenditure to produce the new that will currently purchase the same living standard.

The second primitive point mentioned, expressed by Fisher's Time Reversal Test, would also be an immediate consequence of taking price indices having the form of ratios of price levels, or anyway of some numbers. For if

$$P_{st} = P_s / P_t$$

then

$$P_{ts} = P_t / P_s = \left(P_s / P_t\right)^{-1} = \left(P_{st}\right)^{-1}.$$

When dealing with more than just two periods, beside the Time Reversal (the Fisher "Ideal Index" is a distinguished case among formulae for satisfying this) there can be introduction of the *Chain Test*,

$$P_{rs}P_{st} = P_{rt}$$

(just about never satisfied by any of the one or two hundred usual price index formulae) which implies Time Reversal again, and moreover implies, and obviously is implied by, price indices being expressible as the ratios of a set of numbers associated with the periods—the 'price levels' or whatever. For, bringing in the Identity Test,

$$P_{tt} = 1$$

we have

$$P_{ts}P_{st} = P_{tt} = 1$$

so

$$P_{ts} = \left(P_{st}\right)^{-1}$$

which is Time Reversal, and now, for any fixed r,

$$P_{st} = P_{sr}P_{rt} = P_{sr}\left(P_{tr}\right)^{-1} = P_{sr} / P_{tr}$$

so price indices determined relative to a fixed base can serve as 'price levels' from which all price indices can be determined as their ratios. Evidently now the Chain Test, from first implying Reversal, is equivalent to Fisher's *Circularity Test*,

$$P_{rs}P_{st}P_{tr} = 1.$$

While there has been invariably no prior determination of price levels from which to obtain price indices as their ratios, usually formulae, plain algebraic involving demand data just for the reference periods themselves, and a great number of them, are proposed that go directly to the index without a background of levels. In that approach the great problem is to know what formula to use.

A missing test, in Fisher's list, perhaps not before named and which implies all these others, and which could be called the *Ratio Test*, is simply that the price index be expressed as a ratio of a set of numbers. Among formulae, as such, nowhere is that satisfied, unless the now to be considered method, designated as the *1981-Formula*, be allowed, or another proposed by Bishop William Fleetwood in 1707 and mysteriouly neglected, at least in usual theory of the subject if not actual practice,

$$P_{ts} = p_t a / p_s a,$$

the *inflation rate* for a fixed, perhaps democratically chosen, bundle of goods a.

2 Data and formulae

Reference is made to two spaces, the *budget space* B and *commodity space* C, one the space of non-negative row vectors, and the other column vectors, so with Ω as the non-negative numbers,

$$B = \Omega_n, \quad C = \Omega^n,$$

and any $p \in B, x \in C$ provide $M = px \in \Omega$ as the money cost of the bundle of goods x at the prices p. With such a purchase, making the *demand element* $(p, x) \in B \times C$ of commodities x at the prices p, the associated *budget vector* is $u = M^{-1}p \in B$, for which $ux = 1$. (We follow the rule that a scalar, as if it were a 1×1-matrix, multiplies a row-vector on the left and a column-vector on the right.) Any collection of demand elements makes a *demand correspondence*. A *budget element* is any $(u, x) \in B \times C$ such that $ux = 1$, and an *expenditure correspondence* consists in any collection of these. With any demand correspondence D there is an associated expenditure correspondence E, obtained by taking the associated budget elements.

A fundamental area of discussion involves data provided by a finite demand correspondence D consisting of a series of demand observations

$$(p_t, x_t) \in B \times C \quad (t = 1, 2, \dots, m),$$

as may be associated with different periods described by the index t. Price-levels P_t to be associated with the periods are elements of a vector P in the *price-level space* $\Pi = \Omega^m$. Without altering the price indices determined from their rations, they may be *normalized* to sum to 1, in which case they become barycentric coordinates for a point in the *simplex of reference* Δ, available for graphic representations in case $m = 3$.

Any pair of periods s, t is associated with the *Laspeyres index*

$$L_{ts} = p_t x_s / p_s x_s$$

with s distinguished as the *base* and t the *current* period, so this is simply the inflation rate between the periods for the base-period bundle of goods. There is also the *Paasche index*

$$\begin{aligned} K_{ts} &= p_t x_t / p_s x_t \\ &= (L_{st})^{-1}, \end{aligned}$$

which is the inflation rate for the current bundle.

With any *chain* described by a series of periods

$$s, i, j, \dots, k, t$$

there is associated the *Laspeyes chain product*

$$L_{sij\dots kt} = L_{si} L_{ij} \dots L_{kt}$$

termed the *coefficient* on the chain. Obviously

$$L_{r\dots s\dots t} = L_{r\dots s} L_{s\dots t}$$

A *simple* chain is one without repeated elements, or loops. There are

$$m(m-1)\dots(m-r+1) = m!/r!$$

simple chains of length $r \le m$ and therefore altogether the finite number

$$m!\left(1 + 1/1! + 1/2! + \dots + 1/(m-1)!\right)$$

of simple chains from among m elements.

A chain

$$s, i, j, \ldots, k, t$$

whose extremeties are the same, that is, $s = t$, defines a *cycle*. It is associated with the *Laspeyres cyclical product*

$$L_{tij\ldots kt} = L_{ti} L_{ij} \ldots L_{kt}$$

which is basis for the important *Laspeyres cyclical product test*, or simply the *cycle test*,

$$L_{t\ldots t} \geq 1 \text{ for all cycles } t \ldots t$$

A *simple* cycle is one without loops. There are

$$(m-1)\ldots(m-r+1) = (m-1)!/r!$$

simple cycles of $r \leq m$ elements, and the total number of simple cycles from among m elements is the finite number made up accordingly.

The coefficients $L_{st} L_{ts} = L_{sts}$ on the cycles of two elements define the *intervals* of the system. The *interval test* $L_{st} L_{ts} \geq 1$ is equivalent to

$$(LP) \quad K_{ts} \leq L_{ts}$$

that is, *the Paasche index does not exceed the Laspeyres*, or the *LP-inequality*, a condition very well-known from index number theory based on data for just two periods. Here therefore, with the cyclical test, is a generalization of that condition for any number of periods. J. R. Hicks (without proving anything) calls the *LP*-inequality "The Index Number Theorem" (*Revision*, 1956, p. 181.) One should remember there was a time when there was, briefly, something of a fashion to call almost anything a "Theorem". It is confusing, but perhaps Hicks was just being fashionable.

Another way of stating this condition, of significance since it gives the form for a statement of a direct extension for many periods, is that the 2×2 L-matrix

$$\begin{pmatrix} 1 & L_{st} \\ L_{ts} & 1 \end{pmatrix}$$

be idempotent, or reproduced when multiplied by itself, in the modified arithmetic where + means *min*. In fact, as to be shown, raising the general $m \times m$ L-matrix to powers in this modified arithmetic is a basic process in the price-level computation method.

Introducing the chain Laspeyres and Paasche indices

$$L_{sij\ldots kt} = L_{si} L_{ij} \cdots L_{kt}, \quad K_{sij\ldots kt} = K_{si} K_{ij} \cdots K_{kt},$$

the cycle test $L_{s\ldots t\ldots s} \geq 1$ is equivalently to

$$\text{(chain } LP\text{)} \quad K_{s\ldots t} \leq L_{s\ldots t}$$

for all possible chains … taken separately. Hence introducing the *derived Laspeyres* and *Paasche indices*

$$M_{st} = \min_{ij\ldots k} L_{si} L_{ij} \cdots L_{kt}, \quad H_{st} = \max_{ij\ldots k} K_{si} K_{ij} \cdots K_{kt},$$

subject to the now to be considered conditions required for their existence, for which

$$H_{st} = (M_{ts})^{-1},$$

this is equivalent to

(derived *LP*) $H_{st} \le M_{st}$.

In this case

$$K_{st} \le H_{st} \le M_{st} \le L_{st}$$

showing the relation of the *LP*-interval and the narrower derived version that involves more data.

Here is has been recognized that from

$$K_{ts} = (L_{st})^{-1}$$

follows

$$K_{t\ldots s} = (L_{s\ldots t})^{-1}$$

and therefore

$$H_{ts} = \max_{\ldots} K_{t\ldots s} = (\min_{\ldots} L_{s\ldots t})^{-1} = (M_{st})^{-1},$$

where in each case … is understood so the chain $s\ldots t$ is the reverse of $t\ldots s$.

3 Minimal chains

Any chain can be represented uniquely as a simple chain, with loops at certain of its elements, given by cycles through those elements; and the coefficient on it is then expressed as the product of coefficients on the simple chain and on the cycles.

Also, any cycle can be represented uniquely as a simple cycle, looping in simple cycles at certain of its elements, which loop in cycles at certain of their elements, and so forth, with termination in simple cycles. The coefficient on the cycle is then expressed as a product of coefficients on simple cycles.

Thus out of these generating elements of simple chains and cycles, finite in number, is formed the infinite set of all possible chains.

THEOREM 3.1 *For the chains with fixed extremities to have a minimum the cycle test is necessary and sufficient.*

If any cycle should be below 1, then by taking chains which loop repeatedly round that cycle, chains which have decreasing coefficients are obtained without limit; and so no minimum exists. However, should every cycle be at least 1, then by cancelling the loops on any chain, there can be no increase in the coefficient, so no chain coefficient will be smaller than the coefficient for some simple chain. But there is only a finite number of simple chains on a finite number of elements, and the coefficients on these have a minimum.

THEOREM 3.2 *For the cycle test the simple cycle test is necessary and sufficient.*

For the coefficient on any cycle can be expressed as a product of coefficients on simple cycles.

THEOREM 3.3 *The cycle test implies that a minimal chain with given extremities exists and can be chosen simple.*

For then any chain is then not less than the chain obtained from it by cancelling loops, since the cancelling is then division by a product of numbers all at least 1.

4 System and derived system

The computation of price-levels $P_t\ (t=1,\dots,m)$ depends on solution of the system of inequalities

$$(L) \qquad L_{st} \geq P_s / P_t .$$

Subject to the cyclical product test $L_{t\dots t} \geq 1$ for every cycle, or equivalently every simple cycle, by Theorem 3.2, it is, by Theorem 3.3, possible to introduce

$$M_{st} = \min_{ij\dots k} L_{si} L_{ij} \cdots L_{kt} ,$$

attained for a simple chain. Then

$$L_{sij\dots kt} \geq M_{st}$$

for every chain and, by Theorem 3.3, the equality is attained for some simple chain. In particular,

$$L_{st} \geq M_{st} .$$

The number M_{tt} is the minimum coefficient for the cycles through t, so that

$$L_{tij\dots kt} \geq M_{tt}$$

for every cycle, the equality being attained for some simple cycle. In particular, for a cycle of two elements,

$$L_{ts} L_{st} \geq M_{tt} .$$

The cyclical product test that is the hypothesis now has the statement

$$M_{tt} \geq 1.$$

With the numbers M_{st} so constructed, subject to this hypothesis, it is possible to consider with system L also the *derived system*

$$(M) \qquad M_{st} \geq P_s / P_t .$$

The two systems are said to be *equivalent* if any solution of one is also a solution of the other.

THEOREM 4.1 *The system L and its derived system M, when this exists, are equivalent.*

Let system L have a solution P_t. Then, for any chain of elements

$$s, i, j, \dots, k, t$$

there are the relations

$$L_{si} \geq P_s / P_i,\ L_{ij} \geq P_i / P_j,\ \ldots, L_{kt} \geq P_k / P_t,$$

from which, by multiplication, there follows the relation

$$L_{sij\ldots kt} \geq P_s / P_t.$$

This implies that the derived coefficients M_{st} exist, and

$$M_{st} \geq P_s / P_t.$$

That is, P_t is a solution of system M.

Now suppose the derived coefficients for system M are defined, in which case

$$L_{st} \geq M_{st}.$$

and let P_t be any solution of system M, so that

$$M_{st} \geq P_s / P_t.$$

Then it follows immediately that

$$L_{st} \geq P_s / P_t.$$

or that P_t is a solution of system L. Thus L and M have the same solutions, and are equivalent.

THEOREM 4·2. *If the cycle test holds for L then the interval test holds for the derived system M.*

Since M_{st} is the coefficient of some chain with extremities s, t it appears that the interval coefficient $M_{ts}M_{st}$ of M is the coefficient of some cycle of L through t, and therefore if the cycle test holds for L then so does the interval test hold for the derived system M.

Given any solution for system L, and equivalently system M, necessarily

$$K_{st} \leq H_{st} \leq P_s / P_t \leq M_{st} \leq L_{st},$$

showing how price indices, which on the basis of data just for the reference period are confined to the ordinary Laspeyres-Paasche interval, become confined to the narrower derived Laspeyres-Paasche interval when based on the more extended data.

5 Triangle inequality

From the relation

$$L_{r\ldots s}L_{s\ldots t} = L_{r\ldots t}$$

it follows that the derived coefficients satisfy the multiplicative *triangle inequality*

$$M_{rs}M_{st} \geq M_{rt}$$

the one side being the minimum for chains connecting r, t restricted to include s, and the other side being the minimum without this restriction.

THEOREM 5.1 *Any system subject to the cycle test is equivalent to a system which satisfies the triangle inequality given by its derived system.*

This is true in view of Theorems 3.1, 4.1 and 4.2.

THEOREM 5.2 *The interval test holds for any system that satisfies the triangle inequality.*

Thus, from the triangle inequalities applied to any system M,

$$M_{tr}M_{rs} \geq M_{ts}, \quad M_{ts}M_{sr} \geq M_{tr}$$

there follows, by multiplication, the relation

$$M_{rs}M_{sr} \geq 1$$

or what is the same

$$H_{st} \leq M_{st}$$

or that the derived LP-interval be non-empty.

THEOREM 5.3 *If a system satisfies the triangle inequality then its derived system exists and moreover the two systems are identical.*

From the triangle inequality, it follows by induction that

$$M_{si}M_{ij}\ldots M_{kt} \geq M_{st}$$

that is

$$M_{sij\ldots kt} \geq M_{st}$$

from which it appears that the derived system N exists, with coefficients

$$N_{st} \geq M_{st}$$

so that now

$$N_{st} = M_{st}$$

This shows, what is otherwise evident, that no new system is obtained by repeating the operation of derivation, since the first derived system satisfies the triangle inequality

THEOREM 5.4 *For any system the triangle inequality is equivalent to idempotence of the matrix in the arithmetic where + means* min

That is, the matrix is reproduced in multiplication by itself. For, simply,

$$N_{ij} = \min_k N_{ik}N_{kj}$$

if and only if

$$N_{ij} \leq N_{ik}N_{kj}\,.$$

The triangle inequality

$$M_{rs}M_{st} \geq M_{rt}$$

has the restatement

$$M_{rs} \geq M_{rt}/M_{st}$$

from which it appears that, for any fixed t, taken as *base*, a solution of the system

$$(M) \qquad M_{rs} \geq P_r / P_s$$

for price levels P_r is given by

$$P_r = M_{rt} .$$

Similarly, another solution is

$$P_r = 1 / M_{tr} .$$

These solutions may be distinguished as determinations for the first and second *canonical price-level* systems, with node t as base. Since, by Theorem 5.2,

$$M_{tr} M_{rt} \geq M_{tt} \geq 1$$

they always have the relation

$$1 / M_{tr} \leq M_{rt} ,$$

which is the derived *LP*-relation. However, these are not now price indices, as in that original relation, but here they are price levels from which to derive price indices.

Finding these solutions depends directly on the triangle inequality that is characteristic of the derived sysyem (M), and not on the solution extension property that is a consquence, to which there is appeal in the construction method dealt with in the next Section.

Now established, for every t, are two price-level solutions P_r from which to derive systems of true price indices

$$P_{rs} = P_r / P_s .$$

The two systems, of *canonical price-indices* with base t, are in a way counterparts of the Laspeyres and Paasche endpoints of the *PL*-interval that describes the range of true price indices for the classical case that involves just two periods.

The determinations have reference to periods associated with the data without any dependence on the order $1, \ldots, m$ in which they are taken. This is unlike where there is dependence on the solution extension property for finding solutions, of the next Section. However, they do depend on which period, corresponding to t in the given order, is taken as base. Coming in pairs there are now $2m$ determinations, whose pairwise connections and base references are essential.

When price level solutions are normalized so as to provide barycentric coordinates for a point in the simplex of reference, the set of all solutions is a convex polydron for which these $2m$ solutions are a complete set of vertices from which all solutions may be obtained by taking convex combinations of them.

Note that the findings of this section apply just a well to the approximation method accounted in Section 10, based on relaxing exact cost-efficiency, for the fit of utility to demands, to some degree of partial efficiency.

From the above the following is proved.

THEOREM 5.5 *The derived system (*M*), when it exists, admits the solutions given by the canonical price-levels, so it is always consistent.*

COROLLARY 1 *In that case also the original system (L) is consistent, and admits those same solution.*

For the system and derived system, when this exists, are equivalent, admitting the same solutions, by Theorem 3.1.

COROLLARY 2 *The cycle test is necessary and sufficient for consistency*

For, by Theorem 3.1, the test for system (L) is necessary and sufficient for the existence of the derived system (M), always consistent when it exists, by the present Theorem, and by Theorem 4.2 equivalent to system (L), therefore also consistent.

6 Extension property of solutions

A subsystem M_h of order $h \le m$ of a system M of order m is defined by

$$\left(M_h\right) \quad M_{st} \geq P_s / P_t \quad \left(s,t = 1,\ldots,h\right).$$

Then the systems $M_h \left(h = 2,\ldots,m\right)$ form a nested sequence of subsystems of system M, each being a subsystem of its successor, and $M_m = M$.

Any solution of a system reduces to a solution of any subsystem. But it is not generally true that any solution of a subsystem can be extended to a solution of the original. However, should this be the case, then the system will be said to have the *extension property.*

THEOREM 6.1 *Any system which satisfies the triangle inequality has the extension property.*

Let $P_1, P_2, \ldots, P_{h-1}$ be a solution of M_{h-1}, so that

$$\left(M_{h-1}\right) \qquad M_{st} \geq P_s / P_t \quad \left(s,t = 1,\ldots,h-1\right).$$

It will be shown that, under the hypothesis of the triangle inequality, it can be extended by an element P_h to a solution of M_h.

Thus, there is to be found a number P_h such that

$$M_{hs} \geq P_h / P_s, \quad M_{th} \geq P_t / P_h \quad \left(s,t = 1,\ldots,h-1\right)$$

that is

$$M_{hs} P_s \geq P_h \geq P_t / M_{th}$$

So the condition that such a P_h can be found is

$$M_{hq} P_q \geq P_p / M_{ph}$$

where

$$P_p / M_{ph} = \max_i \left\{P_i / M_{ih}\right\}, \quad P_q M_{hq} = \min_j \left\{P_j M_{hj}\right\}.$$

But if $p = q$ this is equivalent to

$$M_{ph} M_{hp} \geq 1$$

which is verified by Theorem 5.2, and if $p \neq q$ it is equivalent to

$$M_{ph} M_{hq} \geq P_p / P_q$$

which is verified since by hypothesis

$$M_{ph}M_{hq} \geq M_{pq}, \quad M_{pq} \geq P_p / P_q.$$

Therefore, under the hypothesis, the considered extension is always possible. It follows now by induction that any solution of $M_h(h<m)$ can be extended to a solution of $M_m = M$.

This theorem shows how solutions of any system can be practically constructed, step-by-step, by extending the solutions of subsystems of its derived system.

THEOREM 6.2 Any system which satisfies the triangle inequality is consistent.

For, by Theorem 5.2, $M_{12}M_{21} \geq 1$; and this implies that the system M_2 has a solution, which, by Theorem 6.1, can be extended to a solution of M. Therefore M has a solution, and is consistent.

However, this result has already been obtained in Theorem 5.4 without appeal to the extension property, but by direct appeal to the triangle inequality instead of to this consequence.

7 Consistency

THEOREM 7.1 *The cyclical product test is necessary and sufficient for consistency of L, and either* $L^m = M$, *in the modified algebra where + means* min, *is the equivalent derived system with the solution extension property, or system* L *is inconsistent.*

If system L is consistent, let P_t be a solution. Then, for any cycle

$$t, i, j, \ldots, k, t$$

there are the relations

$$L_{ti} \geq P_t / P_i, L_{ij} \geq P_i / P_j, \ldots, L_{kt} \geq P_k / P_t,$$

from which it follows, by multiplication, that

$$\begin{aligned} L_{tij\ldots kt} &= L_{ti}L_{ij}\ldots L_{kt} \\ &\geq (P_t/P_i)(P_i/P_j)\ldots(P_k/P_t) \\ &= 1 \end{aligned}$$

and hence $L_{t\ldots t} \geq 1$. Therefore, if L *is* consistent, all its cycles are at least 1 and the cyclical product test holds.

Conversely, let this test be assumed for L. Then the derived system M is defined, satisfies the triangle inequality, and has the interval test. Hence, by Theorem 6.3, M is consistent. But, by Theorem 4.1, M is equivalent to L. Therefore, L *is* consistent. This shows the converse, so the Theorem is proved.

Now let L denote the actual $m \times m-$matrix of Laspeyres indices for the system, and L^r its r-th power in a modified arithmetic where + means *min*, so

$$L^1 = L, \quad L^{r+1} = L^r L \ (r = 1, 2, \ldots),$$

making

$$L_{ij}^{r+1} = \min_k L_{ik}^r L_{kj} ,$$

where it is seen, since $L_{jj} = 1$ affecting the possibility $k = j$, that

$$L_{ij}^{r+1} \le L_{ij}^r ,$$

which shows what may be termed the *monotonicity* of the process. In any case, for any *r* and *i, j*

$$L_{ik}^r = L_{is\dots tk} ,$$

for some chain $s\dots t$. Subject to the cyclical test, it is proposed that, for $r \le m$ the chain *is...tj* is simple. For otherwise a loop with coefficient at least 1, by hypothesis, can be cancelled, and we have an element from an earlier power which is less, violating the process monotonicity. Then the series of powers either terminates in one not later than the *m*th, when a simple chain cannot be extended further, that is therefore repeated by its successors, or does not terminate. In the first case,

$$L = L^1 \ge L^2 \ge \dots \ge L^t = M \left(= L^{t+1} = \dots \right) \quad \left(t \le m \right),$$

with $\ge$ as between elements, where the terminating matrix M is the matrix of the derived system for *L*. In the second case it is concluded the cyclical product test is violated, system *L* is inconsistent, and there is no derived system. This follows Afriat (1981), Section 13 on "The power algorithm", involving matrix powers in a modified arithmetic where $\times$ means + and + means *min*. There are debts to Jack Edmunds (1973) and S. Bainbridge (1978), for the connection with minimum paths, elaborated in Afriat (1987) where there is also a BASIC computer program pp. 464 ff. applied to "Getting around Berkeley in minimum time" .

Here is how it could go:

0 $x = L,\ t = 1$
1 $y = x,\ x = yL,\ t = t + 1$
2 if $x = y$ then $M = x$ end
3 if $t = m$ then end else 1.

So it appears that either L is inconsistent, or $L^m = M$, for which, as is equivalent to the triangle inequality, there is the idempotence $M^2 = M$ where M is reproduced in multiplication by itself, and which is equivalent to L and has the extension property, so individual price-level solutions can be constructed step-by-step, starting with *any* point in *any derived LP*-interval, which is narrower, because of additional constraints associated with additional data, than the basic or classical *LP*-interval that involves data just for a pair of periods, the reference periods themselves.

Of course, having the canonical price levels of Section 4 available as solutions, there is no need to appeal to the extension property for the existence of solutions. However, with that property it is possible to construct other solutions, step-by-step, beside by taking convex combinations of the canonical solutions.

With any solution for *price-levels* P_t there is, from their ratios, an associated determination of *price-indices*

$$P_{st} = P_s / P_t ,$$

all true, *together*, by reference to the *same* utility, better than merely true separately by reference to different utilities, as in the sense of true usually entertained. Then

$$P_{rs}P_{st} = (P_r / P_s)(P_s / P_t) = P_r / P_t = P_{rt},$$

so that

$$P_{rs}P_{st} = P_{rt},$$

which is Fisher's Chain Test, not satisfied by any of the one or two hundred formulae he dealt with, and so forth with other Tests.

This is a point for the observation that such price-indices, any one for a pair of periods involving data from all the periods, and together giving a realization of all the "Tests" Irving Fisher proposed as proper for price-indices from their nature as such, make a sharp contrast with the established tradition of *algebraical* formulae involving data *just* for the reference periods themselves, without proper compliance with such basic "Tests", or guidance about which of the one or two hundred proposed formulae to use, despite his rankings to decide some as better than others, even "superlative".

After the procedure for finding individual solutions, the further interest is in the collection of all solutions. The solutions describe a polyhedral convex cone in the price-level space of dimension *m*, and the normalized solutions describe a bounded polyhedral convex region in the simplex of reference, with faces or vertices to be determined, the *m* simplex vertices being in correspondence with the *m* data periods, and price-levels. Then there are approximation methods to serve for the case of inconsistency. But first notice will be taken of the price-quantity symmetry inherent in the method, and the utility background that enables all the price-indices so determined to be represented as *altogether true*, that is, all true simultaneously on the basis of the *same* utility.

With any determination of price levels P_t, there is an associated determination of quantity levels X_t, where

$$P_t X_t = p_t x_t \quad (t = 1, \ldots, m).$$

While for price levels,

$$p_t x_s / p_s x_s \geq P_t / P_s,$$

for quantity levels, equivalently,

$$p_t x_s / p_t x_t \geq X_s / X_t,$$

and one could just as well have solved for the quantity levels first, by the same method as for price levels, and then determined the price levels from these. Whichever way,

$$P_s X_t \leq p_s x_t \quad (s, t = 1, \ldots, m),$$

with equality for $s = t$. The introduction of cost-efficiency up to a level *e*, where $0 \leq e \leq 1$, would require

$$P_t X_t \geq e p_t x_t \quad (t = 1, \ldots, m).$$

good also for any lower level, and highest level 1 imposing the equality.

8 Utility basis for the method

First some remarks about terminology. A *ray* is a half-line with vertex the origin, and every point lies on just one ray, the ray *through* it, so

$$\vec{a} = \{at : t \in \Omega\} \subset C$$

is the ray through any $a \in C$. A *cone* is a set described by a set of rays, and every set has a *conical closure*, or cone *through* it, or *projecting* it, described by the set of rays through its points. Hence

$$\vec{A} = \{xt : x \in A, t \in \Omega\} \subset C$$

is the cone through any $A \subset C$.

A function is *conical* if its graph is a cone, or what is the same (just more syllables), *linearly homogeneous*, being such that $\phi(x\lambda) = \phi(x)\lambda$.

With a demand element $(p, x) \in B \times C$, with expenditure $M = px$ and budget vector $u = M^{-1}p$ so that $ux = 1$, there is the *revealed preference* of x over every bundle y which, being such that $uy \leq 1$, is also attainable at no greater cost, as described by the relation $R \subset C \times C$ given by

$$\begin{aligned} R &= \{(x, y) : py \leq px\} \\ &= \{(x, y) : uy \leq 1\}. \end{aligned}$$

Then there would be the transitive closure of a collection of such relations, and a *revealed preference consistency* Samuelson-Houthakker type condition which excludes conflicting preferences.

It may be remembered that originally

$$py \leq px,\ y \neq x \quad \Rightarrow \quad xRy,\ y\bar{R}x$$

going with belief that, in a choice, presumed a maximum and so revealing preferences, it must be more than a mere maximum but moreover a unique maximum—an extra that may be hard to “reveal”. Instead, in the way of revelation without the unsuitable insistence on uniqueness which does not in any way add to preferences, simply

$$py \leq px \quad \Rightarrow \quad xRy$$

has better standing. We take liberty to confine the “revelation” language to this restricted use.

For *conical revealed preference* there would be instead the conical closure of *R*. Then there would be the transitive closure of a collection of such relations, and a *conical revealed preference consistency* which excludes conflicting preferences. *The Laspeyres cyclical product test is exactly such a condition* (a part of the version of so called “Afriat’s Theorem” of Varian (1992) and Fostel *et al.* (2003), originally of Afriat (1961) and (1964)), then for general utility construction and now instead for conical utility).

There are two attributes for a consumption bundle $x \in C$. One is that it has a money cost $M = px \in \Omega$ when the prices are $p \in B$. The other, its use-value or *utility*, is that it is the basis for obtaining a standard of living. Hence there is a link between cost and standard of living, where prices enter. For this link a gap remains between consumption and its utility, made good hypothetically by introduction of the utility function, or utility order.

A *utility function* is any numerical-valued function ϕ defined on the commodity space *B*,

$$\phi : B \to \Omega$$

so $\phi(x) \in \Omega(x \in B)$ is the *utility level* of any commodity bundle *x*. A utility function ϕ determines a *utility order* $R \subset C \times C$ where

$$xRy \equiv \phi(x) \geq \phi(y)$$

A utility function ϕ, with order *R*, *fits* a demand element (p, x), with budget vector *u*, or the demand is *governed* by the utility, if the revealed preferences of it belong to the utility order,

$$uy \leq 1 \Rightarrow xRy\ (y \in C).$$

In other words, if *x* has at least the utility level of every bundle *y* (we do not insist $y \neq x$, see remark above) attainable at no greater expenditure with the prices, or *x* provides the maximum utility $\phi(x)$ for all those bundles *y* under the *budget constraint* $uy \leq 1$, that is

$$py \leq px \Rightarrow \phi(x) \geq \phi(y).$$

The utility system is hypothetical and admitted to the extent that it fits available demand observations. The *cost of a standard of living* is determined as the minimum cost at prevailing prices of getting a consumption that provides it. In terms of a utility function ϕ, this is gathered from the *utility-cost function*

$$\rho(p, x) = \min\{py : \phi(y) \geq \phi(x)\}$$

which tells the minimum cost at given prices *p* of obtaining a consumption *y* that has at least the utility of a given consumption *x*. Since *x* itself, with cost *px*, is a possible such *y*, necessarily

$$\rho(p, x) \leq px \quad \text{for all } p, x$$

while

$$\rho(p, x) = px$$

signifies the admissibility, under government by the utility system, of the demand of *x* at the prices *p*. It shows the demand is *cost effective*, getting the maximum of utility available for the cost, and *cost-efficient*, getting at minimum cost the utility obtained, which conditions would here be equivalent. A case where admissibility does not hold could be attributed to consumption error, described as failure of efficiency, where

$$\rho(p, x) \geq epx, \quad 0 \leq e \leq 1$$

would show attainment of *cost efficiency to a level e*. This idea has use in dealing with demand data inconsistent with government by a utility, by fitting it to a utility that serves only approximately, as reported below, after the account of Afriat (1973).

For the service of a price index this utility-cost should factorize into a product

$$\rho(p,x)=\theta(p)\phi(x),$$

of price-level $P=\theta(p)$ depending on p alone and quantity level $X=\phi(x)$ depending on x alone. This immediately is assured if ϕ is conical, but also the converse is true, showing the following, which we are going to prove, if it was not already, probably long ago. (Samuelson and Swamy 1974, p. 570, attribute theorem and proof to Afriat 1972.)

THEOREM (Utility-Cost Factorization) *For factorization of the utility-cost function it is necessary and sufficient that the utility be conical.*

Given ϕ conical,

$$\begin{aligned}\rho(p,x)&=\min\{py:\phi(y)\geq\phi(x)\}\\&=\min\left\{py(\phi(x))^{-1}:\phi\left(y(\phi(x))^{-1}\right)\geq 1\right\}\phi(x)\\&=\theta(p)\phi(x)\end{aligned}$$

where

$$\theta(p)=\min\{pz:\phi(z)\geq 1\}$$

That shows the sufficiency. Since, for all p,

$$\theta(p)\phi(x)\leq px$$

for all x with equality for some x, as assured with continuous ϕ, it follows that

$$\theta(p)=\min_x px(\phi(x))^{-1}$$

showing θ to be concave conical semi-increasing. Also for x demandable at some prices, as would be the case for any x if ϕ is concave, the inequality holds for all p with equality for some p, showing

$$\phi(x)=\min_p(\theta(p))^{-1}px$$

which, in case every x is demandable at some prices, requires ϕ to be concave conical semi-increasing. But even when not all x are demandable, because they lie in caves and are without a supporting hyperplane, here is a conical function defined for all x that is effectively the same as the actual ϕ as far as any observable demand behaviour is concerned. So it appears that for the cost function factorization the utility function being conical is also necessary, beside being sufficient, as already remarked. Hence, with some details taken for granted, the Theorem is proved.

A pair of functions connected by

$$\theta(p)=\min_x px(\phi(x))^{-1}$$

$$\phi(x)=\min_p(\theta(p))^{-1}px$$

define a *conjugate pair* of price and quantity functions, such that

$$\theta(p)\phi(x) \le px$$

for all p, x and

$$\theta(p)\phi(x) = px$$

signifies efficiency of the demand (p,x), of x at prices p, obtaining maximum utility for the cost and minimum cost for the utility. Instead,

$$\theta(p)\phi(x) \ge epx,$$

where $0 \le e \le 1$, will signify *cost-efficiency to a level* e, as will serve for development of a utility approximation method applicable in case of inconsistency.

The question now is: what utility? A price index being wanted, by the factorization theorem it must be conical, and with given demand data

$$(p_t, x_t) \in B \times C \quad (t = 1, \ldots, m),$$

and belief in efficiency, any utility to be entertained would, to fit the data, have to be such that

$$P_t X_t = p_t x_t \quad ,$$

where

$$P_t = \theta(p_t),\, X_t = \phi(x_t).$$

so in any case

$$P_s X_t \le p_s x_t$$

and now, with

$$L_{st} = p_s x_t \,/\, p_t x_t\,,$$

the Laspeyres index, this condition requires the solubility of the system of inequalities

$$(L) \qquad L_{st} \ge P_s \,/\, P_t\,,$$

for *price levels* $P_t\,(t = 0,1)$. A question is whether a solution exists. If one does, a conical utility can immediately be constructed that fits the given demand data and provides price levels, and consequently also quantity levels X_t, as required, where the X_t are determined from

$$P_t X_t = p_t x_t\,.$$

A worthwhile observation is that these values $X_t = \phi(x_t)$ of the underlying utility ϕ are determined without ever having to actually construct the utility.

Thus, introduce

$$\widehat{\phi}(x) = \min_i P_i^{-1} p_i x$$

so this is a concave conical polyhedral utility function that fits the demand data, with associated price indices as required, to make those prices indices *true*.

Another such function, concave conical, which fits the demand data, again with required values and the same associated price indices, is the polytope type function given by

$$\breve{\phi}(x) = \max\left\{\sum_i X_i t_i : \sum_i x_i t_i \le x, t_i > 0\right\}$$ [3]

and if ϕ is any other concave conical utility that fits the demands and takes the values X_i at the points x_i then

$$\breve{\phi}(x) \le \phi(x) \le \hat{\phi}(x)$$

for all x.

Included in the above is the simple conical precursor of the general theorem on utility construction put in service specifically for price index theory.

Thus, the concave polyhedral function

$$\begin{aligned}\hat{\phi}(x) &= \min_i p_i x / P_i \\ &= \max\{t : t \le p_i x / P_i\}\end{aligned}$$

and the concave polytope function

$$\begin{aligned}\hat{\theta}(p) &= \min\{px : p_i x \ge P_i \text{ for all } i\} \\ &= \max\left\{\sum_i v_i P_i : \sum_i v_i p_i \le p\right\} \text{ by LP duality}\end{aligned}$$

are a conjugate pair of quantity and price functions such that

$$\hat{\theta}(p_t) = P_t, \quad \hat{\phi}(x_t) = X_t$$

where, with

$$a_{st} = p_s x_t / p_t x_t, \quad b_{st} = p_t x_s / p_t x_t$$

P's and X's connected by

$$P_t X_t = p_t x_t$$

are, equivalently, such that

$$a_{st} \ge P_s / P_t, \quad b_{st} \ge X_s / X_t.$$

---

[3] The function of this form introduced by Afriat (1971) is the constant-returns 'frontier production function' that gives a function representation, and at the same time a computational algorithm, for the production efficiency measurement method of Farrell (1957) (Afriat's colleague at DAE Cambridge whose work, done after he left, he at first missed) that marks the beginning of 'data envelope analysis' (DEA). The comment by Afriat attached to Finn R.Førsund and Nikias Sarafoglou (2005) gives a report.

While Afriat is usually given credit for first introduction of the 'non-parametric' approach, here now is opportunity to transfer credit to Farrell who made such an introduction for this case as it were implicitly, with reference to generators for the region bounded by the production function isoquant.

The same type of function but without constant-returns is used for the utility construction in Afriat (1961) but arbitrarily—or for simplicity!, or for the reasons in remarks already made here about over-stringent "revealed preference"—left aside in the account of (1964), where a modified revealed preference condition to avoid the excess of the original and a polyhedral type function are used instead, as again in accounts such as Varian (1992, p. 133) and Fostel *et al.* (2003). It also served for the 1971 extension of Farrell's method by an accidental transfer of ideas from demand analysis.

For another such conjugate pair, instead,

$$\breve{\phi}(x) = \max\left\{\sum_i w_i X_i : \sum_i w_i x_i \le x\right\}$$

$$\breve{\theta}(p) = \min_i px_i / X_i .$$

These pairs of conjugate functions are such that

$$\breve{\theta}(p) \ge \hat{\theta}(p), \quad \breve{\phi}(x) \le \hat{\phi}(x),$$

and any other pair for which

$$\theta(p_t) = P_t, \quad \phi(x_t) = X_t$$

are such that

$$\breve{\theta}(p) \ge \theta(p) \ge \hat{\theta}(p), \quad \breve{\phi}(x) \le \phi(x) \le \hat{\phi}(x).$$

9 Solution structure

The price levels are determined as solutions of the system

$$(M) \qquad M_{st} \ge P_s / P_t ,$$

derived from and equivalent to the system L, subject to the Laspeyres cyclical product test required for consistency. For a restatement of the inequalities affecting P_t,

$$(M_{st}) \qquad M_{st} P_t \ge P_s ,$$

and equivalently

$$(K_{ts}) \qquad P_t \ge K_{ts} P_s .$$

Any positive solution P_r of system M defines a permissible system of price-levels, represented by a point P in the price-level space $\Pi = \Omega^m$ of dimension equal to the number of periods m. The set C of solutions is immediately a polyhedral convex cone in this space.

When price-levels are normalised to have sum 1 they describe a simplex Δ in the space Π. This simplex Δ is cut by the cone C in a bounded convex polyhedron, or polytope, D. The cone C is recoverable from its section D, as the cone through that section projecting it from the origin.

Taking price-levels to be normalised and so represented by points in the simplex Δ is convenient for computation, and for geometrical representation. Only ratios of price-levels are significant and these are unaltered by normalisation. Every point in the normalised solution set D of the system M is a convex combination of a finite set of basic solutions, and so the computational problem requires finding just these. Given any solution P_r we form the matrix of price-indices

$$P_{st} = P_s / P_t ,$$

depending only on the price-level ratios.

Now there will be explorations for a geometrical and diagrammatic understanding of the system M. Dealing with any three periods r, s, t is illustrative of essential features.

While the associated solution cone C_{rst} may be hard to visualise, the normalised solution polytope D_{rst} in the simplex Δ_{rst} is much easier, and can be represented graphically.

We can refer to any constraint of the system M by the two periods involved, so, as already above, let (M_{rs}) denote the general constraint. There has already been some discussion of the case with two periods, in dealing with the *P-L* interval.

Vectors of price-levels for any subset of periods *r*, *s*, ... , understood as representing only the ratios, can be denoted

$$P_{r:s:\ldots} = (P_r : P_s : \ldots).$$

Any period *r* corresponds to the vertex of the simplex Δ where $P_r = 1$, and vertices can all be labelled by the corresponding periods. Any point on the edge *rs* of the simplex corresponds to a ratio $P_r : P_s$, that is, $P_{r:s}$ in the notation just introduced. Similarly any point in a simplex face *rst* specifies the ratios $P_{r:s:t}$ and so forth for any dimension.

The constraint (M_{rs}) cuts the edge *rs* in a point *Z* and requires $P_{r:s}$ to lie in the segment *Zs*, where

$$(rZ : Zs) = (1 : M_{rs}) = (P_s : P_r)$$

Without ambiguity, we can refer to the segment Zs on the edge *rs* as the segment M_{rs}, as in *Figure* 1. At the same time, the constraint (M_{rs}) requires $P_{r:s:t}$ to lie in the simplex *Zst*, and so forth to any dimension.

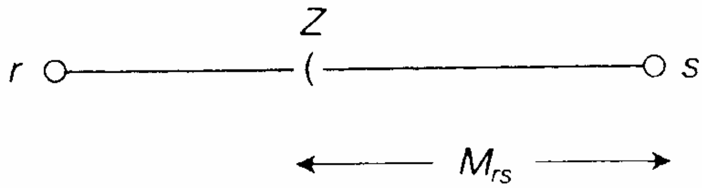


Figure 1

Considering now a pair of constraints (M_{rs}) and (M_{sr}), we have two segments M_{rs} and M_{sr} on the edge *rs*, and they have a nonempty intersection D_{rs} shown in *Figure* 2. This lies within the Paasche-Laspeyres interval, and is a generalisation of that for when data from other periods are involved. It is generally narrower because any effect of extra data must be to reduce indeterminacy.

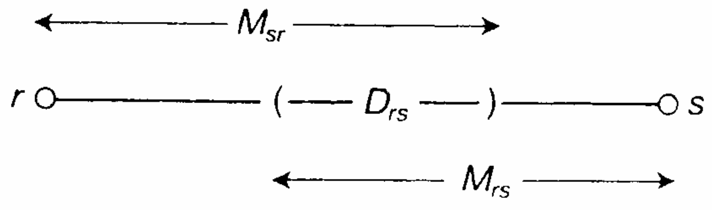


Figure 2

Now consider three constraints associated with the triangle inequality as shown in *Figure* 3. Two of them produce intervals M_{rs} and M_{sr} on *rs* and *st* and, as it were with the triangle equality instead, jointly produce the interval *Yt* on *rt*,. The triangle inequality requires M_{rt} to be a subinterval of this.

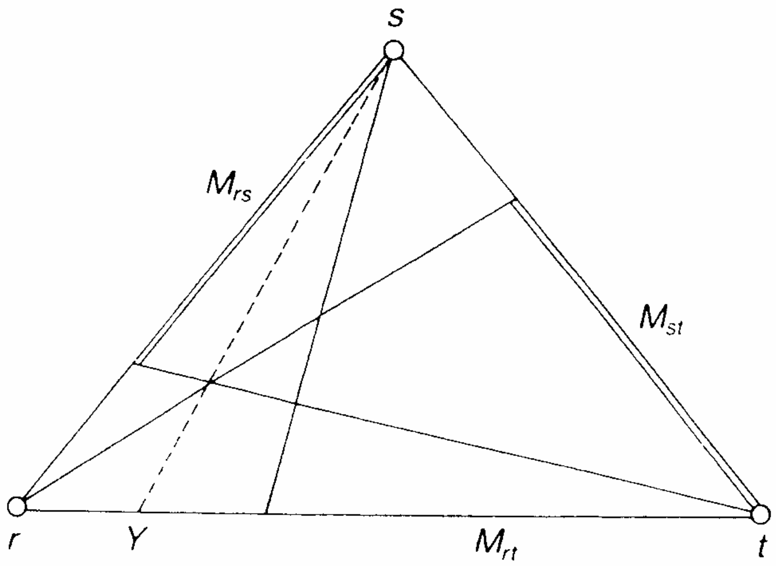


Figure 3

If instead of M_{rt} we take M_{tr} (see Figure 4) cyclically related to the other two, the resulting joint constraint determines a triangle lying within *rst*. The other three cyclically related constraints, associated with the opposite cyclic order, determine another triangle, so configured with the first that their intersection is a hexagon, D_{rst}, as in *Figure* 5, by the triangle inequality assured non-empty.

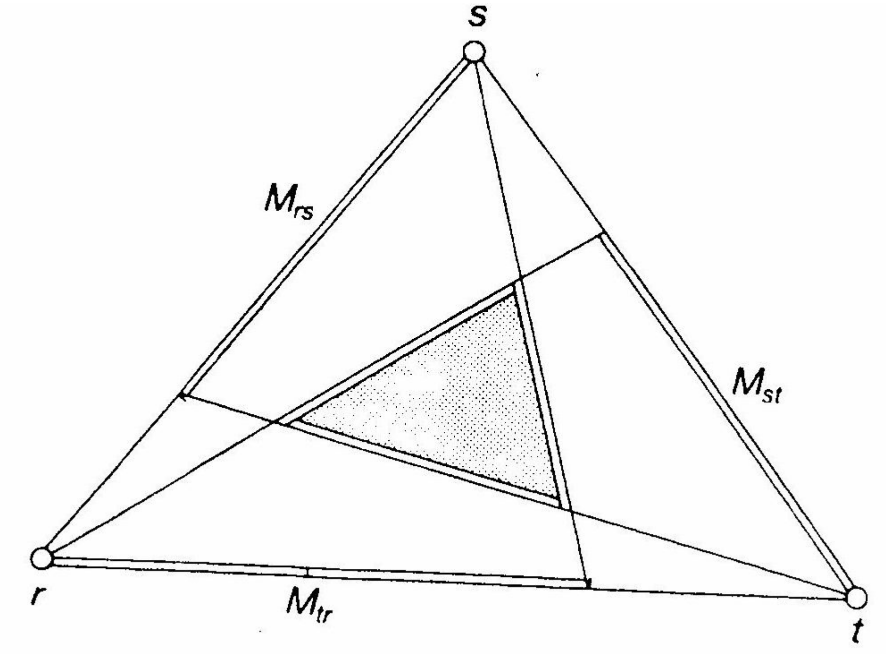


Figure 4

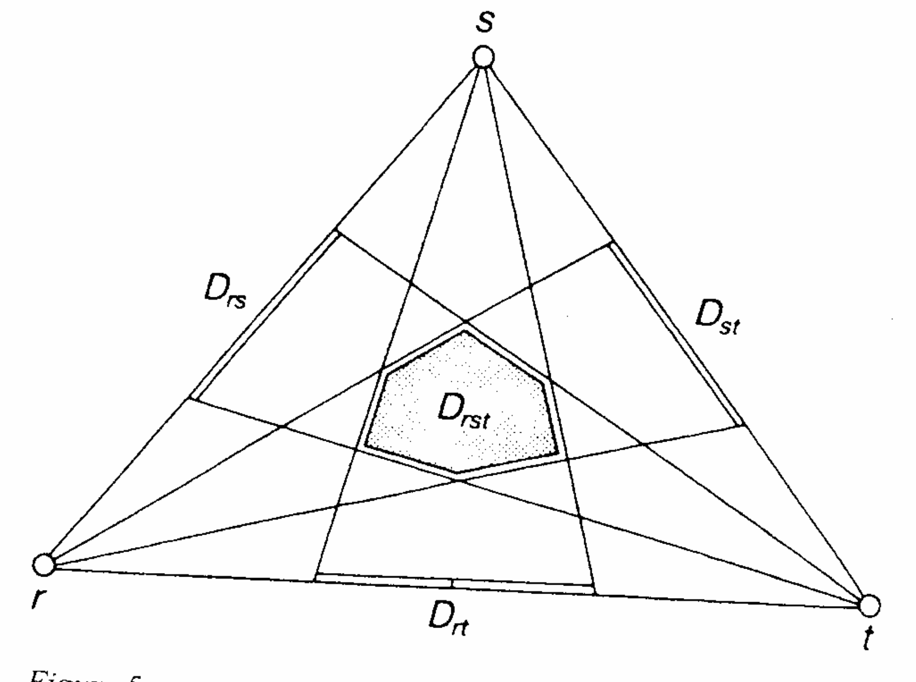


Figure 5

It is seen in this figure that D_{rs} is exactly the projection of D_{rst} from t on to rs. In other words, as $P_{r:s:t}$ describes D_{rst}, $P_{r:s}$ describes D_{rs}. Or again, for any point in D_{rs}, there exists a point in D_{rst} that extends it, in the sense of giving the same ratios concerning r and s. That is the extension property described earlier, a consequence of the triangle inequality, and it continues into higher dimensions indefinitely:

$D_{rs\ldots t}$ is the projection of $D_{rs\ldots tv}$
from the vertex v of the simplex $rs\ldots tv$
onto the opposite face $rs\ldots t$

That shows how price-levels for the periods can be determined sequentially, one further one at a time. Having found any that satisfy the constraints that concern only them, they can be joined by another so that is true again. Starting with two periods and continuing in this way, finally a system of price-levels will have been found for all the periods.

For when the data for a price index between two periods involves data also from other periods, and moreover indices for any subset of periods are to be constructed consistently, these D-polytopes constitute a twofold generalisation of the Paasche-Laspeyres range of indeterminacy of a price index between two periods taken alone.

For a comment on the triangle inequality and equality, along with Z on rs where

$$(rZ : Zs) = (P_s : P_r),$$

now introduce X on st where

$$(sX : Xt) = (P_t : P_s).$$

Let rX and tZ meet in P. Then sP meets tr in Y where

$$(tY : Yr) = (P_r : P_t).$$

So it appears that by choosing the points Z and X for ratios z and x, we arrive at point Y for a ratio y where y = zx. In other words, we have here a geometrical-mechanical multiplication machine, also good for division since from Y and Z for y and z we can arrive at P and so determine X and x for which y = zx, that is, x = y/z.

10 Basic solutions

Taking price-levels to be normalised and so represented by points in the simplex Δ is convenient for computation, as for geometrical representation, when that is possible. Only the ratios of price-levels are significant and these are unaltered by normalisation. The normalised solution set of the system M is a convex polyhedron D in the simplex Δ, every point of which is a convex combination of a finite set of basic solutions, or vertices. The computational problem requires finding just these.

The cases with two periods, or three and four, can serve for a start.

Every conical utility has associated with it a price index, derived from the utility-cost factorization applicable to such a function. A price index is termed *true* if it is connected with a conical utility that fits the demand data.

Every solution for price levels determines true price indices given by their ratios, the existence of a solution requiring the cyclical Laspeyres product test, that requires the cyclical Laspeyres products to be all at least 1. It should be seen what all this has to say in reduction to the classical case of just two periods.

In this case the existence of a solution for price levels is equivalent to the *LP*-inequality, and then any point in the *LP*-interval is representable as a price index, obtained as the ratio of the price levels, which is a true price index from being associated with a conical utility that fits the data.

Hence, as values for the price index, *all points in the LP-interval are true*—all equally, no one more than another (this should dim the aura of extra truth given to Fisher's Ideal Index, especially after it became connected with a—possibly non-existant—quadratic utility). When this was submitted a few decades ago, possibly at the Helsinki Meeting of the Econometric Society, August 1976, it was received with complete disbelief (a proof is in Afriat (1977), 129-30).

Here is a formula to add to Fisher's collection, a bit different from the others:

PRICE-INDEX FORMULA: *Any point in the* LP*-interval, if any*.

However, now we deal rather with price-levels and should put this formula in such terms. Now the simplex Δ is a line segment, so with two vertices. Each point of the segment corresponds to a ratio of price levels in a solution, and so to a price index. A segment in it, corresponding exactly to the *PL*-interval, is the normalized price level solution set, with vertices for *L* and *P*. These are the basic solutions from which all other solutions are determined. There is not much more that can be said about this case, except that it is a generalization of it that makes the present subject.

The case of three periods is already more complex and substantially more interesting, and evocative of the shape of things to come. Already a start was made with that in the last section.

Having the picture there obtained, of the hexagonal boundary of the normalized solution set, the immediate task is to obtain formulae for the six vertices.

The treatment for system (*L*) consists mainly in the power-algorithm for testing consistency and forming the derived system (*M*), equivalent to (*L*), with the triangle inequality and solution extension property that enables solutions to be constructed step-by-step, starting with two variables and following a path for adding variables, to conclude with an individual solution. At each stage the choice to be made can keep the solution as a vertex of the current solution set, so finally there will be arrival at a vertex, making a basic solution. To construct a complete basic solution set this way could be laborious. Firstly the path for adding variables has *m*! possibilities, and with any one path there is a choice between two possibilites at every extension stage. It seems, therefore, there may be about $m!\times 2^{m-1}$ basic solutions, if any, or fewer distinct ones to allow coincidences, with the symbolic description $\left(t_1 t_2 - v_2, t_3 - v_3, \ldots, t_m - v_m\right)$ where $v_i = 1$ or 2

For this discussion, the extension path will simply be $1,\ldots,m$ in that order, though we may not get very far along it.

For P_1 and P_2 referring to periods 1 and 2 (reference denoted 12) there are two basic (non-normalized) solutions

$$\left(12-a\right) \quad P_1 = 1, \quad P_2 = M_{21}.$$

$$\left(12-b\right) \quad P_1 = M_{12}, \quad P_2 = 1.$$

Were we dealing with system (L) these would correspond to the L and P bounds of the LP-interval. For (12-a) there is the verification

$$M_{21} \geq P_2 / P_1 = M_{21}$$
$$M_{12} \geq P_1 / P_2 = \left(M_{21}\right)^{-1}$$

the second line providing confirmation because $M_{12}M_{21} \geq 1$. For (12-b) similarly.

One of these solutions has to be chosen initially, say (12-a). This can be extended to include a third variable, for period 3, relying on the triangle inequality and the solution extension property that follows from it.

Consider

$$(12-a, 3-a) \quad P_1 = 1, \quad P_2 = M_{21}, \quad P_3 = M_{31}.$$

This is a solution that extends the solution (12-a), as may be verified with appeal to $M_{13}M_{31} \geq 1$, and appeals to the triangle inequality, $M_{32}M_{21} \geq M_{31}$ and $M_{23}M_{31} \geq M_{21}$. Similarly

$$(12-a, 3-b) \quad P_1 = 1, \quad P_2 = M_{21}, \quad P_3 = 1/M_{13}$$

is another solution that extends (12-a).

If we identify s, t, r of the last section with 1, 2, 3 in this, we have (12-a,3-a), when normalized, corresponds to the lower of the middle pair of vertices of the hexagon, associated with simplex vertex 1, just as (12-a,3-b) is the upper of the pair. Or something like that. Similarly there are pairs of solution vertices similarly associated with the other two simplex vertices 2 and 3. That makes the six vertices of the hexagon.

Consider

$$(12-a, 3-a, 4-a) \quad P_1 = 1, \quad P_2 = M_{21}, \quad P_3 = M_{31}, \quad P_4 = M_{41}.$$

This is a solution that extends (12-a, 3-a). And so forth.

There may be more to say but for now it may be suitable to submit going further with this approach to the brute computer.

However, there is reassurance to be gained from the circumstance that we already have the canonical solutions, of Section 5, obtained without tedious step-by-step extension but immediate and complete from a reference to the triangle inequality.

None the less there is interest in the determination of all basic solutions, or vertices of the convex polyhedron in the simplex of reference that describes all normalized solutions, illustrated graphically for the case $m = 3$ in Section 8. The $2m$ solutions provided by pairs of canonical solutions in respect to the m possible bases should be the vertices of the convex polyhedron of all price level solutions normalized to make them points in the simplex of reference. For instance in Section 9 we have $2 \times 3 = 6$ vertices of the hexagonal region. This would be, once again, as with the canonical price levels themselves, a providential ready-made solution for what might otherwise have seemed a burdensome abstruse computation.

11 Inconsistency and approximation

A demand correspondence being defined as a correspondence between budget constraints and admitted commodity bundles, here the concern is with a finite correspondence. The approach to constructing a utility that fits such data is most familiar, and now there has been account of the matter where the utility is restricted to be conical, as suits treatment of price-indices.

When the demand data does not have the consistency required for exact admission of a utility, there arises the question of how to admit a utility approximately. Here the impossibility of exactness is treated as due to error, represented as a failure of efficiency.

A theorem will be proved on the existence of a positive solution for a certain system of homogeneous linear inequalities. Such a system can be associated with any finite demand correspondence, together with a number e between 0 and 1 interpreted as a level of cost-efficiency. The existence of a solution is equivalent to the admissibility of the hypothesis that the consumer, whose behavior is represented by the correspondence, (i) *has a definite structure of wants,* represented by an order in the commodity space, as is essential in dealing with price indices, and (ii) *programs at a level of cost-efficiency e.* Any solution permits the immediate construction of a utility function which realizes the hypothesis. When $\boldsymbol{e} = 1$ the utility function fits the data exactly, in the usual sense that its maximum under any budget constraint is at the corresponding commodity point, and when $\boldsymbol{e} < 1$ it can be considered to fit it approximately, to an extent indicated by e. A determination is required for the *critical cost-efficiency*, defined as the upper limit of possible e. Demand analysis which ordinarily knows nothing of approximation and also treats not just a maximum but a strict maximum under the budget constraint, as expressed by the original 'revealed preference' idea, is put in perspective with this approach.

A utility relation is any order in the commodity space Ω^n, that is any $R \subset \Omega^n \times \Omega^n$ which is reflexive and transitive,

$$xRx, \quad xRyR\ldots Rz \Rightarrow xRz$$

A utility function is any

$$\phi : \Omega^n \to \Omega .$$

It represents a utility relation R if

$$xRy \Leftrightarrow \phi(x) \geq \phi(y) .$$

Such representation for R implies it is complete,

$$xRy \vee yRx .$$

Consider a utility relation R and a demand element (p, x) with $px > 0$. A relation between them is defined by the condition

$$(H^*) \quad py \leq px, y \neq x \Rightarrow xRy, y\bar{R}x$$

which is to say x is strictly preferred to every other y which costs no more at the prices p. If R is represented by a utility function this condition is equivalent to

$$(H^*) \quad py \leq px, y \neq x \Rightarrow \phi(x) > \phi(y)$$

With $u = M^{-1}p$ where $M = px$, an equivalent statement, in terms of the associated budget element (u, x), is

$$(H^*) \quad uy \le 1, y \ne x \Rightarrow xRy, y\bar{R}x.$$

This can be called the relation of *strict compatibility* between a utility relation, or function, and a demand, or its associated budget. A demand correspondence being a set D of demand elements, the condition $H_D^*(R)$ of strict compatibility of R with D is defined by simultaneous compatibility of R with all the elements of D. The existence of an order R such that this holds defines the *strict consistency* of D. The original "revealed preference" theory deals with this condition.

Now let further relations between a utility relation R and an demand correspondence D be defined by

$$H'_D(R) \equiv xDp, py \le px \Rightarrow xRy$$
$$H''_D(R) \equiv xDu, yRx \Rightarrow py \ge px$$

with conjunction

$$H_D(R) \equiv H'_D(R) \wedge H''_D(R)$$

by which R and D can be said to be *compatible*. Thus H' signifies that x is as good as any y which costs no more at the prices p, or that maximum utility is obtained for the cost, and H'' signifies any y which is as good as x costs as much, or that the utility has been obtained at minimum cost. In the language of cost-benefit analysis, these are conditions of *cost-efficiency* and *cost-efficacy*. Evidently

$$H_D^*(R) \Rightarrow H_D(R)$$

that is, compatibility is implied by strict compatibility. Let H'_D be defined for H' in the same way as the similar conditions for H^*, and similarly with H'' and H. Then H_D asserts the *consistency* of D.

It is noticed that $H'_D(R)$ derives from $H_D^*(R)$ just by replacing the requirement for an absolute maximum of original "revealed preference" by a requirement for a maximum. But while H_D^*, and similarly H_D, *is* a proper condition, that is there exist D for which it can be asserted and other D for which it can be denied, H'_D is vacuous, since it is always validated by a constant utility function.

It can be remarked, incidentally, that if R *is* semi-increasing,

$$x > y \Rightarrow xRy$$

then

$$H' \Rightarrow H''.$$

Also if R is lower-continuous, that is the sets $xR = [y: xRy]$ are closed, then

$$H'' \Rightarrow H'.$$

Accordingly if, for instance, R is represented by a continuous increasing utility function then H' and H'' are equivalent, so in their conjunction one is redundant, that is mathematically but not economically. But there is no need here to make any assumptions whatsoever about the order R.

It can be granted that as a basic principle H^* requiring an absolute maximum is unwarranted in place of the more standard H' which requires just a maximum. However, while H^* produces the well-known discussion of Samuelson (1948) and Houthakker (1950), described as revealed preference theory—more suitably revealed preference plus revealed non-preference—that discussion is not generalized but its entire basis evaporates when H^* becomes H'. From this circumstance there is a hint that the nature of that theory is not properly gathered in its usual description. The critical feature of it is not that it deals with maxima under budget constraints but that it deals especially with absolute maxima. This might have intrinsic suitability, by mathematical accident, for dealing with continuous demand functions. But it is not a direct expression of normal economic principles, which recognize significance only for a maximum—not that the maximum under the budget should moreover be unique so revealing an additional non-preference significance. If the matter is to be reinitiated, then H' is admitted as such a principle and so equally is H'', so their conjunction H comes into view as an inevitable basis required by normal economic principles. The question of H_D for an expenditure correspondence is proper, that is, capable of being true and false, unlike H'_D which is always true. Also, since $H^* \Rightarrow H$, this provides a generalization of the usual theory with H^*.

It happens, as the mathematical accident just mentioned, that if D *is* a continuous demand function then $H^*_D \Leftrightarrow H_D$. Thus the distinctive revealed preference theory is not lost in this generalization but it just receives a reformulation which puts it in perspective with a normal and broader economic theory not admitting description as revealed preference theory, which moreover is capable of a further simple and necessary extension now to be considered.

With a demand correspondence D interpreted as representing the behavior of the consumer, there is the hypothesis that the consumer (i) *has a definite structure of wants,* represented by a utility relation R, and (ii) *is an efficient programmer.* Then H_D is the condition of the consistency of the data D with that hypothesis. If it is not satisfied, so the data reject the hypothesis, the hypothesis can be modified. If (i) is not to be modified, either because there is no way of doing this systematically or because it is a necessary basic assuunption, as it is for instance in economic index number theory, then (ii) must be modified. Instead of requiring exact efficiency, a form of partial efficiency, signified by a certain level of *cost-efficiency* e where $0 \leq e \leq 1$, will be considered. When $e = 1$ there is return to the original, exact efficiency model.

Thus consider a relation H between a demand (p, x) and a utility relation R together with a number e given by the conjunction of conditions

$$(H') \quad py \leq Me \Rightarrow xRy$$

$$(H'') \quad yRx \Rightarrow py \geq Me$$

where $M = px$. They assert x is as good as any y which costs no more than the fraction eM of the cost M of x, at the prices p, and also any y as good as x costs at least that fraction. In the language of cost-benefit analysis these are conditions of *cost-efficacy* and *cost-efficiency*, but modified to allow a margin of *waste*, which is the fraction

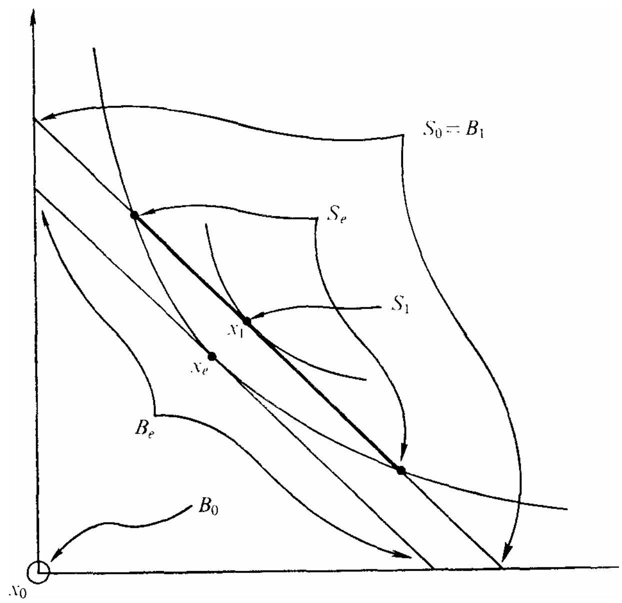
$(1-e)M$ of the outlay M. It is noticed that if H is not to be satisfied vacuously then $e > 0$; and then from H'', with R reflexive necessarily $e \leq 1$.

With R given, for simplicity of illustration say by a continuous increasing strictly quasiconcave function ϕ, and with $p > 0$ and M fixed, it can be seen what varying tolerance this condition gives to x as e increases from 0 to 1. When $e = 0$, x is permitted to be any point in the budget simplex B described by $px = M$, $x \geq 0$. When $e = 1$, x is required to be the unique point x on B for which

$$\phi(x) = \max\{\phi(y) : py = M\}.$$

For $0 \leq e \leq 1$ let x_e be the unique point in the set B_e described by $px = Me$ for which

$$\phi(x_e) = \max\{\phi(y) : py = Me\}.$$



Then x is required to be in the convex set $S_e \subset B$ defined by

$$\phi(x) \geq \phi(x_e), \; px = M.$$

Evidently, if

$$0 \leq e \leq e' \leq 1$$

then

$$B = S_0 \supset S_e \supset S_{e'} \supset S_1 = \{x_1\}$$

That is, the tolerance regions S_e for x form a nested family of convex sets, starting at the entire budget simplex B when $e = 0$ and, as e increases to 1, shrinking to the single point x_1 attained when $e = 1$. The higher the level of cost-efficiency the less the tolerance, and when cost-efficiency is at its maximum 1 all tolerance is removed: the consumer is required, as usual, to purchase just that point which gives the absolute maximum of utility.

For a demand correspondence D, now define *compatibility of D with R at the level of cost-efficiency e* to mean this holds for every element of D. Then *e-consistency* of D, or consistency at the level of cost-efficiency *e,* stated $H_D(e)$, will mean this holds for some *R*. Immediately

$$H_D(1) \Leftrightarrow H_D$$

so 1-consistency of *E* is identical with the formerly defined consistency. Also 0-consistency is valid for every *E*. Further

$$H_D(e), e' \le e \;\Rightarrow\; H_D(e')$$

that is, consistency at any level of cost-efficiency implies it at every lower level. Hence with

$$e_D = \sup\{e : H_D(e)\}$$

defining the *critical cost-efficiency* of any expenditure correspondence D it follows that

$$0 \le e_D \le 1,$$

$$e < e_D \Rightarrow H_D(e), \quad e > e_D \Rightarrow \bar{H}_D(e)$$

The condition $H_D(e)$ *will* now be investigated on the basis of a finite demand correspondence D with elements

$$(p_t, x_t) \in B \times C \;\; (t = 1, \dots, m),$$

and belief in *perfect* efficiency, any utility to be entertained would, to fit the data, have to be such that

$$(PX{=}) \quad P_t X_t = p_t x_t \;,$$

where

$$P_t = \theta(p_t), X_t = \phi(x_t).$$

so in any case

$$(PX \le) \quad P_s X_t \le p_s x_t$$

and now, with

$$L_{st} = p_s x_t / p_t x_t,$$

the Laspeyres index, this condition requires the solubility of the system of inequalities

$$(L) \qquad L_{st} \ge P_s / P_t,$$

for *price levels* $P_t\,(t = 1, \dots, m)$. A question is whether a solution exists. If one does, a conical utility can immediately be constructed that fits the given demand data and provides price levels, and consequently also quantity levels X_t, as required, where the X_t are determined from $(PX{=})$

If instead of perfect efficiency there is to be allowance of partial efficiency, at some level e, then $(PX{=})$ would be replaced by

$$(PXe) \quad P_t X_t \ge e p_t x_t,$$

where $0 \le e \le 1$, which for the perfect efficiency case $e = 1$, in view of $(PX \le)$, becomes again (PX=).

Now from (PXe), with ($PX \le$), follow the systems

$$(a) \qquad a_{st} / e \ge P_s / P_t ,$$

$$(b) \qquad b_{st} / e \ge X_s / X_t ,$$

where

$$a_{st} = p_s x_t / p_t x_t , \quad b_{st} = p_t x_s / p_t x_t$$

with P's and X's connected by

$$P_t X_t = p_t x_t .$$

These systems, even if not consistent for $e = 1$, are *always consistent for sufficiently small* e. From any solution there is obtained a utility that shows demand elements as efficient within the level e.

Thus, with

$$\phi(x) = \min_i X_i p_i x / p_i x_i$$

and antithetical

$$\theta(p) = \min\{px : X_t p_t x \ge P_t\}$$

it appears that

$$p_t x_t \ge \theta(p_t)\phi(x_t) \ge e p_t x_t$$

as required for compatibility at a level of cost efficiency e. In case $e = 1$, then moreover

$$\phi(x_t) = X_t , \quad \theta(p_t) = P_t .$$

Since $a_{st} = p_s x_t / p_t x_t$ is just the Laspeyres index L_{st}, a restatement of system (a) is the system

$$(L/e) \qquad L_{st} / e \ge P_s / P_t .$$

This can be dealt with following exactly the treatment given to the system (L), by replacing the Laspeyres index L_{st} by $L^e_{st} = L_{st} / e$. Then

$$L^e_{s \dots t} = L_{s \dots t} / e \dots e$$

so that

$$L^e_{t \dots t} \ge 1 \Leftrightarrow L_{t \dots t} \ge e \dots e .$$

So it appears that either system (L) is consistent, in which case also system (L/e) is consistent with e = 1, or critical cost efficiency e^* can be determined so that

$$L^e_{t \dots t} \ge 1 \Leftrightarrow e \le e^* .$$

Introducing $L^*_{st} = L_{st} / e^*$, the system

$$(L^*) \qquad L^*_{st} \geq P_s / P_t,$$

is consistent and determines price levels associated with a utility that represents the given demands as together within a cost efficiency at the highest level, in that sense a best approximation to a utility that fits the data, coinciding with a utility that fits the data exactly when that exists. The treatment of (L^*) follows exactly the treatment already accounted for the system (L).

At this point it can be remarked that, with all additional discussion about it put aside, the system (L^*) is the embodiment of the entire method now proposed for the computation of price levels P_t and then price indices $P_{st} = P_s / P_t$ always available and together true in the exact or approximate sense on the basis of demand data for any number of periods.

12 Old and New: an illustration

Some illumination is provided by what this method provides for the classical case of two periods, worked for so long by so many authorities that it may seem unlikely there is anything to add there.

The data consists in a pair of demands

$$(p_t, x_t) \in B \times C \quad (t = 1, 2)$$

in terms of which there are conventional algebraical (not fancy combinatorial) formulae for price indices, especially those associated with Paasche, Laspeyres and Fisher, beside the one or two hundred in Fisher's list.

The Laspeyres is

$$L_{st} = p_s x_t / p_t x_t,$$

Paasche

$$K_{st} = (L_{ts})^{-1},$$

and Fisher

$$F_{st} = (K_{st} L_{ts})^{\frac{1}{2}} = (L_{st} / L_{ts})^{\frac{1}{2}}.$$

For the consistency case $L_{12} L_{21} \geq 1$, where Paasche does not exceed Laspeyres, the *PL*-interval is non-empty and all points in it are accepted as true price indices, all equally true, no one truer than another.

In the contrary case, the data does not admit the existence of true price indices at all, at least not exactly, the *PL*-interval is empty, and now instead for the critical cost-efficincy e^*, that makes the system

$$L_{st} / e \geq P_s / P_t$$

consistent if and only if $e \leq e^*$, which requires

$$L_{12} L_{21} = e^* e^*$$

there is the determination

$$e^* = (L_{12}L_{21})^{\frac{1}{2}}$$

and now

$$L^*_{12} = L_{12} / e^* = (L_{12} / L_{21})^{\frac{1}{2}}, \quad L^*_{21} = L_{21} / e^* = (L_{21} / L_{12})^{\frac{1}{2}}$$

so that, for the Paasche index

$$K^*_{12} = (L^*_{21})^{-1} = L^*_{12}$$

and the system

$$(L^*) \qquad L^*_{st} \geq P_s / P_t \quad (s, t = 1, 2),$$

for determination of approximate price-levels, is equivalent to

$$(L^*) \qquad K^*_{12} \leq P_1 / P_2 \leq L^*_{12},$$

is consistent, but here *the limits are coincident* and the only price-index obtained from a solution is the value

$$P_1 / P_2 = (L_{12} / L_{21})^{\frac{1}{2}}$$

—incidentally, usually known as Fisher "Ideal Index". If the critical e^* is replaced by a more tolerant lower level e, the system is still consistent, with limits now no longer coincident but admitting a range of values, again including the Fisher index but now not unique but just one of its many points.

Hence here we have a New Comment about the Fisher index.

For the Old Comment, in the consistency case, Fisher, being the geometric mean of Laspeyres and Paache, is a point of the now non-empty interval, and so is a true index like any other, and no truer than another. This gives a value to Fisher as being a true index, but also it is deflating from making it no more distinguished than the others. There was a moment of distinction when Fisher became associated with a quadratic utility, which then became put aside, though recently there may have been what may seem to some to be something of a renaissance, see Afriat and Milana (2006).

For the New Comment, in the case of inconsistency, when the *LP*-interval is empty and there are no true indices at all, at least not exactly, at which point in the absence approximation ideas the matter is usually abandoned, Fisher now stands out from being alone associated with *a utility that fits the data as closely as possible*, in the way here approximation is understood that has reference to cost-efficiency criteria.

After the first deflation this gives a real distinction to the Fisher "Ideal" index, and a good reason for the term Fisher gave to it even though not one he entertained. If one does not want to always trouble about consistency and still have an in some way significantly "true" price index, surely this is it—as "superlative" as can be, in the language Irving Fisher invented and has had a perplexed persistence in echoings since. Have latter day pedlars of the superlative ever promoted such a quality in their fancy?

Fisher's index having this new status, its generalization would be quite welcome.

Every point in the entire interval between Laspeyres and Paasche is the possible value for a true index. In this unacceptable indecision the Fisher index, as the geometric mean of the limits, at least picks out one value.

Now with the new method there is again the unfortunate indecision, even expanded since the line segment is now replaced by a multi-dimensional polyhedron. For a fair remedy such as was found before, it may be fair to try some manner of immitation of the original Fisher index.

Here the derived system M may just as well be replaced by M^* is the case of inconsistency, requiring approximation. Everything that follows now applies equally well in either case.

The canonical price-levels, base *t*, are

$$P_i = M_{it}.$$

and

$$P_i = 1/M_{ti},$$

with geometric mean

$$P_i = \left(M_{it}/M_{ti}\right)^{\frac{1}{2}}$$

which is also a price-level solution, determine systems of canonical price-indices

$$P_{ij} = M_{it}/M_{jt}$$

and

$$P_{ij} = M_{tj}/M_{ti}$$

with geometric mean

$$P_{ij} = \left(M_{it}M_{tj}/M_{ti}M_{jt}\right)^{\frac{1}{2}}$$

But this geometric mean price index is identical with the price index determined from the geometric mean price levels,

$$P_{ij} = \left(M_{it}/M_{ti}\right)^{\frac{1}{2}} / \left(M_{jt}/M_{tj}\right)^{\frac{1}{2}}.$$

Going further, similarly, the geometric mean of all the canonical price levels, for all bases, is again a price level solution, the *canonical mean price level solution*, and the price indices derived from it is a price index system where each price index is the geometric mean of the canonical price indices, the *canonical mean price index system.* Any price index in this unique last system is a generalized counterpart of the Fisher index, and in the classical case of just two periods it becomes exactly the Fisher index.

Thus though the price level solutions, and so also price indices they determine, are many, the geometric mean, element by element, of the canonical solutions is again a solution which determines unique price indices that are geometric means of the canonical price indices. Here is a fair conclusion in the quest for elimination of indecision, a multi-period generalization of the Fisher index that even has no conflict with Fisher's own "Tests".

13 Conclusion

Though the mathematics of the method, its theoretical rationalization and computations, require an account, the scheme for applications is simple, and conveys an idea of what could be meant by an answer to “The Index-Number Problem”.

A price-index formula based on a pair of reference periods has conventionally been algebraical and involved data for those periods alone. Then there are inconsistencies between formulae in the treatment of more than two periods, conflicting with the nature of price indices as such, as gathered by Irving Fisher’s “Tests”.

Formulae proposed now are of an entirely different type and are computed simultaneously for any number of periods, involving the data for all of them, without any of the multi-period consistency problems that go with the conventional formulae. There is either exactness, subject to a condition on the data, or approximation, for the hypothetical underlying utility which in any case there is no need to actually construct.

With some m periods listed as 1, …, m and demand data

$$(p_i, x_i) \quad (i = 1, \ldots, m)$$

giving row and column vectors of prices and quantities for some n goods, the first step is to compute the matrix L of Laspeyres indices

$$L_{ij} = p_i x_j / p_j x_j$$

and raise it to the mth power

$$M = L^m$$

in a modified arithmetic where + means *min*.

Diagonal elements $M_{ii} \geq 1$ tell the consistency of the system

$$(L) \quad L_{ij} \geq P_i / P_j$$

for the determination of price-levels P_i, and provide the first and second canonical price-level solutions, with any t as base, given by

$$P_i = M_{it},$$

and

$$P_i = 1 / M_{ti},$$

from which are derived two systems of canonical price indices

$$P_{ij} = P_i / P_j.$$

The price indices in either system, with any base, will all be true together in respect to a utility that fits the data by criterion of cost-efficiency of demand in each period i, so the cost $p_i x_i$ is the minimum cost, at the prices p_i, of the utility of x_i.

Diagonal elements $M_{ii} < 1$ tell the inconsistency of the system, and enable determination of a critical cost efficiency e^* so that the system

$$\left(L/e\right) \quad L_{ij}/e \geq P_i/P_j$$

is consistent if and only if $e \leq e^*$ (features in the computation of e^* remain to be clarified). Then with

$$L^*_{ij} = L_{ij}/e^*$$

the system

$$\left(L^*\right) \quad L^*_{ij} \geq P_i/P_j$$

is consistent, and with

$$M^* = \left(L^*\right)^m$$

there may be obtained canonical price levels and price indices from M^*, as before from M. Now instead the price levels of a canonical system are together true in respect to a utility that fits the data now not exactly, but approximately in the sense of partial cost efficiency at the level e^* in each period, meaning that the fraction e^* of the cost, in the period, is at most the minimum cost at the prices of gaining at least the utility. Hence in the case e^* = 1 that goes with ordinary consistency, the fit would be exact as before.

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