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A Theoretical Model for the Extraction
and Refinement of Natural Resources

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Abstract - The modelling of production in microeconomics has been the subject of heated debate. The controversial issues include the substitutability between production inputs, the role of time and the economic consequences of irreversibility in the production process. A case in point is the use of Cobb-Douglas type production functions. This approach completely ignores the physical process underlying the production of a good. We examine these issues in the context of the production of a basic commodity (such as copper or aluminium). We model the extraction and the refinement of a valuable substance which is mixed with waste material, in a way which is fully consistent with the physical constraints of the process. The resulting analytical description of production unambiguously reveals that perfect substitutability between production inputs fails if a corrected thermodynamic approach is used. We analyze the equilibrium pricing of a commodity extracted in an irreversible way. The thermodynamic model allows for the calculation of the "energy yield" (energy return on energy invested) of production alongside a financial (real) return in a two-period investment decision. The two investment criteria correspond in our economy to a different choice of numeraire and means of payment and corresponding views of the value of energy resources. Under an energy numeraire, energy resources will naturally be used in a more parsimonious way.

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1 Introduction

Two main paradigms have emerged in economics for the role of energy as an input in the production process. On the one hand is the stream of literature that has acknowledged (following the seminal contribution of Gerogescu-Roegen (1971)) the resource degradation connected with the use of energy which gives it a unique role in production processes, while on the other hand, traditional economic analysis of the use of energy in production does not incorporate thermodynamic considerations, focusing instead on the possibilities for substituting it as a factor of production in the presence of energy price shocks or energy shortages¹.

In its current form the debate over the economic significance of irreversibility in the use of energy is far from resolved. The complete substitutability between natural resources (including energy), labor and capital leads to paradoxical consequences. Daly (1997) observes that, if labor and natural resources are substitutes and not complements then it would be possible to make a cake with “[...]only the cook and his kitchen. We do not need flour, eggs, sugar, etc., nor electricity or natural gas, nor even firewood. If we want a bigger cake, the cook simply stirs faster in a bigger bowl and cooks the empty bowl in a bigger oven that somehow heats itself[...]”. A dazzling example of this paradox can be found in the standard representation of funds and flows in the Cobb-Douglas production function model, that is:

$$Q = K^{\alpha_1} L^{\alpha_2} R^{\alpha_3}, \quad (1.1)$$

where Q is the output of the process per unit of time, K represents the stock of capital, L the labor supply and R the flow of natural resources. From expression (1.1) it is evident that we can obtain a fixed amount of output Q_0 even if $R \rightarrow 0$, it is sufficient to choose an amount of K such that:

$$K = \left(\frac{Q_0}{R^{\alpha_3} L^{\alpha_2}} \right)^{\frac{1}{\alpha_1}} \rightarrow +\infty. \quad (1.2)$$

On the other hand, as Ayres and Miller (1980) put it, “[...]There are definite and well-known limits on physical performance in almost every field[...]”. This is one of the key points of the analysis of Gerogescu-Roegen (1979)

¹Abel (1983) proposes a model for the substitution between energy and capital in a generic production function when the energy price is stochastic. A dynamic model is developed by Pindyck (1983) who assumes that the firm has to optimise four level of inputs: capital, labor, energy and materials. The effect of unpredictable energy price variations is analysed in a real business cycle model by Kim and Loungani (1992), explicitly incorporating energy as an input in a CES production function. A vast number of papers are dedicated to the empirical analysis of capital-energy substitution, which is of “great importance in predicting economic disruptions arising from energy shortages” (Field and Grebenstein (1980)). The papers of Berndt and Wood (1975) and Fuss (1977) highlight a negative value for the capital-energy substitution elasticity, indicating that they are not substitutes, while Griffin and Gregory (1976) and Pindyck (1979) find a positive value. The discrepancy is fully explained in Field and Grebenstein (1980) who find that the difference between the two results lies in the way the capital input is handled. A paper by Magnus (1979) remarks that energy inputs interact differently with labor (in a substitutable way) and capital (in a complementary way) and this justifies the introduction of energy as an independent input in the production function. The work of Atkinson and Kehoe (1999) sheds light on the fact that, with respect to the energy prices, energy use is inelastic when dealing with time-series data and elastic with cross-sectional data. They develop a putty-putty model and a putty-clay model which reproduce these differences in elasticities. Finally, the difference between cross-sectional data and time series data in capital-energy substitution elasticity is resolved by Thompson and Taylor (1995) using Hiroshima elasticities. Arrow et al. (1961) provide an exhaustive analysis of capital-labor substitutability, concluding that the substitution elasticity between capital and labor in manufacturing may typically be less than one.

and is widely discussed in Cleveland and Ruth (1997), who provide a survey of the literature arguing against the neoclassical assumption that any factor of production can be substituted by any other factor. The idea they contrast is effectively summarized in Solow's assertion (Solow (1974)) that "the world can, in effect, get along without natural resources". However, the papers that deal with this argument often provide only a qualitative analysis of the limit to substitution between production process inputs. The case can be easily argued either way due to the vagueness of the approach. By contrast, our paper provides an analytical limit to the possibility of substitution between raw material and energy input required to achieve production, starting from a full thermodynamic description of the production process of a commodity. For the specific case, this limit provides an answer to all issues raised by the recent and past literature about substitutability between production process inputs. Total substitution is impossible because there is a physical energy threshold, for a given quantity of raw material input, below which no production exists. This emphasises the importance of a physics-based approach to production modelling as a correct methodological way of resolving the substitutability issue. A standard Cobb-Douglas model would lead to results in contradiction with the laws of nature (Islam (1985))². A list of economic key-objections and economic advantages of the Cobb-Douglas function can be found in Murthy (2002).

An analysis of the impact of thermodynamic limits in microeconomic decisions is put forward by Berry et al. (1978). They highlight the difference between the concepts of thermodynamic optimality of the use of energy in production (i.e. the highest productivity of energy) and overall economic (cost) optimality. Pareto optimality in neoclassical general equilibrium will not in general coincide with the optimal use of available energy from a thermodynamic point of view. However, if energy is in aggregate scarce and becomes a limiting factor because of its lack of substitutability (as argued by Ayres and Miller (1980)), we can show that the two criteria may indeed coincide.

Except for a few papers (Krysiak (2006), Krysiak and Krysiak (2003), Roma (2006), Ruth (1995)) most of microeconomics completely avoids a detailed consideration of the physical constraints which are the essence of every production process. Krysiak and Krysiak (2003) study the impact of the first law of thermodynamics on the standard general equilibrium model of production and consumption. They show that general equilibrium theory is consistent with the mass and energy conservation laws. Krysiak (2006) analyzes the consequences of the second law of thermodynamics on economic equilibrium in a general framework, arguing that if irreversibility is taken into account a non-zero level of inputs and a non zero-level of emissions are necessary to sustain a positive level of consumption³.

We analyze the relevance of thermodynamics and irreversibility for economic decisions. All real world transformations involving energy are in fact irreversible. A thermodynamic transformation or cycle is said to be reversible if it is carried out by varying a state variable with infinitesimal changes that allow the system to be at rest throughout the entire process (Fermi (1956)). Such a transformation is impossible because it would require an infinite amount

²Sav (1984) uses a micro-engineering production function derived from physical laws to model the exploitation of solar energy for domestic water heaters. Substitution elasticities between nonrenewable fuel inputs (oil or natural gas) and capital-intensive solar-produced heat is investigated. In this context Sav (1984) finds that a governmental taxation policy for solar energy incentives could paradoxically result in an increase in consumption of nonrenewable energy resources.

³On the debate about the impact of the entropy law on economic equilibrium see also Young (1991) and Daly (1992) which lead to two completely opposite conclusions. Many authors have investigated Nicholas Georgescu-Roegen's paradigm of ecological economics, again obtaining conflicting conclusions, such as those reported by Khalil (1990), subsequently criticized by Lozada (1991) and finally re-stated by Khalil (1991).

of time. All production processes are therefore irreversible in nature, because they have to be carried out in a finite time-period to bring production to the market. They involve then a strictly positive increase in entropy $\Delta S > 0$. An increase in entropy means a reduction in "useful" energy, that is the part of energy that can be converted into work by an engine. Thus an entropy increase can be interpreted as waste or resource degradation. If the production function does not accommodate the real thermodynamic process which leads to the final consumption good, the impact of the producer's choices on resource depletion and the waste released on the environment will not be evident.

In the production literature a limited amount of attention has been dedicated to the concept of waste as an unavoidable joint product of any production process (Ayres and Kneese (1969), Ethridge (1973) and Kummel (1989)). Gerogescu-Roegen (1971) argued that the production of waste is a direct consequence of the irreversibility postulated by the entropy law.

We develop a specific model for the energy required in the extraction of a commodity, consistent with actual physical constraints. Specifically, we propose an analytic model for a mining operator who refines a mineral from its natural concentration C_0 to a strike concentration k , defining the consumption good. The mining operator faces an initial mixture of useful mineral and waste rock. We suppose that the mineral is initially a minority presence with respect to the waste rock and the mining operator tries to purify the mixture and extract highly concentrated mineral⁴.

Here the production process is modelled analytically, following the thermodynamic theory of an irreversible separation process.

An attempt in this direction can be found in Ruth (1995). This paper contains explicitly thermodynamic constraints on the production function. However these constraints only appear as lower bounds on the inputs, and the production function is again modelled as a Cobb-Douglas function. Moreover (as in Berry et al. (1978)), the lower bounds are calculated for a reversible separation process. As mentioned above, this approach does not reflect the reality of production, which involves only irreversible processes. The degradation of energy is far higher in irreversible processes than in the reversible ones and irreversible processes make a greater contribution to entropic increase, as stated by the second law of thermodynamics.

A production function derived from finite time thermodynamic constraints can be found in Roma (2006), who proposes a microeconomic model for the production of a basic good (hot water) based on a fully consistent thermodynamic description of the process. The production process we analyze here from a physical point of view differs from the hot water production process in Roma (2006) in that the final product obtained will be of homogeneous quality and unused energy can be stored. We incorporate the production model in a simple general equilibrium framework in which we analyze production decisions. We will derive the equilibrium in a simple economy in which our good is produced and consumed and we will address the role of energy efficiency in optimal economic decisions. In a neoclassical equilibrium the optimal allocation of available resources will follow from the non-satiated demand for final goods and the efficiency in the use of available energy will not be driven by its own economic decisions. On the other hand, the negative externalities associated with the degradation of energy, if taken into account, would lead to a thermodynamic efficiency criterion in the use of this scarce resource. However, thermodynamic efficiency

⁴A number of papers deal with the decisions concerning the exploitation of physical resources and the extraction/production of commodities (Brennan and Schwartz (1985), Cortazar et al. (1998), Hartwick (1978), Stiglitz (1976)) but none describes the thermodynamics of extraction.

would have to be imposed on the decision makers (by way, for example, of a "green tax"). We find, similarly to Roma (2006), that the economic equilibrium will be twisted towards higher energy efficiency if energy is forced to be the numeraire and means of payment in the market, in the presence of irreversible thermodynamic processes in production. The neoclassical solution, where a change in numeraire and means of payment would not alter the neoclassical equilibrium, is finally obtained if the production process is carried out over an infinite time, i.e. if it is reversible.

This paper is organized as follows: in section 2 we introduce our concept of thermodynamic production function, starting from the idea by Ruth (1995) of a thermodynamic limit on production inputs. Section 3 is dedicated to a description of the physical process underlying the production of the commodity. Here the thermodynamic production function is derived together with the marginal and average costs of production. Isoquants of the computed production function are compared with those derived from a Cobb-Douglas type production function, in order to highlight the differences between the two approaches. An empirical model for the degradation of the resource due to cumulative extraction is described and incorporated in a two-period production framework. The problem of a producer who uses a thermodynamic production function and faces the problem of the scarcity of energy is presented in section 4. After introducing the concept of Energy Return on Energy Investment (EROEI), energy will be used as the numeraire and means of exchange in our economy. It is shown that, when energy is scarce, and it cannot be substituted, the producer's decision varies drastically if the accounting is made in terms of energy or in terms of another numeraire (such as the product itself). The consumer's problem is also described in this section. Section 5 shows that arbitrage arguments can be stated in a perfectly reversible context to compute the price of the commodity in terms of energy, while the same arguments weaken if irreversibility arises. The producer's choices under a reversible technology are discussed in section 6. Finally, in section 7, we present our conclusions.

2 Ruth (1995) Model Of Mining Technology

Ruth (1995) presents the problem of the mining operator as the maximisation of a value function essentially defined as the time-integral of production growth minus the growth in energy expenditure for production. Hence his model assumes that the energy input is the only cost of production. To model production, Ruth (1995) uses a Cobb-Douglas type function given by:

$$Y_t = \left(\frac{J_t - J_t^*}{J_A - J_A^*} \right)^{\gamma_1} \left(\frac{E_t - E_t^*}{E_A - E_A^*} \right)^{\gamma_2}, \quad (2.1)$$

where J_t is the raw material input, E_t is the energy input and the starred quantities J_t^* and E_t^* are the minimum material input and minimum energy input required for the production of Y_t , respectively. With J_A and E_A (J_A^* and E_A^*), Ruth (1995) indicates the same quantities in a base year.

The Cobb-Douglas production function (2.1) tries to capture the limits to the substitution between energy and raw material that are necessary to produce the quantity of the final good Y_t . The functions E_t^* and J_t^* represent the minimum quantities that are necessary to produce Y_t according to the underlying thermodynamic process and below which production is physically impossible. The energy limit E_t^* is modelled as the minimum energy required to

reversibly separate a mixture of two components. This limit is obtained assuming that the process is carried out in an infinite time although, in fact, all industrial processes must be completed in a finite time to bring production to the market. Still, E_t^* and J_t^* may be partially substituted.

Our approach will be completely different: we will derive the function Y_t from thermodynamic considerations alone. In this approach the limits to the substitution between energy and raw material are represented by an analytical thermodynamic constraint.

As we will see, E_t^* and J_t^* denote the inputs in the limiting case which would occur if production were carried out over an infinite period of time. In particular, Ruth (1995) derives E_t^* from a physical relationship:

$$E_m^{bin} = -RT N_0 [C_0 \ln C_0 + (1 - C_0) \ln (1 - C_0)] \quad (2.2)$$

which is a special case of our model. We show its derivation in Appendix (A). The quantity E_m^{bin} denotes the minimum energy input necessary to completely separate the final product from waste in a fully reversible way. The raw material input does not appear in the expression, and therefore a production function cannot be derived from it. Our alternative theoretical description of the production function keys instead on the thermodynamic limit to production achievable in a finite period of time. A production function that is fully consistent with physical constraints can be directly derived from this limit. This production function has very different properties from a Cobb-Douglas function.

In conclusion, not only does the production function (2.1) fail to accommodate for the real process which leads to the final product Y_t , but its thermodynamic limit is inconsistent with that existing in industrial production.

In the following section we elaborate a theoretical model of production which overcomes these difficulties.

3 The Physical Model

Thermodynamic energetic limits to industrial processes have been analyzed by a vast number of authors. The survey of Sieniutycz (2003) provides an exhaustive analysis of the most commonly used methods for computing thermodynamic energy limits to industrial production, such as separation processes, heat pumps, chemical and electrochemical systems, maximum power in thermal engines, etc. Energy limits in industrial processes are investigated from a classical thermodynamic approach by Forland et al. (1988), Kotas (1985), and Denbigh (1956). Engineers typically face the problem of how to set-up a production plant capable of achieving non-zero production in a finite time and with finite industrial facilities. In most cases this kind of problem is solved using one of two alternative approaches: finite time thermodynamic (Andersen et al. (1977), Berry et al. (2000)) or entropy generation minimization (Bejan (1996), Nummedal et al. (2005), Salamon et al. (1980)), which are usually referred as FTT and EGM, respectively. The idea underlying these approaches is that energy is a scarce resource and its use must be optimized. The physical model we use for the construction of our production function is developed by Tsirlin and Titova (2004). They compute (via EGM) the minimum energy required to achieve the separation of an ideal mixture of components with a specified output.

A (natural) reservoir of raw materials is available in which each valuable element (mineral) is not available in its pure form but is instead mixed with waste material. At time $t = 0$ the composition vector of the reservoir, which describes the (percentage) concentrations of the different components, is (C_{01}, \dots, C_{0n}) . Part of the raw materials are also initially stored in a subsystem (an industrial separation plant) which is in contact with the reservoir and is characterized by the same pressure (P), temperature (T), and chemical potential of the reservoir. The composition vector of the materials inside the subsystem is initially the same as that of the reservoir. We indicate with N_0 the number of moles of materials in the subsystem.

As production is carried out, additional transfer of raw material into the subsystem occurs at the mass transfer coefficient α_i .

The refinement and extraction of valuable materials from waste takes place inside the subsystem in such a way that at the final instant $t = \tau$ the subsystem has a new specified composition $(C_{\tau 1}, \dots, C_{\tau n})$ and contains a number of moles N_τ . Tsirlin and Titova (2004) demonstrate that the minimum energy required to carry out the transformation $[N_0, (C_{01}, \dots, C_{0n})] \rightarrow [N_\tau, (C_{\tau 1}, \dots, C_{\tau n})]$ in a finite time τ (see also Tsirlin et al. (2002)) is:

$$E_{min} = RT N_\tau \sum_{i=1}^n [C_{\tau i} \log C_{\tau i} - C_{0i} \log C_{0i}] + \frac{1}{\tau} \sum_{i=1}^n \frac{\Delta^2 (N C_i)}{\bar{\alpha}_i}, \quad (3.1)$$

where $\Delta (N C_i) = N_\tau C_{\tau i} - N_0 C_{0i}$, $\bar{\alpha}_i$ is the equivalent mass-transfer coefficient for the i th component and R is the universal gas constant. The Appendix (B) provides additional detail on the physical processes that are assumed to take place in the production plant.

Now suppose that we want to refine a substance which has, in nature, a concentration C_0 . For simplicity the substance we are interested in is mixed with only one other, so the initial vector of concentrations is $(C_0, 1 - C_0)$. For example, C_0 can be the concentration of a mineral in the mineral ore and thus $1 - C_0$ can be interpreted as the concentration of the waste rock.

Differently from the approach of Ruth (1995), who adopts the extreme assumption that the mineral can be completely separated from the waste rock, we generalize the definition of the final product and define it more realistically as mineral extracted with a given strike concentration $k > C_0$ but still mixed with a residual concentration $1 - k$ of waste rock. Then the final vector of concentrations is $(k, 1 - k)$. We suppose that we have an engine working at a given energy E to carry out the extraction of the initial number of moles in the subsystem, N_0 .

In the hypothesis that the process is carried out with the maximum thermodynamic efficiency, the number of moles produced is given by equation (3.1), with $n = 2$, $(C_{01}, C_{02}) = (C_0, 1 - C_0)$ and $(C_{\tau 1}, C_{\tau 2}) = (k, 1 - k)$. The energy required for the transformation is:

$$E = RT N_\tau [k \log k + (1 - k) \log (1 - k) - C_0 \log C_0 - (1 - C_0) \log (1 - C_0)] + \frac{1}{\tau \alpha} \left[(N_\tau k - N_0 C_0)^2 + (N_\tau (1 - k) - N_0 (1 - C_0))^2 \right], \quad (3.2)$$

where we have chosen for simplicity $\bar{\alpha}_{11} = \bar{\alpha}_{22} \equiv \alpha$. The second term of expression (3.2) denotes the irreversible use

of energy. It is composed by the square of the difference $N_\tau k - N_0 C_0$ between the refined moles extracted and the moles of pure material in the system, and by the corresponding difference $N_\tau (1 - k) - N_0 (1 - C_0)$ for the waste material.

3.1 The production function

Equation (3.2) implicitly defines a production function $N_\tau(N_0, E)$, where the inputs are energy, E , and raw material, N_0 , and where production time, τ , also determines the amount of final product that can be obtained. We can rearrange expression (3.2) as:

$$N_\tau^2 \frac{1}{\tau \alpha} \left[k^2 + (1 - k)^2 \right] + N_\tau \left[RT \xi_0 - 2 \frac{N_0 \psi_0}{\tau \alpha} \right] + \frac{N_0^2}{\tau \alpha} \left[C_0^2 + (1 - C_0)^2 \right] - E = 0, \quad (3.3)$$

where the constants $\xi_0 \equiv k \log k + (1 - k) \log(1 - k) - C_0 \log C_0 - (1 - C_0) \log(1 - C_0)$ and $\psi_0 \equiv k C_0 + (1 - k)(1 - C_0)$ depend only on the initial and final concentrations. The solutions of (3.3) are:

$$N_\tau^\pm = \frac{-RT \xi_0 + 2 Q_\tau N_0 \psi_0 \pm \sqrt{(RT \xi_0)^2 - 4 Q_\tau^2 (k - C_0)^2 N_0^2 - 4 RT Q_\tau N_0 \psi_0 \xi_0 + 4 G_k Q E}}{2 Q_\tau G_k}, \quad (3.4)$$

where we have introduced the notation $Q_\tau = \frac{1}{\tau \alpha}$ and $G_x = x^2 + (1 - x)^2$. A solution to the problem exists if, and only if, the quantity under the square root of equation (3.4) is greater than or equal to zero:

$$\delta = (RT \xi_0)^2 - 4 Q_\tau^2 (k - C_0)^2 N_0^2 - 4 RT Q_\tau N_0 \psi_0 \xi_0 + 4 G_k Q E \geq 0. \quad (3.5)$$

This condition implies that there is a limited substitutability between process inputs and defines a lower bound on the energy input:

$$E \geq \frac{1}{4 Q_\tau G_k} \left[N_0^2 4 Q_\tau^2 (k - C_0)^2 + N_0 (4 RT \xi_0 \psi_0 Q_\tau) - (RT \xi_0)^2 \right]. \quad (3.6)$$

Inspecting equation (3.6), we can see that if $k \neq C_0$ and $N_0 > 0$, a lower bound on energy always exists for a high enough positive value of N_0 . In our model of production $k > C_0$, because we want to improve the quality of the "valuable" substance, and $N_0 > 0$ because we assume that our basin is not empty at the initial time⁵. This means that we start production only if the energy is over the threshold given by equation (3.6). As pointed out by Islam (1985), such restrictions on production inputs drastically modify the isoquants with respect to those of a Cobb-Douglas function.

Only the solution of (3.4), which increases in the energy input, has a physical sense. We then define our production function as:

⁵As we will show later, energy-efficient production requires a strictly positive value of N_0 which is proportional to the level of production (see equation 3.8).

$$N_\tau(E, N_0) \equiv \frac{-RT \xi_0 + 2 Q_\tau N_0 \psi_0 + \sqrt{(RT \xi_0)^2 - 4 Q_\tau^2 (k - C_0)^2 N_0^2 - 4 RT Q_\tau N_0 \psi_0 \xi_0 + 4 G_k Q E}}{2 Q_\tau G_k}. \quad (3.7)$$

Expression (3.7) explicitly gives the number of moles of the highly refined mixture $(k, 1 - k)$ we obtain starting from N_0 moles of raw material and spending energy E for production in a finite time τ . Figure 3.1 shows the isoquants of $N_\tau(E, N_0)$ for a chosen specification of the model parameters (reported in Table 1 of Appendix C) and for $k = 80\%$. The isoquants of (3.7) display an "economically inefficient" portion, where the same level of production is obtained with more N_0 than necessary. We can disregard this portion.

Isoquants derived from a Cobb-Douglas type production function are also reported as thin dotted lines. The plotted Cobb-Douglas function has the form $N^{C-D}(E, N_0) = A \left(\frac{E}{RT}\right)^{\gamma_1} N_0^{\gamma_2}$ and A , γ_1 and γ_2 are chosen to obtain nearly the same levels of $N_\tau(E, N_0)$ isoquants.

Using $N^{C-D}(E, N_0)$ as a production function would generate perfect substitutability between the two production inputs, that is given a point $(N_0^{(1)}, E^{(1)}/(RT))$ which belongs to an isoquant $N^{C-D}(N_0, E) = K_1$ it is possible to obtain the same level of production K_1 , arbitrarily reducing the quantity of energy entering the process and increasing the number of moles N_0 of raw material.

This property disappears when the production function is obtained via the real thermodynamics process. The constraint (3.6) does not allow perfect substitutability between inputs. This gives a more realistic description of the process and recognizes the different nature of energy as a production input.

Expression (3.7) provides a theoretically consistent production function for the extraction of refined mineral from a low grade mixture. It naturally incorporates a lower bound on the energy input in the spirit of Ruth (1995) and, as anticipated, makes the lack of substitutability between inputs completely clear. However, we find that the lower bound on the quantity of low grade mixture N_0 is zero. In what follows, we assume that there is an unlimited quantity of the raw material N_0 and it is therefore freely available to the producer, while energy is scarce. In this we follow Ayres and Miller (1980), who argue that basic materials will always be available in some concentration in the earth's crust, but the availability of energy (work) needed to extract them is, in fact, the only limit to economic growth⁶.

⁶Gerogescu-Roegen (1979) argued that the degradation of quality materials is, in fact, the real limit to the economic growth, rather than the scarcity of energy itself. This approach is criticised by Ayres and Miller (1980), who argue that technical progress can overcome the scarcity of physical resources. They point out that a finite quantity of resources must always be embodied in capital and a limit on economic growth and technical efficiency always exists, due to the finite availability of renewable resources. Their point is that all resources, no matter how they are distributed on the earth, can be extracted if enough energy is available. Thus they conclude that energy is, in fact, the only resource that could ultimately limit economic growth.

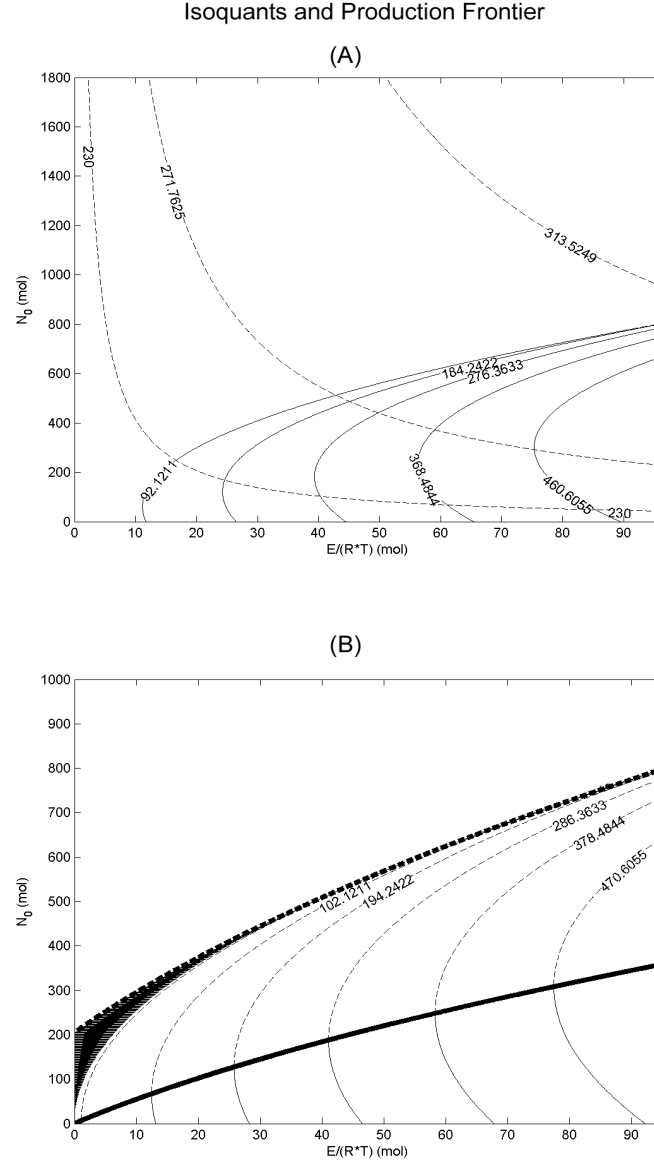


Figure 3.1: (A) The dotted lines are isoquants of a Cobb-Douglas type production function $N^{C-D}(N_0, E) = A \left(\frac{E}{RT}\right)^{\gamma_1} N_0^{\gamma_2}$, with $A = 100$ and $\gamma_1 = \gamma_2 = 0.1$. Continuous lines represent isoquants of the proposed production function $N_\tau(E, N_0)$, where the model parameters are those reported in Table 1 of Appendix C, with a final strike concentration $k = 80\%$. (B) The plot shows the efficient portion of the production function $N_\tau(E, N_0)$ isoquants as black continuous lines. The parameters chosen are those reported in Table 1 of Appendix C. We choose, again, $k = 80\%$ for the strike concentration. The thin dotted lines are the economically inefficient isoquants of $N_\tau(E, N_0)$. The bold dotted line represents the production frontier (given by equation 3.6), which defines the minimum energy required to carry out production for a given value of N_0 . The continuous bold line plots the optimal choice of the freely available raw material N_0 for each value of the energy E (see equation 3.8). The intersection between an isoquant and this line gives the number of moles N_0 that should be used to achieve the chosen level of production with the minimum energy expenditure. The area coloured in black is a zone where, despite energy being over the depicted threshold, it is not enough to have positive solutions of equation (3.3).

3.2 Minimum energy production frontier

If energy is the only scarce input, it is rational to minimize its use and substitute it with the freely available raw material as far as possible. Inspecting figure (3.1) it can be seen that, for a given level of output, there is a value of the input raw material which minimizes the energy required to obtain such a level of production. This value can be computed analytically. From equation (3.2) it can easily be seen that the level of N_0 which minimizes the energy input for a fixed level N_τ of production is given by:

$$N_0(N_\tau) = N_\tau \frac{1 - C_0 - k + 2C_0 k}{1 - 2C_0 + 2C_0^2}. \quad (3.8)$$

By substituting (3.8) into the equation (3.2) and solving it with respect to N_τ , we obtain the production frontier where every level of production is achieved at an absolute minimum energy cost:

$$N_\tau(E) = \frac{\sqrt{4E(C_0 - k)^2 Q_\tau G_{C_0} + (G_{C_0} RT \xi_0)^2} - G_{C_0} RT \xi_0}{2(C_0 - k)^2 Q_\tau}. \quad (3.9)$$

Equation (3.9) defines the maximum amount of mineral refined to the given concentration k that can be extracted in a period of time τ given the energy input E . Under our assumptions a fixed proportion of raw material and energy is required, for each level of production, which only depends on the scale of production and time available for the production process.

Solving relation (3.9) w.r.t. E we find that:

$$E(N) = \frac{(C_0 - k)^2 Q_\tau N^2 + G_{C_0} RT \xi_0 N}{G_{C_0}}. \quad (3.10)$$

The energy $E(N)$ in (3.10) can be interpreted as a conditional energy demand (input) when the other factor is at the optimal level. From expression (3.10) we can compute the marginal and average costs of production in terms of energy, $\frac{\partial E}{\partial N}$ and $\frac{E}{N}$:

$$\frac{\partial E}{\partial N} = \frac{2(C_0 - k)^2 Q_\tau N + G_{C_0} RT \xi_0}{G_{C_0}}, \quad (3.11)$$

$$\frac{E}{N} = \frac{(C_0 - k)^2 Q_\tau N + G_{C_0} RT \xi_0}{G_{C_0}}. \quad (3.12)$$

Within the limit $\tau \rightarrow +\infty$ we have that $Q_\tau \rightarrow 0$ and then:

$$\lim_{\tau \rightarrow +\infty} \frac{\partial E}{\partial N} = \lim_{\tau \rightarrow +\infty} \frac{E}{N} = RT \xi_0. \quad (3.13)$$

This simple result could be also obtained taking the limit $\tau \rightarrow +\infty$ of equation (3.2). Equation (3.13) means that if the process is reversible then the technology is linear, that is marginal and average costs are the same technological constant $RT\xi_0$.

3.3 Quality degradation of the resource

In practice the natural concentration C_0 of the good to be refined is not a constant during extraction. Suppose that the extraction is carried out for an initial period of time τ and then stopped. We assume that when we restart the extraction we will have to deal with a new natural concentration $C'_0 < C_0$ because extraction depletes the resource. Then we need more energy than in the first period of extraction to obtain the same strike concentration k . A quantitative approach based on empirical data can be found in Mudd (2007). This paper reports Australian data on the copper ore grade during the period 1842 – 1992. The ore grade of the copper was about 25% in the year 1842 and decreased to a value $< 5\%$ in the year 1995. Simultaneously the quantity of ore milled starts from nearly $\approx 0 Mt$ and reaches the value of $\approx 80 Mt$ ($1 Mt = 10^3 Kg$). Resource depletion is then strictly correlated with the cumulative quantity of extracted material.

To model resource depletion we follow the approach of Chapman and Roberts (1983) and Ruth (1995) and assume that the relation between the concentration $C(t)$ (of the mineral to be refined) at time t and the cumulative quantity of extracted material $J(t)$ is:

$$\log J(t) = \eta - \rho \log C(t), \quad (3.14)$$

where η reflects the relative abundance of the metal considered and ρ is the degradation velocity. Chapman and Roberts (1983) report different values of ρ , which vary from $\rho = 1.62$ for copper to $\rho = 17.34$ for chromium. Values estimated on historical data for the constant η can be found in Nguyen and Yamamoto (2007). Values for the constant ρ and η are always estimated when $J(t)$ is expressed in metric tons. In our case we measure the quantity of material extracted in moles. Suppose that we have extracted N moles of a substance with atomic mass M_A . Then we have extracted $M_A \times N$ grams of such substance. This means that $J = 10^3 \times M_A N$ is the quantity of material extracted expressed in metric tons. From now on it will be useful to use the convention $\vartheta \equiv 10^3 \times M_A$, namely $J = \vartheta N$.

3.4 Two-period production technology

We assume that the production process by which the mineral is extracted occurs over two periods of time. First we allow our extraction plant to work in a basin, for a period τ_1 , with a number of initial moles N_0 of a mixture $(C_0, 1 - C_0)$, where the first component of the mixture is the useful mineral and the second component is waste rock. We suppose that the concentration of the useful material is lower than that of the waste rock, that is $C_0 < 1 - C_0$. The final consumption good is defined by the mixture $(k, 1 - k)$, with $k > 1 - C_0$. If the mining operator produces N_1 moles of the consumption good in a time τ_1 he needs the energy given by equation (3.10).

In the first time period the producer does not completely deplete the basin, allowing for additional production to be carried out in a second time period, τ_2 , which we assume to be equal to τ_1 .

After the extraction of N_1 moles at time τ_1 , the new concentration C_1 of the useful substance is given by:

$$\log(\vartheta N_1) = \eta - \rho \log C_1 \rightarrow C_1 = \exp\left(\frac{\eta}{\rho}\right) (\vartheta N_1)^{-\frac{1}{\rho}}. \quad (3.15)$$

In the second production period the plant works in a new basin where the natural concentration C_1 is lower than the initial one C_0 . We denote with E_2 and N_2 the energy used and the number of moles produced in the second period. From the previous discussion, it follows that the quantity N_2 depends not only on E_2 but also on N_1 (E_1) through C_1 :

$$N_2(E_2, N_1(E_1)) = \frac{\sqrt{4 E_2 (C_1 - k)^2 Q_\tau G_{C_1} + (G_{C_1} RT \xi_1)^2} - G_{C_1} RT \xi_1}{2 (C_1 - k)^2 Q_\tau}, \quad (3.16)$$

where $\xi_1 = k \log k + (1 - k) \log(1 - k) - C_1 \log C_1 - (1 - C_1) \log(1 - C_1)$ and C_1 is given by (3.15).

4 Effects Of The Absence Of Substitutability In Equilibrium

In what follows we sketch a simple general equilibrium framework in which the qualitative effect of the lack of substitutability can be easily analysed. To keep the structure of the economy as simple as possible, we assume that the production described in this paper is the only industry sector that exists in the economy. This may sound at odds with the basic resource nature of our production, which may itself be an input in other productive sectors of the economy. However, as we only need to highlight the role of the consumer's preferences in the valuation, we avoid any specific connotation of the good in this section, without loss of generality. This will avoid a more complex description of the productive structure of the economy which would obscure appreciation of the effects we will be highlighting.

The limit to the substitutability of useful energy by other factors of production has theoretical implications for the optimal allocation of resources in the economy. General equilibrium theory combines the optimal choice of individual consumers regarding the allocation of their budget, in order to achieve the highest possible utility with the production decision of profit-maximizing firms that produce and supply the consumption good to the economy. By matching of supply and demand a market clearing mechanism determines values (i.e. market prices) according to which resources are allocated. A basic requirement of the general equilibrium analysis is that all values should be comparable, that is expressed in the same unit of account. This is achieved by normalizing prices in terms of one of the goods available in the economy, the numeraire. In a Neoclassical equilibrium, in which every good can be bartered against any other good, the choice of the numeraire is irrelevant as it will not alter the quantities finally produced and consumed. However, a key feature of Neoclassical general equilibrium models is also the perfect substitutability, on the production side, between factors of production. The lack of substitutability of energy implied

by the thermodynamically consistent production function derived in the previous sections affects the equivalence between different factors of production and determines extra rigidity in the firm's decisions. We will provide some insight into the effect that this will have on the equilibrium allocation. We will find that the invariance of the equilibrium with respect to the unit of account may no longer hold. Our discussion will focus on the issue of the metric to be adopted for judging the optimality of resource allocation and production decisions.

An initial discussion of the role of thermodynamic efficiency in general equilibrium decisions can be found in Berry et al. (1978), who model the existence of a minimum energy input in production and identify it (as later done by Ruth (1995)) as that required by a limiting reversible process. In the energy dimension, isoquants of the production function do not approach a zero level within the limit but instead they reach positive constants which depend on the level of production. In their view, a thermodynamic efficiency criterion of resource allocation would select a combination of inputs (a point on the achievable isoquant) as close as possible to the minimum energy asymptote. They acknowledge that the optimal combination of inputs adopted by a firm in general equilibrium will be driven by overall cost minimization and may deviate from thermodynamic efficiency. If other inputs are scarce enough more energy will be used than the minimum thermodynamic limit. The neoclassical Criteria of Pareto Optimality in the allocation of resources (whereby no agent can improve their own welfare without damaging someone else's) would not usually coincide with thermodynamic efficiency in the use of resources. Roma (2006), however, argues that in the presence of negative externalities associated with entropic waste (e.g. global warming) the maximization of thermodynamic efficiency in production may improve global welfare.

We analyse here the problem of a value-maximising producer who operates over two periods under the irreversible technology we have described in the previous sections. The producer is only constrained by scarcity of energy, which does not allow full exploitation of the natural resources available in the time frame provided. The problem is that of allocating the use of available energy, given the prices at which the commodity produced can be sold in the two periods.

Given that the value of the goods produced in the different periods must be expressed in a common numeraire, we take it to be energy rather than the good produced in one of the two periods. That is, we assume that when the producer sells the final product he receives energy in exchange. It is assumed (following Ruth (1995)) that no capital enters the production process and that the raw material (mixture of mineral and rock) is freely available in an unlimited quantity⁷.

When energy is chosen to be the unit of account for the producer, the profitability of the firm will differ from that achievable when the unit of account is a finished good, from which utility can be derived directly. When energy is used as the unit of account, the producer's decision regarding its use in investments will essentially be based on thermodynamic considerations. If these are disregarded, production will be driven entirely by the non-satiated consumer who derives utility from the finished good but not from the energy needed to produce it.

The idea that accounting in terms of energy rather than in standard monetary terms may modify the attractiveness of an investment strategy may seem counter intuitive. However, this concept is firmly established within the literature

⁷Intertemporal production functions that do not explicitly involve the use of capital are common in the financial literature, see for example Cox et al. (1985), Fama (1972) and Hirshleifer (1970).

concerning the concept of EROEI (Energy Return On Energy Investment) of power plants. EROEI is defined as the ratio of gross energy produced by an energy supply process to the total (direct plus indirect) energy cost of its production (Cleveland (1992), Cleveland (2005) and Gately (2007)). Thus the EROEI approach provides the net energy analysis of an energy-based production process and can lead to completely different results from a pure financial assessment. Financial evaluation of investments deals only with market prices for the construction of the power plant plus maintenance costs and consequent financial revenues (Kaldellis et al. (2005), Tian and Wang (2001) and Tsoutsos (2003))⁸.

In our model we recognize that, if energy is taken to be the numeraire and means of exchange, energy itself represents "value", rather than the product. This implies that unused energy balances will enter the producer's value function. These specific features of the model force the producer to account for the profit of his investment in terms of energy. If the additional energy spent on production has a negative return it will not be used but rather retained at its face value as an inventory of the firm. The limit to the substitutability of the energy input will play a central role in this result. The total energy available for production is E_T , which is paid to the firm by the consumer/shareholder in exchange for future dividends before the production process begins. Given that P_e , the price of the good in terms of energy, and B , the real discount factor, are both outside the firm's control in a competitive market, the producer's problem can be written as:

$$\begin{aligned}
 \max_{E_1, E_2} \quad & N_1 P_e + B N_2 P_e - E_1 - E_2, \\
 \text{sub} \quad & N_1 = f^1(E_1), \\
 & N_2 = f^2(E_1, E_2), \\
 & E_1 + E_2 \leq E_T, \\
 & E_1 > 0, \\
 & E_2 > 0,
 \end{aligned} \tag{4.1}$$

where E_1 denotes the energy used to produce N_1 and E_2 the energy used to produce N_2 ⁹. Functions $f^1(E_1)$ and $f^2(E_1, E_2)$ are defined by expressions (3.9) and (3.16) respectively.

In problem (4.1) the producer, who may use up to the amount of energy E_T , may choose not to use part of the available energy for production and retain it as an asset of the firm. The unused energy will be $E_T - E_1 - E_2$ and will add to the value of the firm.

On the consumer's side of the economy we assume, for simplicity, that a representative consumer exists who only

⁸A comparative analysis between financial evaluation of power plants and EROEI can be found in Hall et al. (1979). The authors deal with a problem that arose in the NYSEG (New York State Electric and Gas Corporation) service area in the late 70's, when there was a proposal for a new 870 MW coal-fired generating station. The authors compare the energy (with the EROEI approach) and the dollar cost of building the power plant with the cost of a comprehensive regional program of insulation. Their results show that the insulation program is more efficient in conserving energy than the power plant is in providing it, by a factor of at least 4 in economic terms and a factor of 15 in EROEI terms. This enforces the idea that the decision-making strategy can be quite different if an energy approach is considered.

⁹The energy discount factor would be $Z = B \frac{P_{e1}}{P_{e2}}$, where P_{e1} and P_{e2} are the energy prices of the good at times 1 and 2. Price P_{e2} simplifies in the expression $Z P_{e2} N_2$ and we adopt the notation $P_e \equiv P_{e1}$ for the sake of simplicity.

derives utility from the finished good. In order to define the consumer's budget constraint, we consider two alternative exchange mechanisms. We will initially assume that the amount of scarce energy E_T is in the hands of the consumer, who exchanges it with the firm in order to obtain rights to the consumption good produced. In this case we give energy a special role in the exchange mechanism, by requiring that every good can only be purchased by exchanging it for a pre-specified good, rather than by bartering. This is the essence of a cash in advance constraint (Clower (1967)), which is used to provide a role for fiat money (paper money without any intrinsic value forced onto the economy by a Central Bank) that would otherwise have no economic role in the economy.

Alternatively, we can assume that the same amount of energy E_T is already an endowment of the firm and that the consumer has the right to receive the product of the firm by way of dividends. We will analyze this case below.

However, in a barter economy in which only physical goods with a direct economic use are present the imposition that a specific good should be exchanged against all the others should have no effect. Nevertheless if energy is exchanged for the availability of the good produced then a price of the good in terms of energy will be defined. This will allow the use of energy as a numeraire, and will provide information for its efficient allocation in the economy.

The consumer's problem is:

$$\max_{C_1, C_2} u(C_1, C_2), \quad (4.2)$$

$$\text{sub } P_e (C_1 + B C_2) \leq E_T, \quad (4.3)$$

where C_1 and C_2 denote consumption of the good at the end of the time intervals 1 and 2 respectively. The consumer trades his endowment of energy E_T with the firm before production begins, in exchange for future production. As energy does not enter the utility function in the case in which the endowment of the consumer is energy, there will be no incentive for the consumer to avoid spending some of the available energy in exchange for the good, thus pushing the energy price of the good to the highest level allowed by the budget constraint. This will in turn provide the producer with an incentive to produce more. Hence (4.3) will hold as an equality.

The consumer's first order conditions imply that:

$$B = \frac{\frac{\partial u}{\partial C_2}}{\frac{\partial u}{\partial C_1}}, \quad (4.4)$$

and:

$$P_e = \frac{E_T}{C_1 + B C_2}, \quad (4.5)$$

where in equilibrium $C_1 = N_1$ and $C_2 = N_2$. Hence:

$$P_e = \frac{E_T}{N_1 + B N_2}. \quad (4.6)$$

4.1 Use of energy by the producer.

In the setting described above, where energy is the unit of account, it will not generally be optimal for the producer to use all the available energy, unless specific restrictions are placed on technology and the consumer's preferences. Production will not exceed the level where the marginal energy cost is equal to the price set by the market-clearing condition (4.6). This result derives from the inherent non linearity of the irreversible production function, which displays increasing marginal cost and from the limit to the substitutability of energy by other factors of production. Hence if energy is used as the numeraire, given that there is no alternative technology in the economy under which unused energy balances may efficiently enter production, and therefore the producer will not be able to exchange unused energy with a competitor, equilibrium in the economy will leave some energy resources unused. This would not be the case under perfect substitutability of the two production inputs (e.g. a Cobb-Douglas production function). In such a case the producer would instead use all the available energy, as it would be possible to supplement it with a high enough amount of the other input to achieve an increase in production that justifies the energy spent. Perfect substitutability would be inconsistent with the unlimited availability of the second input, and a price would have to be introduced for the second input which must be assumed to be scarce. In our case we have already substituted in the production frontier (3.9) the optimal amount of the free input, which minimizes the energy cost of production. This may, on the other hand, have positive environmental consequences as less entropy is generated in the economy (Kummel (1989), Roma (2006)).

The general solution of problem (4.1) is obtained by maximizing the Lagrange function:

$$\mathcal{L}(E_1, E_2, \lambda) = N_1(E_1) P_e + B N_2(E_1, E_2) P_e - E_1 - E_2 + \lambda (E_T - E_1 - E_2) \quad (4.7)$$

w.r.t. its arguments. Maximization w.r.t. E_1 and E_2 gives the following pair of equations if the energy bound holds as a strict inequality in an internal solution (namely $E_1 + E_2 < E_T$) and $\lambda = 0$:

$$\begin{aligned} P_e \frac{\partial N_1}{\partial E_1} + B P_e \frac{\partial N_2}{\partial E_1} - 1 &= 0, \\ B P_e \frac{\partial N_2}{\partial E_2} - 1 &= 0. \end{aligned} \quad (4.8)$$

The second condition of (4.8) implies that $\frac{\partial E_2}{\partial N_2} = B P_e$, that is the marginal cost of production in the second period is equal to the discounted price $B P_e$.

In Appendix C.1 we report an analytical solution for the problem (4.1) when depletion is neglected (i.e. $\frac{\partial N_2}{\partial E_1} = 0$) and show that it corresponds to a solution with a non-zero energy slack. It is not possible to find an analytical solution to the producer problem (4.1) when depletion is accounted for (see Appendix C.3). However, in Appendix C.3 we find, numerically, an internal solution (i.e. with a non-vanishing energy slack), for a specification of the model parameters. The chosen depletion parameters ϑ , η and ρ are those of copper (see Nguyen and Yamamoto (2007)). Slack arises because additional use of energy for production does not increase the value of the firm¹⁰.

¹⁰An extension of the production side of the economy to include additional production processes (i.e. other firms producing additional goods)

We now turn to the case in which energy is not the unit of account. In a typical Neoclassical model E_T would be treated as an endowment of the firm which has no value if not used, hence its use would not be a cost and the produced good itself would be the numeraire. If scarce energy did not have to be valued on the market, its full rational use would follow over the time span of the model. Then energy cost does not matter and the producer's problem becomes:

$$\begin{aligned}
 \max_{E_1, E_2} \quad & N_1 + B N_2, \\
 \text{sub} \quad & N_1 = f^1(E_1), \\
 & N_2 = f^2(E_1, E_2), \\
 & E_1 + E_2 \leq E_T, \\
 & E_1 > 0, \\
 & E_2 > 0.
 \end{aligned} \tag{4.9}$$

The objective function is strictly increasing in both N_1 and N_2 . In turn N_1 and N_2 are strictly increasing in E_1 and E_2 , until $E_1 + E_2 = E_T$. So the energy inputs E_1 and E_2 will be increased by the producer until the constraint holds as an equality and $E_2 = E_T - E_1$. This can be seen analytically from equations (4.8). If the maximization problem is (4.9), the first order conditions (4.8) for a solution with an energy slack become:

$$\begin{aligned}
 \frac{\partial N_1}{\partial E_1} + B \frac{\partial N_2}{\partial E_1} &= 0, \\
 B \frac{\partial N_2}{\partial E_2} &= 0.
 \end{aligned} \tag{4.10}$$

Conditions (4.10) are never satisfied because $\forall E_1 > 0$ and $\forall E_2 > 0$ we have that $\frac{\partial N_1}{\partial E_1} > 0$ and $\frac{\partial N_2}{\partial E_2} > 0$. Disregarding the energy cost leads to the full use of energy, E_T .

If all the available energy is used (namely $E_1 + E_2 = E_T$), the problem (4.1), as well as (4.9), reduces to a maximization in one single variable E_1 , that is it can be equivalently rewritten as:

$$\begin{aligned}
 \max_{E_1} \quad & N_1 P_e + B N_2 P_e - E_T, \\
 \text{sub} \quad & N_1 = f^1(E_1), \\
 & N_2 = f^{2'}(E_1), \\
 & E_T \geq E_1 > 0,
 \end{aligned} \tag{4.11}$$

is unlikely to eliminate the potential slack in the use of energy if each additional process is subject to a similar minimum energy threshold. An average minimum energy bound would arise for the production side of the economy, which would prevent substitution of energy by other factors. Similarly, the possibility of including energy as an argument of the consumer's utility function would not rule out slack. If E_c is the portion of the consumer's initial endowment which is consumed directly and therefore not exchanged against the goods produced the resulting energy price $P_e = \frac{E_T - E_c}{N_1 + B N_2}$ will in general be lower, decreasing the incentive to produce even further.

where function $f^{2'}(E_1)$ is defined as $f^{2'}(E_1) \equiv f^2(E_1, E_T - E_1)$. The first order condition is:

$$\frac{\partial N_1}{\partial E_1} + B \frac{\partial N_2}{\partial E_1} = 0. \quad (4.12)$$

That is the equilibrium where all the finite amount of available energy E_T is used for production is independent from the price P_e . The consumer's budget constraint can be rewritten as $C_1 + B C_2 = N_1 + B N_2$. The price P_e plays no role.

Note that if the consumer is forced to exchange energy for the consumption good, no additional constraint is imposed compared to the case in which the budget constraint is $C_1 + B C_2 = N_1 + B N_2$, as the amount of energy available for production is the same. What is modifying the equilibrium is the definition of the energy price P_e , that is the introduction of a market for energy which imposes a thermodynamic efficiency criterion on the firm's decision, whereby energy cost is compared to the energy revenue of the irreversible technology.

It is clear from the previous discussion that the producer's energy efficiency differs between the two cases in which there is an energy slack $E_1 + E_2 < E_T$ or in which all the energy available is spent for production and $E_1 + E_2 = E_T$. Accounting in terms of energy is a technical criterion that leads to different decisions with respect to standard monetary accounting. A key role in this sense is played by the EROEI index, which can be interpreted as the ratio between the energy collected by the producer and the energy spent for production. In the case of our firm, the producer collects an amount of energy equal to $P_e N_1 + B P_e N_2 = E_T$, facing an energy expenditure of $E_1 + E_2$, so that the EROEI for our production is:

$$EROEI = \frac{P_e N_1 + B P_e N_2}{E_1 + E_2} = \frac{E_T}{E_1 + E_2}. \quad (4.13)$$

This will be higher than one if the energy is used efficiently in problem (4.1) and will be equal to one if the energy cost of production is disregarded in the decision.

5 Arbitrage Analysis

The energy price of the good produced will be subjected to arbitrage bounds. Suppose that we observe its market price in terms of energy. We have at our disposal a quantity of moles N_Q of the final good characterized by the highly refined mixture $(k, 1 - k)$. We wonder whether it is possible, in some way, to extract the energy E_Q that has been used to produce the moles N_Q and compare it to the market price. This will be possible under certain conditions. The mining operator has at his disposal an engine which is able to carry out the thermodynamic transformation $(C_0, 1 - C_0) \rightarrow (k, 1 - k)$. We call this transformation the forward process. The operator could plan to extract the energy E_Q spent to produce the moles N_Q by running the engine in a backward process $(k, 1 - k) \rightarrow (C_0, 1 - C_0)$. This is only fully possible, however, if the engine works reversibly, that is within the limit $\tau \rightarrow +\infty$. As discussed above, this thermodynamic limit has no economic sense because production processes must be completed in a finite time and thus the production time τ has a fixed finite value. However, if the transformation (of raw material into

finished good), which occurs in a finite time, is so small that the thermodynamic system remains in equilibrium, it can be assumed that the transformation occurs in a reversible way. Suppose that an infinitesimal quantity dN of the final good can be purchased on the market. It would be possible to perform a reversible transformation in a finite time τ , extracting the infinitesimal quantity of energy dE held in dN . This is possible because the transformation now involves only infinitesimal variation of the thermodynamic variables of the system.

Assume for now that the technology is totally reversible, that is we are within the limit $\tau \rightarrow +\infty$. We suppose that our engine is able to work backward, namely we can make the transformation $(k, 1 - k) \rightarrow (C_0, 1 - C_0)$. We buy N_Q moles of the final good, characterized by a concentration vector $(k, 1 - k)$, and we compute the energy necessary to re-obtain the initial mixture $(C_0, 1 - C_0)$:

$$E_R^{back} = RT N_Q [C_0 \log C_0 + (1 - C_0) \log (1 - C_0) - k \log k - (1 - k) \log (1 - k)]. \quad (5.1)$$

It can be shown, in our setting where $k > 1 - C_0 > C_0$, that the energy variation expressed by formula (5.1) is negative, that is we gain an amount of energy $|E_R^{back}|$. The energy required to obtain N_Q moles of the final good $(k, 1 - k)$ is independent (in the reversible case) from the number of the initial moles of the mixture $(C_0, 1 - C_0)$ and equal to:

$$E_R^{for} = RT N_Q [k \log k + (1 - k) \log (1 - k) - C_0 \log C_0 - (1 - C_0) \log (1 - C_0)] = -E_R^{back} > 0. \quad (5.2)$$

Thus under a fully reversible technology we require, for the forward process, an amount of energy equal, in absolute value, to the energy released by the backward process.

As a consequence, under the reversible technology the energy market price of the good produced must be equal to the marginal cost of production (which is, in this case, a constant independent from the production level). Suppose that $P_e > \frac{\partial E_1}{\partial N_1} = RT \xi_0$; then we can produce and sell N_s moles of the final good with an amount of energy equal to $RT \xi_0 N_s$ and we receive as payment an amount of energy $N_s P_e > RT \xi_0 N_s$ with a net energy gain $N_s (P_e - RT \xi_0) > 0$. If $P_e < \frac{\partial E_1}{\partial N_1} = RT \xi_0$, then we buy N_b moles of the good, paying an amount of energy $P_e N_b$, and then we run our engine backward, extracting an amount of energy $RT \xi_0 N_b$, with a net energy gain of $N_b (RT \xi_0 - P_e) > 0$. It follows that if the technology is reversible $P_e = RT \xi_0$. This is a no-arbitrage condition which is obtained irrespective of the preferences in the economy.

The situation is different if the technology is not reversible. In the irreversible case the energy required to transform N_b moles of concentration $(k, 1 - k)$ into N_b moles of concentration $(C_0, 1 - C_0)$ is:

$$E_{IR}^{back} = E_R^{back} + \frac{2 N_b^2}{\tau \alpha} (k - C_0)^2. \quad (5.3)$$

Since the second term on the right-hand side of expression (5.3) is positive if we want the backward process to release energy, that is $E_{IR}^{back} < 0$, we have to operate our engine for a sufficiently long time τ . More precisely:

$$\tau \geq \frac{2 N_b^2}{R T \xi_0 \alpha} (k - C_0)^2. \quad (5.4)$$

Even if inequality (5.4) holds, part of the energy used to produce N_b cannot be recovered. Moreover the forward process $(C_0, 1 - C_0) \rightarrow (k, 1 - k)$ requires an amount of energy given by:

$$E_{IR}^{for} = E_R^{for} + \frac{2 N_b^2}{\tau \alpha} (k - C_0)^2, \quad (5.5)$$

and then $E_{IR}^{for} > E_{IR}^{back}$ because $E_R^{for} > 0$ and $E_R^{back} < 0$. If the technology is not reversible it is not possible to replicate the argument used in the reversible case to determine P_e for a finite quantity of material and energy. However, the no-arbitrage argument is still partially valid for an infinitesimal variation in production at economic equilibrium. Suppose that we have irreversible technology and we achieve economic equilibrium at some production level N_* (without loss of generality in the first production period). If $P_e > \frac{\partial E_1}{\partial N_1}|_{N_*}$, we can produce and sell an extra infinitesimal quantity of the final good dN , spending an energy $dE = \frac{\partial E_1}{\partial N_1}|_{N_*} dN$, with a net energy gain $dN \left(P_e - \frac{\partial E_1}{\partial N_1}|_{N_*} \right) > 0$ ¹¹. If $P_e < \frac{\partial E_1}{\partial N_1}|_{N_*}$, the producer can buy an extra quantity of refined material dN at a price $dN P_e$ and reversibly extract the energy $dN R T \xi_0$. By inspecting expression (3.11), it can be seen that $\frac{\partial E_1}{\partial N_1}|_{N_*} \geq R T \xi_0, \forall N_*$. Thus the condition $P_e < \frac{\partial E_1}{\partial N_1}|_{N_*}$ is not sufficient to establish that the quantity of energy $dN R T \xi_0 - dN P_e$ delivered by this strategy is positive and therefore there is an arbitrage opportunity. To find an arbitrage opportunity in the irreversible case, the price P_e must be lower than the reversible marginal cost of production. In the range $P_e \in \left[R T \xi_0, \frac{\partial E_1}{\partial N_1}|_{N_*} \right]$ we have no arbitrage restrictions on the price P_e .

The same type of arbitrage arguments are used in Roma (2006) to determine the no-arbitrage price in terms of energy for a more elementary product (hot fluid). This case is quite different from ours and leads to a complete identification of the energy price P_e of the good through no-arbitrage considerations because the product itself is thermal energy (stored in a fluid) and there is no need to operate any thermodynamic engine backward to extract it.

6 The Producer's Problem Under The Reversible Technology

We turn to the analysis of production decisions within the limit of reversible technology. As discussed, this case is unrealistic. We show that the slack (under-utilization) in energy, which characterizes the production decision of a profit-maximising firm that accounts in terms of energy under the irreversible extraction technology, will no longer exist. Moreover, as the price P_e will be set by no-arbitrage considerations in this case, the unconstrained solution will not necessarily be consistent with the investor's preferences. To see this, recall the form that the production function takes under full reversibility. From (3.2) we have a linear technology in this case:

$$N_1 = \frac{E_1}{R T \xi_0}. \quad (6.1)$$

Taking into account the depletion of the mine, in the second period we have:

¹¹Note that this argument is valid without the introduction of the concept of reversibility.

$$N_2 = \frac{E_2}{RT \xi_1}, \quad (6.2)$$

where ξ_1 is defined in section (3.4). Under this technology the firm's profit maximisation problem, with the inequality constraint $E_1 + E_2 < E_T$, will deliver inconsistent results. The first order conditions (4.8) are:

$$\begin{aligned} P_e \frac{1}{RT \xi_0} - B P_e \frac{E_2}{RT \xi_1^2} \frac{C_1}{\rho N_1} \log \left(\frac{C_1}{1 - C_1} \right) \frac{1}{RT \xi_0} - 1 &= 0, \\ B P_e \frac{1}{RT \xi_1} - 1 &= 0, \end{aligned} \quad (6.3)$$

while the consumer's problem (4.2) remains unchanged (the consumer completely ignores whether the process is carried out reversibly or irreversibly).

However if the process is totally reversible and the producer is allowed not to use all the energy E_T , the arbitrage argument of section (5) implies that $P_e = \frac{\partial E_1}{\partial N_1} = RT \xi_0$.

The second equation of (6.3) implies that $B P_e = RT \xi_1$. Substituting these relations in the first equation of (6.3) we obtain:

$$\frac{E_2}{\xi_1} \frac{C_1}{\rho N_1} \log \left(\frac{C_1}{1 - C_1} \right) \frac{1}{RT \xi_0} = 0. \quad (6.4)$$

So, either the quality of the raw material improves from $C_0 < \frac{1}{2}$ to $C_1 = \frac{1}{2}$, which is inconsistent with our depletion assumption, or it must be $E_2 = 0$, which implies that, no matter what the consumer prefers, no production takes place in the second period and energy is only used for production in the first period. Whatever the use of the energy in the first period, the maximised value of the firm will be identically equal to zero and independent of the consumer's preferences:

$$\max_{E_1} N_1 RT \xi_0 - E_1 \equiv 0. \quad (6.5)$$

On the other hand, given the market clearing condition $P_e = \frac{E_T}{N_1 + B N_2} = \frac{E_T}{N_1}$ ($N_2 = 0$ because $E_2 = 0$), we have:

$$RT \xi_0 = \frac{E_T}{N_1} \rightarrow E_1 = N_1 RT \xi_0 = E_T. \quad (6.6)$$

This means that, even if the producer is allowed not to use all the available energy, he will instead use it all in the first period, whatever the consumer's preferences.

If there is no depletion $\xi_1 = \xi_0$ and an equilibrium will exist only if $B \equiv 1$. Again, the maximised value of the firm less the energy cost will be identically equal to zero for whatever choice of E_1 and E_2 and the resource allocation between the two periods will not depend on the consumer's preferences. The producer's profit will not deviate from zero whatever the production choice.

If the process is reversible, the energy constraint is binding ($E_1 + E_2 = E_T$) and the depletion is taken into account, the equilibrium condition of problem (4.11), which applies if all energy is used, now becomes:

$$\frac{1}{\xi_0} + \frac{B Z(E_1) \left\{ E_1 \rho Z(E_1) [\phi_k + \log(1 - V(E_1))] - e^{\frac{\eta}{\rho}} (E_1 - E_T + E_1 \rho) \log\left(\frac{1-V(E_1)}{V(E_1)}\right) \right\}}{E_1 \rho \left\{ Z(E_1) [\phi_k + \log(1 - V(E_1))] + e^{\frac{\eta}{\rho}} \log\left(\frac{V(E_1)}{1-V(E_1)}\right) \right\}^2} = 0, \quad (6.7)$$

where, for the sake of simplicity, we have introduced the notations:

$$Z(E_1) \equiv \left(\frac{E_1 \theta}{RT \xi_0} \right)^{\frac{1}{\rho}}, \quad (6.8)$$

$$V(E_1) \equiv \frac{e^{\frac{\eta}{\rho}}}{Z(E_1)}, \quad (6.9)$$

$$\phi_k \equiv (k - 1) \log(1 - k) - k \log k. \quad (6.10)$$

Equation (6.7) is compatible with an internal solution for N_1 and N_2 determined by the consumer's preferences.

7 Conclusions.

In this paper we have addressed the issue of substitutability between inputs in a production function and, in particular, the existence of a lower bound on the amount of useful energy necessary for production to take place. The hypothesis of perfect substitutability has convenient implications for general equilibrium modelling, but does not plausibly represent many actual production processes.

Our methodological approach to providing a clear cut answer to this much debated question is to resort to finite time thermodynamics and model the input requirements for a given production based on physical principles. In this light we revisit the production problem of a mining operator who refines a mineral from its natural concentration C_0 to a strike concentration k , defining the consumption good. The mining operator faces an initial mixture of useful mineral and waste rock. The production process is modelled analytically, following the thermodynamic theory of an irreversible separation process. The model provides a clear answer to the question of substitutability between natural resources and energy for the specific production considered. It confirms the existence of an energy bound below which production is impossible. This prevents full substitutability between factors of production and contradicts the standard Cobb-Douglas representation of production. If we assume, following the view in Ayres and Miller (1980), that the original mixture from which the commodity has to be extracted is a freely available natural resource, while energy is in aggregate scarce, the inputs for every level of production will be combined with fixed coefficients which depend only on the scale of production and the time available for the production process. The key role of energy as a scarce non-substitutable resource that underlies thermodynamic optimization in the engineering literature can hence

be reconciled with economic optimization in a microeconomic model. The derived production function is non-linear in the inputs as long as the process occurs in a finite time, and displays increasing marginal cost in terms of energy.

This technological description is used within the optimisation problem of a profit-maximising mining operator who produces over two periods and sells the product in a competitive market. As generally highlighted by Berry et al. (1978) Pareto optimality will not necessarily coincide with the most efficient use of energy in a general equilibrium framework. In our economy the only choice available to the producer concerns the use of energy, which provides a framework for the analysis of the relationship between economic and thermodynamic optima. In a neoclassical framework, the optimal choice would imply the full use of energy with an inter-temporal allocation that maximizes the consumer's utility. However, if we impose energy as the means of payment and the unit of account in the economy the equilibrium will be twisted towards thermodynamic efficiency. The lack of substitutability between energy and other inputs implies that the price, in terms of energy, of the finished product established in the market may not be high enough to induce the producer to employ all the energy available in production. Since no other factors of production can be increased in order to improve the marginal productivity of energy, the under-utilization of available energy will follow. In this way the producer will take the irreversibility in the use of energy into account. The choice of numeraire and means of payment will not be irrelevant, in the sense that the creation of a market where goods are exchanged against energy will determine an energy price and provide guidance to the producer on how to use it efficiently. When energy is the unit of account it will represent "value" and will not be wasted. If the producer's objective function is consistent with this criterion, energetically inefficient investments will be ruled out. However, the under-utilization of the energy resources available resulting from energy accounting will not necessarily lead to Pareto suboptimality if the negative externalities of energy degradation during production are taken into account. As highlighted by Kummel (1989) and Roma (2006), the efficient use of energy may have positive environmental effects. On the other hand, if no value is attributed to the energy resources per se (but only the finished product is valued) full utilization of the available energy will follow. This result may be interpreted in the light of the literature on EROEI (Energy Return on Energy Investment).

In the specific case, our assumptions lead to a one-to-one relationship between energy employed and production of the basic commodity considered. Whether this may be considered a reasonable approximation for what will be obtained in the case of more complex production processes, or possibly an abstraction that may apply to the description of aggregate production in a macroeconomic sense, should be the subject of further research. We note here that a vast body of literature devoted to the successful estimation of aggregate production, using electric energy as the key explanatory variable, assumes the existence of such a one-to-one relationship (Bodo and Signorini (1987), Bodo et al. (1991)).

Another matter for further research is the empirical application of thermodynamics based production functions. Only when the production process becomes fully reversible will the producer's analysis in energy or financial (produced good) terms be the same. However, reversibility does not describe any real world production process.

A Reversibility

The reversible increment in the internal energy of the system which is required for production can be computed taking the limit for $\tau \rightarrow +\infty$ of equation (3.1) for the case $n = 2$:

$$E_m^R = RT N_R [k \log k + (1 - k) \log (1 - k) - C_0 \log C_0 - (1 - C_0) \log (1 - C_0)], \quad (\text{A.1})$$

where N_R are the number of moles produced reversibly. Equation (A.1) says that the transformation $[C_0, 1 - C_0] \rightarrow [k, 1 - k]$ requires a positive amount of energy for the pairs (C_0, k) such that:

$$\xi_0 = k \log k + (1 - k) \log (1 - k) - C_0 \log C_0 - (1 - C_0) \log (1 - C_0) \geq 0. \quad (\text{A.2})$$

We suppose that our mineral is initially available at a minority concentration, namely $C_0 < \frac{1}{2}$. We try to purify the substance to obtain a strike concentration k such that $1 - C_0 < k$, which is greater than the initial concentration of the waste rock. In this framework $\xi_0 > 0$ and the number of moles produced when $\tau \rightarrow +\infty$ is:

$$N_R = \frac{E}{RT \xi_0}. \quad (\text{A.3})$$

This approach is slightly different from that used in Ruth (1995), who supposes that the mass flux entering the process is a binary mixture of mineral (with a concentration C_0) and waste rock (with a concentration $1 - C_0$), which are completely separated in the final state. The minimum energy required to separate a binary mixture is computed in Tsirlin and Titova (2004) as a particular case of equation (3.1). They demonstrate that this limit is:

$$E_m^{bin} = -RT N_0 [C_0 \ln C_0 + (1 - C_0) \ln (1 - C_0)] + \frac{N_0^2}{\tau} \left(\frac{C_0^2}{\bar{\alpha}_{11}} + \frac{(1 - C_0)^2}{\bar{\alpha}_{22}} \right), \quad (\text{A.4})$$

where $\bar{\alpha}_{11}$ and $\bar{\alpha}_{22}$ are the parameters characterising the working medium. Within the limit of $\tau \rightarrow +\infty$, equation (A.4) turns into the thermodynamic limit E_t^* used by Ruth (1995) in the construction of the production function (2.1), with the modification that the natural concentration C_0 is allowed to vary in time as the extraction is carried out and the ore grade decreases. The limiting production function resulting from this characterisation does not describe realistic production, it also results in the marginal productivity of energy being constant. Actually for reactions such that $\frac{N_0}{\tau \alpha} \gg RT$ (fast reactions) the second order term in equations (3.1) and (A.4), which represents the irreversible work, dominates the first order term and thus we can neglect the reversible energy. On the other hand, no thermodynamic limit constrains the amount of raw material N_0 .

B The Physical Model: A More Exhaustive Overview

The Separation Process

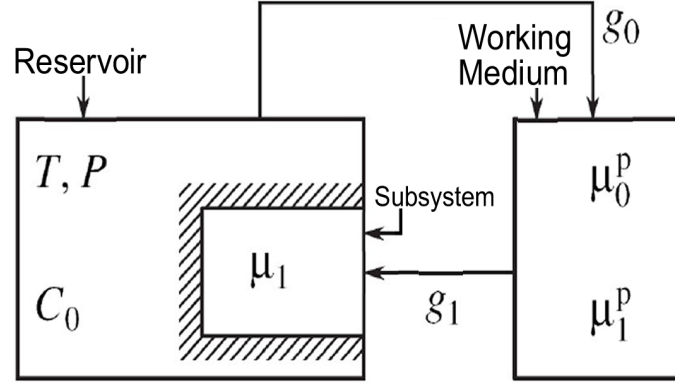


Figure B.1: The scheme used by Tsirlin and Titova (2004) to model the separation of a finite subsystem from an infinite reservoir. The finite subsystem is characterised by a chemical potential μ_1 . The thermodynamic state of the reservoir is identified by the temperature T , the pressure P and the composition vector C_0 . The pumps g_0 and g_1 are the mass-transfer coefficients from the reservoir to the working medium and from the working medium to the reservoir respectively. The working medium has a chemical potential μ_0^P at the contact point with the reservoir and a chemical potential μ_1^P at the contact point with the subsystem.

Figure B.1 illustrates the scheme used to model the process. A subsystem, to be separated out and with a chemical potential μ_1 , is in contact with an infinite reservoir characterised by a temperature T , a pressure P and a chemical potential $\mu_0(T, P)$ which does not vary during the transformation. At time $t = 0$ the composition vector of the reservoir, which describes the (percentage) concentrations of the different components, is (C_{01}, \dots, C_{0n}) . The subsystem is initially at equilibrium with the reservoir and is therefore characterised by the same thermodynamic variables T , P and by the same composition vector (C_{01}, \dots, C_{0n}) as the reservoir. We indicate the number of moles of the subsystem at $t = 0$ with N_0 . Each component of the mixture has a chemical potential $\mu_i(T, P)$ which depends on its concentration C_i and is given by:

$$\mu_i(T, P) = \mu_0(T, P) + RT \log C_i, \quad (\text{B.1})$$

where R is the universal gas constant. The subsystem and the reservoir are in contact with a third medium, a working medium, which has the chemical potentials μ_0^P and μ_1^P at the contact points with the reservoir and the subsystem respectively. The transformation is driven by these differences in chemical potential, which allow for the mass transfer. These differences are generally produced using thermal energy, that is the working medium is taken to be at a higher

temperature when in contact with one source and a lower temperature when in contact with the other one, but the chemical drop could also be driven by electric potentials or by other techniques.

The working medium is characterised by the mass transfer coefficient g_{0i} between the reservoir and the working medium, and g_{1i} between the working medium and the subsystem. In the following it is assumed that there is a linear relation between chemical potential drops $\Delta\mu_i$ and the corresponding mass transfer coefficients g_{0i} , in formula $\Delta\mu_i = \alpha_i g_{0i}$. The proportional constants α_i can be interpreted as the reciprocal of the mass transfer coefficients.

At the final instant $t = \tau$ the subsystem has a new specified composition vector $(C_{\tau 1}, \dots, C_{\tau n})$ and contains a number of moles N_τ . Tsirlin and Titova (2004) demonstrate that the minimum energy required to carry out the transformation $[N_0, (C_{01}, \dots, C_{0n})] \rightarrow [N_\tau, (C_{\tau 1}, \dots, C_{\tau n})]$ in a finite time τ (see also Tsirlin et al. (2002)) is:

$$E_{min} = RT N_\tau \sum_{i=1}^n [C_{\tau i} \log C_{\tau i} - C_{0i} \log C_{0i}] + \frac{1}{\tau} \sum_{i=1}^n \frac{\Delta^2(N C_i)}{\bar{\alpha}_i}, \quad (\text{B.2})$$

where $\Delta(N C_i) = N_\tau C_{\tau i} - N_0 C_{0i}$ and $\bar{\alpha}_i$ is the equivalent mass-transfer coefficient for the i th component.

C Analytical And Numerical Solutions

C.1 Analytical case: no-depletion and positive energy slack

We found analytical solutions for the producer-consumer's problem (4.1)-(4.2) when depletion is neglected (i.e. $\frac{\partial N_2}{\partial E_1} = 0$) and when $u(C_1, C_2) = C_1^\beta C_2^{1-\beta}$, with $\beta \in (0, 1)$. In this situation $B = \gamma \frac{N_1}{N_2}$ where $\gamma = \frac{1-\beta}{\beta}$ and the solutions are:

$$E_1^* = \frac{1}{8(C_0 - k)^2 Q_\tau (1 + \gamma)} \left\{ 4 E_T k^2 Q_\tau + 2 C_0^2 (2 E_T Q_\tau - (1 + \gamma) (RT \xi_0)^2) - \right. \\ \left. - 2 C_0 (4 E_T k Q_\tau - (1 + \gamma) (RT \xi_0)^2) - \right. \\ \left. - RT \xi_0 \left(RT (1 + \gamma) \xi_0 - \sqrt{G_{C_0}} \sqrt{1 + \gamma} \sqrt{8 E_T (C_0 - k)^2 Q_\tau + G_{C_0} (1 + \gamma) (RT \xi_0)^2} \right) \right\}, \quad (\text{C.1})$$

$$E_2^* = \frac{1}{8(C_0 - k)^2 Q_\tau (1 + \gamma)} \left\{ 4 E_T k^2 Q_\tau \gamma + 2 C_0^2 (2 E_T Q_\tau \gamma - (1 + \gamma) (RT \xi_0)^2) - \right. \\ \left. - 2 C_0 (4 E_T k Q_\tau \gamma - (1 + \gamma) (RT \xi_0)^2) - \right. \\ \left. - RT \xi_0 \left(RT (1 + \gamma) \xi_0 - \sqrt{G_{C_0}} \sqrt{1 + \gamma} \sqrt{8 E_T (C_0 - k)^2 Q_\tau \gamma + G_{C_0} (1 + \gamma) (RT \xi_0)^2} \right) \right\}. \quad (\text{C.2})$$

It is possible to verify that $E_1 + E_2 < E_T$, that is (C.1)-(C.2) actually correspond to a solution with an energy slack regardless of the value of E_T . In order to see this, consider the fraction of the total energy spent in the production:

$$\begin{aligned} \frac{E_1^* + E_2^*}{E_T} = & \frac{1}{8 E_T (C_0 - k)^2 Q_\tau (1 + \gamma)} \left\{ 2 (1 + \gamma) (2 E_T (C_0 - k)^2 Q_\tau - G_{C_0} (RT \xi_0)^2) + \right. \\ & + \sqrt{G_{C_0} (1 + \gamma)} RT \xi_0 \left[\sqrt{8 E_T (C_0 - k)^2 Q_\tau + G_{C_0} (1 + \gamma) (RT \xi_0)^2} + \right. \\ & \left. \left. + \sqrt{8 E_T (C_0 - k)^2 Q_\tau \gamma + G_{C_0} (1 + \gamma) (RT \xi_0)^2} \right] \right\}. \end{aligned} \quad (C.3)$$

We will show that:

$$\frac{E_1^* + E_2^*}{E_T} < 1, \quad (C.4)$$

whatever is E_T . Straightforward computations show that inequality (C.4) holds if and only if:

$$\begin{aligned} \Psi(E_T) \equiv & -2 (1 + \gamma) (2 E_T (C_0 - k)^2 Q_\tau + G_{C_0} (RT \xi_0)^2) + \\ & + \sqrt{G_{C_0} (1 + \gamma)} RT \xi_0 \left[\sqrt{8 E_T (C_0 - k)^2 Q_\tau + G_{C_0} (1 + \gamma) (RT \xi_0)^2} + \right. \\ & \left. + \sqrt{8 E_T (C_0 - k)^2 Q_\tau \gamma + G_{C_0} (1 + \gamma) (RT \xi_0)^2} \right] < 0. \end{aligned} \quad (C.5)$$

It is easy to see that $\Psi(E_T) \rightarrow -\infty$ when $E_T \rightarrow +\infty$. Moreover, the equation $\Psi(E_T) = 0$ has only complex solutions, given by:

$$E_T^\pm = \frac{-G_{C_0} (C_0 - k)^2 Q_\tau (1 + \gamma)^2 (RT \xi_0)^2 \pm \sqrt{-[G_{C_0}^2 (C_0 - k)^4 Q_\tau^2 (\gamma^2 - 1)^2 (RT \xi_0)^4]}}{2 (C_0 - k)^4 Q_\tau^2 (1 + \gamma)^2}, \quad (C.6)$$

i.e. the function $\Psi(E_T)$ does not intercept the real axis and then $\Psi(E_T) < 0, \forall E_T$. This shows inequality (C.4).

It is interesting to analyze the producer's choice within the limit of infinite energy available (i.e. $E_T \rightarrow +\infty$). From expressions (C.1)-(C.2) it is easy to see that:

$$\lim_{E_T \rightarrow +\infty} \frac{E_1^*}{E_T} = \frac{1}{2 (1 + \gamma)}, \quad (C.7)$$

$$\lim_{E_T \rightarrow +\infty} \frac{E_2^*}{E_T} = \frac{\gamma}{2 (1 + \gamma)}. \quad (C.8)$$

This means that the fractions of the energy spent in the two production periods depend only on the consumer's preference γ . Moreover:

$$\lim_{E_T \rightarrow +\infty} \frac{E_1^* + E_2^*}{E_T} = \frac{1}{2}. \quad (C.9)$$

That is, only half of the available energy is used for production.

C.2 Analytical case: no-depletion and zero energy slack

Analytical solutions can be also found for the slackless case (4.11) if depletion is neglected. Straightforward computations show that, in this case, the producer's choice is:

$$E_1^{no-slack} = \frac{E_T}{(1+B^2)} - \frac{(B^2-1) G_{C_0} (RT \xi_0)^2}{4(1+B^2)(C_0-k)^2 Q_\tau}, \quad (C.10)$$

$$E_2^{no-slack} = \frac{B^2 E_T}{(1+B^2)} + \frac{(B^2-1) G_{C_0} (RT \xi_0)^2}{4(1+B^2)(C_0-k)^2 Q_\tau}. \quad (C.11)$$

Obviously, $E_1^{no-slack} + E_2^{no-slack} = E_T$. Similarly to the case of Appendix (C.1), if the producer is provided with an infinite amount of energy (i.e. $E_T \rightarrow +\infty$), the fractions of energy spent in first and second period depend only on the consumer's preferences (contained in parameter B). In formula:

$$\lim_{E_T \rightarrow +\infty} \frac{E_1^{no-slack}}{E_T} = \frac{1}{1+B^2} \quad (C.12)$$

$$\lim_{E_T \rightarrow +\infty} \frac{E_2^{no-slack}}{E_T} = \frac{B^2}{1+B^2} \quad (C.13)$$

C.3 Numerical solutions in the case of depletion

We now investigate the solutions to problem (4.1) when depletion is taken into account. As in Appendix (C.1), we model the consumer's utility function choosing $u(C_1, C_2) = C_1^\beta C_2^{1-\beta}$, with $\beta \in (0, 1)$, so that $B = \gamma \frac{N_1}{N_2}$ where $\gamma = \frac{1-\beta}{\beta}$.

It is more opportune, in the case under analysis, to rephrase problem (4.1) in terms of N_1 and N_2 for the sake of simplicity. We then use the following equivalent formulation:

$$\begin{aligned} \max_{E_1, E_2} \quad & N_1 P_e + B N_2 P_e - E_1 - E_2, \\ \text{sub} \quad & E_1 = f^{1*}(N_1), \\ & E_2 = f^{2*}(N_1, N_2), \\ & E_1 + E_2 \leq E_T, \\ & N_1 > 0, \\ & N_2 > 0. \end{aligned} \quad (C.14)$$

From discussion of sections 3, 3.3 and 3.4, the functions f^{1*} and f^{2*} are explicitly given by:

$$E_1 = f^{1*}(N_1) = \frac{(C_0 - k)^2 Q_\tau N_1^2 + G_{C_0} RT \xi_0 N_1}{G_{C_0}}, \quad (C.15)$$

$$E_2 = f^{2*}(N_1, N_2) = \frac{(C_1 - k)^2 Q_\tau N_2^2 + G_{C_1} R T \xi_1 N_2}{G_{C_1}}. \quad (\text{C.16})$$

If we are looking for an internal solution ($E_1 + E_2 < E_T$) then the first order conditions of problem (C.14) are:

$$P_e - \frac{\partial E_1}{\partial N_1} - \frac{\partial E_2}{\partial N_1} = 0, \quad (\text{C.17})$$

$$B P_e - \frac{\partial E_2}{\partial N_2} = 0. \quad (\text{C.18})$$

Using expressions (C.15)-(C.16) and the explicit dependence of C_1 , G_{C_1} and ξ_1 on N_1 (see sections 3, 3.3 and 3.4), the system of equations (C.17)-(C.18) becomes:

$$\begin{aligned} & \frac{E_T}{1+\gamma} - N_1 \left[\frac{2(C_0 - k)^2 N_1 Q_\tau}{1 + 2(C_0 - 1) C_0} + RT \xi_0 \right] + \frac{2e^{\frac{3\eta}{\rho}} N_2^2 Q_\tau \left[e^{\frac{\eta}{\rho}} (4k - 2) + (1 - 2k^2) (N_1 \vartheta)^{\frac{1}{\rho}} \right]}{\left[2e^{\frac{2\eta}{\rho}} - 2e^{\frac{\eta}{\rho}} (N_1 \vartheta)^{\frac{1}{\rho}} + (N_1 \vartheta)^{\frac{2}{\rho}} \right]^2 \rho} + \\ & + \frac{2e^{\frac{\eta}{\rho}} N_2^2 Q_\tau \left[e^{\frac{\eta}{\rho}} (1 - 2k) + (k - 1) k (N_1 \vartheta)^{\frac{1}{\rho}} \right]}{\left[2e^{\frac{2\eta}{\rho}} - 2e^{\frac{\eta}{\rho}} (N_1 \vartheta)^{\frac{1}{\rho}} + (N_1 \vartheta)^{\frac{2}{\rho}} \right]^2 \rho} - \frac{e^{\frac{\eta}{\rho}} N_2 RT \left[\log \left(\frac{e^{\frac{\eta}{\rho}}}{(N_1 \vartheta)^{\frac{1}{\rho}}} \right) - \log \left(1 - \frac{e^{\frac{\eta}{\rho}}}{(N_1 \vartheta)^{\frac{1}{\rho}}} \right) \right]}{(N_1 \vartheta)^{\frac{1}{\rho}} \rho} = 0, \quad (C.19) \end{aligned}$$

$$\begin{aligned} & \frac{E_T \gamma}{N_2 (1 + \gamma)} - \frac{2 N_2 Q_\tau \left[e^{\frac{\eta}{\rho}} - k (N_1 \vartheta)^{\frac{1}{\rho}} \right]^2}{2e^{\frac{2\eta}{\rho}} - 2e^{\frac{\eta}{\rho}} (N_1 \vartheta)^{\frac{1}{\rho}} + (N_1 \vartheta)^{\frac{2}{\rho}}} + (k - 1) RT \log (1 - k) - k RT \log k + \\ & + \frac{RT \left\{ e^{\frac{\eta}{\rho}} \log \left(\frac{e^{\frac{\eta}{\rho}}}{(N_1 \vartheta)^{\frac{1}{\rho}}} \right) + \left[(N_1 \vartheta)^{\frac{1}{\rho}} - e^{\frac{\eta}{\rho}} \right] \log \left(1 - \frac{e^{\frac{\eta}{\rho}}}{(N_1 \vartheta)^{\frac{1}{\rho}}} \right) \right\}}{(N_1 \vartheta)^{\frac{1}{\rho}}} = 0. \quad (C.20) \end{aligned}$$

There are no analytical (closed form) solutions to the system (C.19)-(C.20). From equation (C.20) it is possible to derive N_2 as a function of N_1 and then equation (C.19) only has the variable N_1 . It can be easily solved via a numerical method for a particular specification of the model parameters.

With this substitution we obtain an equation of the form $F(N_1) = 0$. We choose to solve this equation by computing the function $F(N_1)$ on a lattice of the type $N_1^i = i \Delta N_1$, with $i = 0, 1, 2, \dots$.

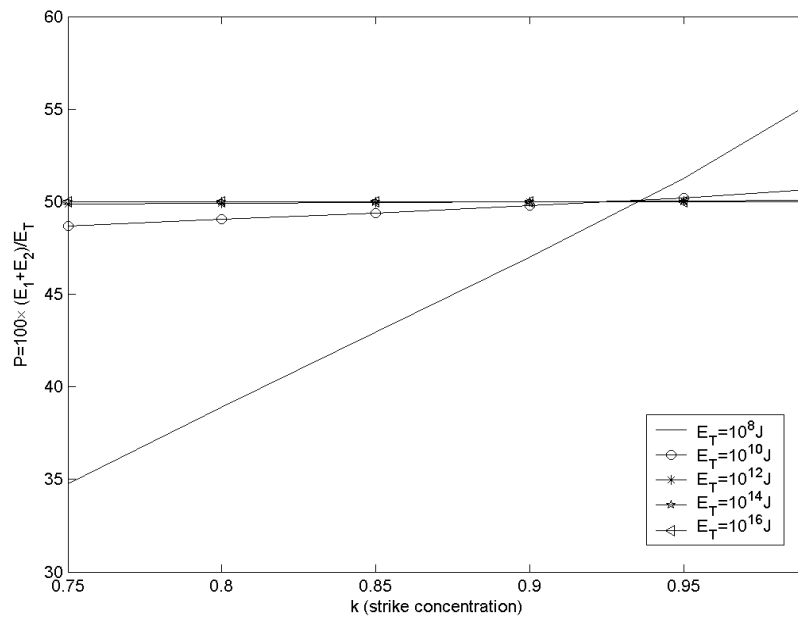
We take as an approximation for the solution the point N_1^{i*} such that $|F(N_1^{i*})| = \min_i |F(N_1^i)|$. The less ΔN_1 , the more accurate the estimation of the solution. We choose a value of ΔN_1 such that $\frac{\Delta N_1}{N_1^{i*}} \approx 10^{-3}$.

We solve system (C.19)-(C.20) with the parameter set in Table 1 and for different values of k and E_T . More precisely, we choose $k = [0.75, 0.80, 0.85, 0.90, 0.95, 0.99]$ for the strike concentration and $E_T = [10^8, 10^{10}, 10^{12}, 10^{14}, 10^{16}] J$ for the total energy available. We focus on the case of copper, for which $\eta = 7.84$ and $\rho = 1.80$ (see Nguyen and Yamamoto (2007)).

Figure C.1 shows the numerical estimate for the percentage $P \equiv 100 \times \frac{E_1 + E_2}{E_T}$ of energy spent in the whole production process as a function of the strike concentration and for different values of E_T . The producer is inclined to spend half of his available energy, provided that he has a sufficient energy supply ($E_T \geq 10^{14}$), no matter what the strike concentration k is. If provided with less energy, the higher the strike concentration, the higher is the fraction of the total energy used by the producer to carry out the whole production process.

Parameter	Value
T	1250 K
C_0	30%
τ	3600 sec
α	$10^{-4} \text{ mol}^2 / (\text{J sec})$
η	7.84
ρ	1.80
β	0.3

Table 1: A set of chosen parameters.

Figure C.1: Numerical estimate of the total percentage of energy spent for production as a function of the strike concentration and for different values of the total energy E_T . The set of chosen parameters is reported in Table 1.

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