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On the Economic Interpretation of Imprimitive Leontiev-vonNeumann-Sraffa Matrices: Cyclical Input-Output Relationships

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Abstract - Starting from the known results of Perron (1907) and Frobenius (1912) I apply graph theory to give an economically intuitive characterization of imprimitivity. Such property implies cyclical vertical relationships among groups of industries which, either directly or indirectly, use each others' products as inputs. More precisely, if the index of imprimitivity is *h*, then industries may be sorted in h groups such that i) each group produces the inputs of one and only one other group and ii) there is no direct flow of commodities between industries of the same group. A sufficient condition for primitivity is provided which offers some reasons to expect non-basic industries to have more vertical cyclical flows than basic ones.

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1. Introduction

In the Leontiev-vonNeumann-Sraffa (LNS) framework, indecomposable matrices constitute a particular class of non-negative square matrices which have a straightforward economic meaning. An indecomposable matrix represents an economy, or part of it, where each industry produces a commodity which is used as an input by any other industry either directly or indirectly. Primitive matrices are a sub-class of indecomposable matrices. A primitive matrix has a single eigenvalue of maximum modulus in its spectrum while an imprimitive indecomposable matrix has more than one.

To the best of my knowledge, the economic meaning of primitivity/imprimitivity has not yet been investigated. In this note I provide an interpretation of such properties in terms of the structure of flows of commodities among industries. More precisely, imprimitive matrices represent economies where the flow of commodities takes the form of a cycle between groups of industries. In this respect, the fundamental parameter turns out to be the number of eigenvalues of maximum modulus which determines both the number of groups and the period of cycle.[1] On the contrary, primitive matrices represent economies where the flow of commodities does not show any such *cyclicity*.

In Section 2 I introduce the definitions of indecomposability and primitivity, providing an intuitive economic characterization of both by means of graph theory. In Section 3, starting from the results of Perron (1907) and Frobenius (1912), I study the case of indecomposable imprimitive matrices from an economic perspective. Then, under the assumption that at least a group of basic industries exists (Sraffa (1960)), I also characterize the more general case of a decomposable matrix with some indecomposable imprimitive sub-matrices. Again, graph theory is applied to provide both intuition and easily understandable examples. Finally, I state and prove a sufficient condition for primitivity whose economic meaning seems to support the idea that the cyclicity implied by imprimitivity is more likely to occur in non-basic industries than in basic ones.

2. Primitive Matrices

1. Indecomposability

Let $A \equiv \{a_{ij}\}_{i,j=1}^{n}$ be a square matrix of order n. I use the convention that element a_{ij} represents the amount of commodity i required to produce one unit of commodity j. Hence, row indices may be interpreted as commodity indices and column indices as industry indices. Furthermore, I denote by $\underline{a}_i$ the i-th row and by $\underline{a}^j$ the j-th column. Joint production and technical change are not considered.

Definition 1 (Decomposable/Indecomposable Matrices)
A non-negative square matrix $A \equiv \{a_{ij}\}_{i,j=1}^{n}$ is decomposable if the indices $1, 2, ..., n$ can be divided into two disjoint non-empty sets $I \equiv \{i_1, i_2, ..., i_l\}$ and $J \equiv \{j_1, j_2, ..., j_m\}$ with $l + m = n$, such that $a_{i_\alpha j_\beta} = 0$ for any $i_\alpha \in I$ and $j_\beta \in J$. Otherwise the matrix A is indecomposable.

[1] It is shown that all such eigenvalues are single root of the characteristic polynomial and therefore there is no ambiguity in not making references to eigenvalues' multiplicities.

Graph representation offers a more intuitive way to characterize matrix indecomposability. Let $g(N, E)$ be a directed graph, where N is the set of nodes and E the set of directed edges. The shorthand ij indicates the directed edge going from node i to node j. By a *path* of length $m \geq 1$ connecting i to j it is meant a finite sequence of m directed edges $\{e\}_{h=1}^{m} \equiv \{(r_h s_h)\}_{h=1}^{m}$ where $e_h \in E$, and $r_h \in N$ and $s_h \in N$ indicate, respectively, the root (or starting point) and the sink (or ending point) of each directed edge.[2] When there is no need to specify the edges composing a path, a path of length m connecting node i to node j is denoted by $p^m(i, j)$. Moreover, a graph is *connected* if and only if there exists $i \in N$ such that, for every $j \neq i$ there exists $p^m(i, j)$ for some m. Finally, a graph is *strongly connected* if and only if for any pair $i, j \in N$, $i \neq j$, there exists $p^m(i, j)$ for some m.

Suppose matrix A represents an economy composed of n industries. We define the graph induced by A as $g_A(N, E)$ where $N \equiv \{1, 2, ..., n\}$ and $ij \in E$ whenever $a_{ij} > 0$. The following result is well known.

Lemma 1
Let A be a non-negative square matrix. Then, A is decomposable if and only if g_A is not strongly connected.

Proof. Suppose A is of order n. If A is decomposable then there exist two disjoint sets of indices $I \equiv \{i_1, i_2, ..., i_u\}$ and $J \equiv \{j_1, j_2, ..., j_v\}$ with $u + v = n$, such that $a_{i_\alpha j_\beta} = 0$ for any $i_\alpha \in I$ and $j_\beta \in J$. It is straightforward to see that $p^m(i_\alpha, j_\beta)$ does not exist for any $i_\alpha \in I$ and $j_\beta \in J$ and $m \geq 1$.

Conversely, suppose that there are no paths of any length going from node i to node j, $i \neq j$. Define $I \equiv \{i\} \cup \{k \in N : p^m(i, k), m \geq 1\}$ and $J \equiv N \setminus I$. Notice that J contains at least j. By construction, if $r \in I$ then, for any $m \geq 1$ and $s \in J$, $p^m(r, s)$ does not exists, otherwise there would exist $p^{m+l}(i, s)$ for some $l \geq 1$ and s would belong to I; in particular, $p^1(r, s)$ does not exist. Therefore, we have that $a_{rs} = 0$ whenever $r \in I$ and $s \in J$.■

Figure 1 shows an example of two economies – one decomposable, the other indecomposable – and their associated graphs.

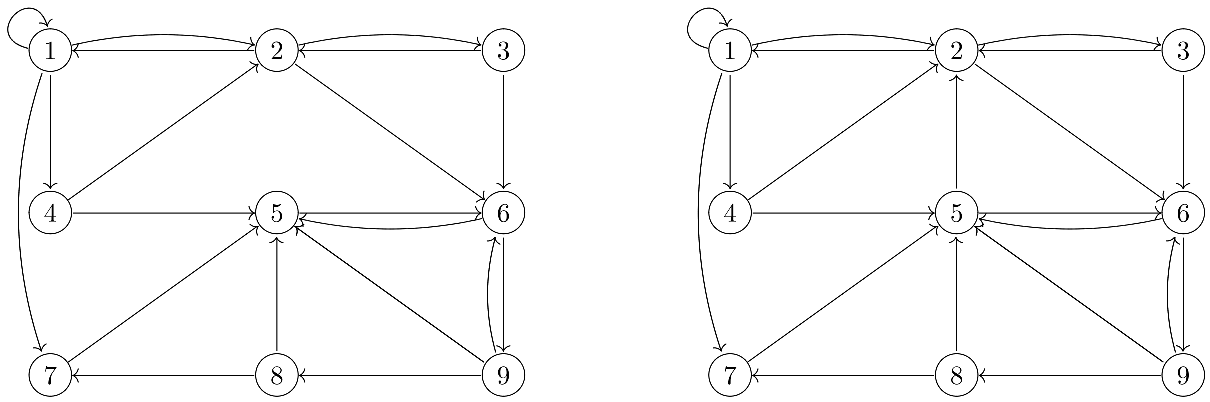
2. Primitivity

Primitive matrices are a special case of indecomposable matrices. They are characterized by the spectrum being strictly dominated by a single eigenvalue of multiplicity one.

Definition 2 (Primitive/Imprimitive Matrices)
Let A be an indecomposable matrix and S the set of its eigenvalues where each eigenvalue appears a number of times equal to its multiplicity as a root of the characteristic polynomial of A. Define $\lambda^* \equiv \max_{\lambda \in S} |\lambda|$ and $h \equiv ||\{\lambda \in S : |\lambda| = \lambda^*\}||$. If $h = 1$ then A is primitive; otherwise, A is imprimitive and h is its index of imprimitivity.

From Definition 2, however, the economic meaning of primitivity may not be evident. The following proposition provides a better insight and allows a further graph-based characterization.[3]

[2] A path can have edges repeated as long as it is consistent with the definition (for instance $\{ij, ji, ij\}$ or $\{ij, jj, jj, jk\}$).
[3] A standard proof of Proposition 1 can be found, among others, in Gantmacher (1959).



Economy A Economy B

Figure 1. Economies A and B are composed by nine industries and show qualitatively similar flows of commodities. They differ only for industry 2 using or not using the output of industry 5. However, Economy A is decomposable while Economy B is indecomposable. Indeed, in the graph representing Economy A there is no directed chain connecting industries 5, 6, 7, 8 and 9 to any of the industries 1, 2, 3 and 4. On the contrary, in the graph representing Economy B there is a directed chain connecting any two industries.

Proposition 1
An indecomposable matrix A is primitive if and only if there exists a positive integer k such that $A^k > 0$.

Corollary 1
An indecomposable matrix A is primitive if and only if there exists an integer $gk > 0$ such that, in the graph g_A, there exists $p^k(i,j)$ for every $i \neq j$.

Proof. Let $\{a_{ij}\}^k$ denote the entry of matrix A^k belonging to the i-th row and j-th column. Corollary follows from Proposition 1 once it is proved that $\{a_{ij}\}^k > 0$ if and only if there exists a path $p^k(i,j)$ in g_A. I show this by induction. The condition is trivially satisfied for $k = 1$. Suppose it holds for $k > 1$. Since $\{a_{ij}\}^{k+1} = \{\underline{a}_i\}^k \underline{a}^j$ we have that $\{a_{ij}\}^{k+1} > 0$ if and only if there exists some $h \in N$ such that $\{a_{ih}\}^k a_{hj} > 0$. Then, by the inductive hypothesis, $\{a_{ij}\}^{k+1} > 0$ if and only if there exists $p^k(i,h)$ and $p^1(h,j)$ in g_A for some $h \in N$, which in turn implies the existence of $p^{k+1}(i,j)$ in g_A.■

As defined above, the j-th column of matrix A, $\underline{a}^j$, gives the input required to produce 1 unit of commodity j. Thus, a vector of commodities q requires a total quantity of inputs equal to $Aq = \sum_{j=1}^n \underline{a}^j q_j$ to be produced. In particular, $A\underline{a}^j$ gives the inputs required to produce the inputs that, in a subsequent round of production, will be required to produce 1 unit of commodity j. Similarly, $A^2 = (A\underline{a}^1, \ldots, A\underline{a}^n)$ gives, industry by industry, the quantities of commodities required to produce the inputs which, in a subsequent production period, will be necessary to produce 1 unit of each commodity. Let me refer to a single round of production activity as *production lag*. Then, A^k represents the quantities of inputs that are required

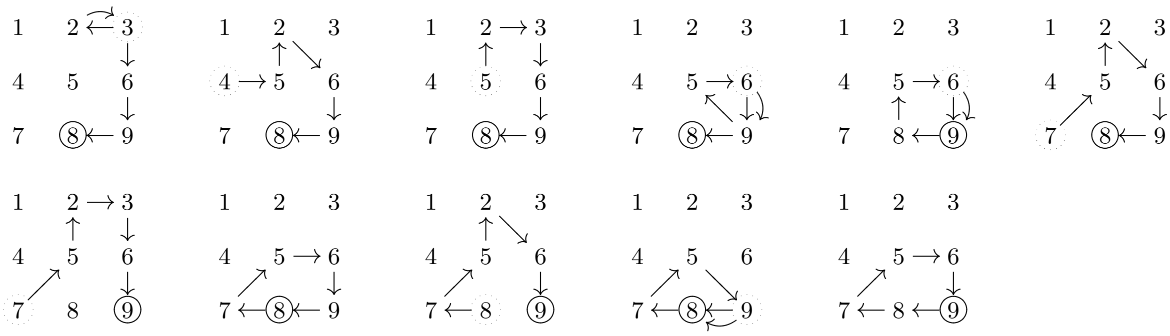
Example: Primitivity of Economy B by graph representation

In order to show that Economy B of Figure 1 is primitive, it is proved that there exists a path of length 5 between any two nodes in $N = \{1, \ldots, 9\}$. Since among the edges of g_B we find 11 – i.e. industry 1 uses its output as an input – we can conclude that, if there exists a path of length not greater than 5 connecting i to j and passing through industry 1, then there exists a path of length 5 connecting i to j. The table below shows the length of the shortest path connecting any industry to 1 and 1 to any industry.

Length of shortest paths from and to industry 1 in g_B

Industry label	1	2	3	4	5	6	7	8	9
Length of paths to 1	1	1	2	2	2	3	3	3	3
Length of paths from 1	1	1	2	1	2	2	1	4	3

By combining the elements of the last two rows we obtain the length of the shortest path connecting any two industries and passing through industry 1. All paths but the following have shortest path not longer than 5: 3 to 8, 4 to 8, 5 to 8, 6 to 8, 6 to 9, 7 to 8, 7 to 9, 8 to 8, 8 to 9, 9 to 8 and 9 to 9. The existence of a path of length 5 between the latter is shown one by one in the the following eleven graphs. Paths go from industries with a dotted round to industries with full rounds (when the two do not coincide).



to produce 1 unit of each commodity in exactly k production lags – each column referring to the inputs required by the corresponding industry.

Therefore, by Proposition 1, we have a straightforward economic interpretation of imprimitivity: there must be some number of production lags k such that the output of each industry shows up indirectly as input of any other industry. Moreover, since indecomposability implies that $A^k A^l > 0$ for any integer $l > 0$, we easily conclude that if this holds for k production lags then it also holds for any number of production lags greater than k. Corollary 1 characterizes such property in terms of the graph $g_{A^{kg}}$. Indeed, primitivity of A is equivalent to the existence in g_A of n^2 paths of length k connecting any two (not necessarily distinct) nodes. This is illustrated in detail by an example referring to Economy B of Figure 1.

3. An Economic Interpretation of Imprimitivity

1. Imprimitive Indecomposable Matrices

Thanks to the seminal work of Perron (1907) and the subsequent refinements by Frobenius (1912), several spectral properties of indecomposable matrices have been established. If A is an indecomposable matrix and S and λ^* are defined as in Definition 2, we know that there exists a positive real eigenvalue $\tilde{\lambda} \in S$ which is equal to λ^*. Moreover, we know that $\tilde{\lambda}$ is a single root of the characteristic polynomial of A and that it is strictly greater than any other positive real element of S. As anticipated by Definition 2, however, A may have negative or complex eigenvalues with a modulus equal to λ^*. The following results by Frobenius (1912) illustrate the relationship between the structure of a matrix and the number of elements in S of modulus equal to λ^*.[4]

Proposition 2
Let A be an indecomposable matrix and S the set of its eigenvalues where each eigenvalue appears a number of times equal to its multiplicity as a root of the characteristic polynomial of A. Define $\lambda^* \equiv \max_{\lambda \in S} |\lambda|$ and $h \equiv ||\{\lambda \in S : |\lambda| = \lambda^*\}||$. Then,

i) $\lambda_1, ..., \lambda_h \in S$ *are distinct solutions of the equation* $\lambda^h - \lambda^* = 0$,

ii) if $h > 1$ then A is imprimitive and there exists a permutation of both rows' and columns' indices such that A is reduced to the following cyclic form with square blocks along the diagonal

$$A = \begin{bmatrix} 0 & A_{12} & 0 & \dots & 0 \\ 0 & 0 & A_{23} & \dots & 0 \\ 0 & 0 & 0 & \dots & A_{h-1,h} \\ A_{h,1} & 0 & 0 & \dots & 0 \end{bmatrix}$$

Proposition 2 tells us that whenever an indecomposable matrix is imprimitive it shows a cyclical structure whose period is equal to the index of imprimitivity. In particular, we have the following

Corollary 2
Let A be an indecomposable imprimitive matrix with index of imprimitivity h. Then, there exists a permutation of both rows' and columns' indices such that A^h has the following form with square blocks along the diagonal

$$A^h = \begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ 0 & 0 & \dots & A_h \end{bmatrix}$$

Proof. Corollary follows by computing A^h in the form of point ii) of Proposition 2. ■

Corollary 2 implies that no power of an imprimitive indecomposable matrix is strictly positive which is consistent with Proposition 1.

[4] Again, Proposition 2 can be found in Gantmacher (1959). Proof is omitted.

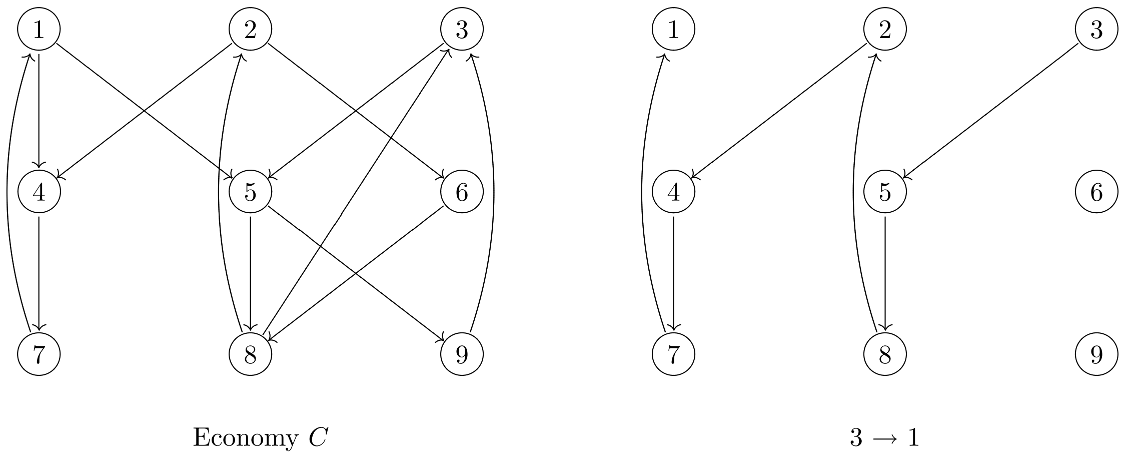


Figure 2. Economy C is imprimitive with index of imprimitivity 3. The sets $N_1 = \{1, 2, 3\}$, $N_2 = \{4, 5, 6\}$ and $N_3 = \{7, 8, 9\}$ identify the three groups of industries among which there is a cyclical flow of indirect inputs. The cycle's period is 3. Notice that, however, the shortest path between 3 and 1 has a length of 6.

2. Cyclical Production Lags in an Indecomposable Economy

Again, graph representation offers a better intuition than matrix representation.

COROLLARY 3
Suppose A is an imprimitive indecomposable matrix of order n and index of imprimitivity $h > 1$. Then, there exist a partition $\Pi \equiv \{N_1, ..., N_h\}$ of the set N such that $g_A(N, E)$ satisfies

i) $\forall i, j \in N_k$, $ij \notin E$, $k = 1, ..., h$

ii) $\forall N_k \in \Pi$, $\exists N_l \in \Pi : \big(\forall i \in N_k, \exists j \in N_l : ij \in E\big) \wedge \big(\forall i \in N_k, j \notin N_l \Rightarrow ij \notin E\big)$

Proof. Corollary follows by checking the properties of g_A where A is in the form of point ii) of Proposition 2. ■

An imprimitive indecomposable economy with index of imprimitivity h can be divided in h groups of industries $N_1, ..., N_h$ such that each group's outputs constitute all direct inputs of one and only one other group. In particular, there exists a cyclic linkage among the h groups of industries whose period is h. Without loss of generality, I adopt the convention that N_k directly produces the inputs of N_{k+1} and $h + 1 \equiv 1$. Then, from a groups' point of view the output of each N_k is indirectly used as input by N_{k+r} every $r > 1$ production lags, with $1 \leq r \leq h$. Instead, from an industry's point of view the output of $i \in N_k$ is indirectly used as an input by $j \in N_{k+r}$ every $r + hz > 1$ productions lags, where $z \geq 1$ is an integer. Indeed, the shortest path between $i \in N_k$ and $j \in N_{k+r}$ may not be r because, although i must be linked by a path of length h to some $u \in N_{k+r}$, the latter may differ from j. So, it could take one or more additional full rounds of length h to get to j (see Figure 2).

Define the equivalence relation $\sim$ such that $i \sim j$ if and only if $i, j \in N_k$. The equivalence classes induced by $\sim$ represent the h groups of industries while the quotient graph $g_{A|\sim}$ exemplifies the structure of vertical relationships among them. In the case A is imprimitive but indecomposable $g_{A|\sim}$ takes the form of a cycle (Figure 3). If A is primitive then $g_{A|\sim}$ is a single point with no edges. Indeed, if A is imprimitive there is a clear vertical structure of production since the flow of indirect inputs among industries has a regular period, namely a multiple of h. On the contrary, if A is primitive such a sharp distinction cannot be made because the flow of indirect inputs goes from each industry to any other for any production lags greater than a certain finite number.

3. Cyclical Production Lags in a Decomposable Economy

Decomposable matrices are neither primitive nor imprimitive. However, decomposable matrices contain indecomposable sub-matrices which may or may not be primitive. Therefore, by looking at the index of imprimitivity of these sub-matrices we can gather useful information about inter-industry relationships. To this aim, let me introduce the concept of the normal form of a matrix.

Definition 3 (Normal Form of a Matrix)
Let A be a square matrix of order n. Its normal form is

$$\hat{A} = \begin{bmatrix} A_1 & 0 & \dots & \dots & 0 & 0 & \dots & 0 \\ 0 & A_2 & \dots & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & A_g & 0 & 0 & \dots & 0 \\ A_{g+1,1} & A_{g+1,2} & \dots & A_{g+1,g} & A_{g+1} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ A_{s,1} & A_{s,2} & \dots & A_{s,g} & \dots & A_{s,g+1} & \dots & A_s \end{bmatrix}$$

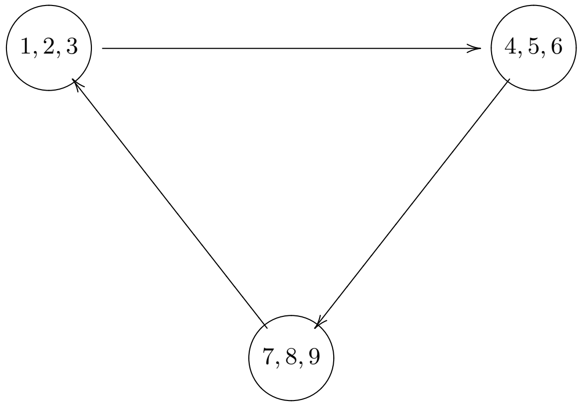
where sub-matrices $A_1, ...A_s$ are indecomposable and in each row $f = g+1, ..., s$ at least one of the matrices $A_{f,1}, ..., A_{f,f-1}$ is different from zero.

The normal form of a matrix is unique up to permutations of the blocks of indices.[5] Clearly, if A is indecomposable then $s = 1$ while if A is completely reducible then $g = s$.

Following Sraffa (1960) I restrict the analysis to the case where there exists a group of *basic* industries, namely a group of industries whose products are either direct or indirect means of production of any industry in the economy. Let A represent such an economy. In terms of the graph g_A, we have that there exists a set $B \subseteq N$ such that, for any $i \in B$ and $j \in N$, there exists $p^k(i, j)$ for some $k > 0$. In terms of the normal form $\hat{A}$, we have that $g = 1$.

For the sake of notation, assume A is a Sraffa matrix already in normal form. Let $h_1, ..., h_s$ be the indices of imprimitivity of the indecomposable blocks along the diagonal of A, where $h_i = 1$ means A_i is primitive. Each A_i identifies a set $N_i \subseteq N$ of industries. For each N_i, consider the partition $\Pi_i \equiv \{N_{i,1}, ..., N_{i,h_i}\}$ of the type described in Corollary 3. Since $\cup_{i=1}^n N_i = N$, the partitions $\Pi_1 ..., \Pi_s$

[5]More precisely, the blocks $1, ..., g$ can be always be permuted without modifying the normal form while permutations of the blocks $g+1, ..., s$ are allowed only in certain cases. See Gantmacher (1959) for a detailed proof.



Economy C: Flow of Commodities among Groups of Industries

Figure 3. Since Economy C of Figure **??** is imprimitive with index of imprimitivity 3, the relation $\sim$ induces three equivalence classes.

induce the partition $\Pi \equiv \{N_{1,1}, ..., N_{1,h_1}, ..., N_{1,1}, ..., N_{1,h_s}\}$ on N. Given this, the equivalence relation $\sim$ is naturally reinterpreted in such a way that $i \sim j$ if and only if $i, j \in N_{k,l}$.

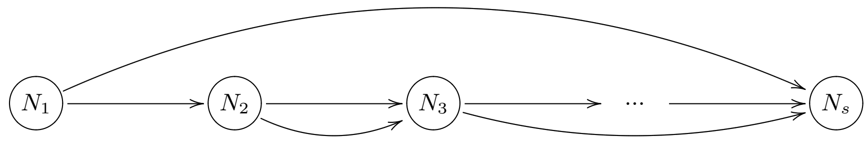
Again, the quotient graph $g_{A|\sim}$ illustrates the qualitative structure of vertical relationships among groups of industries. Since there are s indecomposable sub-matrices we can have up to s cycles which may be connected among themselves in various ways. In particular, for each block N_k, industries belonging to the same group $N_{k,l}$, besides producing for some industries in $N_{k,l+1}$ and using products of some industries in $N_{k-1,l}$, may also directly use the products of any industry belonging to N_q, where $q < k$, or produce the direct inputs for any industry in N_r, where $r > k$. Therefore, $g_{A|\sim}$ can take quite different shapes. On one extreme, if $h_1 = ... = h_s = 1$, then $g_{A|\sim}$ is composed by s nodes and one or more paths of the type $\{(r_j, s_j)\}_{j=1}^{m}$ where $m \leq s$ and $r_j < s_j$ (see Figure 3). This is the case of no cyclicity as indirect inputs flow from basic industries to non-basic ones with no regular period. On the other extreme, if $h_i > 1$ for all $i = 1, ..., s$, then $g_{A|\sim}$ is composed by s cycles which have, respectively, $h_1, ..., h_s$ nodes. Such cycles are encompassed by paths of the type $\{(r_j, s_j)\}_{j=1}^{m}$ where $r_j \geq s_j$ (see Figure 4). This is the case of maximal cyclicity as indirect inputs flow according to regular periods both within and between the s blocks of industries. Any other case is just a combination of the two extremes described.

4. A Sufficient Condition for Primitivity

The following proposition provides a sufficient condition for the primitivity of indecomposable matrices.

Corollary 4

Let A be an indecomposable square matrix of order n. If $a_{ii} > 0$ for some $i \in N$ then A is primitive.



Flow of Commodities among Industries: $A_1, ..., A_s$ Primitive

Figure 4. The sets $N_1, ..., N_s$ identify a partition of N according to the indecomposable matrices along the diagonal of the normal form. All flows goes to the right.

Proof. Although this corollary is a trivial consequence of Proposition 2, I shall prove it with reference to the graph $g_A(N, E)$ in order to make clear why the presence of positive elements along the diagonal gets rid of cyclicity.

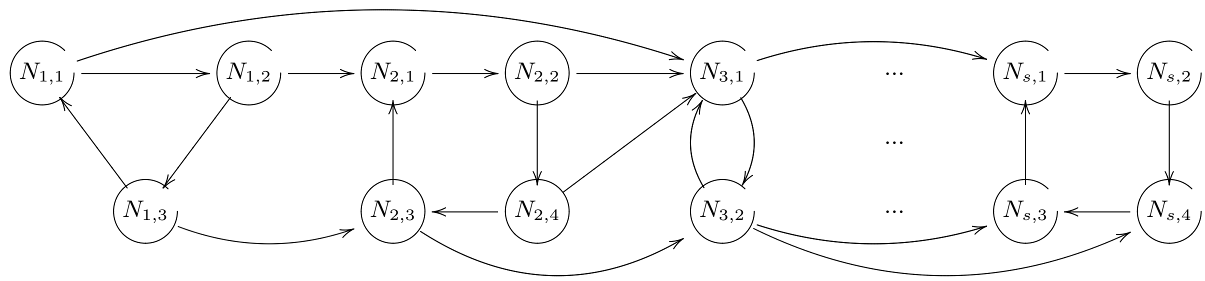
Suppose $a_{ii} > 0$. Then, for every $r > 0$, in g_A we have the path $p^r(i, i)$. Since A is indecomposable, by Lemma 1 we have that $g_A(N, E)$ is strongly connected. In particular, for each $j, l \in N$ in g_A there exist $p^{k_j}(j, i)$ and $p^{k_l}(i, l)$ for some $k_j > 0$ and $k_l > 0$. Consider the n^2 paths connecting any $j \in N$ to any $l \in N$ which are obtained by joining $p^{k_j}(j, i)$, $p^r(i, i)$ and $p^{k_l}(i, l)$, that is $p^{k_j+r+k_l}(j, l)$. Since r can be any positive integer and k_j, k_l are finite for any $j, l \in N$, we have that there exists an integer $\tilde{k} > 0$ such that, for any $j, l \in N$, the graph g_A shows the path $p^{\tilde{k}}(j, l)$. Then, the result follows by applying Corollary 1.■

So, the presence of a single industry which uses its own output as an input suffices to make acyclical the vertical relationships of an entire indecomposable set of industries. Indeed, as suggested by the proof of Corollary 4, if a part of i's product at time t is used as a direct input to produce i's output at time $t + 1$ then, from a certain period $t + k$ onwards, i's product at time t is used as an indirect input in the production of all commodities.

It may be argued that it is very likely that some basic industries use their own output as an input and, hence, cyclicity rarely happens among basic industries. Indeed, output reuse seems particularly likely for agriculture, low-tech heavy industry and constructions, among others. However, since much depends on the exact definition of industries and commodities we cannot exclude that cyclicity happens for basic industries. Moreover, there is no particular reason to expect that the indecomposable diagonal blocks representing non-basic industries are primitive. Therefore, given the characteristics of actual production processes, Corollary 4 might suggest, at most, that non-basic industries are more susceptible to have a cyclical flow of commodities among themselves than basic ones.

4. Conclusions

In this note I have provided an economic interpretation of imprimitive indecomposable matrices in the Leontiev-vonNeumann-Sraffa framework. Imprimitivity implies cyclical vertical relationships among groups of industries which, either directly or indirectly, use each others' products as inputs. If the index of imprimitivity is h, then industries may be divided in h groups such that each group produces the inputs of one and only one other group and there is no direct flow of commodities between industries of



Flow of Commodities among Industries: $A_1, ..., A_s$ Imprimitive

Figure 5. The sets $N_{1,1}, ..., N_{1,3}, N_{2,1}, ..., N_{2,4}, ..., N_{s,1}, ..., N_{s,4}$ identify a partition of N. There are s groups of industries, one for each block along the diagonal of the normal form. Each group k is constituted by h_k sub-groups of industries encompassed by a cycle, where h_k is the index of imprimitivity of the k-th block. Flows which are not part of inner cycles go from a sub-group to another belonging to a group with a greater index.

the same group.

Starting from the known results of Perron (1907) and Frobenius (1912) I have applied graph theory in order to give an economically intuitive characterization of both primitivity and imprimitivity. As a byproduct I have showed that, when an economy has several cyclic flows of commodities among groups of industries, the analysis of imprimitivity helps to identify them in an easy way. Finally, I have stated a sufficient condition for primitivity which offers some reasons to expect non-basic industries to have more vertical cyclical flows than basic ones.

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