QUADERNI

Università degli Studi di Siena
DIPARTIMENTO DI ECONOMIA POLITICA

MARCO P. TUCCI
DAVID A. KENDRICK
HANS M. AMMAN

Expected optimal feedback with Time-Varying Parameters

n. 497 - Febbraio 2007

Abstract - In this paper we derive, by using dynamic programming, the closed loop form of the Expected Optimal Feedback rule with time varying parameter. As such this paper extends the work of Kendrick (1981, 2002, Chapter 6) for the time varying parameter case. Furthermore, we show that the Beck and Wieland (2002) model can be cast into this framework and can be treated as a special case of this solution.

JEL classification: C63, E61.

Marco P. Tucci (tucci@unisi.it), Dipartimento di Economia Politica, Università di Siena
David A. Kendrick (kendrick@eco.utexas.edu), Department of Economics, University of Texas
Hans M. Amman (amman@uu.nl), Utrecht School of Economics, Utrecht University

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1 Introduction

The introduction of stochastic parameters in a control theory framework frequently leads to the use of approximations of the dynamic programming algorithm. For this reason researchers in the control field are often induced to discuss in great details the approximation of their choice and treat implicitly the alternative methods. For instance, Kendrick (1981, 2002) and Tucci (1989, 1997, 2004) discuss at length the *DUAL* algorithm, but they fail to spell out the effects of the introduction of system equations with time-varying parameters on the computation of the familiar expected optimal feedback control. This paper aims to fill the gap. It is therefore an extension of Kendrick (1981, 2002, Chapter 6) which provides a similar derivation for models with constant parameters.[1]

In the first section of this paper the problem is stated. Then the approximate optimal cost for periods N, $N-1$ and a generic period j are derived. The approximation is based on the information available at the beginning of the planning horizon, that is time 0. It is worthwhile to point out that the formulae associated with the time-varying parameters problem look exactly the same as those in Kendrick (1981, 2002, Chapter 6), except for the fact that now the expectation on the random quantities is conditional on the information available at time 0, thus E_0. Section 5 shows that this minor notational difference has substantial computational consequences. Finally the Beck and Wieland (2002) model is cast in the framework of this paper. It is observed that for the parameter set used in Amman et al. (2007) and Beck and Wieland (2002) the approximated optimal control is indeed the optimal control, because the feedback matrices are independent of the future path of the time-varying parameters.[2]

2 Statement of the Problem

A general *quadratic linear control model* can be stated as follows: select the control vectors $\mathbf{u}_0, \ldots, \mathbf{u}_{N-1}$ to minimize the criterion functional

$$J = E\left\{C_N\right\} = E\left\{L_N\left(\mathbf{x}_N\right) + \sum_{k=0}^{N-1} L_k\left(\mathbf{x}_k, \mathbf{u}_k\right)\right\} \tag{1}$$

with E the expectation operator and with,

$$L_N(\mathbf{x}_N) = \frac{1}{2}\mathbf{x}_N' \mathbf{W}_N \mathbf{x}_N + \mathbf{w}_N' \mathbf{x}_N \tag{2}$$

and

$$L_k\left(\mathbf{x}_k, \mathbf{u}_k\right) = \frac{1}{2}\mathbf{x}_k' \mathbf{W}_k \mathbf{x}_k + \mathbf{w}_k' \mathbf{x}_k + \mathbf{x}_k' \mathbf{F}_k \mathbf{u}_k + \frac{1}{2}\mathbf{u}_k' \boldsymbol{\Lambda}_k \mathbf{u}_k + \boldsymbol{\lambda}_k' \mathbf{u}_k \tag{3}$$

[1]In a discussion paper Amman and Kendrick (2001) consider the case where a subset of parameters is stochastic and follows a first order Markov process with a time-varying transition matrix, $\mathbf{D}$, and covariance matrix. They suggest finding the *EOF* control using an augmented state vector including both the states and the stochastic parameters. This paper suggests an alternative approach to solve the same problem.

[2]This result is the same as in Beck and Wieland (2002, page 1365).

subject to the system equations

$$\mathbf{x}_{k+1} = \mathbf{A}_k\mathbf{x}_k + \mathbf{B}_k\mathbf{u}_k + \mathbf{c}_k + \mathbf{v}_k \quad k = 0,\ 1,\ ..., N-1 \tag{4}$$

where $\mathbf{x}_k$ is the n-dimensional vector of states, $\mathbf{u}_k$ the m-dimensional vector of controls, the $\mathbf{W}_k$, $\mathbf{\Lambda}_k$, $\mathbf{w}_k$ and $\boldsymbol{\lambda}_k$ are penalty weights, $\mathbf{A}_k$, $\mathbf{B}_k$ and $\mathbf{c}_k$ are arrays of parameters of appropriate dimension and $\mathbf{v}_k$ an additive noise term.[3] The expectation in (1) is taken over $\mathbf{v}_k$, $\mathbf{A}_k$, $\mathbf{B}_k$ and $\mathbf{c}_k$. It is assumed that these parameters follow a first-order Markov process of the form

$$\theta_{k+1} = \mathbf{D}\theta_k + \eta_k \tag{5}$$

where $\mathbf{D}$ is a known matrix, η_k is a random vector and[4]

$$\theta_k = \begin{bmatrix} vec(\mathbf{A}_k) \\ vec(\mathbf{B}_k) \\ vec(\mathbf{c}_k) \end{bmatrix} \tag{6}$$

is of dimension $(s\ \times\ 1)$, with $s = n \times n + n \times m + n$.[5]

The noise vectors $\mathbf{v}_k$ and η_k are assumed independently distributed with[6]

$$\begin{aligned} \mathbf{v}_k &\sim N(\mathbf{0},\ \mathbf{Q}) \\ \eta_k &\sim N(\mathbf{0},\ \mathbf{G}) \end{aligned} \tag{7}$$

[3] As discussed in Kendrick (1981, 2002, Chapter 2), the $\mathbf{W}_k$ and $\mathbf{\Lambda}_k$ may be interpreted as penalty matrices on the deviations of the states and controls, respectively, from their desired paths and the $\mathbf{w}_k$ and $\boldsymbol{\lambda}_k$ as some known functions of the desired paths of the state and controls, respectively. The $\mathbf{w}_k$ and $\boldsymbol{\lambda}_k$ are zero when the desired paths of the state and controls, respectively, are 0. In the engineering literature it is usually assumed that the $\mathbf{W}_k$ are positive semidefinite symmetric matrices and the $\mathbf{\Lambda}_k$ are positive definite symmetric matrices. See Bertsekas (2005, Chapter 4).

[4] It should be noticed that in Kendrick (1981, 2002) only the unknown parameters, either time-varying or constant, are included in θ_k. To go from the θ_k as defined in the paper to that used in Kendrick, say θ_k^K, it suffices to pre-multiply θ_k by the matrix $\mathbf{T}$ of dimension $r \times s$ where r is the number of unknown parameters and s is as in the text. Each row in $\mathbf{T}$ has 1 in the position associated with a certain unknown parameter and zero elsewhere. As an example consider a situation where $s = 5$ but only the second and fourth parameter of vector θ_k are unknown. Then the matrix $\mathbf{T}$ is $2\times\ 5$ with 1's in position (1,2) and (2,4) and 0 elsewhere.

[5] This formulation is general enough to model both time-varying and constant parameters. When a certain parameter, say the i-th parameter in θ, is assumed constant the corresponding row in $\mathbf{D}$ has 1 in the i-th column and zero elsewhere and the corresponding element in η_k is zero.

[6] When some of the parameters in θ_k are assumed known the corresponding elements in η_k are zero and the associated variances and covariances are zero. Therefore, in general, the matrix $\mathbf{G}$ is symmetric and positive semidefinite in (7).

Furthermore, they are independent of the initial condition $\mathbf{x_0}$, assumed given, and[7]

$$\theta_0 \sim N(\theta_{0|0},\ \Sigma_{0|0}^{\theta\theta}). \tag{8}$$

The $s \times s$ covariance matrix looks like

$$\Sigma_{0|0}^{\theta\theta} = \begin{bmatrix} \Sigma_{0|0}^{\mathrm{AA}} & \Sigma_{0|0}^{\mathrm{AB}} & \Sigma_{0|0}^{\mathrm{A}\mathbf{c}} \\ \bullet & \Sigma_{0|0}^{\mathrm{BB}} & \Sigma_{0|0}^{\mathrm{B}\mathbf{c}} \\ \bullet & \bullet & \Sigma_{0|0}^{\mathbf{cc}} \end{bmatrix}$$

with[8]

$$\begin{aligned}
\Sigma_{0|0}^{\mathrm{AA}} &= \text{the } \left(n^2 \times n^2\right) \text{ covariance matrix of the parameters in } \mathbf{A}_0; \\
\Sigma_{0|0}^{\mathrm{AB}} &= \text{the } \left(n^2 \times nm\right) \text{ matrix of covariances between the parameters in } \\
& \quad\ \mathbf{A}_0 \text{ and } \mathbf{B}_0; \\
\Sigma_{0|0}^{\mathrm{A}\mathbf{c}} &= \text{the } \left(n^2 \times n\right) \text{ matrix of covariances between the parameters in } \\
& \quad\ \mathbf{A}_0 \text{ and } \mathbf{c}_0; \\
\Sigma_{0|0}^{\mathrm{BB}} &= \text{the } (nm \times nm) \text{ covariance matrix of the parameters in } \mathbf{B}_0; \\
\Sigma_{0|0}^{\mathbf{Bc}} &= \text{the } (nm \times n) \text{ matrix of covariances between the parameters in } \\
& \quad\ \mathbf{B}_0 \text{ and } \mathbf{c}_0; \\
\Sigma_{0|0}^{\mathbf{cc}} &= \text{the } (n \times n) \text{ covariance matrix of the parameters in } \mathbf{c}_0
\end{aligned}$$

[7] When a certain parameter is constant and known the relative row and column in $\Sigma_{0|0}^{\theta\theta}$ have zeroes. When it is constant but unknown the same row and column contain the covariances of its estimate at time zero included in $\theta_{0|0}$. Some authors, for instance Harvey (1981, pages 104-106), prefer to use the notation $(\theta_\mathbf{0} - \theta_\mathbf{0|0}) \sim \mathbf{N}(\mathbf{0}, \mathbf{\Sigma}_{\mathbf{0|0}}^{\theta\theta})$, in place of (8), to indicate the distribution of a vector containing both fixed and random parameters. In the presence of measurement error $\mathbf{x}_0$ is usually assumed normally distributed with mean $\mathbf{x}_{0|0}$ and covariance $\Sigma_{0|0}^{\mathbf{xx}}$. See, e.g., Kendrick (1981, 2002, Chapter 10).

[8] Equation (8) can be put in Kendrick's (1981, 2002) notation and using the $\mathbf{T}$ matrix defined in Footnote (3), that is $\theta_k^K \equiv \mathbf{T}\theta_k \sim N\left(\mathbf{T}\theta_k, \mathbf{T}\Sigma_{0|0}^{\theta\theta}\mathbf{T}'\right)$. In the example discussed in that footnote the vector θ_k^K is defined as

$$\mathbf{T}\theta_k = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \end{bmatrix}$$

and it should be noticed that $\theta_k \equiv \mathbf{T}'\theta_k^K$. Therefore the same matrix can be used to go from the notation of this paper to Kendrick's (1981, 2002) and notation and in the opposite direction. This is extremely convenient from a computational point of view.

In the following pages this problem is solved by using dynamic programming methods and working backward in time following the procedures used in Kendrick (1981, 2002, Chapter 6) but with time varying parameters. Given $k = 0$, first the problem is solved for period N and then for period $N - 1$. This leads to the solution for a generic period j in the planning horizon. Then the optimal control for period zero is determined and the system is moved forward.

3 Period N

Using the notation in Kendrick (1981, 2002, Chapter 6) the optimal expected cost to go at period 0, with N periods remaining, is written as

$$J_N^* = \min_{\mathbf{u}_0} E \left\{ \dots \min_{\mathbf{u}_{N-2}} E \left\{ \min_{\mathbf{u}_{N-1}} E \left\{ C_N | \mathcal{P}^{N-1} \right\} | \mathcal{P}^{N-2} \right\} \cdots | \mathcal{P}^0 \right\} \tag{9}$$

where $\mathcal{P}^j$, for $j = 0, \dots, N-1$, is defined as the means and covariances of the unknown parameters at time j. Alternatively equation (9) can be written as[9]

$$J_N^* = \min_{\mathbf{u}_0} E_0 \left\{ \cdots \min_{\mathbf{u}_{N-2}} E_{N-2} \left\{ \min_{\mathbf{u}_{N-1}} E_{N-1} \left\{ C_N \right\} \right\} \cdots \right\} \tag{10}$$

where the subscript on the expectation operator indicates that the expectation is conditional on the information available at that time, that is

$$E_{N-1} \left\{ C_N \right\} \equiv E \left\{ C_N | \mathcal{P}^{N-1} \right\}$$

From the nested expression (10) it follows that each control $\mathbf{u}_j$ must be chosen with the information available through time j.

The typical situation when $\mathcal{P}^j$, for $j = 0, \dots, N-1$, is known at time

[9] In general at time j, with $N - j$ periods remaining, the summation in Equation (1) goes from $k = j$ to $N - 1$ and the associated cost is denoted by C_{N-j}. Then Equation (10) looks like

$$J_{N-j}^* = \min_{\mathbf{u}_j} E_j \left\{ \cdots \min_{\mathbf{u}_{N-2}} E_{N-2} \left\{ \min_{\mathbf{u}_{N-1}} E_{N-1} \left\{ C_{N-j} \right\} \right\} \cdots \right\}$$

0, is when the parameters are identically and independently distributed.[10] When the parameters are modeled as in Equations (5)-(6) this is clearly not true and an approximation to the dynamic programming algorithm is needed. The approximation presented in these pages uses all the information available at time zero, namely $\mathbf{x}_0$ and the distribution associated with θ_0, and replaces Equation (10) with[11]

$$J_N^* = \min_{\mathbf{u}_0} E_0 \left\{ \cdots \min_{\mathbf{u}_{N-2}} E_0 \left\{ \min_{\mathbf{u}_{N-1}} E_0 \left\{ C_N \right\} \right\} \cdots \right\}. \tag{11}$$

As discussed in Kendrick (1981, Chapter 2) in dynamic programming problems, for any arbitrary time period j, the optimal cost-to-go with $N-j$ periods remaining will equal the minimum over the choice of the control at time j of the cost incurred during period j plus the optimal cost-to-go with $N-(j+1)$ periods remaining. Therefore the approximate optimal feedback rule for problem (1)-(8) is solved starting from the last period and working backward toward the initial period.

In period N no control is chosen and from Equation (2) it follows that the optimal cost is

$$J_0^* = \frac{1}{2}\mathbf{x}_N'\mathbf{W}_N\mathbf{x}_N + \mathbf{w}_N'\mathbf{x}_N \tag{12}$$

In general, see e.g. Kendrick (1981, 2002, Chapter 2), the optimal cost-to-go for the quadratic linear problem, sometimes called the *regulatory problem*, in a certain period is a quadratic function of the state of the system in that period. So the expected cost-to-go with zero periods to go may be written as

$$J_0^* = \frac{1}{2}\mathbf{x}_N'\mathbf{K}_N\mathbf{x}_N + \mathbf{p}_N'\mathbf{x}_N + \nu_N \tag{13}$$

where the scalar ν_N, the vector $\mathbf{p}_N$, and the matrix $\mathbf{K}_N$ are the parameters of the quadratic function to be determined recursively in the optimization procedure.[12]

Then comparing Equation (12) with Equation (13) one obtains the terminal conditions for the Riccati equations, namely

[10] This is the case considered in Kendrick (1981, 2002, Chapter 6) and usually discussed in the engineering literature. See Bertsekas (2005, Chapter 4).

[11] Chow (1973) uses a similar approximation when dealing with unknown constant parameters.

[12] The term ν is sometimes omitted because "it does not affect the optimal control path but only the optimal cost-to-go" (Kendrick (1981, page 48))

$$\mathbf{K}_N = \mathbf{W}_N,\ \mathbf{p}_N = \mathbf{w}_N \text{ and } \nu_N = 0 \tag{14}$$

4 Period $N-1$

The optimal cost-to-go in period $N-1$ can be written as

$$J_1^* = \min_{\mathbf{u}_{N-1}} E_0 \left\{ J_0^* + L_{N-1}\left(\mathbf{x}_{N-1}, \mathbf{u}_{N-1}\right)\right\} \tag{15}$$

where J_0^* is the optimal cost-to-go with 0 periods remaining and

$$L_{N-1}\left(\mathbf{x}_{N-1}, \mathbf{u}_{N-1}\right)$$

is the cost incurred in period $N-1$. Substituting Equation (3) and Equation (13), with $\nu_N = 0$, into Equation (15) yields

$$\begin{aligned} J_1^* = \min_{\mathbf{u}_{N-1}} E_0 \Big\{ & \tfrac{1}{2}\mathbf{x}'_N \mathbf{K}_N \mathbf{x}_N + \mathbf{p}'_N \mathbf{x}_N + \tfrac{1}{2}\mathbf{x}'_{N-1}\mathbf{W}_{N-1}\mathbf{x}_{N-1} + \mathbf{w}'_{N-1}\mathbf{x}_{N-1} \\ & + \mathbf{x}'_{N-1}\mathbf{F}_{N-1}\mathbf{u}_{N-1} + \tfrac{1}{2}\mathbf{u}'_{N-1}\mathbf{\Lambda}_{N-1}\mathbf{u}_{N-1} + \boldsymbol{\lambda}'_{N-1}\mathbf{u}_{N-1} \Big\} \end{aligned} \tag{16}$$

This expression gives the optimal cost-to-go in terms of $(\mathbf{x}_N,\ \mathbf{x}_{N-1},\ \mathbf{u}_{N-1})$. After replacing $\mathbf{x}_N$ with the system equations given in Equation (4), Equation (16) looks like

$$\begin{aligned} J_1^* \quad = \quad & \min_{\mathbf{u}_{N-1}} E_0 \Big\{ \frac{1}{2}(\mathbf{A}_{N-1}\mathbf{x}_{N-1} + \mathbf{B}_{N-1}\mathbf{u}_{N-1} + \mathbf{c}_{N-1} + \mathbf{v}_{N-1})'\mathbf{K}_N \\ & \times (\mathbf{A}_{N-1}\mathbf{x}_{N-1} + \mathbf{B}_{N-1}\mathbf{u}_{N-1} + \mathbf{c}_{N-1} + \mathbf{v}_{N-1}) \\ & + \mathbf{p}'_N \left(\mathbf{A}_{N-1}\mathbf{x}_{N-1} + \mathbf{B}_{N-1}\mathbf{u}_{N-1} + \mathbf{c}_{N-1} + \mathbf{v}_{N-1}\right) + \frac{1}{2}\mathbf{x}'_{N-1}\mathbf{W}_{N-1}\mathbf{x}_{N-1} \\ & + \mathbf{w}'_{N-1}\mathbf{x}_{N-1} + \mathbf{x}'_{N-1}\mathbf{F}_{N-1}\mathbf{u}_{N-1} + \frac{1}{2}\mathbf{u}'_{N-1}\mathbf{\Lambda}_{N-1}\mathbf{u}_{N-1} + \boldsymbol{\lambda}'_{N-1}\mathbf{u}_{N-1} \Big\} \end{aligned} \tag{17}$$

which depends only on $\mathbf{x}_{N-1}$ and $\mathbf{u}_{N-1}$. Multiplying the various terms in (17) and taking expectations conditional on the information available at time 0, that is based on $\mathbf{x}_0$ and θ_0, yields

$$
\begin{aligned}
J_1^* = \min_{u_{N-1}} & \left\{ \frac{1}{2} \left[\mathbf{x}'_{N-1} E_0 \left(\mathbf{A}'_{N-1} \mathbf{K}_N \mathbf{A}_{N-1} \right) \mathbf{x}_{N-1} \right. \right. \\
& + \mathbf{x}'_{N-1} E_0 \left(\mathbf{A}'_{N-1} \mathbf{K}_N \mathbf{B}_{N-1} \right) \mathbf{u}_{N-1} \\
& + \mathbf{x}'_{N-1} E_0 \left(\mathbf{A}'_{N-1} \mathbf{K}_N \mathbf{c}_{N-1} \right) + \mathbf{u}'_{N-1} E_0 \left(\mathbf{B}'_{N-1} \mathbf{K}_N \mathbf{A}_{N-1} \right) \mathbf{x}_{N-1} \\
& + \mathbf{u}'_{N-1} E_0 \left(\mathbf{B}'_{N-1} \mathbf{K}_N \mathbf{B}_{N-1} \right) \mathbf{u}_{N-1} + \mathbf{u}'_{N-1} E_0 \left(\mathbf{B}'_{N-1} \mathbf{K}_N \mathbf{c}_{N-1} \right) \\
& + E_0 \left(\mathbf{c}'_{N-1} \mathbf{K}_N \mathbf{A}_{N-1} \right) \mathbf{x}_{N-1} + E_0 \left(\mathbf{c}'_{N-1} \mathbf{K}_N \mathbf{B}_{N-1} \right) \mathbf{u}_{N-1} \\
& \left. + E_0 \left(\mathbf{c}'_{N-1} \mathbf{K}_N \mathbf{c}_{N-1} \right) + E_0 \left(\mathbf{v}'_{N-1} \mathbf{K}_N \mathbf{v}_{N-1} \right) \right] \\
& + \mathbf{p}'_N E_0 \left(\mathbf{A}_{N-1} \right) \mathbf{x}_{N-1} + \mathbf{p}'_N E_0 \left(\mathbf{B}_{N-1} \right) \mathbf{u}_{N-1} + \mathbf{p}'_N E_0 \left(\mathbf{c}_{N-1} \right) \\
& + \frac{1}{2} \mathbf{x}'_{N-1} \mathbf{W}_{N-1} \mathbf{x}_{N-1} + \mathbf{w}'_{N-1} \mathbf{x}_{N-1} + \mathbf{x}'_{N-1} \mathbf{F}_{N-1} \mathbf{u}_{N-1} \\
& \left. + \frac{1}{2} \mathbf{u}'_{N-1} \mathbf{\Lambda}_{N-1} \mathbf{u}_{N-1} + \boldsymbol{\lambda}'_{N-1} \mathbf{u}_{N-1} \right\}
\end{aligned} \tag{18}
$$

with the expectations involving only $\mathbf{v}_{N-1}$ and the covariances between $\mathbf{v}_{N-1}$ and the time-varying parameters omitted because they are 0 by assumption.

Minimizing Equation (18) with respect to the vector of controls yields the first order condition, namely

$$
\begin{aligned}
& E_0 \left(\mathbf{B'}_{N-1} \mathbf{K}_N \mathbf{A}_{N-1} \right) \mathbf{x}_{N-1} + E_0 \left(\mathbf{B'}_{N-1} \mathbf{K}_N \mathbf{B}_{N-1} \right) \mathbf{u}^*_{N-1} \\
& \quad + E_0 \left(\mathbf{B'}_{N-1} \mathbf{K}_N \mathbf{c}_{N-1} \right) + E_0 \left(\mathbf{B'}_{N-1} \right) \mathbf{p}_N + \\
& \quad \mathbf{F}'_{N-1} \mathbf{x}_{N-1} + \mathbf{\Lambda}_{N-1} \mathbf{u}^*_{N-1} + \boldsymbol{\lambda}_{N-1} = 0,
\end{aligned} \tag{19}
$$

which implies that the cost minimizing control, or feedback rule, for time $N-1$ is

$$
\mathbf{u}^*_{N-1} = \mathbf{G}_{N-1} \mathbf{x}_{N-1} + \mathbf{g}_{N-1} \tag{20}
$$

where

$$
\begin{aligned}
\mathbf{G}_{N-1} &= -[\mathbf{\Lambda}_{N-1} + E_0(\mathbf{B}'_{N-1} \mathbf{K}_N \mathbf{B}_{N-1})]^{-1} \\
& \qquad \times [E_0(\mathbf{B}'_{N-1} \mathbf{K}_N \mathbf{A}_{N-1}) + \mathbf{F}'_{N-1}] & (21) \\
\mathbf{g}_{N-1} &= -[\mathbf{\Lambda}_{N-1} + E_0(\mathbf{B}'_{N-1} \mathbf{K}_N \mathbf{B}_{N-1})]^{-1} \\
& \qquad \times [E_0(\mathbf{B}'_{N-1} \mathbf{K}_N \mathbf{c}_{N-1}) + E_0(\mathbf{B}'_{N-1}) \mathbf{p}_N + \boldsymbol{\lambda}_{N-1}] & (22)
\end{aligned}
$$

which resembles the 'standard' stochastic case, that is the case where the parameter matrices are assumed either identically and independently distributed or unknown but constant, except for the fact that here the expectations are conditional on the information available at time 0. The feedback rule (20), (21) and (22) provide the optimality condition sought for period $N-1$. The optimal cost-to-go is obtained replacing the feedback rule in the cost functional. Then substituting Equation (20) into Equation (18) one obtains

$$
\begin{aligned}
J_1^* \;=\; & \Bigg\{ \frac{1}{2}[\mathbf{x}'_{N-1} E_0(\mathbf{A}'_{N-1}\mathbf{K}_N\mathbf{A}_{N-1})\mathbf{x}_{N-1} \\
& + \mathbf{x}'_{N-1} E_0(\mathbf{A}'_{N-1}\mathbf{K}_N\mathbf{B}_{N-1})(\mathbf{G}_{N-1}\mathbf{x}_{N-1} + \mathbf{g}_{N-1}) \\
& + \mathbf{x}'_{N-1} E_0(\mathbf{A}'_{N-1}\mathbf{K}_N\mathbf{c}_{N-1}) \\
& + (\mathbf{G}_{N-1}\mathbf{x}_{N-1} + \mathbf{g}_{N-1})' E_0(\mathbf{B}'_{N-1}\mathbf{K}_N\mathbf{A}_{N-1})\,\mathbf{x}_{N-1} \\
& + (\mathbf{G}_{N-1}\mathbf{x}_{N-1} + \mathbf{g}_{N-1})' E_0(\mathbf{B}'_{N-1}\mathbf{K}_N\mathbf{B}_{N-1})(\mathbf{G}_{N-1}\mathbf{x}_{N-1} + \mathbf{g}_{N-1}) \\
& + (\mathbf{G}_{N-1}\mathbf{x}_{N-1} + \mathbf{g}_{N-1})' E_0(\mathbf{B}'_{N-1}\mathbf{K}_N\mathbf{c}_{N-1}) \\
& + E_0(\mathbf{c}'_{N-1}\mathbf{K}_N\mathbf{A}_{N-1})\mathbf{x}_{N-1} + E_0(\mathbf{c}'_{N-1}\mathbf{K}_N\mathbf{B}_{N-1})(\mathbf{G}_{N-1}\mathbf{x}_{N-1} + \mathbf{g}_{N-1}) \\
& + E_0(\mathbf{c}'_{N-1}\mathbf{K}_N\mathbf{c}_{N-1}) + E_0(\mathbf{v}'_{N-1}\mathbf{K}_N\mathbf{v}_{N-1})] \\
& + \mathbf{p}'_N E_0(\mathbf{A}_{N-1})\mathbf{x}_{N-1} + \mathbf{p}'_N E_0(\mathbf{B}_{N-1})(\mathbf{G}_{N-1}\mathbf{x}_{N-1} + \mathbf{g}_{N-1}) \\
& + \mathbf{p}'_N E_0(\mathbf{c}_{N-1}) + \frac{1}{2}\mathbf{x}'_{N-1}\mathbf{W}_{N-1}\mathbf{x}_{N-1} + \mathbf{w}'_{N-1}\mathbf{x}_{N-1} \\
& + \mathbf{x}'_{N-1}\mathbf{F}_{N-1}(\mathbf{G}_{N-1}\mathbf{x}_{N-1} + \mathbf{g}_{N-1}) \\
& + \frac{1}{2}(\mathbf{G}_{N-1}\mathbf{x}_{N-1} + \mathbf{g}_{N-1})'\boldsymbol{\Lambda}_{N-1}(\mathbf{G}_{N-1}\mathbf{x}_{N-1} + \mathbf{g}_{N-1}) \\
& + \boldsymbol{\lambda}'_{N-1}(\mathbf{G}_{N-1}\mathbf{x}_{N-1} + \mathbf{g}_{N-1}) \Bigg\}
\end{aligned} \tag{23}
$$

At this point, using Equation (14) and simplifying and rearranging the terms gives

$$
\begin{aligned}
J_1^* \;=\; & \frac{1}{2}\mathbf{x}'_{N-1}\Big\{\mathbf{W}_{N-1} + E_0(\mathbf{A}'_{N-1}\mathbf{W}_N\mathbf{A}_{N-1}) \\
& 2\,[E_0(\mathbf{A}'_{N-1}\mathbf{W}_N\mathbf{B}_{N-1}) + \mathbf{F}_{N-1}]\mathbf{G}_{N-1} \\
& +\mathbf{G}'_{N-1}[E_0(\mathbf{B}'_{N-1}\mathbf{W}_N\mathbf{B}_{N-1}) + \mathbf{\Lambda}_{N-1}]\mathbf{G}_{N-1}\Big\}\mathbf{x}_{N-1} \\
& +\mathbf{x}'_{N-1}\Big\{[E_0(\mathbf{A}'_{N-1}\mathbf{W}_N\mathbf{B}_{N-1}) + \mathbf{F}_{N-1}]\mathbf{g}_{N-1} \\
& +\mathbf{G}'_{N-1}[E_0(\mathbf{B}'_{N-1}\mathbf{W}_N\mathbf{B}_{N-1}) + \mathbf{\Lambda}_{N-1}]\mathbf{g}_{N-1} \\
& +E_0(\mathbf{A}'_{N-1}\mathbf{W}_N\mathbf{c}_{N-1}) + E_0(\mathbf{A}'_{N-1})\mathbf{w}_N \\
& +\mathbf{G}'_{N-1}[E_0(\mathbf{B}'_{N-1}\mathbf{W}_N\mathbf{c}_{N-1}) + E_0(\mathbf{B}'_{N-1})\mathbf{w}_N + \boldsymbol{\lambda}_{N-1}] + \mathbf{w}_{N-1}\Big\} \\
& +\frac{1}{2}\Big\{\mathbf{g}'_{N-1}[E_0(\mathbf{B}'_{N-1}\mathbf{W}_N\mathbf{B}_{N-1}) + \mathbf{\Lambda}_{N-1}]\mathbf{g}_{N-1} \\
& +2\,\mathbf{g}'_{N-1}[E_0(\mathbf{B}'_{N-1}\mathbf{W}_N\mathbf{c}_{N-1}) + E_0(\mathbf{B}'_{N-1})\mathbf{w}_N + \boldsymbol{\lambda}_{N-1}] \\
& +2\,\mathbf{w}'_N E_0(\mathbf{c}_{N-1}) + E_0(\mathbf{c}'_{N-1}\mathbf{W}_N\mathbf{c}_{N-1}) + E_0(\mathbf{v}'_{N-1}\mathbf{W}_N\mathbf{v}_{N-1})\Big\}
\end{aligned} \tag{24}
$$

But from Equations (21) and (22) it follows that

$$
\begin{aligned}
&\mathbf{G}'_{N-1}\left[E_0\left(\mathbf{B}'_{N-1}\mathbf{W}_N\mathbf{B}_{N-1}\right) + \mathbf{\Lambda}_{N-1}\right]\mathbf{G}_{N-1} = \\
&\qquad -\left[E_0\left(\mathbf{B}'_{N-1}\mathbf{K}_N\mathbf{A}_{N-1}\right) + \mathbf{F}'_{N-1}\right]'\mathbf{G}_{N-1}
\end{aligned}
$$

$$
\begin{aligned}
&\mathbf{G}'_{N-1}\left[E_0\left(\mathbf{B}'_{N-1}\mathbf{W}_N\mathbf{B}_{N-1}\right) + \mathbf{\Lambda}_{N-1}\right]\mathbf{g}_{N-1} = \\
&\qquad -\left[E_0\left(\mathbf{B}'_{N-1}\mathbf{K}_N\mathbf{A}_{N-1}\right) + \mathbf{F}'_{N-1}\right]'\mathbf{g}_{N-1}
\end{aligned}
$$

$$
\begin{aligned}
&\mathbf{G}'_{N-1}\left[E_0\left(\mathbf{B}'_{N-1}\mathbf{W}_N\mathbf{c}_{N-1}\right) + E_0\left(\mathbf{B}'_{N-1}\right)\mathbf{w}_N + \boldsymbol{\lambda}_{N-1}\right] = \\
&\qquad \left[E_0\left(\mathbf{B}'_{N-1}\mathbf{K}_N\mathbf{A}_{N-1}\right) + \mathbf{F}'_{N-1}\right]'\mathbf{g}_{N-1}
\end{aligned}
$$

$$
\begin{aligned}
&\mathbf{g}'_{N-1}\left[E_0\left(\mathbf{B}'_{N-1}\mathbf{W}_N\mathbf{B}_{N-1}\right) + \mathbf{\Lambda}_{N-1}\right]\mathbf{g}_{N-1} = \\
&\qquad -\mathbf{g}'_{N-1}\left[E_0\left(\mathbf{B}'_{N-1}\mathbf{W}_N\mathbf{c}_{N-1}\right) + E_0\left(\mathbf{B}'_{N-1}\right)\mathbf{w}_N + \boldsymbol{\lambda}_{N-1}\right]
\end{aligned}
$$

Therefore Equation (24) can be rewritten as

$$
J_1^* = \frac{1}{2}\mathbf{x}'_{N-1}\mathbf{K}_{N-1}\mathbf{x}_{N-1} + \mathbf{x}'_{N-1}\mathbf{p}_{N-1} + \nu_{N-1} \tag{25}
$$

with[13]

$$
\begin{aligned}
\mathbf{K}_{N-1} \quad &= \mathbf{W}_{N-1} + E_0(\mathbf{A}'_{N-1}\mathbf{K}_N\mathbf{A}_{N-1}) \\
&-[E_0(\mathbf{A}'_{N-1}\mathbf{K}_N\mathbf{B}_{N-1}) + \mathbf{F}_{N-1}][\mathbf{\Lambda}_{N-1} + E_0(\mathbf{B}'_{N-1}\mathbf{K}_N\mathbf{B}_{N-1})]^{-1} \\
&\times[E_0(\mathbf{B}'_{N-1}\mathbf{K}_N\mathbf{A}_{N-1}) + \mathbf{F}'_{N-1}] \qquad (26) \\
\mathbf{p}_{N-1} \quad &= E_0(\mathbf{A}'_{N-1}\mathbf{K}_N\mathbf{c}_{N-1}) + E_0(\mathbf{A}'_{N-1})\mathbf{p}_N + \mathbf{w}_{N-1} \\
&-[E_0(\mathbf{A}'_{N-1}\mathbf{K}_N\mathbf{B}_{N-1}) + \mathbf{F}_{N-1}][\mathbf{\Lambda}_{N-1} + E_0(\mathbf{B}'_{N-1}\mathbf{K}_N\mathbf{B}_{N-1})]^{-1} \\
&\times[E_0(\mathbf{B}'_{N-1}\mathbf{K}_N\mathbf{c}_{N-1} + E_0(\mathbf{B}'_{N-1})\mathbf{p}_N + \boldsymbol{\lambda}_{N-1}] \qquad (27) \\
\nu_{N-1} \quad &= \frac{1}{2}\Big\{ - [E_0(\mathbf{B}'_{N-1}\mathbf{K}_N\mathbf{c}_{N-1}) + E_0(\mathbf{B}'_{N-1})\mathbf{p}_N + \boldsymbol{\lambda}_{N-1}]' \\
&\times[\mathbf{\Lambda}_{N-1} + E_0(\mathbf{B}'_{N-1}\mathbf{K}_N\mathbf{B}_{N-1})]^{-1} \\
&\times[E_0(\mathbf{B}'_{N-1}\mathbf{K}_N\mathbf{c}_{N-1}) + E_0(\mathbf{B}'_{N-1})\mathbf{w}_N + \boldsymbol{\lambda}_{N-1}] \\
&+2\,\mathbf{p}'_N E_0(\mathbf{c}_{N-1}) + E_0(\mathbf{c}'_{N-1}\mathbf{K}_N\mathbf{c}_{N-1}) + E_0(\mathbf{v}'_{N-1}\mathbf{K}_N\mathbf{v}_{N-1})\Big\} (28)
\end{aligned}
$$

Again the Riccati equations (26), (27) and (28) are the same as in the *standard* stochastic parameter case except for the expectation being conditional on the information available at time 0, that is $\mathbf{x}_0$ and θ_0.[14] This process can be repeated backward for periods $N-1$, $N-2$, ... and so on and so forth.

5 Period j

For a generic period j in the planning horizon, from period 0 to $N-1$, the optimal cost-to-go can be written as

$$J^*_{N-j} = \min_{\mathbf{u}_j} E_0 \left\{ J^*_{N-(j+1)} + L_j(\mathbf{x}_j, \mathbf{u}_j) \right\} \qquad (29)$$

where $J^*_{N-(j+1)}$ is the optimal cost-to-go with $N-(j+1)$ periods remaining. Proceeding as in the case $j = N-1$ yields

$$\mathbf{u}^*_j = \mathbf{G}_j\mathbf{x}_j + \mathbf{g}_j \qquad (30)$$

where

[13]The term $E_0(\mathbf{c}\prime_{N-1}\mathbf{K}_N\mathbf{c}_{N-1})$ in equation (28) is incorrectly given as $2E_0(\mathbf{c}\prime_{N-1}\mathbf{K}_N\mathbf{c}_{N-1})$ in Kendrick (1981, page 46).

[14]See, e.g., Equation (6.29) in Kendrick (1981, 2002).

$$\mathbf{G}_j = -\left[\boldsymbol{\Lambda}_j + E_0\left(\mathbf{B'}_j\mathbf{K}_{j+1}\mathbf{B}_j\right)\right]^{-1}\left[E_0\left(\mathbf{B'}_j\mathbf{K}_{j+1}\mathbf{A}_j\right) + \mathbf{F'}_j\right] \tag{31}$$

$$\mathbf{g}_j = -\left[\boldsymbol{\Lambda}_j + E_0\left(\mathbf{B'}_j\mathbf{K}_{j+1}\mathbf{B}_j\right)\right]^{-1} \times \left[E_0\left(\mathbf{B'}_j\mathbf{K}_{j+1}\mathbf{c}_j\right) + E_0\left(\mathbf{B'}_j\right)\mathbf{w}_{j+1} + \boldsymbol{\lambda}_j\right] \tag{32}$$

And the optimal cost-to-go can be rewritten as

$$J_j^* = \frac{1}{2}\mathbf{x'}_{N-j}\mathbf{K}_{N-j}\mathbf{x}_{N-j} + \mathbf{x'}_{N-j}\mathbf{p}_{N-j} + \nu_{N-j} \tag{33}$$

with

$$\begin{aligned}\mathbf{K}_j &= \mathbf{W}_j + E_0(\mathbf{A}'_j\mathbf{K}_{j+1}\mathbf{A}_j) \\ &\quad -[E_0(\mathbf{A}'_j\mathbf{K}_{j+1}\mathbf{B}_j) + \mathbf{F}_j][\boldsymbol{\Lambda}_j + E_0(\mathbf{B}'_j\mathbf{K}_{j+1}\mathbf{B}_j)]^{-1} \\ &\quad \times[E_0\mathbf{B}'_j\mathbf{K}_{j+1}\mathbf{A}_j) + \mathbf{F}'_j]\end{aligned} \tag{34}$$

$$\begin{aligned}\mathbf{p}_j &= E_0(\mathbf{A}'_j\mathbf{K}_{j+1}\mathbf{c}_j) + E_0(\mathbf{A}'_j)\mathbf{p}_{j+1} + \mathbf{w}_j - [E_0(\mathbf{A}'_j\mathbf{K}_{j+1}\mathbf{B}_j) + \mathbf{F}_j] \\ &\quad \times[\boldsymbol{\Lambda}_j + E_0(\mathbf{B}'_j\mathbf{K}_{j+1}\mathbf{B}_j)]^{-1}[E_0(\mathbf{B}'_j\mathbf{K}_{j+1}\mathbf{c}_j) + E_0(\mathbf{B}'_j)\mathbf{p}_{j+1} + \boldsymbol{\lambda}_j]\end{aligned} \tag{35}$$

$$\begin{aligned}\nu_j &= \frac{1}{2}\Big\{ -[E_0(\mathbf{B}'_j\mathbf{K}_{j+1}\mathbf{c}_j) + E_0(\mathbf{B}'_j)\mathbf{p}_{j+1} + \boldsymbol{\lambda}_j]' \\ &\quad \times[\boldsymbol{\Lambda}_j + E_0(\mathbf{B}'_j\mathbf{K}_{j+1}\mathbf{B}_j)]^{-1}[E_0(\mathbf{B}'_j\mathbf{K}_{j+1}\mathbf{c}_j) + E_0(\mathbf{B}'_j)\mathbf{p}_{j+1} + \boldsymbol{\lambda}_j] + \\ &\quad 2\,\mathbf{p}'_{j+1}E_0(\mathbf{c}_j) + E_0(\mathbf{c}'_j\mathbf{K}_{j+1}\mathbf{c}_j) + E_0(\mathbf{v}'_j\mathbf{K}_{j+1}\mathbf{v}_j)\Big\}\end{aligned} \tag{36}$$

As in the previous section, the only difference with respect to the *standard* stochastic case is that the expectation is conditional on the information available at the beginning of the planning horizon.

In summary, similarly to the constant parameter case considered in Kendrick (1981, page 49), the problem at period zero is solved using the terminal conditions $\mathbf{K}_N = \mathbf{W}_N$, $\mathbf{p}_N = \mathbf{w}_N$ and $\mathrm{v}_N = 0$ in Equations (34), (35) and (36) to integrate the Riccati equations backward in time. Then the $\mathbf{G}$ and $\mathbf{g}$ elements can be computed for all time periods. Using the initial conditions for the state vector $\mathbf{x}_0$ and the parameters, the feedback rule is applied to compute $\mathbf{u}_0$. As soon as the new observation on the state $\mathbf{x}_1$ becomes available the estimate of the parameter vector can be updated[15] and the exercise

[15]For comparison reasons, DUALPC uses the same procedure both in EOF and DUAL

is repeated for period $k = 1$ with all the expectations conditional on the information on $\mathbf{x}_1$ and θ_1 available at time 1, then for period $k = 2$ with all the expectations conditional on the information on $\mathbf{x}_2$ and θ_2 available at time 2 and so on until $k = N - 1$.

6 Computing the conditional expectations

To compute the mean of the product of matrices appearing in the feedback rule, in the Riccati equations and in the optimal cost-to-go it is customary to exploit the fact that the Riccati matrices are not stochastic. When the parameters in $\mathbf{A}$ and $\mathbf{B}$ are assumed identically and independently distributed, it is possible to show that the mean of each element of the resulting matrix, say $\mathbf{R}$ with $\mathbf{R} = \mathbf{A}'\mathbf{K}\mathbf{B}$, takes the form[16]

$$E(r_{ij}) \equiv E(\mathbf{a}_i'\mathbf{K}\mathbf{b}_j) = E(\mathbf{a}_i')\mathbf{K}E(\mathbf{b}_j) + tr[\mathbf{K}\Sigma^{\mathbf{b}_j\mathbf{a}_i}] \tag{37}$$

where $E(\mathbf{a}_i')$ is the mean of the elements appearing in the i-th row of matrix $\mathbf{A}'$, or in the i-th column of matrix $\mathbf{A}$, $E(\mathbf{b}_j)$ the mean of the j-th column of $\mathbf{B}$, $\Sigma^{\mathbf{b}_j\mathbf{a}_i}$ the covariance between the elements in $\mathbf{b}_j$ and $\mathbf{a}_i'$ and $tr[.]$ the trace operator. On the other hand if the parameters in $\mathbf{A}$ and $\mathbf{B}$ are modeled as in (5) and (6), the expectations needed to compute $\mathbf{K}_{N-1}$ at time 0 take the form $E_0(\mathbf{A}_{N-1}'\mathbf{K}_N\mathbf{B}_{N-1})$ and Equation (37) is replaced by

$$\begin{aligned} E_0\left(r_{ij,N-1}\right) &\equiv E_0\left(\mathbf{a}_{i,N-1}'\mathbf{K}_N\mathbf{b}_{j,N-1}\right) \\ &= E_0(\mathbf{a}_{i,N-1}')\mathbf{K}_N E_0(\mathbf{b}_{j,N-1}) + tr[\mathbf{K}_N\Sigma_{0,N-1}^{\mathbf{b}_j\mathbf{a}_i}] \end{aligned} \tag{38}$$

where $E_0\left(\mathbf{a}_{i,N-1}'\right)$ is the mean of the elements appearing in the i-th row of matrix $\mathbf{A}_{N-1}'$, or in the i-th column of matrix $\mathbf{A}$, conditional on the information on the parameters available at time 0, $E_0(\mathbf{b}_{j,N-1})$ the mean of the j-th column of $\mathbf{B}_{N-1}$ similarly defined, $\mathbf{K}_N = \mathbf{W}_N$ a deterministic matrix by assumption and $\Sigma_{0,N-1}^{\mathbf{b}_j\mathbf{a}_i}$ is defined as

$$\Sigma_{0,N-1}^{\mathbf{b}_j\mathbf{a}_i} \equiv E_0\left\{[\mathbf{b}_{j,N-1} - E_0(\mathbf{b}_{j,N-1})][\mathbf{a}_{i,N-1}' - E_0(\mathbf{a}_{i,N-1}')]\right\} \tag{39}$$

to update the estimate and the covariance of the parameters. This procedure is based on Kalman Filter. See, for instance Kendrick (1981, page 104), for details.

[16]See, e.g., pp. 49-50 and Appendix B in Kendrick (1981, 2002).

The mean and variance of the rows and columns of $\mathbf{A}$ and $\mathbf{B}$ appearing in Equation (38) and (39) have not been explicitely defined so far. However it is apparent that the i-th column of matrix $\mathbf{A}$ can be written as $\mathbf{S}_i\theta$ with $\mathbf{S}_i$ a selecting matrix of dimension $n{\times}s$ defined as

$$\mathbf{S}_i = [\mathbf{S}_{i,1} \ ... \ \mathbf{S}_{i,n} \ \mathbf{S}_{i,n+1} \ ... \ \mathbf{S}_{i,n+m} \ \mathbf{S}_{i,n+m+1}] \tag{40}$$

where the $\mathbf{S}_{i,j}$ block of dimension $n{\times}n$ is equal to the identity matrix if i $=j$ and the null matrix $\mathbf{O}$ otherwise. Then for i going from 1 to n, $\mathbf{S}_i$ selects the elements of θ corresponding to the i-th column of $\mathbf{A}$, for i going from $n+1$ to $n+m$ it selects the $(i-n)$-th column of $\mathbf{B}$ and for $i = n+m+1$ it selects the parameters in $\mathbf{c}$.

Equations (5) and (6) describe the behavior of all the parameters and can be used to compute the mean and variance of the parameters at time $N-1$, given the mean and variance of θ_0 at time 0, namely[17]

$$E_0\left(\theta_{N-1}\right) = \mathbf{D}^{N-1}E_0\left(\theta_0\right) = \mathbf{D}^{N-1}\theta_{0|0} \tag{41}$$

$$\begin{aligned} &E_0\left\{\left[\theta_{N-1} - E_0\left(\theta_{N-1}\right)\right]\left[\theta_{N-1} - E_0\left(\theta_{N-1}\right)\right]'\right\} = \\ &\qquad \mathbf{D}^{N-1}\Sigma_{0|0}^{\theta\theta}\left(\mathbf{D}^{N-1}\right)' + \mathbf{D}^{N-2}\mathbf{G}\left(\mathbf{D}^{N-2}\right)' + ... + \mathbf{G} \end{aligned} \tag{42}$$

Then using the $\mathbf{S}_i$ matrix, the mean and variance of the individual columns of $\mathbf{A}$ and $\mathbf{B}$ can be promptly isolated, that is

$$\begin{aligned} E_0\left(\mathbf{a}'_{i,N-1}\right) &= \mathbf{S}_i\mathbf{D}^{N-1}E_0\left(\theta_0\right) \\ E_0\left(\mathbf{b}'_{j,N-1}\right) &= \mathbf{S}_{n+j}\mathbf{D}^{N-1}E_0\left(\theta_0\right) \end{aligned}$$

for $i = 1, \ldots \ n$ and $j = 1, ..., m$. Similarly Equation (39) can be rewritten as

$$\begin{aligned} \Sigma_{0,N-1}^{\mathbf{b}_j\mathbf{a}_i} &\equiv E_0\left\{\mathbf{S}_{n+j}\left[\theta_{N-1} - \mathbf{D}^{N-1}E_0\left(\theta_0\right)\right]\left[\theta_{N-1} - \mathbf{D}^{N-1}E_0\left(\theta_0\right)\right]'\mathbf{S}'_i\right\} \\ &= \mathbf{S}_{n+j}\left[\mathbf{D}^{N-1}\Sigma_{0|0}^{\theta\theta}\left(\mathbf{D}^{N-1}\right)' + \mathbf{D}^{N-2}\mathbf{G}\left(\mathbf{D}^{N-2}\right)' + ... + \mathbf{G}\right]\mathbf{S}'_i \end{aligned} \tag{43}$$

[17]Equation (41) and (42) follow directly from (5). In the special case $N = 3$, they look like
$E_0\left(\theta_{3-1}\right) = \mathbf{D}\left(\mathbf{D}\theta_{0|0}\right) = \mathbf{D}^2\theta_{0|0}$
$E_0\left\{\left[\theta_2 - E_0\left(\theta_2\right)\right]\left[\theta_2 - E_0\left(\theta_2\right)\right]'\right\} = \mathbf{D}^2\Sigma_{0|0}^{\theta\theta}\left(\mathbf{D}^2\right)' + \mathbf{DGD}' + \mathbf{G}$.

Again the role of the **S** matrix is to isolate, in this case from the $s \times s$ covariance matrix $\Sigma^{\theta\theta}$ associated with the whole parameter vector θ, the $n \times n$ matrix of covariances associated with the parameters in the i-th column of **A** and the j-th column of **B**.

At this point the riccati matrix $\mathbf{K}_{N-1}$, and $\mathbf{p}_{N-1}$, can be computed. Both $\mathbf{K}_{N-1}$ and $\mathbf{p}_{N-1}$ are deterministic because they are functions of the means and variances of random variables. Therefore the procedure sketched in this section can be used to compute $\mathbf{K}_{N-2}$, and $\mathbf{p}_{N-2}$, and so on and so forth until $\mathbf{K}_1$, and $\mathbf{p}_1$, needed to compute the feedback rule for the control at time 0.

7 The Beck and Wieland model

In this section we will the Beck and Wieland (2002) model, can be cast into the above framework. Furthermore we will show that, when the parameters are as in Beck and Wieland (2002) and Amman et al. (2007), this model is a special case and the optimal control is identical to that obtained following the presentation of Kendrick (1981, 2002, Chapter 6 and 7).

Following Beck and Wieland (2002) the decision maker is faced with a linear stochastic optimization problem of the form[18]

$$\underset{[u_k]_{k=0}^{N-1}}{Min} \; E\Bigg[\delta^N (x_N - \hat{x}_N)^2 + \sum_{k=0}^{N-1} \delta^k \{(x_k - \hat{x}_k)^2 + \omega(u_k - \hat{u}_k)^2\} \,|\, (x_0, b_0, v_0^b)\Bigg] \quad (44)$$

subject to the equations

$$\begin{aligned} x_{k+1} &= \alpha + \beta_k u_k + \gamma x_k + \epsilon_k && (45) \\ \beta_{k+1} &= \beta_k + \zeta_k && (46) \end{aligned}$$

where δ is a discount factor, $\epsilon_k \sim N(0, \sigma_\epsilon)$ and $\zeta_k \sim N(0, \sigma_\zeta)$. It is assumed that x_0 is given and the model contains one uncertain parameter

[18] In an earlier strand of literature, going back to the early Seventies, a similar model and approach is discussed. See, MacRae (1972, 1975).

β, with an initial estimate of its value at $k = 0$, $E_0(\beta_0) = b_0$, and an initial estimate of its variance at time $k = 0$, $VAR_0(\beta_0) = v_0^b$. The parameters α and γ are constant. Beck and Wieland assume in their paper that $N \to \infty$. In contrast we will assume that the planning horizon is finite, hence $N < \infty$. Furthermore, we have adopted the *timing* convention from Kendrick (1981, 2002) where the control, u_k, has a lagged response on the state, x_k. Moreover the desired path for the state and the control, and the penalty weight on the latter, is zero.

With this set of assumptions, the above model can be fitted with little effort into the format of Equations (2)-(8) when $\mathbf{A}_k = \gamma$, $\mathbf{B}_k = \beta_k$, $\mathbf{c}_k = \alpha$, $\mathbf{v}_k = v_k$, $\eta_k = \zeta_k$ and

$$\theta_k = \begin{bmatrix} \gamma \\ \beta_k \\ \alpha \end{bmatrix}, \quad \eta_k = \begin{bmatrix} 0 \\ \zeta_k \\ 0 \end{bmatrix} \tag{47}$$

and $\mathbf{D}$ is an identity matrix. In this case the covariance matrices are $\mathbf{Q} \equiv \sigma_v^2$ and

$$\mathbf{G} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sigma_\eta^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{48}$$

Furthermore it is assumed that $\mathbf{x}_0$ is given and

$$\theta_{0|0} = \begin{bmatrix} \gamma \\ b_0 \\ \alpha \end{bmatrix}, \quad \Sigma_{0|0}^{\theta\theta} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & v_0^b & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{49}$$

In this case the only relevant $\mathbf{S}_i$ is $\mathbf{S}_2 = [0\ 1\ 0]$ and the optimal cost can be expressed as

$$J_j^* = \frac{1}{2}\mathbf{x}'_{N-j}\mathbf{K}_{N-j}\mathbf{x}_{N-j} \tag{50}$$

because the desired paths for the state and control are 0, $\alpha = 0$, $\omega = \mathbf{\Lambda}_j = 0$ and $\mathbf{F}_j = 0$. The optimal control at time 0 is

$$\mathbf{u}_0^* = \mathbf{G}_0\mathbf{x}_0 \tag{51}$$

where

$$\begin{aligned}\mathbf{G}_0 &= -\left[E_0\left(\mathbf{B}_0'\mathbf{K}_1\mathbf{B}_0\right)\right]^{-1} E_0\left(\mathbf{B}_0'\mathbf{K}_1\mathbf{A}_0\right) \\ &= -\left[\mathbf{K}_1\left(b_0^2+v_0^b\right)\right]^{-1}\left[\mathbf{K}_1\gamma b_0\right] = -\left(b_0^2+v_0^b\right)^{-1}\gamma b_0 \qquad (52)\end{aligned}$$

This means that the optimal control is solely a function of the current information about the stochastic parameter. Hence, in the Beck and Wieland (2002) case, the time varying parameter solution can be obtained using the framework of Kendrick (1981, 2002, Chapter 6 and 7).

8 Summary

In this paper we derived the closed loop form of the *Expected Optimal Feedback* rule with time varying parameter. As such this paper extents the work of Kendrick . Furthermore, we showed that the Beck and Wieland model can be cast into this framework and basically can be treated as a special case of this solution.

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