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Global Coalitional Games

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Abstract - Global coalitional games are TU cooperative games intended to model situations where the worth of coalitions varies across different partitions of the players. Formally, they are real-valued functions whose domain is the direct product of the subset lattice and the lattice of partitions of a finite player set. Therefore, the dimension of the associated vector space grows dramatically fast with the cardinality of the player set, inducing flexibility as well as complexity. Accordingly, some reasonable restrictions that reduce such a dimension are considered. The solution concepts associated with the Shapley value and the core are studied for the general (i.e., unrestricted) case.

Key words: lattice, lattice function, coalition, partition, Shapley value, core.

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1 Introduction

Traditional TU cooperative game theory deals with situations where each player may decide whether to cooperate or not. Furthermore, attention is usually focused on grand coalitions (i.e., player sets) within which monotonicity holds, so that such grand coalitions must eventually form. Therefore, it is natural to concentrate solely on the solution problem (i.e., how to share the 'grand cake'). In particular, the core concept is concerned with those additive coalitional games (i.e., sharing rules) that, when considered as payoff vectors, make it rational for each player to join the grand coalition (and for each coalition to join its complement). Conversely, when monotonicity is relaxed, it comes natural to ask what largest coalitions will form (and how will their worth get shared among members). In other words, endogenous coalition formation enters the picture, and nontrivial coalition structures (i.e., different from the grand coalition) do result. In this paper, endogenous coalition formation is not directly approached; yet, most of the concern is on the partition lattice of a finite player set.

Coalition structures are partitions of the players, and constitute the outcome of many situations where cooperation displays synergies and yet the choice of individual players does not merely reduce to whether to cooperate or not, but also concerns the degree of cooperation. For example, in voting situations voters may furnish a finite number of different support levels to the various bills, so that these latter pass whenever a given majority rule is fulfilled by the partition of the voter set obtained by putting any two voters furnishing the same support level into the same block. More generally, Gilboa and Lehrer (1991a) defined global games as real-valued functions whose domain is the partition lattice of the player set, and proposed to use them to model situations where the global (i.e., the world's) welfare level depends on the (strategic) choice of each player over what block to join. Nevertheless, such a modeling choice does not allow to consider the possibility that for each admissible global welfare level there exist several different distributions of such a welfare over the global population. In order to do so, it is necessary to consider functions taking values on pairs consisting of a coalition of players and a partition of the (whole) player set. Such an approach was firstly adopted by Thrall and Lucas (1963) and subsequently by Myerson (1977), even though they restricted attention to those pairs consisting of a coalition and a partition embedding the former as one of its blocks. Such a restriction is here abandoned, so that for each coalition a global game gets defined. This implies that each coalition has the strategic possibility not only of forming a corresponding unique block (of some 'final' partition), but also of spreading its members over different blocks, possibly containing nonmembers as well. From another viewpoint, this can be seen as an attempt to consider situations where cooperation may occur at both quantitatively as well as qualitatively different levels. In other words, in global coalitional games players may be seen as cooperating at qualitatively different levels when considered as members of coalitions or else as members of blocks of partitions.

An important remark (applying to the approach here proposed as well as to those of Thrall and Lucas (1963), Myerson (1977) and Gilboa and Lehrer

(1991a)) is the following. For any arbitrarily large (finite integer) number n of players, the cardinality of the associated subset lattice (that is, the power set of the player set endowed with the set inclusion order relation) is 2^n. Nevertheless, there does not exist (as known to the author) an equivalent closed form in n for the cardinality of the associated partition lattice (that is, the set of partitions of the player set endowed with the 'coarser than' order relation). In number theory, a partition of any integer n is any collection $\lambda_1, \ldots, \lambda_m$ such that $\sum_{j=1}^{m} \lambda_j = n$, all λ_j's being integers as well. The number $p(n)$ of all such partitions was determined by Hardy and Ramanujan, and can be found in Andrews (1976), ch. 5. It is clearly much greater than 2^n; furthermore, the number of partitions of an n-set is much greater than $p(n)$, in that each partition $\lambda_1, \ldots, \lambda_m$ of n corresponds to $n! - m! \prod_{j=1}^{m} \lambda_j!$ different partitions of an n-set. In fact, the number $\mathcal{B}_n$ of partitions of an n-set is determined through recursion by $\mathcal{B}_n = \sum_{k=0}^{n-1} \binom{n-1}{k} \mathcal{B}_k$, with $\mathcal{B}_0 := 1$ (see Aigner (1979) on the Bell numbers).

Given the above remark, it is easily understood that any kind of n-players game that makes use of the partition lattice becomes rapidly intractable as n increases. In turn, this implies that, at least in terms of conceivable applications, such games may be useful solely for modeling situations involving a reasonably small number of players. In particular, Gilboa and Lehrer (1991a) propose to model as global games "questions of art and historical treasures preservation, a cure for cancer and AIDS, indeed, the progress of science and art in general, and many other issues [that] -though not unrelated to nations' political interests-seem to be 'global', at least as a first approximation. [Their] paper models such games and tries to cope with the question of their 'solution'. [Their idea is that any partition of the player set has an associated global worth (i.e., a worth for the grand coalition), and that, in particular,] the payoff is defined for all players together. (Or, if you will, that the utilities of the players coincide.)" (p. 129). Firstly note that, as previously mentioned, by switching from global to global coalitional games, one may relax this last assumption, i.e., that the utilities of the players coincide. Secondly, consider that for all the above mentioned global situations, one may assume that countries, and not individuals, are the actual players, so that their number allows to model such situations in terms of (games defined on) the partition lattice. Thirdly, in many of such situations it may well be that coalitions of countries find it strategically optimal to cooperate, at the partition level, by spreading their members over different blocks containing nonmembers too.

Example 1: environmental clean-up and preservation. For air and water pollutions migrate from polluting to nonpolluting countries, the globe deals with environmental clean-up and preservation by means of international agreements. Furthermore, in practice, at any given time each country signs at most one such agreement and thus a partition of the country set results[1]. Nevertheless, the 'worth' of any agreement (i.e., its efficacy) depends on the residual configura-

[1] Alternatively, if some countries had signed more than one agreement, it would still be possible to define a partition of the country set by ranking agreements from tighter to looser (i.e., in terms of the regulations they encode), and then considering each country only as a member of the tighter agreement it has signed.

tions of agreements (i.e., signed by nonmembers of the former). Furthermore, consider (a coalition of) two important oil-producer countries facing the choice of what agreement to sign. By signing each a different one, they may (try to) achieve some loose regulation for the whole globe, rather than a looser one for a restricted region only. Note that this does not entirely depend on strategic matters. More precisely, such two countries have a common interest they can pursue only at the international level. Furthermore, such an interest may be best pursued by spreading over distinct agreements. In other words, the two countries need not necessarily form a coalition in the usual sense, even though they can definitely agree to do so.

Example 2: currency unions (and areas). When trying to model some 'international monetary game', it is rapidly realized that any configuration of currency unions (and areas) within the world economy definitely results in both: (i) a partition of the country set (simply regard any country with a free exchange rate with respect to all other currencies as a one-player union), and (ii) some level of global welfare level. Nevertheless, here again, the worth (however measured) of any currency union does depend on the residual configuration of currency unions. Furthermore, considering the current situation, the United States on the one side, and the European 'euro-countries' on the other, typically constitute (disjoint) coalitions, as their currencies are both anchors of two different currency areas (and these latter define the blocks of the partition). Nevertheless, (in the absence of altruism) the utility level attained by their citizens is definitely higher in such a situation than it would be if a unique currency for the world economy was used. Yet, $\{BCE, FED\}$ cannot be considered a typical two-player coalition. In particular, it can be considered as a coalition that may form at a level of cooperation that differs from that defining the partition[2].

The paper is organized as follows. Section 2 contains a formalization of the setting, identifies the vector space of global coalitional games and recalls some general results of Gilboa and Lehrer (1991a) on lattice functions. Section 3 considers both the global games of Gilboa and Lehrer (1991a) and the games in partition function form of Thrall and Lucas (1963) and Myerson (1977); in particular, the Möbius transform of the latter is derived and expressed in terms of the former's one, so to show what vector (proper) subspace (i.e., of the vector space of global coalitional games) the latter games belong to. Sections 4 and 5 deal, respectively, with the Shapley value and the core concepts. The Shapley (1953) axioms are adapted to global coalitional games and shown (following Weber (1988)) to characterize a unique value function. As previously mentioned, any global coalitional game associates to each coalition a global game (i.e., a number of real quantities that equals the number of partitions of the players), so that defining the core may be (and has been so far) approached as a typical aggregation problem, here solved by means of the Choquet integral. Section 6 contains some concluding remarks and possible future research topics.

[2] The application of global coalitional games (graph-restricted and in partition function form; see section 3) to currency unions is done in a forthcoming paper. Furthermore, most likely currency areas could be dealt with by means of the MLE (multilinear extension) of global coalitional games, to be defined.

2 Preliminary notations and results

A complete lattice is any set X endowed with an order (binary) relation $\succcurlyeq$ satisfying reflexivity ($x \succcurlyeq x$ for all $x \in X$), antisymmetry ($y \succcurlyeq x, x \succcurlyeq y \Rightarrow y = x$ for all $x, y \in X$) and transitivity ($z \succcurlyeq y, y \succcurlyeq x \Rightarrow z \succcurlyeq x$ for all $x, y, z \in X$), and such that there exist the least upper bound $\vee_{x \in S} \in X$ and the greatest lower bound $\wedge_{x \in S} \in X$ (with respect to $\succcurlyeq$) for all $S \subseteq X$, in which case $x_\perp, x^\top \in X$ such that $x_\perp \preccurlyeq x, x^\top \succcurlyeq x$ for all $x \in X$ are bottom and top elements respectively.

Any finite player set $N = \{1, \ldots, n\}$ defines two main complete lattices. One is the subset lattice, that is $2^N = \{A \mid N \supseteq A\}$, which is in fact endowed with the set inclusion relation $\supseteq$ by definition. The other is the partition lattice, i.e., $\mathcal{P} = \left\{\{A_1, \ldots, A_m\} \subset 2^N \backslash \emptyset \mid \underset{1 \leq j \leq m}{\cup} A_j = N, \underset{j \in J \subseteq \{1, \ldots, m\}}{\cap} A_j = \emptyset\right\}$ endowed with the coarser than relation $\geq$. Recall that for any $P, P' \in \mathcal{P}$, with $P = \{A_1, \ldots, A_m\}$ and $P' = \{A'_1, \ldots, A'_{m'}\}$, the order relation $\geq$ is defined by: $P \geq P'$ if for each $j' \in \{1, \ldots, m'\}$ it holds $A'_{j'} \subseteq A_j$ for some $j \in \{1, \ldots, m\}$. Also recall that $A \supset B \Leftrightarrow A \supseteq B, A \neq B$ for all $A, B \subseteq N$, as well as $P > Q \Leftrightarrow P \geq Q, P \neq Q$ for all $P, Q \in \mathcal{P}$. The subset lattice may be denoted $\left(2^N, \cap, \cup\right)$. Similarly, the partition lattice is usually denoted $(\mathcal{P}, \wedge, \vee)$, where $P \wedge P'$ ($P \vee P'$) denotes the coarsest (finest) partition finer (coarser) than both $P, P' \in \mathcal{P}$. Formally, $P \wedge P' = \left\{A_j \cap A'_{j'} \mid A_j \cap A'_{j'} \neq \emptyset\right\}$ and $P \vee P' = \left\{\underset{1 \leq j \leq m, 1 \leq j' \leq m'}{\cup} \left(A_j \cup A'_{j'}\right) \mid A_j \cap A'_{j'} \neq \emptyset\right\}$. The bottom elements are $\emptyset \in 2^N$ and $P_0 = \{\{1\}, \ldots, \{n\}\} \in \mathcal{P}$, while the top one is[3] $N \in 2^N, \mathcal{P}$. Coalitional and global games on N are 0-normalized and real-valued lattice functions with domains 2^N and $\mathcal{P}$ respectively, so that $C^N = \left\{v : 2^N \to \mathbb{R} \mid v(\emptyset) = 0\right\}$ and $G^N = \{f : \mathcal{P} \to \mathbb{R} \mid f(P_0) = 0\}$ denote the sets of such games.

Now consider the Cartesian product $2^N \times \mathcal{P}$, and endow it with the order relation[4] $\succcurlyeq$ such that $(A, P) \succcurlyeq (B, Q)$ if $A \supseteq B, P \geq Q$ for any two pairs $(A, P), (B, Q) \in 2^N \times \mathcal{P}$. This assures that $\succcurlyeq$ satisfies reflexivity, antisymmetry and transitivity, and thus that $\left(2^N \times \mathcal{P}, \succcurlyeq\right)$ is a well defined complete lattice, its bottom and top elements being $(\emptyset, P_0)$ and (N, N) respectively. The additional binary relation[5] $\succ^*$, defined by $(A, P) \succ^* (B, Q)$ if $(A, P) \succ (B, Q)$ but for no (B', Q') it holds $(A, P) \succ (B', Q') \succ (B, Q)$, will also be useful.

A global coalitional game (grounded) on N is a 0-normalized lattice function $h : 2^N \times \mathcal{P} \to \mathbb{R}$; thus $h(\emptyset, P_0) = 0$. It is easily seen that h is an application naturally embedding both: a number $\mid \mathcal{P} \mid$ of coalitional games (i.e., on 2^N), each consisting of the collection $\left\{h(A, P) \mid A \in 2^N\right\}$ for $P \in \mathcal{P}$, and a number 2^n of global games (i.e., on $\mathcal{P}$), each consisting of the collection $\{h(A, P) \mid P \in \mathcal{P}\}$ for $A \in 2^N$. In the sequel, the restriction $h(\emptyset, P) = 0$ for all $P \in \mathcal{P}$ is shown

[3] Parentheses for the coarsest partition $\{N\} = N \in \mathcal{P}$ are omitted whenever no confusion is possible.

[4] The order relation $\succcurlyeq$ is denoted $\gg$ in Myerson (1977).

[5] The binary relations $\supset^*$ on 2^N and $>^*$ on $\mathcal{P}$ used in the sequel are defined analogously. In fact, $\succ^*$ results to be the 'sum' of $\supset^*$ and $>^*$.

to result in no loss of generality. Let GC^N denote the set of global coalitional games on N, with the embedding implying $C^N \subset GC^N \supset G^N$.

C^N, G^N and GC^N can be treated as vector spaces. Concerning their dimensions, Shapley (1953) showed that the set $\left\{u_A \mid A \in 2^N \backslash \emptyset\right\}$ of unanimity (coalitional) games, defined by $u_A(B) = 1$ if $B \supseteq A$ and 0 otherwise, constitutes a basis of $C^N \subseteq \mathbb{R}^{2^n-1}$. Similarly, Gilboa and Lehrer (1991a) showed that $\{g_P \mid P \in \mathcal{P} \backslash P_0\}$, where $g_P(Q) = 1$ if $Q \geq P$ and 0 otherwise, is a basis of $G^N \subseteq \mathbb{R}^{|\mathcal{P}|-1}$. Here the dimension of GC^N clearly is $\mid 2^N \times \mathcal{P} \mid -1$, and a similar result obtains[6].

Proposition 1 *The set of games* $\left\{g_{A,P} \mid (\emptyset, P_0) \neq (A, P) \in 2^N \times \mathcal{P}\right\}$*, defined by* $g_{A,P}(B, Q) = 1$ *if* $(B, Q) \succcurlyeq (A, P)$ *and* 0 *otherwise for any two pairs* $(A, P), (B, Q) \in 2^N \times \mathcal{P}$*, is a linear basis of* GC^N*.*

Proof. Following Gilboa and Lehrer (1991a), linear independence is shown by considering that $\sum_{(\emptyset,P_0)\neq(A,P)\in 2^N\times\mathcal{P}} \alpha_{A,P} g_{A,P} = 0$ iff $\alpha_{A,P} = 0$ for all pairs $(\emptyset, P_0) \neq (A, P) \in 2^N \times \mathcal{P}$, in that $\sum_{(\emptyset,P_0)\neq(A,P)\in 2^N\times\mathcal{P}} \alpha_{A,P} g_{A,P}(B, Q) = 0$ iff $\alpha_{B,Q} = 0$ for all $(B, Q) \succ^* (\emptyset, P_0)$, and the argument proceeds by induction. For its cardinality is in fact $\mid 2^N \times \mathcal{P} \mid -1$, the set $\left\{g_{A,P} \mid (\emptyset, P_0) \neq (A, P) \in 2^N \times \mathcal{P}\right\}$ is now easily seen to be a basis of GC^N. ■

Treating global coalitional games as generic lattice functions leads to define any $h \in GC^N$ to be

nonnegative: if $h(A, P) \geq 0$ for all $(A, P) \in 2^N \times \mathcal{P}$;

monotone: if $(A, P) \succcurlyeq (B, Q)$ implies $h(A, B) \geq h(B, Q)$,

convex: if $h(A, P) + h(B, Q) \leq h(A \cup B, P \vee Q) + h(A \cap B, P \wedge Q)$,

additive: if its convexity holds with equality,

for all $(A, P), (B, Q) \in 2^N \times \mathcal{P}$;

m-positive: if for every $(A_1, P_1), \ldots, (A_m, P_m) \in 2^N \times \mathcal{P}$

$$h\left(\bigcup_{1\leq i\leq m} A_i, \bigvee_{1\leq i\leq m} P_i\right) \geq \sum_{\emptyset\neq I\subseteq\{1,\ldots,m\}} (-1)^{|I|+1} h\left(\bigcap_{i\in I} A_i, \bigwedge_{i\in I} P_i\right),$$

totally positive: if m-positivity holds for all $m \in \{2, \ldots, 2^n \times \mid \mathcal{P} \mid\}$.

Monotonicity is a standard assumption in cooperative game theory, for it translates the idea that the larger the number of players that cooperate, the greater the worth that gets produced through cooperation. Thus, such an assumption is innocuous when referred to coalitional games. On the other hand,

[6] In fact, the results reported in this section have been shown to hold for lattice functions in general by Gilboa and Lehrer (1991a), and thus they are here merely reproduced for the sake of completeness.

global games associate a worth to each partition, so that coarser partitions may be seen as corresponding to situations where cooperation occurs within (collections of) larger coalitions. Yet, global coalitional games allow to compare pairs such as $\left(A, \{A\} \cup P^{A^c}\right)$ and $\left(A, \{A\} \cup Q^{A^c}\right)$, where $A^c = N \backslash A$ and $\mathcal{P}(A^c) \ni P^{A^c}, Q^{A^c}$ denotes the set of partitions of any $A^c \subset N$. Consider the case $P_0^{A^c} \leq P^{A^c} < Q^{A^c} \leq \{A^c\}$, where $P_0^{A^c}, \{A^c\} \in \mathcal{P}(A^c)$ constitute, respectively, the finest and coarsest partition of A^c. For such pairs monotonicity requires $h\left(A, \{A\} \cup P^{A^c}\right) \leq h\left(A, \{A\} \cup Q^{A^c}\right)$, even though the worth of A, every time it constitutes a block on its own, might (intuitively) be greater when the remaining players $i \in A^c$ are more dispersed (i.e., over more blocks). In fact, most likely $h\left(A, \{A\} \cup P^{A^c}\right) < h\left(N, \{A\} \cup P^{A^c}\right) < h\left(N, \{A\} \cup Q^{A^c}\right)$. Yet, players $i \in A^c$ appear better organized for bargaining (i.e., displaying more cohesion) under Q^{A^c} rather than under P^{A^c}.

GC^N can clearly be endowed with addition and (positive) scalar multiplication in the usual manner, and any subset of global coalitional games that is closed under such two operations constitutes a cone in $\mathbb{R}^{2^n \times |\mathcal{P}|-1} \supseteq GC^N$. In particular, following Shapley (1971), it can be noticed that the set of convex games constitutes a cone that contains the subspace of additive games. Similarly, following Gilboa and Lehrer (1991a), it can be noticed that the set of totally positive and monotone games constitutes a cone that includes that of convex games. The next two sections are devoted to define reasonable subspaces of (and cones in) GC^N.

Theorem 2 *$h \in GC^N$ is totally positive and monotone iff $\alpha_{A,P}(h) \geq 0$ for all $(A, P) \in 2^N \times \mathcal{P}$.*

Proof. Consider the cone $GC^N_{TPM} \subset GC^N$ of totally positive and monotone global coalitional games. As in Gilboa and Lehrer (1991a), if global coalitional unanimity games are such that $g_{A,P} \in GC^N_{TPM}$, then the 'if' part is proved. For games $g_{A,P}$ are monotone by definition, they must be shown to be totally positive, that is, any collection $(A, P), (B_1, Q_1), \ldots, (B_m, Q_m) \in 2^N \times \mathcal{P}$ must satisfy

$$g_{A,P}\left(\bigcup_{1 \leq i \leq m} B_i, \bigvee_{1 \leq i \leq m} Q_i\right) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,m\}} (-1)^{|I|+1} g_{A,P}\left(\bigcap_{i \in I} B_i, \bigwedge_{i \in I} Q_i\right).$$

Let $J \subseteq \{1 \leq j \leq m \mid (A, P) \preccurlyeq (B_j, Q_j)\}$, noting that if $J = \emptyset$ the right side vanishes and the inequality holds. Otherwise,

$$g_{A,P}\left(\bigcup_{1 \leq i \leq m} B_i, \bigvee_{1 \leq i \leq m} Q_i\right) - \sum_{\emptyset \neq I \subseteq J} (-1)^{|I|+1} g_{A,P}\left(\bigcap_{i \in I} B_i, \bigwedge_{i \in I} Q_i\right) =$$

$= 1 - \sum_{\emptyset \neq I \subseteq J} (-1)^{|I|+1} = \sum_{I \subseteq J} (-1)^{|I|} = (1-1)^{|J|} = 0$.

Let $\{(B_1, Q_1), \ldots, (B_m, Q_m)\} = \left\{(B, Q) \in 2^N \times \mathcal{P} \mid (A, P) \succ^* (B, Q)\right\}$ for each pair $(A, P) \in 2^N \times \mathcal{P}$ and consider any $h \in GC^N$. The 'only if' part firstly requires the following result:

$$h(A,P) - \sum_{\emptyset \neq I \subseteq \{1,\ldots,m\}} (-1)^{|I|+1} h\left(\bigcap_{i \in I} B_i, \bigwedge_{i \in I} Q_i \right) = \alpha_{A,P}(h) \qquad (*) .$$

To see this, recall the representation

$$h(A,P) = \sum_{(B,Q) \in 2^N \times \mathcal{P}} \alpha_{B,Q}(h)\, g_{B,Q}(A,P) = \sum_{(B,Q) \preccurlyeq (A,P)} \alpha_{B,Q}(h),$$

so that $(*)$ is immediately seen to hold true as

$$\sum_{(B,Q) \prec (A,P)} \alpha_{B,Q}(h) - \sum_{\emptyset \neq I \subseteq \{1,\ldots,m\}} (-1)^{|I|+1} \sum_{(B',Q') \preccurlyeq \left(\bigcap_{i \in I} B_i, \bigwedge_{i \in I} Q_i \right)} \alpha_{B',Q'}(h) =$$

$$= \sum_{(B,Q) \prec (A,P)} \alpha_{B,Q}(h) \left(1 - \sum_{\emptyset \neq I \subseteq \{1,\ldots,m\} | (B,Q) \preccurlyeq \left(\bigcap_{i \in I} B_i, \bigwedge_{i \in I} Q_i \right)} (-1)^{|I|+1} \right) =$$

$= \sum_{(B,Q) \prec (A,P)} \alpha_{B,Q}(h) \left(1 - \sum_{\emptyset \neq I \subseteq J_{B,Q}} (-1)^{|I|+1} \right) = 0$, in that $J_{B,Q} \neq \emptyset$ for all $(B,Q) \neq (\emptyset, P_0)$, where $J_{B,Q} = \{1 \leq j \leq m \mid (B,Q) \preccurlyeq (B_j, Q_j)\}$.

The proof ends by considering a totally positive and monotone game h and $(A,P) \succ^* (B_1,Q_1), \ldots, (B_m,Q_m)$. If $m = 1$, then nonnegativity is implied by monotonicity, as $(*)$ yields $\alpha_{A,P}(h) = h(A,P) - h(B_1,Q_1)$. Otherwise, nonnegativity is implied by $(*)$ and total positivity. ■

The product lattice $\left(2^N \times \mathcal{P}, \succcurlyeq\right)$ is rich, that is, whenever the level $l(A,P)$ of a pair is strictly greater than unity its degree $d(A,P)$ also is so, where $l(A,P) = k$ such that

$$\begin{aligned} (A,P) &\succ^* (B,Q)_1 \succ^* (B,Q)_2 \succ^* \ldots \succ^* (B,Q)_k = (\emptyset, P_0), \\ \text{while } d(A,P) &= \left| \left\{ (B,Q) \in 2^N \times \mathcal{P} \mid (A,P) \succ^* (B,Q) \right\} \right| . \end{aligned}$$

For $d(\emptyset, P_0) = l(\emptyset, P_0) = 0$, each (A,P) has $l(A,P) > 1 \Rightarrow d(A,P) > 1$ holding. In particular, observation 3.4 of Gilboa and Lehrer (1991a) applies.

Proposition 3 *For any $h \in GC^N$ such that $h(\emptyset, P) \neq 0$ for at least one $P \in \mathcal{P}$, there exists a $\emptyset$-normalized game $h_\emptyset \in GC^N$ such that $h_\emptyset(\emptyset, P) = 0$ for all $P \in \mathcal{P}$ and $h_\emptyset(A,P) - h_\emptyset(B,P) = h(A,P) - h(B,P)$ for all $P \in \mathcal{P}$ and $A, B \subseteq N$.*

Proof. For $h \in GC^N$, define $h_\emptyset(A,P) = h(A,P) - \sum_{Q \leq P} \alpha_{\emptyset,Q}(h)$ for all $(A,P) \in 2^N \times \mathcal{P}$; then $h_\emptyset(\emptyset,P) = h(\emptyset,P) - h(\emptyset,P) = 0$ for all $P \in \mathcal{P}$, as well as $h_\emptyset(A,P) - h_\emptyset(B,P) =$

$$= h(A,P) - h(\emptyset,P) - (h(B,P) - h(\emptyset,P)) = h(A,P) - h(B,P)$$

for all $P \in \mathcal{P}$ and $A, B \subseteq N$. ■

Let $\mathbb{R}_+^{(2^n-1)\times|\mathcal{P}|} \supseteq GC_\emptyset^N \subset GC^N$ denote the set of $\emptyset$-normalized and non-negative global coalitional games; its dimension is $(2^n - 1) \times \mid \mathcal{P} \mid$ for a basis clearly is $\left\{g_{A,P} \mid (A \neq \emptyset, P) \in 2^N \times \mathcal{P}\right\} \subset \left\{g_{A,P} \mid (\emptyset, P_0) \neq (A, P) \in 2^N \times \mathcal{P}\right\}$.

3 Global games and the partition function form

Different reasonable approaches may be adopted for reducing the dimension of $GC_\emptyset^N$, one of whose basis is in fact $\left\{\widetilde{g}_{A,P} \mid (A \neq \emptyset, P) \in 2^N \times \mathcal{P}\right\}$ defined by $\widetilde{g}_{A,P}(B,Q) = 1$ if $(B,Q) = (A,P)$ and 0 otherwise. An important reduction occurs through the partition function form approach adopted by Thrall and Lucas (1963) and Myerson (1977). Let $\mathcal{E} = \left\{(A,P) \in 2^N \times \mathcal{P} \mid A \in P\right\}$ denote the subset of pairs where the partition embeds the coalition. Then, an 'n-person game in partition function form' is any function $\widehat{h} : \mathcal{E} \to \mathbb{R}$, and it can be extended as $\widehat{h}^{ext} : 2^N \times \mathcal{P} \to \mathbb{R}$ to the whole domain of global coalitional games by letting $\{(B_1, Q_1), \ldots, (B_m, Q_m)\} = \{(B,Q) \in \mathcal{E} \mid (A,P) \succcurlyeq (B,Q)\}$ for each $(A,P) \in 2^N \times \mathcal{P}$. In fact, this allows to define both the set of indices

$$J_{A,P} = \left\{1 \leq j \leq m \mid (B_i, Q_i) \underset{\text{or } \not\succ}{\preccurlyeq} (B_j, Q_j), i \in \{1, \ldots, m\} \setminus \{j\}\right\}$$

and the extension $\widehat{h}^{ext}(A,P) = \sum_{j \in J_{A,P}} \widehat{h}(B_j, Q_j)$ for all $(A,P) \in 2^N \times \mathcal{P}$. Thus, $\{(B_j, Q_j) \mid j \in J_{A,P}\}$ denotes the set of pairs such that: (i) $B_j \in Q_j$, (ii) $(A,P) \succcurlyeq (B_j, Q_j)$, and (iii) there is no $(B_i, Q_i) \succ (B_j, Q_j), i \in \{1, \ldots, m\}$ satisfying (i) and (ii). In other terms, the pairs $(B_j, Q_j), j \in J_{A,P}$ are $\succcurlyeq$-maximals of $\{(B_1, Q_1), \ldots, (B_m, Q_m)\}$. First note that, apart from the trivial case $J_{\emptyset,P} = \emptyset$ for all $P \in \mathcal{P}$, it may be either $\mid J_{A,P} \mid= 1$, or else $\mid J_{A,P} \mid> 1$. In particular, for any $(A \neq \emptyset, P) \in 2^N \times \mathcal{P}$, let $P^A \in \mathcal{P}(A)$ denote the partition of A induced by $\left\{B_1, \ldots, B_{|P|}\right\} = P \in \mathcal{P}$, that is $\left\{A \cap B_1, \ldots, A \cap B_{|P|}\right\}$. Note that $\mid P^A \mid=\mid J_{A,P} \mid= 1$ either when $(A,P) \in \mathcal{E}$, or else when $A \subset B_j \in P$ for some $1 \leq j \leq\mid P \mid$. In the former case $\widehat{h}^{ext}(A,P) = \widehat{h}(A,P)$, while in the latter $\widehat{h}^{ext}(A,P) = \widehat{h}\left(A, P^A \cup P^{A^c}\right)$, in that $P^A = \{A\} \cup \{B_j \backslash A\}$ and thus $\left(A, P^A \cup P^{A^c}\right) \in \mathcal{E}$. On the other hand, $\mid P^A \mid=\mid J_{A,P} \mid> 1$ implies $\widehat{h}^{ext}(A,P) = \sum_{\widehat{A} \in P^A} \widehat{h}\left(\widehat{A}, P^A \cup P^{A^c}\right)$. Also note that, from another perspective, one may regard the extension $\widehat{h}^{ext}$ of games $\widehat{h}$ in partition function as the restriction of global coalitional games $h \in GC^N$ that coincide with $\widehat{h}$ on $\mathcal{E}$.

Definition 4 *The $\mathcal{E}$-restriction (in partition function form) of $h \in GC^N$ is*

$$h_{/\mathcal{E}}(A,P) = \sum\nolimits_{j \in J_{A,P}} h(B_j, Q_j) \text{ for all } (A,P) \in 2^N \times \mathcal{P}.$$

Clearly, $h_{/\mathcal{E}} \in GC_\emptyset^N$ for any $h \in GC^N$. Also, the vector space $GC_\mathcal{E}^N$ of $\mathcal{E}$-restricted global coalitional games constitutes a (proper) subspace of $GC_\emptyset^N$; in fact, its dimension is shown to be $\mid \mathcal{E} \mid< (2^n - 1) \times \mid \mathcal{P} \mid$ in the sequel.

Global games have been already introduced as 0-normalized partition functions $f : \mathcal{P} \rightarrow \mathbb{R} \mid f(P_0) = 0$. Note that a $\mathcal{E}$-restricted global coalitional game $h_{/\mathcal{E}}$ can be treated as a mapping that associates to each nonvoid coalition A a global subgame $f_A : \mathcal{P}(A^c) \rightarrow \mathbb{R}$ defined by $f_A\left(P^{A^c}\right) = h_{/\mathcal{E}}\left(A, \{A\} \cup P^{A^c}\right)$ (and $h_{/\mathcal{E}}\left(A, \{A\} \cup P^{A^c}\right) = h\left(A, \{A\} \cup P^{A^c}\right))$ for all $P^{A^c} \in \mathcal{P}(A^c)$. Nevertheless, this clearly requires to abandon the assumption of 0-normalization, i.e., $f_A\left(P_0^{A^c}\right) \gtreqless 0$. The remainder of this section focuses on the Möbius transforms of these latter global subgames (i.e., the sets of reals $\left\{\alpha_{P^{A^c}}(f_A) \mid P^{A^c} \in \mathcal{P}(A^c)\right\}$ for each nonvoid A; see Gilboa and Lehrer (1991a)), and on the Möbius transform of $h_{/\mathcal{E}}$ (i.e., the set of reals $\left\{\alpha_{A,P}\left(h_{/\mathcal{E}}\right) \mid (A,P) \in 2^N \times \mathcal{P}\right\}$ as defined by $(*)$ in the proof of theorem 2).

For any $\left\{B_1, \ldots, B_{|P|}\right\} = P \in \mathcal{P}$, let $\{Q_1, \ldots, Q_{k_P}\} = \{Q \in \mathcal{P} \mid P >^* Q\}$ denote the set of partitions covered by P. Then, any $Q_j, 1 \leq j \leq k_P$ has form $Q_j = P^{B_i^c} \cup Q_j^{B_i}$, with $Q_j^{B_i} \in \mathcal{P}(B_i)^{(2)}$ and $\mathcal{P}(B_i)^{(2)} = \{Q' \in \mathcal{P}(B_i) \mid 2 = \mid Q' \mid\}$ denoting the set of 2-block partitions of $B_i \in P$ (and $B_i^c = N \backslash B_i$). In words, any Q_j covered by P must equal this latter for all blocks $B_{i' \neq i}$, while dividing some block $B_i, 1 \leq i \leq \mid P \mid$ in two (new) blocks, i.e., $\{B_i', B_i \backslash B_i'\} = Q_j^{B_i}$ with $\emptyset \neq B_i' \subset B_i$. In particular, each $P \in \mathcal{P}$ covers $k_P = - \mid P \mid + \sum_{B \in P} 2^{|B|-1}$ partitions $Q_j, 1 \leq j \leq k_P$ (see Aigner (1979), ex. I.4 #6, p. 29), and clearly the same applies to any $P^A \in \mathcal{P}(A)$ for nonvoid A. Thus, any $(A,P) \in 2^N \times \mathcal{P}$ covers pairs either of the form $(B,Q) = (A \backslash i, P)$ with $i \in A$ (i.e., $A \sqsupset^* B$), or else of the form $(B,Q) = (A, Q_j)$ with $P >^* Q_j$ and $1 \leq j \leq k_P$ as above. For there are $\mid A \mid + k_P$ such covered pairs, let $\{(B_j, Q_j) \mid 1 \leq j \leq \mid A \mid + k_P\} =$

$$\{(B,Q) \mid (A,P) \succ^* (B,Q)\} = \left\{(B_1, P), \ldots, \left(B_{|A|}, P\right), (A, Q_1), \ldots, (A, Q_{k_P})\right\}$$

for each $(A,P) \in 2^N \times \mathcal{P}$. Also recall that any coalitional game $v \in C^N$ has Möbius transform $\left\{\alpha_A(v) \mid A \in 2^N\right\}$ given by

$$\alpha_A(v) = \sum_{B \subseteq A} (-1)^{|A \backslash B|} v(B) = v(A) - \sum_{\emptyset \neq I \subseteq \{1, \ldots, |A|\}} (-1)^{|I|+1} v\left(\underset{i \in I}{\cap} B_i\right),$$

where $A \sqsupset^* B_i = A \backslash i$ for each $i \in A$ when $A = \{1, \ldots, \mid A \mid\} \subseteq N$. Analogously, any global game $f \in G^N$ has Möbius transform $\{\alpha_P(f) \mid P \in \mathcal{P}\}$ given by

$$\alpha_P(f) = f(P) - \sum_{\emptyset \neq I \subseteq \{1, \ldots, k_P\}} (-1)^{|I|+1} f\left(\underset{i \in I}{\wedge} Q_i\right),$$

where $\{Q_1, \ldots, Q_{k_P}\} = \{Q \in \mathcal{P} \mid P >^* Q\}$ as above.

Theorem 5 *Any $h \in GC^N$ is $\mathcal{E}$-restricted (that is, $h \in GC_{\mathcal{E}}^N$ or, equivalently, $h = h_{/\mathcal{E}}$) iff $\alpha_{A,P}(h) = \alpha_{P^{A^c}}(f_A) - \sum_{B \subset A} \alpha_{P^{B^c}}(f_B)$ for all $(A,P) \in \mathcal{E}$ and $\alpha_{A,P}(h) = 0$ for all $(A,P) \notin \mathcal{E}$.*

Proof. Firstly consider the 'if' part for $(A,P)\in\mathcal{E}$, in which case[7]

$$\alpha_{A,P}\left(h_{/\mathcal{E}}\right)=h\left(A,P\right)-\sum_{k=1}^{|A|+k_P}\left(-1\right)^{k+1}\sum_{I\in\{1,\dots,|A|+k_P\}^{(k)}}h_{/\mathcal{E}}\left(\underset{i\in I}{\cap}B_i,\underset{i\in I}{\wedge}Q_i\right)=$$

$$=h\left(A,P\right)-\sum_{k=1}^{k_{P^{A^c}}}\left(-1\right)^{k+1}\sum_{I\in\left\{1,\dots,k_{P^{A^c}}\right\}^{(k)}}h\left(A,\underset{i\in I}{\wedge}Q_i\right)+$$

$$-\sum_{k=1}^{|A|+k_P}\left(-1\right)^{k+1}\sum_{\substack{I\in\{1,\dots,|A|+k_P\}^{(k)}\\ A\notin\left(\underset{i\in I}{\wedge}Q_i\right)^{\underset{i\in I}{\cap}B_i}}}h_{/\mathcal{E}}\left(\underset{i\in I}{\cap}B_i,\underset{i\in I}{\wedge}Q_i\right),$$

where each $Q_i, 1\leq i\leq k_{P^{A^c}}$ has form $Q_i=\{A\}\cup Q_i^{A^c}$ with $P^{A^c}>^* Q_i^{A^c}$. Thus, the first line equals $\alpha_{P^{A^c}}\left(f_A\right)$. On the other hand, the second line groups all terms such that $h_{/\mathcal{E}}\left(\underset{i\in I}{\cap}B_i,\underset{i\in I}{\wedge}Q_i\right)\neq h\left(A,\widetilde{Q}\right)$ for some $\widetilde{Q}\in\mathcal{P}$, in that $\left(\underset{i\in I}{\wedge}Q_i\right)^{\underset{i\in I}{\cap}B_i}$ denotes the partition of $\underset{i\in I}{\cap}B_i$ induced by $\underset{i\in I}{\wedge}Q_i$. Now, for any $B\subset A$, consider all $I\subseteq\{1,\dots,\mid A\mid+k_P\}$ such that $\pm h\left(B,P^B\cup P^{B^c}\right)$ appears as a summand in the second line. Firstly, this occurs for $\mid I\mid=1$, i.e., $I=\{i_1'\}$, when (a) $B_{i_1'}=A$ and $Q_{i_1'}=P^B\cup P^{B^c}=\{B\}\cup\{A\backslash B\}\cup P^{A^c}$, and thus with sign $-$. Secondly, it occurs for $\mid I\mid=\mid A\backslash B\mid$, i.e., $I=\left\{i_1,\dots,i_{|A\backslash B|}\right\}$ (and thus with sign $(-1)^{|A\backslash B|+1}$), when (b) $B_{i_j}=A\backslash j$, with $j\in A\backslash B$ and $1\leq j\leq\mid A\backslash B\mid$ (and thus $A\backslash B=\{1,\dots,\mid A\backslash B\mid\}\subset N$), while $Q_{i_j}=P$ for all $i_j\in I$. Thirdly (and lastly), it occurs for (c) $\mid I\mid=\mid A\backslash B\mid+1$, i.e., $I=\{i_1'\}\cup\left\{i_1,\dots,i_{|A\backslash B|}\right\}$ (and thus with sign $(-1)^{|A\backslash B|+2}$). Clearly, cases (b) and (c) cancel out, in that they display opposite sign for any $\mid B\mid$. Therefore, $\alpha_{A,P}\left(h_{/\mathcal{E}}\right)=\alpha_{P^{A^c}}\left(f_A\right)-\sum_{B\subset A}h\left(B,P^B\cup P^{B^c}\right)+$

$$-\sum_{k=1}^{|A|+k_P}\left(-1\right)^{k+1}\sum_{\substack{I\in\{1,\dots,|A|+k_P\}^{(k)}\\ A\notin\left(\underset{i\in I}{\wedge}Q_i\right)^{\underset{i\in I}{\cap}B_i}\\ \left(B,P^B\cup P^{B^c}\right)\neq\left(\underset{i\in I}{\cap}B_i,\underset{i\in I}{\wedge}Q_i\right)\text{ for all }B\subset A}}h_{/\mathcal{E}}\left(\underset{i\in I}{\cap}B_i,\underset{i\in I}{\wedge}Q_i\right)=$$

[7] For any finite set S, in combinatorics $S^{(k)}$ denotes the k-th level set of 2^S, i.e., the set of all k-cardinal subsets of S, thus $\mid S^{(k)}\mid=\binom{|S|}{k}$. Also, note that $(\emptyset,P)\notin\mathcal{E}$ for all $P\in\mathcal{P}$.

$= \alpha_{P^{A^c}}(f_A) +$

$$-\sum_{B\subset A}\left[h\left(B,P^B\cup P^{B^c}\right)-\sum_{k=1}^{k_{P^{B^c}}}(-1)^{k+1}\sum_{\substack{I\in\{1,\dots,|A|+k_P\}^{(k)}\\ \left(\underset{i\in I}{\wedge}Q_i\right)^B=P^B\\ P^{B^c}>^*Q_i^{B^c}\text{ for all }i\in I}}h\left(B,\underset{i\in I}{\wedge}Q_i\right)\right]=$$

$= \alpha_{P^{A^c}}(f_A) - \sum_{B\subset A}\alpha_{P^{B^c}}(f_B)$ as desired.

Secondly, the 'if' part for $(A,P)\notin\mathcal{E}$ requires to distinguish between case (i) $\mid P^A\mid=\mid J_{A,P}\mid=1$, and case (ii) $\mid P^A\mid=\mid J_{A,P}\mid>1$. For the former case, let $J_{A,P}=\{j_{A,P}\}$ and $\widehat{J}_{A,P}=\left\{1\le j\le\mid A\mid+k_P\mid(B_j,Q_j)\not\succcurlyeq\left(B_{j_{A,P}},Q_{j_{A,P}}\right)\right\}$, noting that $\widehat{J}_{A,P}\neq\emptyset$ for all $(A,P)\notin\mathcal{E}$ such that $\mid P^A\mid>1$. Therefore,

$$\alpha_{A,P}\left(h_{/\mathcal{E}}\right)=(1-1)^{|\widehat{J}_{A,P}|}\alpha_{B_{j_{A,P}},Q_{j_{A,P}}}\left(h_{/\mathcal{E}}\right)=0,$$

where $\alpha_{B_{j_{A,P}},Q_{j_{A,P}}}\left(h_{/\mathcal{E}}\right)=$

$$=h\left(B_{j_{A,P}},Q_{j_{A,P}}\right)-\sum_{\emptyset\neq I\subseteq\left\{1,\dots,|B_{j_{A,P}}|+k_{Q_{j_{A,P}}}\right\}}(-1)^{|I|+1}h_{/\mathcal{E}}\left(\underset{i\in I}{\cap}B_i,\underset{i\in I}{\wedge}Q_i\right),$$

while $(1-1)^{|\widehat{J}_{A,P}|}=1+\sum_{\emptyset\neq I\subseteq\widehat{J}_{A,P}\neq\emptyset}(-1)^{|I|}=0$. Similarly, for case (ii), let $J_{A,P}=\left\{j^1_{A,P},\dots,j^{|P^A|}_{A,P}\right\}$ as well as

$$\widehat{J}^i_{A,P}=\left\{1\le j\le\mid A\mid+k_P\mid(B_j,Q_j)\not\succcurlyeq\left(B_{j^i_{A,P}},Q_{j^i_{A,P}}\right)\right\}$$

for $1\le i\le\mid P^A\mid$, noting that $\widehat{J}^i_{A,P}\neq\emptyset$, all i. Therefore,

$$\alpha_{A,P}\left(h_{/\mathcal{E}}\right)=\sum_{i=1}^{|P^A|}(1-1)^{|\widehat{J}^i_{A,P}|}\alpha_{B_{j^i_{A,P}},Q_{j^i_{A,P}}}\left(h_{/\mathcal{E}}\right)=0$$

as for the previous case.

Eventually, concerning the 'only if' part, note that if $h \in GC^N$ has Möbius transform as defined by the theorem, then $h(A,P) =$

$$= \sum_{(B,Q)\preccurlyeq(A,P)} \alpha_{B,Q}(h) = \sum_{j\in J_{A,P}} \sum_{\mathcal{E}\ni(B',Q')\preccurlyeq(B_j,Q_j)} \alpha_{B',Q'}(h) =$$

$$= \sum_{j\in J_{A,P}} \sum_{\mathcal{E}\ni(B',Q')\preccurlyeq(B_j,Q_j)} \left(\alpha_{Q'^{B'^c}}(f_{B'}) - \sum_{B''\subset B'} \alpha_{Q'^{B''^c}}(f_{B''})\right) =$$

$$= \sum_{j\in J_{A,P}} \left(\begin{array}{c} h(B_j,Q_j) - \sum_{\emptyset\neq I\subseteq\left\{1,\dots,k_{Q_j^{B_j^c}}\right\}} (-1)^{|I|+1} h\left(B_j, \bigwedge_{i\in I} Q_i\right) + \\ - \sum_{B'\subset B_j} \alpha_{Q_j^{B'^c}}(f_{B'}) + \\ + \sum_{\mathcal{E}\ni\left(\widehat{B},\widehat{Q}\right)\prec(B_j,Q_j)} \left(\alpha_{\widehat{Q}^{\widehat{B}^c}}\left(f_{\widehat{B}}\right) - \sum_{\widehat{\widehat{B}}\subset\widehat{B}} \alpha_{\widehat{Q}^{\widehat{\widehat{B}}^c}}\left(f_{\widehat{\widehat{B}}}\right)\right) \end{array} \right) =$$

$= \sum_{j\in J_{A,P}} h(B_j,Q_j)$ as wanted. ■

The idea that the larger cooperation the greater the worth that gets produced, which is roughly captured by monotonicity, is made more precise and enforced by convexity, and even more by total positivity. In particular, in a convex global coalitional game any two disjoint coalitions A, B, considered under any two partitions P, Q, always find it convenient to merge under $P \vee Q$. Thus, merging must be (weakly) convenient even when $(P \vee Q)^{A\cup B} = P^{A\cup B} = Q^{A\cup B}$. Nevertheless, in such a case solely $(A \cup B)^c$, and not $A \cup B$, increases its cohesion by switching from P, Q to $P \vee Q$. More generally, the representation coefficients $\alpha_{A,P}(h)$ quantify the net surplus of worth produced by (A,P) with respect to all $(B,Q) \prec (A,P)$. Therefore, a totally positive and monotone game describes a situation where cooperation most widely displays synergies, making it interesting to consider the intersection of GC^N_{TPM} with subspaces of $GC^N_\emptyset$. In fact, $GC^N_{TPM} \cap GC^N_{\mathcal{E}}$ is the cone consisting of those h that for all $(A,P) \in 2^N \times \mathcal{P}$ satisfy $\alpha_{A,P}(h) \geq 0$ if $A \in P$ and $\alpha_{A,P}(h) = 0$ if $A \notin P$. In particular, if $a \in \mathbb{R}_+$ and $h, h' \in GC^N_{TPM} \cap GC^N_{\mathcal{E}}$, then $ah(A,P) = \sum_{j\in J_{A,P}} ah(B_j,Q_j)$ and $(h+h')(A,P) = \sum_{j\in J_{A,P}} (h+h')(B_j,Q_j)$, as well as $\alpha_{A,P}(ah) = a\alpha_{A,P}(h)$ and $\alpha_{A,P}(h+h') = \alpha_{A,P}(h) + \alpha_{A,P}(h')$ for all $(A \neq \emptyset, P) \in 2^N \times \mathcal{P}$, implying $ah, (h+h') \in GC^N_{TPM} \cap GC^N_{\mathcal{E}}$. It is also evident that $GC^N_{\mathcal{E}}$ corresponds to the subspace spanned by $\left\{g_{A,P} \text{ (or } \widetilde{g}_{A,P}) \mid (A \neq \emptyset, P) \in \mathcal{E} \subset 2^N \times \mathcal{P}\right\}$, and thus with dimension $\sum_{P\in\mathcal{P}} |\, P \,| = |\, \mathcal{P} \,| + \sum_{P\in\mathcal{P}} (|\, P \,| - 1) = |\, \mathcal{E} \,| < (2^n - 1) \times |\, \mathcal{P} \,|$. Interestingly, $h \in GC^N_{TPM} \cap GC^N_{\mathcal{E}}$ implies $\alpha_{P^{A^c}}(f_A) \geq \sum_{B\subset A} \alpha_{P^{B^c}}(f_B)$ for all $(A,P) \in \mathcal{E}$, as well as monotonicity and total positivity (i.e., total monotonicity) of coalitional games $v \in C^N$ imply $v(A) \geq \sum_{B\subset A} \alpha_B(v)$ for all $A \subseteq N$.

4 The Shapley value

A solution is any function $\phi : GC^N_\emptyset \to AC^N$. In words, a solution ϕ associates an n-dimensional payoff vector $\phi(h) = (\phi_i(h))_{i\in N} \in \mathbb{R}^n$ (that is, an additive coalitional game $\phi(h) \in AC^N$) to each $h \in GC_{\emptyset}^N$. Attention is usually placed on solutions satisfying certain requisites; in particular, consider the following axioms:

linearity: $\phi(h+h') = \phi(h) + \phi(h')$ and $\phi(ah) = a\phi(h)$ for all $h, h' \in GC_{\emptyset}^N$ and $a \in \mathbb{R}$;

nonnegativity: $\phi_i(h) \geq \phi_i(h') \geq 0$ for all $i \in N$ and for all $h, h' \in GC_{WM}^N$ such that $h \geq h'$ (i.e., $h(A,P) \geq h'(A,P)$ for all $(A,P) \in 2^N \times \mathcal{P}$);

symmetry: $\phi(h) = \pi\phi(\pi h)$ for all $\pi \in \Pi^N$, where Π^N denotes the set of permutations of N and game πh is defined by $\pi h(\pi A, \pi P) = h(A,P)$ for all $(A,P) \in 2^N \times \mathcal{P}$;

dummy: if $h(A,P) = h\left(A\backslash i, \{i\} \cup P^{N\backslash i}\right) + h(\{i\}, P_0)$ for $i \in N$ and all $(A,P) \in 2^N \times \mathcal{P}$ such that $i \in A$, then $\phi_i(h) = h(\{i\}, P_0)$, and $i \in N$ is a dummy player in h;

efficiency: $\sum_{i \in N} \phi_i(h) = h(N,N)$.

Linearity, symmetry, efficiency and dummy are as usual. In particular, this latter axiom simply formalizes the idea of a dummy player in a global coalitional game. Also note that $\pi P = \{\pi B_1, \ldots, \pi B_m\}$ when $P = \{B_1, \ldots, B_m\}$ as well as $\pi\phi(\pi h) = (\phi_{\pi i}(\pi h))_{i \in N}$. On the other hand, nonnegativity is somehow new. It is clearly intended to substitute the traditional monotonicity axiom, in that restricting attention to monotone global coalitional games has the above mentioned undesired implications. In particular, in the presence of nonnegativity of $h \in GC_{\emptyset}^N$, weak monotonicity seems sufficient for requiring nonnegativity of each player's payoff, while the implication $h \geq h' \Rightarrow \phi(h) \geq \phi(h')$ is theoretically acceptable as well as technically useful in a proof. Following Weber (1988), solutions satisfying linearity, dummy and nonnegativity may be defined to be probabilistic. Furthermore, probabilistic solutions that also satisfy symmetry may be defined to be semivalues, while adding efficiency leads to the class of values. In the sequel, let $\mathbf{C}^+ \subseteq GC_{\emptyset}^N$ be any cone.

Theorem 6 *If $\phi : \mathbf{C}^+ \to \mathbb{R}^n$ satisfies linearity, then there exist real constants $\left\{\widehat{p}_{A,P}^i \mid (A \neq \emptyset, P) \in 2^N \times \mathcal{P}\right\}$ for each $i \in N$ such that*

$$\phi_i(h) = \sum_{(A \neq \emptyset, P) \in 2^N \times \mathcal{P}} \widehat{p}_{A,P}^i h(A,P).$$

Proof. See Weber (1988), theorem 1, p. 104. ■

This enables to immediately concentrate on the dummy axiom.

Theorem 7 *If $\phi : \mathbf{C}^+ \to \mathbb{R}^n$ satisfies linearity and dummy, then there exist real constants $\left\{p_{A,P}^i \mid (A,P) \in 2^N \times \mathcal{P}, A \ni i\right\}$ for each $i \in N$ such that*

$$\phi_i(h) = \sum_{(A,P) \in 2^N \times \mathcal{P} \mid A \ni i} p_{A,P}^i \left[h(A,P) - h\left(A\backslash i, \{i\} \cup P^{N\backslash i}\right)\right]$$

and $\sum_{(A,P) \in 2^N \times \mathcal{P} \mid A \ni i} p_{A,P}^i = 1$.

Proof. Firstly note that linearity alone implies $\phi_i(g_{N,N}) = \widehat{p}^i_{N,N}$, while dummy and linearity together imply

$$\phi_i\left(g_{N\backslash i,\{i\}\cup\{N\backslash i\}}\right) = 0 = \widehat{p}^i_{N,N} + \widehat{p}^i_{N\backslash i,N} + \widehat{p}^i_{N,\{i\}\cup\{N\backslash i\}} + \widehat{p}^i_{N\backslash i,\{i\}\cup\{N\backslash i\}},$$

that is $\widehat{p}^i_{N,N} = -\left(\widehat{p}^i_{N\backslash i,N} + \widehat{p}^i_{N,\{i\}\cup\{N\backslash i\}} + \widehat{p}^i_{N\backslash i,\{i\}\cup\{N\backslash i\}}\right)$, for all $i \in N$ (where $\{i\}\cup\{N\backslash i\} = \{i, N\backslash i\} \in \mathcal{P}^{(2)} = \mathcal{P}^{(2)}(N)$). Secondly, consider any $A \subseteq N\backslash i$ and $P \in \mathcal{P}$ with $\{i\} \in P$, so that

$$\begin{aligned} 0 &= \phi_i(g_{A,P}) = \sum_{(B,Q)\succcurlyeq(A,P)} \widehat{p}^i_{B,Q} = \\ &= \sum_{(B,Q)\succcurlyeq(A,P)|B\ni i} \left(\widehat{p}^i_{B,Q} + \widehat{p}^i_{B\backslash i,Q} + \widehat{p}^i_{B,\{i\}\cup Q^{N\backslash i}} + \widehat{p}^i_{B\backslash i,\{i\}\cup Q^{N\backslash i}}\right), \end{aligned}$$

and thus, by induction, $\widehat{p}^i_{B,Q} = -\left(\widehat{p}^i_{B\backslash i,Q} + \widehat{p}^i_{B,\{i\}\cup Q^{N\backslash i}} + \widehat{p}^i_{B\backslash i,\{i\}\cup Q^{N\backslash i}}\right)$ for all $(B,Q) \in 2^N \times \mathcal{P}$ such that $B \ni i$. Also, $\{i\} \in Q$ implies $\widehat{p}^i_{B,Q} = -\widehat{p}^i_{B\backslash i,Q}$, and in particular $\widehat{p}^i_{\{i\},P_0} = -\widehat{p}^i_{\emptyset,P_0}$, where $\widehat{p}^i_{\emptyset,P_0}$ may be arbitrarily set. Therefore, simply set $p^i_{\{i\},P_0} = \widehat{p}^i_{\{i\},P_0}$ for all $i \in N$ as well as

$$p^i_{A,P} = \widehat{p}^i_{A,P} = -\left(\widehat{p}^i_{A,\{i\}\cup P^{N\backslash i}} + \widehat{p}^i_{A\backslash i,P} + \widehat{p}^i_{A\backslash i,\{i\}\cup P^{N\backslash i}}\right)$$

for all $(A,P) \in 2^N \times \mathcal{P}$ such that $A \ni i$. Eventually, noting that

$$\phi_i\left(g_{\{i\},P_0}\right) = 1 = \sum_{(A,P)\in 2^N\times\mathcal{P}|A\ni i} p^i_{A,P}$$

for all $i \in N$ completes the proof. ■

Theorem 8 *If $\phi : \mathbf{C}^+ \to \mathbb{R}^n$ satisfies linearity, dummy and nonnegativity, then there exist real constants $\left\{p^i_{A,P} \mid (A,P) \in 2^N \times \mathcal{P}, A \ni i\right\}$ for each $i \in N$ such that*

$$\phi_i(h) = \sum_{(A,P)\in 2^N\times\mathcal{P}|A\ni i} p^i_{A,P}\left[h(A,P) - h\left(A\backslash i, \{i\}\cup P^{N\backslash i}\right)\right],$$

$\sum_{(A,P)\in 2^N\times\mathcal{P}|A\ni i} p^i_{A,P} = 1$ and $p^i_{A,P} \geq 0$ for all $P \in \mathcal{P}$.

Proof. For any player $i \in N$, consider any (possibly void) coalition $A \subseteq N\backslash i$ and any partition $P \in \mathcal{P}$, and let games $\widehat{g}_{A,P}$ and $\widehat{\widehat{g}}_{A,P}$ be defined by

$$\widehat{g}_{A,P}(B,Q) = \begin{cases} 1 \text{ if } B \supset A, Q \geq P \\ 0 \text{ otherwise} \end{cases}, \quad \widehat{\widehat{g}}_{A,P}(B,Q) = \begin{cases} 1 \text{ if } B \supset A, Q > P \\ 0 \text{ otherwise} \end{cases}$$

for all $(B \neq \emptyset, Q) \in 2^N \times \mathcal{P}$. Note that both $\widehat{g}_{A,P}$ and $\widehat{\widehat{g}}_{A,P}$ are weakly monotone and that $\widehat{g}_{A,P}(B,Q) \geq \widehat{\widehat{g}}_{A,P}(B,Q)$ for all $(B \neq \emptyset, Q) \in 2^N \times \mathcal{P}$. Thus, when $\{i\} \in P$, nonnegativity requires

$$\phi_i(\widehat{g}_{A,P}) = \sum_{Q \geq P} p^i_{A\cup i,Q} \geq \phi_i\left(\widehat{\widehat{g}}_{A,P}\right) = \sum_{Q > P} p^i_{A\cup i,Q} \geq 0,$$

implying $p^i_{A\cup i,P} \geq 0$ for all $A \subseteq N\backslash i, P \in \mathcal{P}, \{i\} \in P$. Similarly, $\{i\} \notin P$ yields

$$\phi_i(\widehat{g}_{A,P}) = \sum_{\substack{B \supset A, B \ni i \\ Q \geq P}} p^i_{B,Q} \geq \phi_i\left(\widehat{\widehat{g}}_{A,P}\right) = \sum_{\substack{B \supset A, B \ni i \\ Q > P}} p^i_{B,Q} \geq 0,$$

so that $p^i_{B,P} \geq 0$ for all $B \subseteq N, B \ni i, P \in \mathcal{P}, \{i\} \notin P$ as wanted. ■

As already mentioned, these last three results enable to characterize the class of probabilistic solutions. In fact, such solutions may actually be defined as those satisfying linearity, dummy and nonnegativity (see Weber (1988), theorems 4 and 5), and clearly identify situations where i's payoff is the expectation of random variable $\left\{h(A,P) - h\left(A\backslash i, \{i\} \cup P^{N\backslash i}\right) \mid (A,P) \in 2^N \times \mathcal{P}, A \ni i\right\}$ with respect to i's subjective probability $\left\{p^i_{A,P} \mid (A,P) \in 2^N \times \mathcal{P}, A \ni i\right\}$, all $i \in N$.

Attention is now focused on the symmetry and efficiency axioms. In particular, the former characterizes semivalues by requiring the $p^i_{A,P}$'s to depend solely on A's, P's, and P's blocks' cardinalities (i.e., $\mid A \mid, \mid P \mid$ and $\mid B_j \mid$ for $1 \leq j \leq \mid P \mid$), while the latter characterizes values by means of an additional normalization condition. In particular, this latter results to be very much similar to the normalization condition found by Gilboa and Lehrer (1991a) as characterizing their Shapley value of global games.

Theorem 9 *Any probabilistic solution satisfying symmetry has constants* $p^i_{A,P}$ *that depend solely on* $\mid A \mid= a, \mid P \mid= m, \{\mid B_1 \mid, \ldots, \mid B_m \mid\} = \{b_j\}_{1\leq j\leq m}$, *i.e.,*

$$p^i_{A,P} = \begin{cases} p^{\{i\}\in}_{\left(a,m,\{b_j\}_{1\leq j\leq m}\right)} & \text{for all } A \subseteq N, A \ni i \text{ and } P \in \mathcal{P} \text{ such that } \{i\} \in P, \\ p^{\{i\}\notin}_{\left(a,m,\{b_j\}_{1\leq j\leq m}\right)} & \text{for all } A \subseteq N, A \ni i \text{ and } P \in \mathcal{P} \text{ such that } \{i\} \notin P. \end{cases}$$

Furthermore,

$$\begin{array}{lll} p^{\{i\}\in}_{\left(a,m,\{b_j\}_{1\leq j\leq m}\right)} = p^{\{j\}\in}_{\left(a,m,\{b_j\}_{1\leq j\leq m}\right)} = p^{\in}_{\left(a,m,\{b_j\}_{1\leq j\leq m}\right)} & & \text{for all } i,j \in N \\ p^{\{i\}\notin}_{\left(a,m,\{b_j\}_{1\leq j\leq m}\right)} = p^{\{j\}\notin}_{\left(a,m,\{b_j\}_{1\leq j\leq m}\right)} = p^{\notin}_{\left(a,m,\{b_j\}_{1\leq j\leq m}\right)} & & \text{and } A \subseteq N, P \in \mathcal{P}. \end{array}$$

Proof. Let $A \subseteq N\backslash i, P \in \mathcal{P}$ and consider a permutation $\pi \in \Pi^N$ such that $\pi i = i$. Also let $B = \pi A, Q = \pi P$ and games $\widehat{g}, \widehat{\widehat{g}}$ be defined as before. Then

$$\phi_{\pi i}\left(\pi\widehat{\widehat{g}}_{A,P}\right) = \phi_{\pi i}\left(\widehat{\widehat{g}}_{\pi A,\pi P}\right) = \phi_i\left(\widehat{\widehat{g}}_{B,Q}\right) = \phi_i\left(\widehat{\widehat{g}}_{A,P}\right),$$

with the last equality following from symmetry. Firstly consider the case where i constitutes a block on her own in P (and thus in Q as well), i.e., $P \ni \{i\} \in Q$, yielding

$$\phi_i\left(\widehat{\widehat{g}}_{B,Q}\right) = \sum_{Q'>Q} p^i_{B\cup i,Q'} = \phi_i\left(\widehat{\widehat{g}}_{A,P}\right) = \sum_{P'>P} p^i_{A\cup i,P'}.$$

Nevertheless, for game $\widehat{g}$ symmetry imposes

$$\phi_i\left(\widehat{g}_{B,Q}\right) = \sum_{Q'\geq Q} p^i_{B\cup i,Q'} = \phi_i\left(\widehat{g}_{A,P}\right) = \sum_{P'\geq P} p^i_{A\cup i,P'},$$

so that

$$\phi_i\left(\widehat{g}_{B,Q}\right) - \phi_i\left(\widehat{\widehat{g}}_{B,Q}\right) = \phi_i\left(\widehat{g}_{A,P}\right) - \phi_i\left(\widehat{\widehat{g}}_{A,P}\right)$$

implies $p^i_{B\cup i,Q} = p^i_{A\cup i,P}$ for all $A, B \subseteq N\backslash i$ such that $\mid A \mid=\mid B \mid$ and for all $P, Q \in \mathcal{P}$ such that $Q = \pi P, \{i\} \in P, Q$. On the other hand, if $\{i\} \notin P, Q$, then

$$\begin{aligned}
\phi_i\left(\widehat{\widehat{g}}_{B,Q}\right) &= \sum_{\substack{B'\supset B, B'\ni i\\ Q'>Q}} p^i_{B',Q'} = \phi_i\left(\widehat{\widehat{g}}_{A,P}\right) = \sum_{\substack{A'\supset A, A'\ni i\\ P'>P}} p^i_{A',P'}\\
\phi_i\left(\widehat{g}_{B,Q}\right) &= \sum_{\substack{B'\supset B, B'\ni i\\ Q'\geq Q}} p^i_{B',Q'} = \phi_i\left(\widehat{g}_{A,P}\right) = \sum_{\substack{A'\supset A, A'\ni i\\ P'\geq P}} p^i_{A',P'},
\end{aligned}$$

so that $\phi_i\left(\widehat{g}_{B,Q}\right) - \phi_i\left(\widehat{\widehat{g}}_{B,Q}\right) = \phi_i\left(\widehat{g}_{A,P}\right) - \phi_i\left(\widehat{\widehat{g}}_{A,P}\right)$ implies $p^i_{B',Q} = p^i_{A',P}$ for all $A', B' \subseteq N, A', B' \ni i$ such that $\mid A' \mid=\mid B' \mid$ and for all $P, Q \in \mathcal{P}$ such that $\{i\} \notin P, Q$. These results show that both when $\{i\} \in P$ as well as when $\{i\} \notin P$ the $p^i_{A,P}$'s depend solely on A's, P's and P's blocks' cardinalities for all $i \in N$. Therefore, the next step consist in showing that $p^j_{A\cup j,P} = p^i_{A\cup i,P}$ for all $i, j \in N$ whenever $(A, P) \in 2^N \times \mathcal{P}$ is such that $i, j \notin A$ and either $\{i\}, \{j\} \in P$, or else $\{i\}, \{j\} \notin P$. In words, i and j must be considered when each of them (separately) joins some (possibly void) coalition $A \subseteq N\backslash \{i, j\}$, and for any partition P where either they both constitute a 1-element block, or else each of them belongs to some larger block (that is, $\{i, j\} \subseteq B \in P$ or $i \in B \in P \ni B' \ni j$, with $\mid B \mid, \mid B' \mid\geq 2$). Let $\pi \in \Pi^N$ be such that $\pi i = j, \pi j = i, \pi k = k$ for $k \in N\backslash \{i, j\}$. Also let $A \subseteq N\backslash \{i, j\}$, implying $\pi A = A$, and firstly consider the case where $\{i\}, \{j\} \in P \in \mathcal{P}$. Then symmetry requires

$$\phi_{\pi i}\left(\pi\widehat{\widehat{g}}_{A,P}\right) = \phi_{\pi i}\left(\widehat{\widehat{g}}_{\pi A,\pi P}\right) = \phi_j\left(\widehat{\widehat{g}}_{A,P}\right) = \phi_i\left(\widehat{\widehat{g}}_{A,P}\right) = \phi_{\pi j}\left(\pi\widehat{\widehat{g}}_{A,P}\right),$$

where the third equality follows from symmetry. Therefore,

$$\begin{aligned}
\phi_j\left(\widehat{\widehat{g}}_{A,P}\right) &= \sum_{P'>P} p^j_{A\cup j,P'} = \phi_i\left(\widehat{\widehat{g}}_{A,P}\right) = \sum_{P'>P} p^i_{A\cup i,P'}\\
\phi_j\left(\widehat{g}_{A,P}\right) &= \sum_{P'\geq P} p^j_{A\cup j,P'} = \phi_i\left(\widehat{g}_{A,P}\right) = \sum_{P'\geq P} p^i_{A\cup i,P'}.
\end{aligned}$$

Thus, $\phi_j(\widehat{g}_{A,P}) - \phi_j\left(\widehat{\widehat{g}}_{A,P}\right) = \phi_i(\widehat{g}_{A,P}) - \phi_i\left(\widehat{\widehat{g}}_{A,P}\right)$ implies $p^j_{A\cup j,P} = p^i_{A\cup i,P}$ for all $A \subseteq N\backslash\{i,j\}, P \in \mathcal{P}, \{i\}, \{j\} \in P$. Now consider that when $\{i\}, \{j\} \notin P$ symmetry imposes

$$\begin{aligned}
\phi_j\left(\widehat{\widehat{g}}_{A,P}\right) &= \sum_{\substack{A'\supset A, A'\ni j\\ P'>P}} p^j_{A',P'} = \phi_i\left(\widehat{\widehat{g}}_{A,P}\right) = \sum_{\substack{A'\supset A, A'\ni i\\ P'>P}} p^i_{A',P'} \\
\phi_j(\widehat{g}_{A,P}) &= \sum_{\substack{A'\supset A, A'\ni j\\ P'\geq P}} p^j_{A',P'} = \phi_i(\widehat{g}_{A,P}) = \sum_{\substack{A'\supset A, A'\ni i\\ P'\geq P}} p^i_{A',P'},
\end{aligned}$$

so that $p^j_{A\cup j,P} = p^i_{A\cup i,P}$ for all $A \subseteq N\backslash\{i,j\}, P \in \mathcal{P}, \{i\}, \{j\} \notin P$. Clearly, the coarsest partition such that $\{i\}, \{j\} \notin P$ is $P = N \in \mathcal{P}$, thus let

$$p^i_{N,N} = p^j_{N,N} = \phi_i(g_{N,N}) = \phi_j(g_{N,N}) = p^{\notin}_{(n,1,\{1\})}$$

for all $i, j \in N$. On the other hand, the finest partition such that $\{i\}, \{j\} \in P$ is $P = P_0$. Thus, for each $i \in N$ let $p^i_{\{i\},P_0} = \phi_i\left(g_{\{i\},P_0}\right) - \sum_{\substack{(A,P)\in 2^N\times\mathcal{P}\\ (A,P)\succ(\{i\},P_0)}} p^i_{A,P} =$

$$= 1 - \left(\sum_{\substack{(A,P)\succ(\{i\},P_0)\\ \{i\}\in P}} p^{\in}_{\left(a,m,\{b_j\}_{1\leq j\leq n}\right)} + \sum_{\substack{(A,P)\succ(\{i\},P_0)\\ \{i\}\notin P}} p^{\notin}_{\left(a,m,\{b_j\}_{1\leq j\leq n}\right)}\right).$$

By symmetry, $\phi_i\left(g_{\{i\},P_0}\right) = \phi_j\left(g_{\{j\},P_0}\right)$ for all $i, j \in N$, so that the proof gets complete simply by setting $p^i_{\{i\},P_0} = p^{\in}_{\left(1,n,\left\{\underbrace{1,\ldots,1}_{n}\right\}\right)}$ for all $i \in N$. ■

Theorem 10 *Any probabilistic solution satisfying efficiency has constants $p^i_{A,P}$ satisfying*

$$\sum_{i\in A} p^i_{A,P} = \begin{cases} 1 \text{ if } A = N, P = N \\ 0 \text{ if } \begin{cases} A = N, P \neq N \text{ or} \\ A \neq N, \{j \in A^c \mid \{j\} \in P\} = \emptyset \end{cases} \\ \sum\limits_{\substack{j\in A^c\\ \{j\}\in P}} \sum\limits_{\substack{Q\geq P\\ \{j\}\cup Q^{N\backslash j} = P}} p^j_{A\cup j,Q} \text{ if } A \neq N, \{j \in A^c \mid \{j\} \in P\} \neq \emptyset \end{cases}$$

for all $(A \neq \emptyset, P) \in 2^N \times \mathcal{P}$.

Proof. Efficiency implies $1 = \sum_{i\in N} \phi_i(g_{N,N}) = \sum_{i\in N} p^i_{N,N}$. Furthermore, $\phi_i(g_{N,P}) = p^i_{N,N} + \sum_{N\neq Q\geq P} p^i_{N,Q}$ for all $i \in N$ and $\mathcal{P} \ni P < N$, so that

$$1 = \sum_{i\in N} \phi_i(g_{N,P}) = \sum_{i\in N} p^i_{N,N} + \sum_{i\in N} \sum_{N\neq Q\geq P} p^i_{N,Q} = 1 + \sum_{i\in N} \sum_{N\neq Q\geq P} p^i_{N,Q},$$

and thus $\sum_{i\in N}\sum_{N\neq Q\geq P}p^i_{N,Q}=0$ by efficiency, and $\sum_{i\in N}p^i_{N,P}=0$ for all $P\neq N$ by induction. Next consider that

$$\begin{aligned}\sum_{i\in N}\phi_i(h) &= \sum_{i\in N}\sum_{\substack{(A,P)\in 2^N\times\mathcal{P}\\ A\ni i}} p^i_{A,P}\left[h(A,P)-h\left(A\backslash i,\{i\}\cup P^{N\backslash i}\right)\right]=\\ &= \sum_{(A,P)\in 2^N\times\mathcal{P}} h(A,P)\left[\sum_{i\in A}p^i_{A,P}-\sum_{\substack{j\in A^c\\ \{j\}\in P}}\sum_{\substack{Q\geq P\\ \{j\}\cup Q^{N\backslash j}=P}} p^j_{A\cup j,Q}\right]\end{aligned}$$

for all h, so that any probabilistic solution satisfying the conditions of the theorem is also efficient. Now consider any $(A,P)\in 2^N\times\mathcal{P}$ such that $A^c\neq\emptyset$ and $\{j\in A^c\mid\{j\}\in P\}=\emptyset$, so that $(g_{A,P}-\widehat{g}_{A,P})(N,N)=0=$

$$\begin{aligned}&\sum_{i\in A}\sum_{(B,Q)\succcurlyeq(A,P)}p^i_{B,Q}+\sum_{j\in A^c}\sum_{\substack{(B,Q)\succ(A,P)\\ B\ni j}}p^j_{B,Q}+\\ &-\sum_{i\in A}\sum_{\substack{(B,Q)\succ(A,P)\\ B\supset A}}p^i_{B,Q}+\sum_{j\in A^c}\sum_{\substack{(B,Q)\succ(A,P)\\ B\ni j}}p^j_{B,Q},\end{aligned}$$

and therefore $\sum_{i\in A}\sum_{Q\geq P}p^i_{A,Q}=0$, implying $\sum_{i\in A}p^i_{A,P}=0$ for all such pairs (A,P) by induction. On the other hand, if $A^c\neq\emptyset\neq\{j\in A^c\mid\{j\}\in P\}$, then define $\widetilde{\widetilde{g}}_{A,P}$ by $\widetilde{\widetilde{g}}_{A,P}(B,Q)=1$ if $B\supset A, Q\geq P$ or $B\supseteq A, Q>P$ or both, and 0 otherwise, so that $\sum_{i\in N}\phi_i\left(\widetilde{\widetilde{g}}_{A,P}\right)=$

$$\begin{aligned}&\sum_{i\in A}\left(\sum_{\substack{(B,Q)\succ(A,P)\\ B\supseteq A,Q>P}}p^i_{B,Q}+\sum_{\substack{(B,Q)\succ(A,P)\\ B\supset A,Q\geq P}}p^j_{B,Q}-\sum_{\substack{(B,Q)\succ(A,P)\\ B\supset A,Q>P}}p^j_{B,Q}\right)+\\ &+\sum_{\substack{j\in A^c\\ \{j\}\notin P}}\sum_{\substack{(B,Q)\succ(A,P)\\ B\ni j}}p^j_{B,Q}+\sum_{\substack{j\in A^c\\ \{j\}\in P}}\sum_{\substack{Q\geq P\\ \{j\}\cup Q^{N\backslash j}=P}}p^j_{A\cup j,Q}+,\end{aligned}$$

and therefore

$$\sum_{i\in N}\phi_i\left(g_{A,P}-\widetilde{\widetilde{g}}_{A,P}\right)=0=\sum_{i\in A}p^i_{A,P}-\sum_{\substack{j\in A^c\\ \{j\}\in P}}\sum_{\substack{Q\geq P\\ \{j\}\cup Q^{N\backslash j}=P}}p^j_{A\cup j,Q}$$

for all such pairs (A,P) as wanted. ■

Theorem 11 *Any probabilistic solution satisfying symmetry and efficiency has* *constants* $p^i_{A,P}$ *satisfying*

$$p^i_{A,P} = \begin{cases} \frac{1}{n} = p^{\notin}_{(n,1,\{1\})} \textit{ if } (A,P) = (N,N) \\ 0 \textit{ if } \begin{cases} A = N, P \neq N \textit{ or} \\ A \neq N, P \neq \{A\} \cup P_0^{A^c} \end{cases} \\ \frac{1}{n\binom{n-1}{n-a}} = p^{\notin}_{\left(a, n-a+1, \left\{\underbrace{1,\ldots,1}_{n-a}, a\right\}\right)} \textit{ if } A \neq N, P = \{A\} \cup P_0^{A^c} \end{cases}$$

for all $(A \neq \emptyset, P) \in 2^N \times \mathcal{P}, i \in A$.

Proof. It is clear that efficiency and symmetry together imply

$$\begin{array}{l} \frac{1}{n} = p^{\notin}_{(n,1,\{1\})} = p^i_{N,N} = \phi_i\left(g_{N,N}\right) \text{ and} \\ 0 = p^{\notin \text{or} \in}_{\left(n,m,\{b_j\}_{1\leq j\leq m}\right)} = p^i_{N,P} \text{ if } P \neq N \end{array} \quad \text{for all } i \in N.$$

Thus, let $j \in N$, noting that $\sum_{i\in N\backslash j} p^i_{N\backslash j,\{j\}\cup\{N\backslash j\}} = p^j_{N,\{j\}\cup\{N\backslash j\}} + p^j_{N,N}$ by efficiency. Adding symmetry leads to $p^{\notin}_{(n-1,2,\{1,n-1\})} = \frac{1}{n-1}\left(0 + \frac{1}{n}\right) = \frac{1}{n\binom{n-1}{1}}$. On the other hand, efficiency alone clearly implies $p^i_{N\backslash j,P} = 0$ for all $P \in \mathcal{P}$ such that $\{j\} \notin P$, for all $i \in N\backslash j$. Eventually, note that

$$\sum_{i\in N\backslash j} p^i_{N\backslash j,\{j\}\cup P^{N\backslash j}} = p^j_{N,\{j\}\cup P^{N\backslash j}} + \sum_{B\in P^{N\backslash j}} p^j_{N,\{B\cup j\}\cup P^{N\backslash(B\cup j)}} = 0$$

for all $\{N\backslash j\} \neq P^{N\backslash j} \in \mathcal{P}\left(N\backslash j\right)$. Therefore, $p^i_{N\backslash j,P} = 0$ for all $P \neq \{j\}\cup\{N\backslash j\}$, for all $i \in N\backslash j$. In order to use induction, assume the theorem holds true for all $A' \neq N$ such that $\mid A' \mid\geq a'$, and consider any $A \neq N$ such that $\mid A \mid= a = a'-1$, with $P \in \mathcal{P}$. Then $\sum_{i\in A} p^i_{A,P} = \sum_{j\in A^c\mid\{j\}\in P} \sum_{Q\geq P\mid\{j\}\cup Q^{N\backslash j}=P} p^j_{A\cup j,Q}$ by efficiency, but $p^j_{A\cup j,Q} = \frac{1}{n\binom{n-1}{n-a-1}}$ if $Q = \{A\cup j\} \cup P_0^{A^c\cup j}$ and 0 otherwise by assumption, therefore $\sum_{i\in A} p^i_{A,P} = \sum_{j\in A^c} p^j_{A\cup j,\{A\cup j\}\cup P_0^{A^c\cup j}}$ if $P = \{A\} \cup P_0^{A^c}$ and 0 otherwise. Adding symmetry leads to

$$a p^{\notin}_{\left(a,n-a+1,\left\{\underbrace{1,\ldots,1}_{n-a},a\right\}\right)} = (n-a)\frac{1}{n\binom{n-1}{n-a-1}},$$

and thus $p^{\notin}_{\left(a,n-a+1,\left\{\underbrace{1,\ldots,1}_{n-a},a\right\}\right)} = \frac{n-a}{a}\frac{1}{n\binom{n-1}{n-a-1}} = \frac{1}{n\binom{n-1}{n-a}}$ as wanted. ■

Let $\phi^{Sh} : GC^N_\emptyset \to \mathbb{R}^n$ denote the Shapley value of global coalitional games as obtained above. It is clear that $\phi_i^{Sh}(h) =$

$$= \sum_{\substack{A\subseteq N\\ A\ni i}} \frac{(a-1)!\,(n-a)!}{n!}\left[h\left(A,\{A\}\cup P_0^{A^c}\right)-h\left(A\backslash i,\{A\backslash i\}\cup P_0^{A^c\cup i}\right)\right]$$

$$= \sum_{\substack{A\subseteq N\\ A\ni i}} \frac{(a-1)!\,(n-a)!}{n!}\left[\sum_{\left(A\backslash i,\{A\backslash i\}\cup P_0^{A^c\cup i}\right)\prec(B,Q)\preccurlyeq\left(A,\{A\}\cup P_0^{A^c}\right)} \alpha_{B,Q}(h)\right]$$

for all $i\in N$ and all $h\in GC_\emptyset^N$. It is also evident that

$$\phi_i^{Sh}(g_{A,P})=\phi_i^{Sh}\left(g_{D_{A,P}^c,\{D_{A,P}^c\}\cup P_0^{D_{A,P}}}\right)=\begin{cases}\frac{1}{|D_{A,P}^c|} \text{ if } i\in D_{A,P}^c\\ 0 \text{ otherwise}\end{cases}$$

for all $(A\neq\emptyset,P)\in 2^N\times\mathcal{P}$, where $D_{A,P}=\{j\in A^c \mid \{j\}\in P\}$ is the set of dummy players in $g_{A,P}$, while $D_{A,P}^c=N\backslash D_{A,P}$.

Thus, as in Gilboa and Lehrer (1991a)[8], the Shapley value of any global coalitional game $h\in GC_\emptyset^N$ coincides with the Shapley value of the coalitional game $\widetilde{v}_h\in C^N$ defined by $\widetilde{v}_h(A)=h\left(A,\{A\}\cup P_0^{A^c}\right)$ for all $\emptyset\neq A\subset N$ and $\widetilde{v}_h(N)=h(N,N)$. As shown above, this means that ϕ^{Sh} is determined by a proper subset of $\left\{\alpha_{A,P}(h)\mid (A,P)\in 2^N\times\mathcal{P}\right\}$. Therefore, such a solution seems inappropriate when applied to global coalitional games, in that it (arbitrarily) reduces these latter to coalitional games, making it somehow useless to formalize the cooperative situation in terms of the partition lattice.

5 The core

The core $\mathcal{C}_{C^N}(v)$ of any coalitional game $v\in C^N$ is the set of additive coalitional games $\phi=(\phi_i)_{i\in N}\in AC^N\subseteq\mathbb{R}^n$ such that $\phi(A)=\sum_{i\in A}\phi_i\geq v(A)$ for $A\subseteq N$, with equality for $A=N$, that is $\mathcal{C}_{C^N}(v)=\left\{\phi\in AC^N\mid v\leq\phi, v(N)=\phi(N)\right\}$ (thus $\mathcal{C}_{C^N}: C^N\twoheadrightarrow AC^N$, with $\twoheadrightarrow$ denoting a correspondence). In terms of bargaining, this means that each coalition A will accept to cooperate so to form the grand coalition $N=A\cup A^c$ only if its (coalitional) payoff is greater than 'the best payoff it can achieve without help from other players' (Shapley (1971), p. 13; see also footnote 3 p. 11), that is $v(A)$. Nevertheless, such a definition does not straightforwardly apply here. In fact, as observed by Gilboa and Lehrer (1991a), any partition of N involves all the n players, and thus defining what coalitions can achieve without help from other players becomes arbitrary. In general, defining the core of any global coalitional game h amounts to define some lower-bound coalitional game v_h such that $v_h(A)$ represents the minimum coalitional payoff required by any coalition $A\subseteq N$ for joining any coalition $B\subseteq A^c$. This general problem was obviously dealt

[8]See theorem 5.1.1 and remarks 5.1.2 and 5.1.3, pp. 143-4.

with by Gilboa and Lehrer (1991a), who mapped any $f \in G^N$ into the (lower-bound) coalitional game v_f defined by $v_f(A) = f\left(\{A\} \cup P_0^{A^c}\right)$. In fact, such a mapping is the analogue of one of the three methods considered by Myerson (1977)[9] for obtaining coalitional games from (global coalitional) games $h \in G_{\mathcal{E}}^N$ in partition function form, i.e., $\widetilde{v}_h(A) = h\left(A, \{A\} \cup P_0^{A^c}\right)$. The other two such methods are $\widetilde{\widetilde{v}}_h(A) = h(A, \{A, A^c\})$ and $\widetilde{\widetilde{\widetilde{v}}}_h(A) = \min_{P \in \mathcal{P} | A \in P} h(A, P)$, with $\widetilde{v}_h(N) = \widetilde{\widetilde{v}}_h(N) = \widetilde{\widetilde{\widetilde{v}}}_h(N) = h(N, N)$. Note that, for all $A \subset N$, in each of the three methods a unique element from the set $\{h(A, P) \mid A \in P\}$ is selected and used for defining the corresponding lower-bound coalitional game. Thus, an alternative approach may consist in using some aggregation function, i.e., the Choquet (discrete) integral, for mapping the whole set above (or even the larger one $\{h(A, P) \mid P \in \mathcal{P}\}$) into a unique real quantity.

When any coalition $\emptyset \neq A \subset N$ considers what best payoff it can achieve without help from other players, a first (stricter) interpretation of these words leads to the following reasoning. If each player $i \in A$ does not cooperate (neither coalitionally, nor at the partition level) with any player $j \in A^c$, then A (as a whole) forces the 'final' partition to be some $P \leq \{A, A^c\}$ by choosing some $P^A \in \mathcal{P}(A)$. Correspondingly, this will determine some final partition $P^A \cup P^{A^c} = P$, with $P^{A^c} \in \mathcal{P}(A^c)$ depending on A^c's choice. In other words, without any kind of cooperation with players $j \in A^c$, coalition A may force its (coalitional) payoff to lie between $\min\{h(A, P) \mid P \leq \{A, A^c\}\}$ and $\max\{h(A, P) \mid P \leq \{A, A^c\}\}$, that is

$$\min\{h(A, P) \mid P \leq \{A, A^c\}\} \leq v_h(A) \leq \max\{h(A, P) \mid P \leq \{A, A^c\}\}.$$

Thus, in general, a number $| \{h(A, P) \mid P \leq \{A, A^c\}\} | = | \mathcal{P}(A) | \times | \mathcal{P}(A^c) |$ of real quantities may be considered for defining $v_h(A)$, and therefore some reasonable operation of aggregation is required. In particular, the problem may be conceptually approached by defining some expectation (to be placed $= v_h(A)$) of random variable $\{h(A, P) \mid P \leq \{A, A^c\}\}$, and the 'probability' to be used should depend on the features displayed by the strategic game between A and A^c, with strategy spaces $\mathcal{P}(A)$ and $\mathcal{P}(A^c)$ respectively. Nevertheless, weak monotonicity has strong strategic implications, as shown hereafter.

First of all, consider the set $\mathcal{P}_*^{A,A^c} \subset \{P \in \mathcal{P} \mid P \leq \{A, A^c\}\}$ of best responses, or pure-strategy Nash equilibria, defined by

$$\mathcal{P}_*^{A,A^c} = \left\{ \begin{array}{c} P_* = P_*^A \cup P_*^{A^c} \mid P_*^A \in \mathcal{P}(A), P_*^{A^c} \in \mathcal{P}(A^c) \text{ such that} \\ \max\left\{h\left(A, P^A \cup P_*^{A^c}\right) \mid P^A \in \mathcal{P}(A)\right\} = h\left(A, P_*^A \cup P_*^{A^c}\right) \text{ and} \\ \max\left\{h\left(A^c, P_*^A \cup P^{A^c}\right) \mid P^{A^c} \in \mathcal{P}(A^c)\right\} = h\left(A^c, P_*^A \cup P_*^{A^c}\right) \end{array} \right\}.$$

Note that weak monotonicity of h entails not only $\{A, A^c\} \in \mathcal{P}_*^{A,A^c}$, but also

[9] See example pp. 26-7.

strategic independence, that is

$$\begin{aligned}\max\left\{h\left(A,P^{A}\cup\widetilde{P^{A^c}}\right)\mid P^{A}\in\mathcal{P}\left(A\right)\right\} &= h\left(A,\{A\}\cup\widetilde{P^{A^c}}\right)\\ \max\left\{h\left(A^c,\widetilde{P^{A}}\cup P^{A^c}\right)\mid P^{A^c}\in\mathcal{P}\left(A^c\right)\right\} &= h\left(A^c,\widetilde{P^{A}}\cup\{A^c\}\right)\end{aligned}$$

for all $\widetilde{P^{A^c}}\in\mathcal{P}\left(A^c\right),\widetilde{P^{A}}\in\mathcal{P}\left(A\right)$. Therefore, it seems particularly desirable to have $v_h\left(A\right)=h\left(A,\{A,A^c\}\right)$ whenever h is weakly monotone. To this end, for each $\emptyset\neq A\subset N$, consider the fuzzy measure $\mu^A:2^{\mathcal{P}}\rightarrow[0,1]$ defined by

$$\mu^A\left(\{P_1,\ldots,P_m\}\right)=\begin{cases}\frac{\max\{h(A^c,P)\mid P\in\{P_1,\ldots,P_m\}\}}{\max\left\{h(A^c,P)\mid P\in\mathcal{P}_*^{A,A^c}\right\}} \text{ if } \{P_1,\ldots,P_m\}\subseteq\mathcal{P}_*^{A,A^c}\\ 0 \text{ otherwise}\end{cases},$$

so that the lower-bound coalitional game v_h may be defined as

$$v_h\left(A\right)=\sum_{j=1}^{|\mathcal{P}_*^{A,A^c}|}\left[h\left(A,P_{(j)}\right)-h\left(A,P_{(j-1)}\right)\right]\mu^A\left(\left\{P_{(j)},P_{(j+1)},\ldots,P_{\left(|\mathcal{P}_*^{A,A^c}|\right)}\right\}\right)$$

with $h\left(A,P_{(0)}\right)=0$ and $h\left(A,P_{(1)}\right)\leq h\left(A,P_{(2)}\right)\leq\cdots\leq h\left(A,P_{\left(|\mathcal{P}_*^{A,A^c}|\right)}\right)$. It may be recognized that the right-hand side equals $C_{\mu^A}\left(h\left(A,\bullet\right)\right)$, that is the Choquet integral of $h\left(A,\bullet\right):\mathcal{P}_*^{A,A^c}\rightarrow\mathbb{R}_+$ with respect to $\mu^A:2^{\mathcal{P}_*^{A,A^c}}\rightarrow[0,1]$. Such an aggregation function has well-known, interesting properties[10], and in particular here satisfies

$$h \text{ weakly monotone } \Rightarrow C_{\mu^A}\left(h\left(A,\bullet\right)\right)=v_h\left(A\right)=h\left(A,\{A,A^c\}\right).$$

As already explained, $v_h\left(A\right)$ must quantify the minimum payoff coalition A requires for cooperating with any $\emptyset\neq B\subseteq A^c$. In the context of global coalitional games such a minimum payoff may be interpreted as an expectation of random variable $\left\{h\left(A,P\right)\mid P\in\mathcal{P}_*^{A,A^c}\right\}$. In particular, each segment $h\left(A,P_{(j)}\right)-h\left(A,P_{(j-1)}\right)$ is expected (and therefore required) by A with (subjective and fuzzy) probability $\mu^A\left(\left\{P_{(j')}\mid j\leq j'\leq\mid\mathcal{P}_*^{A,A^c}\mid\right\}\right)$. Furthermore, μ^A is defined by considering the profitability for A^c (and not A, for this latter always plays best responses) of choosing any $P\in\{P_1,\ldots,P_m\}\subseteq\mathcal{P}_*^{A,A^c}$. Also set $v_h\left(N\right)=\max_{P\in\mathcal{P}}h\left(N,P\right)$, noting that $h\in GC_{WM}^N$ entails $\max_{P\in\mathcal{P}}h\left(N,P\right)=h\left(N,N\right)$. This allows to define a first core $\mathcal{C}_{GC^N}:GC_{\emptyset}^N\twoheadrightarrow AC^N$ of global coalitional games as $\mathcal{C}_{GC^N}\left(h\right)=\left\{\phi\in AC^N\mid v_h\leq\phi,v_h\left(N\right)=\phi\left(N\right)\right\}$.

[10]Note that here μ^A varies with $h\in GC_{\emptyset}^N$, and thus $C_{\mu^A}\left(h\left(A,\bullet\right)\right)$ fails to satisfy the SPL (i.e., stability under admissible positive linear affine transformations) condition used by Marichal (2000), pp. 251 and 256, for characterizing the (traditional) Choquet integral.

The idea of an expected lower-bound coalitional payoff (that characterizes $\mathcal{C}_{GC^N}$) may be used for extending the above argument to the whole set $\{h(A,P) \mid P \in \mathcal{P}\}$ (and such an extension is particularly desirable for those global coalitional games $h \in GC_{\emptyset}^N \backslash GC_{\mathcal{E}}^N$ that are not in partition function form). In fact, if $\max\{h(A,P) \mid P \in \mathcal{P}\} > \max\{h(A,P) \mid P \leq \{A, A^c\}\}$, then players $i \in A$ may decide to cooperate, at the partition level, with players $j \in A^c$ so to get some coalitional payoff $\widehat{v}_h(A) > \max\{h(A,P) \mid P \leq \{A, A^c\}\}$. Formally, for all $\emptyset \neq A \subset N$, let $\mathcal{P}_{\backslash \mathcal{P}(A,A^c)} = \{P \in \mathcal{P} \mid P \nleq \{A, A^c\}\}$ denote the set of partitions that intersect $\{A, A^c\}$ (i.e., such that $P \vee \{A, A^c\} = N \in \mathcal{P}$), and define $\widehat{\mu}^A : 2^{\mathcal{P}_{\backslash \mathcal{P}(A,A^c)}} \to [0,1]$ by $\widehat{\mu}^A(\{P_1, \ldots, P_m\}) = \frac{\max\{h(A^c,P) \mid P \in \{P_1, \ldots, P_m\}\}}{\max\left\{h(A^c,P) \mid P \in \mathcal{P}_{\backslash \mathcal{P}(A,A^c)}\right\}}$ for all $\{P_1, \ldots, P_m\} \subseteq \mathcal{P}_{\backslash \mathcal{P}(A,A^c)}$. Eventually, let $\widehat{v}_h(A) =$

$$= \sum_{j=1}^{|\mathcal{P}_{\backslash \mathcal{P}(A,A^c)}|} \left[h\left(A, P_{(j)}\right) - h\left(A, P_{(j-1)}\right)\right] \widehat{\mu}^A \left(\left\{P_{(j)}, P_{(j+1)}, \ldots, P_{\left(|\mathcal{P}_{\backslash \mathcal{P}(A,A^c)}|\right)}\right\}\right)$$

with $h\left(A, P_{(0)}\right) = 0$ and $h\left(A, P_{(1)}\right) \leq h\left(A, P_{(2)}\right) \leq \cdots \leq h\left(A, P_{\left(|\mathcal{P}_{\backslash \mathcal{P}(A,A^c)}|\right)}\right)$. This allows to consider a further core correspondence $\widehat{\mathcal{C}}_{GC^N} : GC_{\emptyset}^N \twoheadrightarrow AC^N$ defined by $\widehat{\mathcal{C}}_{GC^N}(h) = \left\{\phi \in AC^N \mid \max\{v_h, \widehat{v}_h\} \leq \phi, v_h(N) = \phi(N)\right\}$. Note that $\phi \geq \max\{v_h, \widehat{v}_h\} \Rightarrow \phi \geq v_h$, thus $\widehat{\mathcal{C}}_{GC^N} \subseteq \mathcal{C}_{GC^N}$ refines[11] $\mathcal{C}_{GC^N}$.

$\mathcal{C}_{GC^N}$ might be regarded as the core of weakly monotone games in partition function form, in that $\mathcal{C}_{GC^N}(h) = \mathcal{C}_{C^N}(v_h)$ and weak monotonicity entails $v_h(A) = h(A, \{A, A^c\})$; also, $(A, \{A, A^c\}), (A^c, \{A, A^c\}) \in \mathcal{E}$ for all $\emptyset \neq A \subset N$. On the other hand, $\widehat{\mathcal{C}}_{GC^N}$ may be regarded as the core of global coalitional games $h \in \left(GC_{\emptyset}^N \backslash GC_{\mathcal{E}}^N\right) \cap GC_{WM}^N$, in which case $\phi \in \widehat{\mathcal{C}}_{GC^N}(h)$ entails $\phi(A) \geq h(A, N)$ for all $\emptyset \neq A \subset N$, and $\phi(N) = h(N, N)$. Apart from weak monotonicity, $\mathcal{C}_{GC^N}(h)$ and $\widehat{\mathcal{C}}_{GC^N}(h)$ result to be nonempty iff the Bondareva-Shapley conditions (see Shapley (1967)) are satisfied by coalitional games v_h and $\max\{v_h, \widehat{v}_h\}$ respectively (in that $\mathcal{C}_{GC^N}, \widehat{\mathcal{C}}_{GC^N} : GC_{\emptyset}^N \to C^N \twoheadrightarrow AC^N$). On the other hand, it may well be that such coalitional games are (both) convex, and yet the Shapley value $\phi^{Sh}(h)$ of the originating global coalitional game does not belong to neither $\mathcal{C}_{GC^N}(h)$ nor $\widehat{\mathcal{C}}_{GC^N}(h)$ (see Shapley (1971)), in that $\phi^{Sh}(h) = \phi^{Sh}(\widetilde{v}_h)$, and $v_h(A) \neq \widetilde{v}_h(A) \neq \max\{v_h(A), \widehat{v}_h(A)\}$ for $\emptyset \neq A \subset N$. Eventually, it is evident that, in the absence of weak monotonicity, determining the lower-bound coalitional game involves an high level of computational complexity. In particular, for $\mathcal{C}_{GC^N}$ such a level depends on the number $|\mathcal{P}_*^{A,A^c}|$ (for all $\emptyset \neq A \subset N$) of pure-strategy Nash equilibria defined above.

[11] Notice that players $i \in A$, after collectively deciding not to coalitionally cooperate with any $j \in A^c$, must decide whether or not to (try and) cooperate with players $j \in A^c$ at the partition level. In particular, they cannot do both (i.e., cooperating and noncooperating at the partition level), so that $(v_h + \widehat{v}_h)$ cannot substitute $\max\{v_h, \widehat{v}_h\}$.

6 Concluding Remarks

A first remark concerns weak monotonicity. As already mentioned, such a condition enables to avoid the undesired implications of monotonicity. Yet, it is sufficient (as the traditional monotonicity of coalitional games) for pushing the system towards the formation of the grand coalition and coarsest partition, in that it entails $h(N,N) \geq h(A,P)$ for all $(A,P) \in 2^N \times \mathcal{P}$. Furthermore, both convexity and total positivity might be weakened by defining (additional) conditions to be satisfied by both (i) the unions and intersections of coalitions, and/or (ii) by the partitions of the unions and intersections of coalitions induced by the finest coarser than and the coarsest finer than (whole) partitions. Furthermore, even though global coalitional games probably constitute a better instrument than (traditional) coalitional games for studying endogenous coalition formation, yet, in order to do so, it is necessary either to abandon weak monotonicity, or to assume some cooperation restrictions preventing the grand coalition and the coarsest partition to form. In this latter case, it might be interesting to start by considering matroid-restricted games. In particular, assume that only coalitions $A \in \mathcal{M} \subset 2^N$ are feasible, where $\mathcal{M}$ is a matroid. This allows to define the set $\mathcal{A} \subset 2^N \times \mathcal{P}$ of admissible pairs, where $(A,P) \in \mathcal{A}$ if $A \in \mathcal{M}$ and $P = \{B_1, \ldots, B_m\}$ satisfies $B_j \in \mathcal{M}$ for all $1 \leq j \leq m$. As a special case, the intersection $\mathcal{A} \cap \mathcal{E}$ might also be considered as identifying a (more restricting) set of feasible pairs. Eventually, note that studying endogenous coalition formation by means of global coalitional games does not need any so-called 'rule of order' identifying who plays before who.

Secondly, concerning 'value theory with efficiency only', consider the solution $\phi_i^{My}(h) = \sum_{(A,P)|A \ni i} \frac{\alpha_{A,P}(h)}{|A|}$. In fact, such a solution is easily seen to satisfy efficiency, and parallels the Shapley value of games in partition function form derived by Myerson (1977). It depends on the whole set of values assumed by the Möbius transform (i.e., as a lattice function), and not only on a proper subset like the Shapley values derived here and in Gilboa and Lehrer (1991a). Nevertheless, Myerson (1977) did not have the dummy and nonnegativity axioms, thus these latter (and the probabilistic form in general) might be responsible for the undesired features displayed by the Shapley value derived here. (Note that $\phi_i^{My}(g_{A,P}) = (|A|)^{-1}$ if $i \in A$ and 0 otherwise for all global coalitional 'unanimity' games $g_{A,P}$.)

A final remark concerns the core. In fact, it would be tempting to define the core of any global coalitional game h as the set of additive global coalitional games $\phi \in AGC_\emptyset^N$ satisfying $\phi(A,P) \geq h(A,P)$ for all $(A,P) \in 2^N \times \mathcal{P}$, with equality if $(A,P) = (N,N)$. Nevertheless, it is not clear if such a definition would have any meaning at all. In particular, it is not known (to the author) what is the dimension of the vector (sub)space $AGC_\emptyset^N$ of ($\emptyset$-normalized) additive global coalitional games. And even if such a dimension was a multiple β of n, with $\beta > 1$ finite integer, how should one interpret the generic $\phi \in \mathbb{R}^{\beta n}$ in the core? (Note that exactly the same remark, with $\beta' < \beta$ and AG^N in place of β and $AGC_\emptyset^N$ respectively, applies to global games $f \in G^N$.)

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