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Computation of the eigenvalues of convexity preserving matrices

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ABSTRACT

A direct method using $O(kn^2)$ elementary operations to compute the k largest eigenvalues of an r-convexity preserving $n \times n$ matrix, for all $r = 0, 1, \ldots, k$, is presented.

1. Introduction

The computation of the eigenvalues of a matrix is one of the main topics of numerical linear algebra. This work considers a class of matrices (r-convexity preserving matrices for all $r \leq k$) that are important in many applications (see [1,4,5] or [6], and Section 2). We present a direct method for computing the k largest eigenvalues of these matrices. In fact, we obtain explicit formulae for the obtaining of these eigenvalues, and the computational cost of the corresponding method is $O(kn^2)$ elementary operations for an $n \times n$ matrix. In the particular case of an r-convexity preserving matrix for all $r \leq n-1$ we provide a direct method using $O(n^3)$ elementary operations to compute all of its eigenvalues.

2. Explicit formulae and the direct method

Let us first present some basic definitions. Let k be a nonnegative integer. A vector $v = (v_1, v_2, \ldots, v_n)^{\mathrm{T}} \in \mathbf{R}^n$ is said to be *k-convex* if $\Delta^k v_i \geq 0$ for all $i \in \{1, \ldots, n-k\}$, where

$$\Delta^k v_i := \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} v_{i+j}.$$

Observe that a vector is 0-convex if and only if it is nonnegative and a vector is 1-convex if and only if it is monotonically increasing. A matrix A is said to be *k-convexity preserving* if for any k-convex vector v, the vector Av is also k-convex. Let us observe that A is 0-convexity preserving if and only if it transforms nonnegative vectors into nonnegative vectors, which is equivalent to $A \geq 0$. A matrix A is 1-convexity preserving if and only if it is monotonicity preserving.

We shall denote by E the lower triangular matrix

$$E := \begin{pmatrix} 1 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 1 & \ldots & 1 & 0 \\ 1 & \ldots & 1 & 1 \end{pmatrix}, \qquad E^{-1} = \begin{pmatrix} 1 & 0 & \ldots & \ldots & 0 \\ -1 & 1 & \ddots & & \vdots \\ 0 & -1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & -1 & 1 \end{pmatrix}. \tag{1}$$

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For each $j \in \{1, \dots, n\}$, let E_j be the following $n \times n$ matrix: $E_1 := E$ and for $j \geq 2$,

$$E_j := \begin{pmatrix} I_{j-1} & | & 0 \\ \hline 0 & | & E \end{pmatrix}, \qquad E_j^{-1} = \begin{pmatrix} I_{j-1} & | & 0 \\ \hline 0 & | & E^{-1} \end{pmatrix}.$$

where I_{j-1} is the $(j-1) \times (j-1)-$ identity matrix and E is the $(n-j+1) \times (n-j+1)$-matrix given by (1).

The following result corresponds to Corollary 4.4 of [1] and shows an important property of r-convexity preserving matrices for $r = 0, 1, \dots, k$ $(k \geq 1)$.

Proposition 2.1. *Let A be a r-convexity preserving matrix for $r = 0, 1, \dots, k(k \geq 1)$. Then*

$$(E_1 \cdots E_k)^{-1} A (E_1 \cdots E_k) = \begin{pmatrix} \Lambda_k & | & * \\ \hline 0 & | & A_k \end{pmatrix}$$

where Λ_k is an upper triangular matrix whose diagonal elements $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k (\geq 0)$ are the largest eigenvalues of A, and A_k is a nonnegative matrix with $\rho(A_k) \leq \lambda_k$.

Observe that, by the previous result, the first k columns of $E_1 \cdots E_k$ form a basis of the invariant subspace corresponding to the k dominant eigenvalues of a matrix A which is r-convexity preserving for $r = 0, 1, \dots, k(k \geq 1)$. Let us recall that a vector v is said to be *k-concave* if the vector $-v$ is k-convex. By Remark 2.6 of [1], the first k columns of $E_1 \cdots E_k$ form a basis of the vector space formed by the vectors which are simultaneously k-convex and k-concave.

Matrices which are r-convexity preserving for $r = 0, \dots, k$ arise in many practical and theoretical problems. The case $k = 1$ corresponds to the important case of nonnegative matrices which are monotonicity preserving. A wide family of totally positive matrices (matrices with all their minors nonnegative) which are r-convexity preserving for $r = 0, 1, \dots, k$ is presented in Corollary 3.5 of [1]. A source of many examples of r-convexity preserving matrices is provided by the collocation matrices of r-convexity preserving systems of functions. A function $u : [a, b] \to \mathbf{R}$ is called *k-convex* if all kth-order divided differences of u are nonnegative, that is, $u[t_0, \dots, t_k] \geq 0$ for any $a \leq t_0 < \cdots < t_k \leq b$. In particular, 0-convexity is synonymous with u nonnegative, 1-convexity is synonymous with u increasing and 2-convexity coincides with the usual assertion that u is convex. A system of functions $(u_0, \dots, u_n)$ defined on $[a, b]$ is called *k-convexity preserving* if for any k-convex vector $c = (c_0, \dots, c_n)^{\mathrm{T}}$ the function $\sum_{i=0}^{n} c_i u_i$ is k-convex.

Given a system of functions $(u_0, \dots, u_n)$ defined on $[a, b]$, the matrices

$$M \begin{pmatrix} u_0, \dots, u_n \\ t_0, \dots, t_m \end{pmatrix} := (u_j(t_i))_{i=0,\dots,m; j=0,\dots,n}, \quad a \leq t_0 < t_1 < \cdots < t_m \leq b,$$

are usually called *collocation matrices*.

The next proposition, corresponding to Proposition 2.10 of [1], shows the relationship between k-convexity preserving system of functions and k-convexity preserving matrices.

Proposition 2.2. *Let $u_0, \dots, u_n \in C[a, b]$. Then $(u_0, \dots, u_n)$ is k-convexity preserving if and only if all the collocation matrices*

$$M \begin{pmatrix} u_0, u_1, \dots, u_n \\ s_0, s_0 + h, \dots, s_0 + nh \end{pmatrix}, \quad h \in \left(0, \frac{b-a}{n}\right], \quad s_0 \in [a, b - nh],$$

are k-convexity preserving.

Many systems used in approximation theory, statistics or computer aided geometric design (GAGD) are r-convexity preserving (see [1,4,7,5] or [6]). In CAGD, the fact that $(u_0, \dots, u_n)$ is k-convexity preserving can be interpreted as follows. If the control polygon $P_0 \cdots P_n$ is the graph of a k-convex function, then the graph of $u = \sum_{i=0}^{n} u_i P_i$ is a k-convex function. Finally, let us recall that the Bernstein basis is r-convexity preserving for all r and the B-spline basis of the space of polynomial splines of degree m with equally spaced knots is r-convexity preserving for $r = 0, \dots, m$ (see [1]). Let us also recall that spectral properties of the Bernstein operator have been deeply studied in the literature (see [2]).

First we need an auxiliary result on combinatorial numbers.

Lemma 2.3. *For all $m \geq 0$,*

$$\sum_{j=0}^{m} \binom{k+j}{k} = \binom{k+1+m}{k+1}. \tag{2}$$

Proof. Let us prove the result by induction on $m \geq 0$. For $m = 0$ it is obvious. Now let us suppose that formula (2) holds for $m \geq 0$ and let us show that it also holds for $m + 1$. We can write

$$\sum_{j=0}^{m+1} \binom{k+j}{k} = \left[\sum_{j=0}^{m} \binom{k+j}{k}\right] + \binom{k+m+1}{k}.$$

By the induction hypothesis we have that

$$\sum_{j=0}^{m} \binom{k+j}{k} = \binom{k+1+m}{k+1}.$$

Then by the two last formulas and the usual properties of combinatorial numbers we deduce

$$\sum_{j=0}^{m+1} \binom{k+j}{k} = \binom{k+1+m}{k+1} + \binom{k+m+1}{k} = \binom{(k+1)+m+1}{k+1}$$

and (2) holds for $m + 1$. ■

Given an square matrix A of order n the following results provide, respectively, explicit expressions for the columns of $B_k := AE_1 \cdots E_k$ and the rows of $(E_1 \cdots E_k)^{-1} A (E_1 \cdots E_k)$ in terms of the columns of A and of the rows of B_k.

Proposition 2.4. *Let A be an square matrix of order n with columns $A^1, \ldots, A^n$, $k < n$ and $B_r := A\, E_1 \cdots E_r$ for $r = 1, \ldots, k$. Let us denote by $B_r^1, \ldots, B_r^n$ the columns of B_r. Then, for each $j \in \{1, \ldots, r-1\}$,*

$$B_r^j = \sum_{i=j}^{n} \binom{i}{j-1} A^i \tag{3}$$

and, for each $j \in \{r, \ldots, n\}$,

$$B_r^j = \sum_{i=j}^{n} \binom{r-1+i-j}{r-1} A^i. \tag{4}$$

Proof. Let us prove formulas (3) and (4) by induction on $r \in \{1, \ldots, k\}$. For $r = 1$ we have by the definition of E_1 that

$$B_1 = A\, E_1 = (A^1, A^2, \ldots, A^n)\, E_1 = \left(\sum_{i=1}^{n} A^i, \sum_{i=2}^{n} A^i, \ldots, \sum_{i=n}^{n} A^i\right).$$

Then $B_1^j = \sum_{i=j}^{n} A^i$ for all $j \in \{1, \ldots, n\}$ and therefore formulas (3) and (4) hold for $r = 1$. Let us suppose that (3) and (4) hold for $r \in \{1, \ldots, k-1\}$ and let us prove them for $r + 1$. We can write

$$B_{r+1} = A\, E_1 \cdots E_{r+1} = (A\, E_1 \cdots E_r) E_{r+1}.$$

Then by the induction hypothesis we have

$$B_{r+1} = (B_r^1, B_r^2, \ldots, B_r^n) E_{r+1}.$$

where B_r^j are given by (3) for $j \in \{1, \ldots, r-1\}$ and by (4) for $j \in \{r, \ldots, n\}$. By the definition of E_{r+1} and the induction hypothesis we have that

$$B_{r+1}^j = B_r^j = \sum_{i=j}^{n} \binom{i}{j-1} A^i$$

for $j \in \{1, \ldots, r\}$. Therefore (3) holds for $j \in \{1, \ldots, r\}$. Analogously, by the definition of E_{r+1} and the induction hypothesis we deduce

$$B_{r+1}^j = \sum_{i=j}^{n} B_r^i = \sum_{i=j}^{n} \sum_{m=i}^{n} \binom{r-1+m-i}{r-1} A^m$$

for $j \in \{r+1, \ldots, n\}$. Reordering the terms in the previous formula it can be written as

$$B_{r+1}^j = \sum_{m=j}^{n} \left[\sum_{i=j}^{m} \binom{r-1+i-j}{r-1}\right] A^m.$$

Changing the index of the inner sum in the previous formula we have that

$$B_{r+1}^{j}=\sum_{m=j}^{n}\left[\sum_{i=0}^{m-j}\binom{r-1+i}{r-1}\right]A^{m}.$$

Finally, applying Lemma 2.3 we deduce

$$B_{r+1}^{j}=\sum_{m=j}^{n}\binom{r+m-j}{r}A^{m}.$$

Hence formula (4) also holds for $j\in\{r+1,\ldots,n\}$ and the induction follows. ■

Proposition 2.5. *Let $k<n$, $N_0:=B_k$ be the matrix defined in Proposition 2.4, with rows $B_{k,1},\ldots,B_{k,n}$, and let $N_r:=(E_1\cdots E_r)^{-1}B_k$ for $r\in\{1,\ldots,k\}$. Let us denote by $N_{r,1},\ldots,N_{r,n}$ the rows of N_r. Then, for each $i\in\{1,\ldots,r\}$,*

$$N_{r,i}=\nabla^{i-1}B_{k,i}=\sum_{j=0}^{i-1}(-1)^{j}\binom{i-1}{j}B_{k,i-j} \tag{5}$$

and, for each $i\in\{r+1,\ldots,n\}$,

$$N_{r,i}=\nabla^{r}B_{k,i}=\sum_{j=0}^{r}(-1)^{j}\binom{r}{j}B_{k,i-j}, \tag{6}$$

where ∇ is the usual backward difference $\nabla f_i:=f_i-f_{i-1}$.

Proof. Performing $E_s^{-1}N_{s-1}$ for all $s\in\{1,\ldots,k\}$ consists of subtracting from the rows $s+1,\ldots,k$ of N_{s-1} the rows $s,\ldots,k-1$ of N_{s-1}, respectively. Therefore, we have that, for each $i\in\{1,\ldots,r\}$, $N_{r,i}=\nabla^{i-1}B_{k,i}$, and that, for each $i\in\{r+1,\ldots,n\}$, $N_{r,i}=\nabla^{r}B_{k,i}$. Finally, the combinatorial formulas in (5) and (6) are well known formulas for the backward difference ∇. ■

From Propositions 2.1, 2.4 and 2.5 we derive the following result.

Corollary 2.6. *Let k be a positive integer less than n, let A be an r-convexity preserving matrix for $r=0,1,\ldots,k$ $(k\geq 1)$ and let $N_k=(m_{ij})_{1\leq i,j\leq n}$ be the matrix defined in Proposition 2.5. Then $m_{11},m_{22},\ldots,m_{kk}$ are the largest eigenvalues of A satisfying $m_{11}\geq m_{22}\geq\cdots\geq m_{kk}\geq 0$, and the remaining $n-k$ eigenvalues of A are the $n-k$ eigenvalues of the nonnegative matrix $(m_{ij})_{k+1\leq i,j\leq n}$.*

Let us observe that, by Corollary 2.6, the explicit formulae of Propositions 2.4 and 2.5 allow us to calculate the k largest eigenvalues of a matrix A r-convexity preserving for $r=0,1,\ldots,k$. By Corollary 2.6, the k largest eigenvalues of A are the k first diagonal entries of the matrix N_k. By Proposition 2.5, each of the last $k-1$ mentioned diagonal entries can be obtained as a linear combination of the entries above them in the matrix B_k of Proposition 2.4 and the first mentioned diagonal entry coincides with the corresponding one in B_k. Finally, the involved elements (i,j) (with $1\leq i\leq j\leq k$) of B_k can be obtained by Proposition 2.4 as a linear combination of the entries to the right of them in the matrix A. In conclusion, the k largest eigenvalues of $A=(a_{ij})_{1\leq i,j\leq n}$ only depend on the entries a_{ij} with $1\leq i\leq k$ and $i\leq j\leq n$. If A is an r-convexity preserving matrix for $r=0,1,\ldots,n-1$, then by Corollary 2.6 all its eigenvalues only depend on the entries of the upper triangular part of A.

The explicit formulae of Propositions 2.4 and 2.5 for calculating the k largest eigenvalues of a matrix A r-convexity preserving for $r=0,1,\ldots,k$ comprise two phases: the first phase corresponds to the calculation of entries above and to right of the first diagonal entries of $B_k=A(E_1\cdots E_k)$ (Phase I) and the second one corresponds to the calculation of the first diagonal entries of $N_k=(E_1\cdots E_k)^{-1}B_k$ (Phase II).

Let us now analyze the computational cost of both phases through the explicit formulae of Propositions 2.4 and 2.5. Observe that both formulae use combinatory numbers of the form $\binom{m}{i}$, $0\leq i\leq m\leq n$. Nowadays, we can assume we have stored such numbers with as high an accuracy as we want. Anyway, we can also calculate them directly through the well known construction of the Tartaglia triangle, which needs $n(n-1)/2$ sums. Taking into account that, for $j=1$, all coefficients in formula (3) are 1, the additional cost of Phase I corresponds to the calculation of the first k entries of the first row of B_k, the entries $2,\ldots,k$ of the second row of B_k, and so on until the k-th entry of the k-th row of B_k is reached, having a computational cost of $(2n-k-1)k/2$ sums and $(2n-k)(k-1)/2$ products, $(2n-k-2)(k-1)/2$ sums and $(2n-k)(k-1)/2$ products, and so on up to $n-k$ sums and $n-k+1$ products, respectively. So, Phase I requires $O(nk^2)$ sums and products. The cost of Phase II corresponds to the calculation of the entries $(2,2),\ldots,(k,k)$ of N_k, which require $k(k-1)/2$ sums and $(k-1)^2/2$ products.

We say that we perform an *accurate* computation if the relative error of the computation is bounded by $\mathcal{O}(\varepsilon)$, where ε is the machine precision. Given an algebraic expression defined by additions, subtractions, multiplications and divisions and assuming that each initial real datum is known to high relative accuracy, then it is well known that the algebraic expression can be computed accurately if it is defined by sums of numbers of the same sign, products and quotients (cf. p. 52 of [3]). Observe that the formulae of Proposition 2.4 corresponding to Phase I can be computed accurately.

As for Phase II, the size of the elements appearing when using formulae of Proposition 2.5 can be very large in comparison with the elements of B_k. In order to avoid this negative property, which could lead to overflow, let us present an alternative to Phase II. Let us first recall that the growth factor is an indicator of the numerical stability of a numerical algorithm and it measures the size of intermediate and final quantities relative to initial data. Then we can obtain the elements of the first k diagonal entries of N_k without increasing the size of the corresponding elements of B_k. Since A is 0-convexity preserving, it is clearly nonnegative, and so $B_k = A(E_1 \cdots E_k)$ is also nonnegative. Performing $E_1^{-1}B_k$ consists of subtracting from each row of B_k the previous one. Since by applying Proposition 2.1 for $r = 0, 1$, we obtain that the $(2, 2)$ entry of N_k (which coincides with the $(2, 2)$ entry of $E_1^{-1}B_k$) is nonnegative, and since it has been obtained by subtracting two nonnegative numbers, we deduce that this entry is not greater than the same entry of B_k. In fact this happens with all entries of $E_1^{-1}B_k$. We can continue analogously with $E_2^{-1}(E_1^{-1}B_k)$ to deduce that its $(3, 3)$ entry (which coincides with the $(3, 3)$ entry of N_k) is not greater than the same entry of B_k. We can continue the previous procedure until we obtain the (k, k) entry of N_k. This direct method controls the growth factor because all intermediate quantities are nonnegative numbers obtained through the subtraction of nonnegative numbers. This method has k steps of the elimination procedure. In the first step, we subtract from each row $2, \ldots, k$ of B_k the previous one. In the second step, we subtract from each row $3, \ldots, k$ of the obtained matrix the previous one. We continue up to step $k - 1$, in which we subtract from the k-th row of the matrix obtained in the previous step the previous row. This alternative procedure has clearly a computational cost of higher order: $O(k^3)$ elementary operations, instead of $k(k-1)/2$ sums and $(k-1)^2/2$ products required by performing Phase II through the formulae of Proposition 2.5.

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