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On Multidimensional Inequality with Variable Distribution Mean

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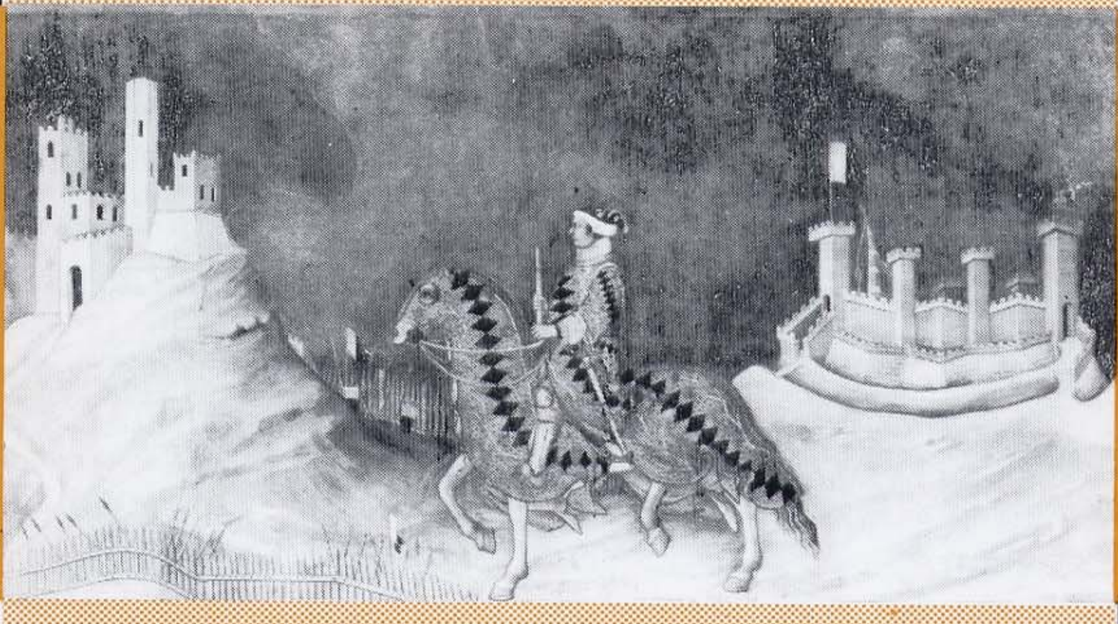

Abstract: We compare alternative populations of individuals, who differ for many characteristics besides income, in terms of inequality. In order to achieve our aim, we extend the notion of Generalized Lorenz Preorder to a context of multivariate distributions with different marginals. Finally, we show, by using convex analysis, that some conditions, relevant in the analysis of multidimensional inequality, are equivalent to the ordering we introduced.

Keywords: Multidimensional Inequality, Generalized Lorenz Preordering, Price-majorization;

JEL classification: D31, D63

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1. Introduction

The present work is devoted to the problem of how to determine the relative desirability of different social outcomes, namely we achieve comparability between populations of individuals with different characteristics (such as a vector of different goods) with emphasis placed on their uneven distribution. Economic literature on inequality measurement is mainly concerned with the comparison of *univariate* indices of well-being, which record the differences in distribution of income (and/or wealth) within and between populations. However, such an approach is considered an inadequate basis for comparing individual disparities because people differ in many aspects besides income. The analysis of different individual attributes is indeed crucial to understand and evaluate inequality among persons. Therefore, a very recent research trend is focused on the development of criteria for ranking *multivariate* distributions of individual attributes.[1] Unfortunately, few progress has been made on extending the theory of inequality measurement from univariate to the multivariate case (see e.g. [2], [3], [4], [7], [8]), the works on multidimensional disparity comparisons are rather sparse and the problem is really complex. In the present work, we follow the literature trend analyzing inequality in a context of more than one (income) variable and addressing the problem to compare multidimensional alternative distributions. In order to achieve our aim, we extend the notion of generalized Lorenz (or dually weak majorization) preorder (see [5], [9]) to a context of multiple individual attributes. In particular, we compare multivariate distributions in terms of inequality when the means of their marginals differ. We represent a multidimensional distribution as a matrix, whose generic entry consists in the quantity of the kth good, $k = 1, ..., m$, allocated to the ith group of individuals, $i = 1, ..., n$. A preorder of different distribution matrices is defined according to their level of inequality and the properties of the preorder are provided. Using certain tools of convex analysis, we show that such a preorder can be replaced by the order defined as the inclusion of the columns (and of course of the rows) of a distribution matrix in the convex hull defined by the set of all convex combinations of the columns (and of course of the rows) of another distribution matrix and analogously by a social evaluation function that records the level of inequality of alternative individual distributions of goods.

Finally, we compare our preorder of matrix distributions with the main inequality criteria for ranking multidimensional distributions discussed in economic literature (see e.g. [2], [3], [4], [5], [7]).

2. Notation and definitions

We consider a fixed population of individuals $N = \{1, ..., n\}$, with $n \geq 2$, distinguished according to their level of income as follows: $y_1 \geq y_2 \geq \cdots \geq y_n$, with y_i the income of the ith individual in the distribution $\mathbf{y} \in \mathbb{R}^n$. The concept that the components of a distribution y are 'more spread out' than the components of a distribution x, has been studied, among others by Hardy, Littlewood and Polya (HLP [1]), who showed that for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\mathbf{y}$ is 'less unequal than' $\mathbf{x}$, denoted $\mathbf{y} \preceq \mathbf{x}$ if and only if $\mathbf{y} = \mathbf{x}P$ for some doubly stochastic matrix P,[2] or equivalently

[1]See Savaglio [8] for a survey.

[2]A doubly stochastic matrix is a square semipositive matrix, with the sum of all components of each row or column equals to one.

if and only if $\sum_{i=1}^{n} g(x_i) \leq \sum_{i=1}^{n} g(y_i)$ holds for all continuous convex functions $g : \mathbb{R} \to \mathbb{R}$. It is well-known that the crucial assumption for the comparison of those two distributions is that $\mathbf{x}$ and $\mathbf{y}$ have identical means.

Considerations on efficiency (higher incomes are more desiderable than lower incomes) and international comparisons (GDP in France differs from GDP in Italy) forced researchers to check for conditions that enable us to compare distributions where the total income distributed over different populations can differ. Scholars have isolated the problem of comparing distributions with different mean by the issue of analyzing income inequality among population with different sizes. Indeed, let us suppose $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$ are two vector distributions with means $\overline{x}$ and $\overline{y}$ respectively, where $\overline{x} \neq \overline{y}$, and whose n components are ordered in terms of decreasing incomes, then distribution $\mathbf{x}$ can be considered more unequal than $\mathbf{y}$, denoted as $\mathbf{y} \preceq_{weak} \mathbf{x}$, if and only if

$$\sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k} y_i, \qquad \text{for } k = 1, ..., n. \tag{2.1}$$

Condition 2.1 is referred to as distribution $\mathbf{y}$ is *weakly majorized* by $\mathbf{x}$ (see [5] chapter 1) or dually we say that $\mathbf{y}$ is dominated by $\mathbf{x}$ according to the *generalized Lorenz preordering* (see among others [9]). Morever, 2.1 is equivalent to say that:

(i)**:** the set $\{\mathbf{y} : \mathbf{y} \preceq_{weak} \mathbf{x}\}$ is the convex hull of points obtained by permuting the components of $\mathbf{x}$, or that

(ii)**:** the inequality $\sum_{i=1}^{n} g(y_i) \geq \sum_{i=1}^{n} g(x_i)$ holds for all continuous convex functions $g : \mathbb{R} \to \mathbb{R}$ (see [5] chapter 3).

In what follows, we extend the notion of ordering between vectors in 2.1 and the results mentioned above as (i) and (ii), to the case of rectangular matrices, namely multidimensional distributions representing a population of n agents among which real-valued attributes are distributed.

Indeed, we consider a fixed population of individuals $N = \{1, ..., n\}$ with $n \geq 2$, distinguished for a set $M = \{1, ..., m\}$ of attributes with $m \geq 2$ in order to avoid trivial qualifications. A distribution matrix, denoted as $\mathbf{X} = (x_1, ..., x_m)$, which is a collection of m column vectors, is a matrix where x_j are all column vectors of lenght n, of the following form:

$$
\begin{array}{rl}
 & \quad a \qquad b\ \ldots\ l\ \ldots\ m \qquad\qquad \longleftarrow \text{attributes} \\
\text{people} & \overbrace{\hphantom{x_{1,a}\quad x_{1,b}\quad \cdots\quad x_{1,m}}} \\
\downarrow & \\
\mathbf{X} = & \begin{bmatrix}
x_{1,a} & x_{1,b} & \cdots & x_{1,m} \\
. & . & & . \\
. & . & x_{i,l} & . \\
. & . & & . \\
. & . & . & . \\
x_{n,a} & \cdots & . & x_{n,m}
\end{bmatrix}
\end{array}
\tag{2.2}
$$

As mentioned above, we suppose that the element $x_{i,j} \in \mathbf{X}$, represents the quantity of the jth real-valued attributes (as e.g. the net annual flow of the jth commodity) belonging the the ith individual. The ith row of $\mathbf{X}$ is denoted row_i or $x_{i,\cdot}$, the jth column col_j or $x_{\cdot,j}$, and $\mathcal{A} \subset \mathbb{R}^{n,m}$ is the real vector space of (n, m)

matrices with non-negative real entries. A nonegative square matrix $\mathbf{X}$ (i.e. a matrix such that $x_{i,j} \geq 0$ for every $x_{i,j} \in \mathbf{X}$ and with the same number of rows and columns) with all its row sums equal to 1 is said to be *row-stochastic* or *Markovian* for its role in the theory of discrete Markov chains. When the sum of all components of each row or column of a matrix is equal to one, a nonnegative square matrix $\mathbf{X}$ is said to be *double stochastic.* If each row and column of a doubly stochastic matrix has a single unit and all other entries are zero, matrix Π is said to be a *permutation* matrix.[3]

In order to establish when a given distribution $\mathbf{X}$ is more unequal than $\mathbf{Y}$, the economic literature on multidimensional inequality have introduced several notions of matrix-ranking as the following:

Definition 1. *Let* $\mathbf{X},\ \mathbf{Y} \in \mathcal{A}$*, then* $\mathbf{X}$ *is said to majorize* $\mathbf{Y}$*, written* $\mathbf{Y} \prec \mathbf{X}$*, if there exists a* $n \times n$ *doubly stochastic matrix* $\mathbf{P}$*, such that* $\mathbf{PX} = \mathbf{Y}$*.*

The foregoing definition extends the notion of majorization on integers and the idea of transfer first introduced by Muirhead [6] for the unidimensional case to multivariate distributions. It essentially means that the average is a *smoothing-operation*, which makes the components of $\mathbf{Y}$ more spread out than components of $\mathbf{X}$. It is now well established (see e.g. Marshall and Olkin [5] and Koshevoy [3]) that $\mathbf{Y} \prec \mathbf{X}$ is tantamount to require that the following inequality:

$$\sum_{i=1}^{n} f(x_i) \geq \sum_{i=1}^{n} f(_i) \tag{2.3}$$

holds for any convex function $f : \mathbb{R}^m \to \mathbb{R}$ defined on the real-vector space of the columns of distribution matrix.

3. Results

In contrast to the ordering $\prec$ discussed in the previous section, we introduce a binary relational system that compares multidimensional distributions of individual attributes with different means in terms of their relative inequality:

Definition 2. *Let* $\mathbf{X},\ \mathbf{Y} \in \mathcal{A}$ *be two matrices. Then* $\mathbf{Y}$ *is said to be w-majorized by* $\mathbf{X}$*, written* $\mathbf{Y} \ll_w \mathbf{X}$*, if there exists a* $n \times n$ *row-stochastic matrix* $\mathbf{P}$*, such that* $\mathbf{PX} = \mathbf{Y}$*.*

We illustrate the foregoing definition by considering the following:

Example 1. *Let* $\mathbf{X},\ \mathbf{Y} \in \mathbb{R}^{2,3}$ *be two matrices representing a population of two individuals endowed with the same three goods with different affluence and distribution:*

$$\mathbf{X} = \begin{pmatrix} 2 & 5 & 1 \\ 1 & 1 & 7 \end{pmatrix} \text{ and } \mathbf{Y} = \begin{pmatrix} 1.8 & 4.2 & 2.2 \\ 1.5 & 3 & 4 \end{pmatrix}$$

Since there exists a row-stochastic matrix:

$$\mathbf{P} = \begin{pmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{pmatrix}$$

such that $\mathbf{PX} = \mathbf{Y}$*,* $\mathbf{Y} \ll_w \mathbf{X}$*, then we conclude that* $\mathbf{X}$ *has a greater level of disparity than* $\mathbf{Y}$ *as each row of latter is a convex combinationof the rows of the former.*

[3]Notice that the permutation matrices represent the extremal points of the set of doubly stochastic matrices (see [5]).

In words,if we interpret the rows of a matrix as individuals, it is as if we compare a population with a set of attributes (the columns) in terms of inequality at different periods of time, when individual relative affluence is changed. The ordering $\ll_w$ is a *preorder*, i.e. a reflexive and transitive binary relation. It differs from the ordering $\ll$ studied in [7], which compares matrices with a different number of rows (therefore with different population units), but the same number of columns. Moreover, *w-majorization* is equivalent, in the univariate case, to 2.1 for distributions with different means and amounts to the replacement of each entry of a distribution matrix by averages of the column distribution components of the latter. Finally, it will be useful to notice that the following equivalence:

Claim 1. $\mathbf{Y} \ll_w \mathbf{X}$ *if and only if there exists a row-stochastic matrix* $\mathbf{P}$ *such that* $\mathbf{P}x_{\cdot,j} = y_{\cdot,j}$ *for any* $j = 1, ..., m$*,*

always holds true.

In fact, it is easy to prove that the binary relation $\ll_w$ is nested between the matrix majorization $\prec$ and *vp-majorization* $\ll$ analyzed in [7] (i.e. $\prec$ implies $\ll_w$, which implies $\ll$). But, $\ll_w$ does not in general imply $\prec$ as it is straightforward to check in the Example 1 above. On the contrary, if $\mathbf{X}$ and $\mathbf{Y}$ have the same column marginal distributions then $\ll_w$ implies $\prec$ as it is shown by the following:

Proposition 1. *Let* $\mathbf{X}$*,* $\mathbf{Y} \in \mathcal{A}$ *and* $\mathbf{P}$ *be a square matrix with non-negative entries and such that* $\mathbf{PX} = \mathbf{Y}$*. If* $\mathbf{Y} \ll_w \mathbf{X}$ *and* $e^t\mathbf{X} = e^t\mathbf{Y}$*, then* $\mathbf{Y} \prec \mathbf{X}$*.*

Proof. It is known (see e.g. [5] chapter 15 and [3]) that in a binary relational system comparing rectangular matrices $\mathbf{X}$, $\mathbf{Y} \in \mathbb{R}^{n,m}$, if it adds a suitable column vector $\mathbf{e} = (1, 1, ..., 1)$ to both matrices under consideration, then their relative ranking is preserved. In particular, if $\mathbf{X}$, $\mathbf{Y} \in \mathcal{A}$, then $\mathbf{Y} \ll_w \mathbf{X}$ if and only if $\mathbf{A} = [\mathbf{Y}, \mathbf{e}] \ll_w [\mathbf{X}, \mathbf{e}] = \mathbf{B}$. Now, suppose $\mathbf{A} \ll_w \mathbf{B}$, then there exists a row-stochastic matrix $\mathbf{P}$ such that $\mathbf{PB} = \mathbf{A}$. Postmultiplying both sides for the so-called Moore-Penrose pseudoinverse $\mathbf{A}^{-1}$ of $\mathbf{A}$, we get $\mathbf{BA}^{-1} = \mathbf{PAA}^{-1}$. But $\mathbf{AA}^{-1} = \mathbf{O}$ is the orthogonal projection matrix onto $\mathbf{A}$, then $\mathbf{BA}^{-1} = \mathbf{PO}$, with $\mathbf{PO}$, that has non-negative entries by assumption. Thus, we show tha $\mathbf{PO}$ (and then, of course, $\mathbf{BA}^{-1}$) is doubly stochastic in order to get the result required, namely that $\mathbf{POe} = \mathbf{e}$ and $\mathbf{e}^t\mathbf{PO} = \mathbf{e}^t$. Since $\mathbf{Oe} = \mathbf{e}$, then $\mathbf{Pe}$, but $\mathbf{Pe} = \mathbf{e}$, because $\mathbf{P}$ is row-stochastic, hence the thesis. On the other hand, $\mathbf{e}^t\mathbf{PO} = \mathbf{e}^t\mathbf{PAA}^{-1} = \mathbf{e}^t\mathbf{BA}^{-1}$, but $e^t\mathbf{A} = e^t\mathbf{B}$ by assumption, then $\mathbf{e}^t\mathbf{AA}^{-1} = \mathbf{e}^t\mathbf{O} = \mathbf{e}^t$, i.e the thesis. Therefore, $\mathbf{PO} = \mathbf{BA}^{-1}$ is doubly stochastic. Recalling that $\mathbf{A} \ll_w \mathbf{B}$ is equivalent to $\mathbf{Y} \ll_w \mathbf{X}$ the required result. $\square$

Recently, Koshevoy [3] has introduced standard tools of convex analysis in the theory of inequality measurement. In fact, he defines the multivariate extension of the Lorenz preordering as the inclusion between the zonotopes of two multivariate distributions.[4] More in general, if $\mathbb{H} = co\left\{(\mathbf{z}_{i,1}, ..., \mathbf{z}_{i,m}),\ i = 1, ..., n\right\}$ denotes the convex hull of a matrix $\mathbf{Z} \in \mathbb{R}^{n,m}$, namely the set of convex combinations of the row-vectors of $\mathbf{Z}$, then we say, given two matrices $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{n,m}$, that $\mathbf{Y}$ has lower inequality than $\mathbf{X}$ if it lies in the convex hull of all column (and, of course, all row) permutations of $\mathbf{X}$. In particular, it holds true that:

[4] Notice that a zonotope is the finite Minkowski sum of line segments in $\mathbb{R}^m$, generated by the column vectors of a matrix distribution.

Proposition 2. *Let* $\mathbf{X}$, $\mathbf{Y} \in \mathcal{A}$, *then* $\mathbf{Y} \ll_w \mathbf{X}$ *if and only if* $\mathbf{Y} \subseteq co(\mathbf{X})$.

Proof. ($\Rightarrow$) Take $\mathbf{Y} \ll_w \mathbf{X}$ then $\mathbf{PX} = \mathbf{Y}$ which is tantamount to:

$$\text{row}_i(\mathbf{Y}) = [\mathbf{r}_{i,1}, ..., \mathbf{r}_{i,d}] = \sum_{i=1}^{d} p_{i,k} \,\text{col}_k(\mathbf{X}) = \sum_{i\ j} p_{i,k} \,\text{col}_{i\ k} \quad \text{for } i = 1, ..., n,$$

i.e. $\text{row}(\mathbf{Y}) \in co(\text{col}(\mathbf{X}))$ as required.

($\Leftarrow$) Assume $\mathbf{Y} \subseteq co(\mathbf{X})$, that is equivalent to $\text{row}_i(\mathbf{Y}) = \sum_{i\ j} p_{i,k} \,\text{col}_{i\ k}$, for $i = 1, ..., n$. That is a system of n linear equation in d variables (with $n \geq d$), a solution of which is a matrix $\mathbf{P}$ (with the constraint that $\sum_k \mathbf{p}_{i,k} = 1$), such that $\mathbf{PX} = \mathbf{Y}$, namely $\mathbf{Y} \ll_w \mathbf{X}$. □

According to Kolm [2], a matrix $\mathbf{Y}$ is said to be price-majorized by $\mathbf{X}$ if $\mathbf{Xp} \preceq \mathbf{Yp}$ for all $\mathbf{p} \in \mathbb{R}^m$, where $\mathbf{Xp} = ((\mathbf{x}_{1,\cdot}, \mathbf{p}), ..., (\mathbf{x}_{n,\cdot}, \mathbf{p}))$ and $\preceq$ is interpreted in the sense of standard majorization between vector distributions. Hence:

Corollary 1. *Let* $\mathbf{X}$, $\mathbf{Y} \in \mathcal{A}$, *if* $\mathbf{X}$ *price-majorized* $\mathbf{Y}$ *then* $\mathbf{Y} \ll_w \mathbf{X}$.

Proof. Suppose $\mathbf{X}$ price-majorized $\mathbf{Y}$ but that $\mathbf{Y}$ is not w-majorized by $\mathbf{X}$. This implies that there at least a $\text{row}_i(\mathbf{Y}) \notin co(\text{col}(\mathbf{X}))$. Then, there exists a hyperplane such that: $\langle \text{row}_i(\mathbf{Y}), \mathbf{p} \rangle \geq t$ and $\langle \text{row}_j(\mathbf{X}), \mathbf{p} \rangle < t$, for all $j = 1, ..., n$, $\mathbf{p} \in \mathbb{R}^m$ and $t > 0$, and such that $\langle \cdot \rangle$ is the inner product of two vectors.

However, by definition of price-majorization we have that

$$y_1 = \text{row}_1(\mathbf{Y})\,\mathbf{p} \geq y_i = \text{row}_i(\mathbf{Y})\,\mathbf{p} \geq t > \text{row}_1(\mathbf{X})\,\mathbf{p} = x_1$$

a contradiction and therefore the required result. □

For each set $S \subseteq \{1, ..., m\}$, let $\mathbf{Z}_S$ denote the submatrix of $\mathbf{Z}$ induced by the columns indexed by the elements in S. Then, the preordering induced by the *w-majorization* on set $\mathcal{A}$ satisfies the following properties:

Property D**:** If $\mathbf{Y} \ll_w \mathbf{X}$, then $\mathbf{Y}_S \ll_w \mathbf{X}_S$ for each $S \subseteq \{1, ..., m\}$;
Property C**:** If $\mathbf{Y} \ll_w \mathbf{X}$, then $rank(\mathbf{Y}) \leq rank(\mathbf{X})$;
Property S**:** If $\mathbf{Y} \ll_w \mathbf{X}$ and $\mathbf{R}, \mathbf{Q} \in \mathbb{R}^{n,n}$ are two *permutation* matrices, then $\mathbf{RY} \ll_w \mathbf{QX}$.

The foregoing properties are provided using elementary arguments on stochastic matrix as it is shown by the following:

Proposition 3. *For each* $\mathbf{X}$, $\mathbf{Y} \in \mathcal{A}$, w-majorization $\ll_w$ *satisfies Properties* D, C *and* S.

Proof. All properties can be deduced directly from the definition of $\ll_w$, considering that the set of row-stochatic matrices is closed under matrix products. □

Following the traditional approach in economic literature on inequality measurement, we now introduce welfare considerations into the present multidimensional framework. We define a set $\Upsilon(\mathbb{R}^m)$ as the set of all real-valued convex functions defined over a nonempty compact convex set $C \subset \mathbb{R}^{k,m}$. We establish a counterpart of the result 2.3 mentioned above:

Proposition 4. *Let* $\mathbf{X}$, $\mathbf{Y} \in \mathcal{A}$. *Then the following statements are equivalent:*

(i) $\mathbf{Y} \ll_w \mathbf{X}$;

(*ii*) *For each integer* $1 \le k \le n$ *and* $\gamma \in \Upsilon_k(\mathbb{R}^m)$, *we have that*

$$\max_k \gamma(x_k) \ge \max_k \gamma(y_k) \tag{3.1}$$

Proof. ($\Rightarrow$) According to Proposition 2 above, *w-majorization* $\ll_w$ is equivalent to $\mathrm{row}(\mathbf{Y}) \subseteq co(\mathrm{row}(\mathbf{X}))$. Now, $\mathrm{row}(\mathbf{Y})$ is convex, because any $\mathrm{row}_i(\mathbf{Y}) = \sum_{j=1}^n p_{i,j}\,\mathrm{row}_j(\mathbf{X})$, where $p_{i,j}$ are the entries of some row-stochastic matrix $\mathbf{P}$ such that $\mathbf{Y}\ll_w\mathbf{X}$, and $co(\mathrm{row}(\mathbf{X}))$ is also convex by definition. Then, following [10], we know that given two convex sets A and B, $A \subseteq B$ if and only if $\max_{z\in A}\phi(z) \le \max_{z\in B}\phi(z)$ for every convex function defined over $A \cup B \subseteq \mathbb{R}^{k,m}$ and $z \in \mathbb{R}^m$. Hence, the result required.

($\Leftarrow$) Let us assume that $\max_k \gamma(x_k) \ge \max_k \gamma(y_k)$ for every real-valued convex function $\gamma \in \Upsilon_k(\mathbb{R}^m)$. That is equivalent to say that the set of all rows of $\mathbf{Y}$ is convex and it is a subset of the convex set of all rows of $\mathbf{X}$. But, this is tantamount to $co(\mathrm{row}(\mathbf{Y})) \subseteq co(\mathrm{row}(\mathbf{Y}))$, namely $\mathbf{Y}\ll_w\mathbf{X}$. □

Inequality 3.1 can be interpreted as a function that evaluates the inequality of each person in a society. Moreover, it could be considered the dual of a corresponding (multidimensional) inequality index. Finally, convexity of γ captures the inequality aversion when consider a distribution $\mathbf{X}$ more "spreading out" than distribution $\mathbf{Y}$.

In [7], we have characterized the support functions, namely real-valued sublinear functions, as the class of functions preserving the preordering $\ll$ (see [7], Therem 3) . In fact, *w-majorization* $\ll_w$ represents a subordering of the ranking induced by $\ll$ (i.e. $\ll_w$ implies $\ll$). Then, it implies that the class of functions preserving the binary relation $\ll_w$ must be larger than the latter one, as it is provided by the following:

Proposition 5. *Let* $\mathbf{X}$, $\mathbf{Y} \in \mathcal{A}$, *then the following conditions are equivalent:*

(*a*) $\mathbf{Y}\ll_w\mathbf{X}$;

(*b*) $\Psi(\mathbf{Y}) \le \Psi(\mathbf{X})$ *holds for any* $\Psi : \mathcal{A} \to \mathbb{R}$ *of the following form,for each* $\mathbf{Z} \in \mathcal{A}$, $\Psi(\mathbf{Y}) = \max_k \varphi_z(x_k)$, *with* $1 \le k \le n$, *where* $\varphi_z : \mathbb{R}^m \to \mathbb{R}$, *is a linear function defined by* $\varphi_z(z) = \langle z, v\rangle$ *for every* $v \in \mathbb{R}^m$.

Proof. ($\Rightarrow$) According to Proposition 2 above, *w-majorization* $\ll_w$ is equivalent to $\mathrm{row}(\mathbf{Y}) \subseteq co(\mathrm{row}(\mathbf{X}))$, which is tantamount to $\max_i \langle x_i, v\rangle \ge \langle \mathrm{row}(\mathbf{Y}), v\rangle$ for any $v \in \mathbb{R}^m$. The latter expression implies $\max_i \langle x_i, v\rangle \ge \max_i \langle y_i, v\rangle$ or equivalently that $\varphi_z(x_i) \ge \varphi_z(y_i)$ as required.

($\Leftarrow$) $\max_i \langle x_i, v\rangle \ge \max_i \langle \mathbf{y}_i, v\rangle$ for any $v \in \mathbb{R}^m$ implies that the rows of matrix $\mathbf{Y}$ rely in the convex hull of all rows of $\mathbf{X}$, hence the thesis. □

An important problem to be considered concerns the characterization of the linear operators that preserve the *w-majorization* $\ll_w$ in the sense explained by the following:

Definition 3. *Let* $\mathcal{L}$ *be a linear space of rectangular matrices, let* $\Im$ *be a linear operator defined on* $\mathcal{L}$ *and let* $\Re$ *be a relation on* $\mathcal{L}$. *We say* $\Im$ preserves $\Re$ *if:*

$$\Re(\Im(\mathbf{X}), \Im(\mathbf{Y})) \qquad \textit{whenever } \Re(\mathbf{X}, \mathbf{Y}).$$

In fact, we guess, for the moment, that:

Theorem 1. *A linear operator* $\Im : \mathcal{A} \to \mathcal{A}$ *preserves* w-majorization *if and only if* $\Im(\mathbf{Z}) = \mathbf{SZP}$ *for all* $\mathbf{Z} \in \mathcal{A}$ *and* $\mathbf{S} \in \mathbb{R}^{n,n}$ *and* $\mathbf{P}$ *a* $m \times m$ *permutation matrix.*

4. Conclusions and final remarks

We explored the possibility to theoretically extend inequality analysis from the univariate to the multivariate setting. Our main aim was to provide a multidimensional counterpart of the generalized Lorenz preorder for the case in which people differ in many attributes besides income.

We know that the approach that analyzes multidimensional inequality by using measure indices is very problematic because it requires judgements regarding the relative importance of the various individual attributes, the degree of substitution between them and the degree of inequality aversion in society. At the same time, the approach to multidimensional inequality via stochastic dominance introduces further layers of complexity to the measurement of disparity in several dimensions as pointed out in [8], drawbacks that need to be resolved in order to use of multidimensional inequality analysis in comparative studies.

On the contrary, this work, using simple tools of convex analysis, shows how our approach can be of some interest for the analysis of inequality when several individual characteristics are simultaneously considered and offers a significant scope for further developments in the study of multidimensional inequality.

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