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Concept Lattices and Convexity of Coalitional Game Forms

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Abstract - The concept lattice of a coalitional game form is introduced and advocated as a structural classificatory tool. The basic properties of such lattices are studied. Sufficient concept-latticial properties for convexity of the underlying coalitional game form are identified. Spectral issues concerning widths and lengths of concept lattices of convex CGFs are also addressed.

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1 Introduction

In the last few decades a huge flow of literature on implementation theory and mechanism design has provided an impressive amount of detailed knowledge on the behaviour of the games induced by certain game forms on a few typical preference domains, and with respect to a wide variety of both non-cooperative and cooperative solution concepts. As an implicit by-product of this work, a lot has been learnt on ways of classifying game forms from that behavioural perspective. By contrast, virtually no attention has been devoted to the task of classifying *game forms as such,* with no reference to the behaviour of the games they may induce on certain preference domains. To be sure, in some previous work of mine (see e.g. Vannucci (1999)) concept lattices of effectivity functions were introduced and used in order to provide structural classifications of the latter in terms of the allocation of a priori coalitional power they induce, quite independently of the preference domain they are meant to be applied to. It goes without saying that, generally speaking, the ensuing 'structural' classifications of game forms need not be related to the former, 'behavioural' ones, which arise from their performance with respect to certain solution concepts on prespecified environments. However, it is also not clear why such 'structural' and 'behavioural' classifications of game forms should be indeed 'orthogonal' or totally unrelated. Indeed, one should not rule out that *some* connections -perhaps interesting ones- do exist between those two types of classifications of game forms.

In this paper, concept lattices of *general* coalitional game forms are first introduced, and their basic properties studied. Then, a preliminary exploration of relationships between concept-latticial properties of game forms and their behavioural performance is pursued. In particular, it is shown that if the concept lattice of a coalitional game form is a chain and its closure systems obey certain closure conditions w.r.t. the meet operation then the given coalitional game form for must be convex, hence strongly core-stable.

The paper is organized as follows. Section 2 is devoted to introducing the concept lattice of a coalitional game form with related notions. Section 3 includes the main result of the paper relating some concept-latticial properties to convexity of the relevant coalitional game form. Section 4 consists of a few concluding remarks.

2 The concept lattice of a coalitional game form

Let (N, X) be a pair of non-empty sets (the sets of players and outcomes, respectively; we also assume that N is *finite*, and that $\#N \geq 2$ and $\#X \geq 2$ in order to avoid trivialities). A *(monotonic) simple game* on N is a set pair (N, W) where W is a non-empty *order filter* of $(P(N), \supseteq)$, the inclusion-ordered power set of N (recall that an order filter of a partially ordered set $(Y, \geq)$ is a set $U \subseteq Y$ such that for all $u, y \in Y$ if $u \in U$ and $y \geq u$ then $y \in U$): the coalitions belonging to W are meant to represent the *winning* or all-powerful ones. A simple game (N,W) is said to be *principal* if W is a *principal order filter* of $(P(N),\supseteq)$ i.e. there exists $S\subseteq N$ such that $W=\{T\subseteq N : T\supseteq S\}$. A *coalitional game form(CGF)* on (N,X) (with total domain) is a triple $\mathbf{G}=(N,X,E)$ where $E:P(N)\rightarrow P(P(X))$ is a function. A CGF $\mathbf{G}$ is said to be *normalized* if $E(\emptyset)=\emptyset$, *souvereign* if there exists $T\subseteq N$ such that $E(T)\supseteq P(X)\backslash\{\emptyset\}$, *exhaustive* if $X\in E(S)$ for any S, $\emptyset\neq S\subseteq N$, and *non-empty-valued* if $\emptyset\notin E(S)$ for any S , $\emptyset\subset S\subseteq N$. An *effectivity function* (EF) on (N,X) is a coalitional game form $\mathbf{G}=(N,X,E)$ which is *normalized, souvereign, exhaustive, and non-empty-valued.*

A CGF $\mathbf{G}$ =(N,X,E) is *monotonic* if for any $S,T\subseteq N$ and any $A,B\subseteq X$

$[A\in E(S)$ and $S\subseteq T$ entail $A\in E(T)]$ and

$[A\in E(S)$ and $A\subseteq B$ entail $B\in E(S)]$.

In what follows we shall be mainly concerned with *monotonic* EFs.

A monotonic CGF $\mathbf{G}$ =$(N,X,E))$ is *regular* if $\emptyset\neq A\in E(S)$ entails $X\backslash A\notin E(N\backslash S)$ for any $S\subseteq N$ and $B\subseteq X$, *maximal* if $A\notin E(S)$ entails $(X\backslash A)\in E(N\backslash S)$ for any $\emptyset\neq S\subseteq N$ and $\emptyset\neq A\subseteq X$, *essential* if for any $i\in N$: $E(\{i\})\neq\{A\subseteq X : A\neq\emptyset\}$, and *consensual* if for any $S\subset N$: $E(S)\neq\{A\subseteq X : A\neq\emptyset\}$. Moreover, a CGF $\mathbf{G}$ =(N,X,E) is *superadditive* if for any $S,T\subseteq N$ and $A,B\subseteq X$, $A\in E(S)$, $B\in E(T)$ and $S\cap T=\emptyset$ entail $A\cap B\in E(S\cup T)$, *convex* if for any $S,T\subseteq N$ and $A,B\subseteq X$ if $A\in E(S)$ and $B\in E(T)$ then $A\cap B\in E(S\cup T)$ or $A\cup B\in E(S\cap T)$,and *additive* if there exist positive probability measures p,q on N,X respectively s.t. $A\in E(S)$ iff $p(S)+q(A)>1$. We shall also say that a CGF $\mathbf{G}=(N,X,E)$ is *Ferrers* if for any $S,T\subseteq N$, $E(S)\subseteq E(T)$ or $E(T)\subseteq E(S)$.

Finally, an EF $\mathbf{G}$ =(N,X,E) is *simple* if there exists a simple game (N,W) such that for any $S\subseteq N$, $A\subseteq X$, $A\in E(S)$ if and only if either $A=X$ and $S\neq\emptyset$ or $A\neq\emptyset$ and $S\in W$ (notice that a simple EF is –by definition– both well-behaved and monotonic). Indeed, a simple EF $\mathbf{G}$ =(N,X,E) is induced by a simple game (N,W) as endowed with a fixed outcome set X, and will also be denoted $\mathbf{G}$ =(N,X,E_W).

We are specially interested in those CGFs -indeed EFs- that can represent the decision power of coalitions under a certain decision mechanism, or *strategic game form* . A *strategic game form* on (N,X) is a tuple $\mathbf{\Gamma}=(N,X,(S_i)_{i\in N},h)$ where S_i is the set of strategies available to player $i\in N$ and $h:\prod_{i\in N}S_i\rightarrow X$ denotes the strategic outcome function.

Now, the notion of decision power admits at least two distinct interpretations, namely "guaranteeing power" and "counteracting power" that in turn correspond to the ability to force maximin and minimax outcomes, respectively. Thus, the allocation of "guaranteeing power" under strategic game form Γ is represented by the $\alpha-EF$ of Γ - denoted by $E_\alpha(\Gamma)$- as defined by the following rule: for any non-empty $S\subseteq N$,

$$(E_\alpha(\Gamma))(S)=\left\{\begin{array}{c} A\subseteq X\text{: there exists } t^S\in\prod_{i\in S}S_i \text{ exists such that} \\ h(t^S,s^{N\backslash S})\in A \\ \text{for any } s^{N\backslash S}\in\prod_{i\in N\backslash S}S_i, \end{array}\right\}.$$

Conversely, the allocation of "counteracting power" under strategic game form G with domain D is represented by the $\beta - EF$ of G, denoted by $E_\beta(G)$ and defined as follows :

for any non-empty $S \subseteq N$

$$(E_\beta(\Gamma))(S) = \left\{ \begin{array}{c} A \subseteq X : \text{ for any } s^{N\setminus S} \in \prod_{i\in N\setminus S} S_i \text{ there exists } t^S \in \prod_{i\in S} S_i \\ \text{such that } h(t^S, s^{N\setminus S}) \in A \end{array} \right\}.$$

It is easily checked that $E_\alpha(\Gamma)$ is regular, $E_\beta(\Gamma)$ is maximal, and both of them are *monotonic* and - provided that G is non-empty valued- *well-behaved* . Also, it is well-known that superadditivity and monotonicity of an EF $\mathbf{G} = (N, X, E)$ is *α-strategically playable* i.e. there exists a strategic game form Γ such that $E = E_\alpha(\Gamma)$: see e.g. Otten,Borm,Storcken and Tijs(1995)) . Indeed, monotonicity of $\alpha - EFs$ and $\beta - EFs$ of strategic game forms is our main reason for confining the ensuing analysis to monotonic EFs (as mentioned previously).

It is easily checked that : i) the set of all EFs on (N, X) is bijective to a set of binary relations on $(P(N), P(X))$: hence any EF $\mathbf{G} = (N, X, E)$ can be equivalently regarded as a binary relation ; ii) therefore, the classic Birkhoff theorem on so called Galois connections applies. It follows that the functions $f_E : P(P(N)) \to P(P(X))$, $g_E : P(P(X)) \to P(P(N))$ as defined by the rules

$f_E(\mathbf{S}) = \{A \subseteq X : A \in E(S) \text{ for any } S \in \mathbf{S}\}$ for any $\mathbf{S} \subseteq P(N)$, and

$g_E(\mathbf{A}) = \{S \subseteq N : A \in E(S) \text{ for any } A \in \mathbf{A}\}$

enjoy the following list of properties:

a) the functions $K_E = g_E \circ f_E$ and $K_E^* = f_E \circ g_E$ are *closure operators* on $(P(N), \supseteq)$ and $(P(X), \supseteq)$, respectively (we recall here that a (Moore) closure operator on a preordered set $(Y, \geq)$ is a function $K : Y \to Y$ such that for any $y, z \in Y : K(y) \geq y$; $y \geq z$ entails $K(y) \geq K(z)$; $K(y) \geq K(K(y))$).

b) the corresponding *closure systems* - i.e. sets of closed sets - $\mathbf{C}(K_E) = \{\mathbf{S} \subseteq P(N) : \mathbf{S} = K_E(\mathbf{S})\}$, $\mathbf{C}(K_E^*) = \{\mathbf{A} \subseteq P(X) : \mathbf{A} = K_E^*(\mathbf{A})\}$ are (dually isomorphic) *complete lattices* under the join and meet operations defined as follows:

for any $\{\mathbf{S}_i\}_{i\in I} \subseteq \mathbf{C}(K_E)$, $\{\mathbf{A}_i\}_{i\in I} \subseteq \mathbf{C}(K_E^*)$,

$\vee_{i\in I}\mathbf{S}_i = K_E(\cup_{i\in I}\mathbf{S}_i)$, $\wedge_{i\in I}\mathbf{S}_i = \cap_{i\in I}\mathbf{S}_i$, $\vee^*_{i\in I}\mathbf{A}_i = K_E^*(\cup_{i\in I}\mathbf{A}_i)$, $\wedge^*_{i\in I}\mathbf{A}_i = \cap_{i\in I}\mathbf{A}_i$

(we recall that a lattice is a partially ordered set $(L, \geq)$ such that for any pair $\{x, y\} \subseteq L$, both a greatest lower bound (glb) -or meet- $\wedge \{x, y\}$and a lowest upper bound (lub) - or join - $\vee \{x, y\}$ exist; a lattice is *complete* if any subset of L has both a glb and a lub).

c) the lattices under b) are *dense,* i.e. have a unique atom and - if E is well-behaved- *co-dense*, i.e. have a unique co-atom (an *atom* of a lattice $(L, \geq)$ is a $\geq$-minimal non-bottom element of L, and a *co-atom* is-dually- a $\geq$-maximal non-top element of L).

The *concept or Galois lattice* of a CGF $\mathbf{G} = (N, X, E)$ is

$\mathbf{L}(\mathbf{G}) = (L(\mathbf{G}) = \{(\mathbf{S}, \mathbf{A}) : \mathbf{S} = g_E(\mathbf{A}) \text{ and } \mathbf{A} = f_E(\mathbf{S})\}, \supseteq)$, and for any $\{(\mathbf{S}_i, \mathbf{A}_i)_{i\in I}\} \subseteq L(\mathbf{G})$

$\bigvee_{i\in I} (\mathbf{S}_i, \mathbf{A}_i) = (K_E(\cup_{i\in I}\mathbf{S}_i), \cap_{i\in I}\mathbf{A}_i)$, $\bigwedge_{i\in I}(\mathbf{S}_i, \mathbf{A}_i) = (\cap_{i\in I}\mathbf{S}_i, K_E^*(\cup_{i\in I}\mathbf{A}_i))$.

Clearly enough, the concept lattice $\mathbf{L}(\mathbf{G})$ -that is also sometimes called the *Galois lattice* of $\mathbf{G}$- is lattice-isomorphic to the lattices of inclusion-ordered closure systems of $\mathbf{G}$ (see e.g. Barbut and Monjardet (1970) and Ganter and Wille (1999)). Hence, $\mathbf{L}(\mathbf{G})$ is complete, has a unique atom if $\mathbf{G}$ is normalized and a unique co-atom if $\mathbf{G}$ is non-empty-valued. Moreover, if $\mathbf{G}$ is Ferrers then $\mathbf{L}(\mathbf{G})$ is a chain. Finally, it is clearly the case that finiteness of the player set or of the outcome set entails finiteness of $\mathbf{L}(\mathbf{G})$.Those basic facts concerning $\mathbf{L}(\mathbf{G})$ can be summarized by the following proposition (see also Vannucci(1999) for an earlier specialized version of the same statement, and Ganter and Wille(1999) for some related results on general concept lattices):

Proposition 1 *Let* $\mathbf{G} =(N, X, E)$ *be a CGF. Then, a complete lattice* $\mathbf{L}(\mathbf{G})$ *- the concept lattice of* $\mathbf{G}$ *- uniquely defined up to isomorphisms- can be canonically attached to* $\mathbf{G}$. *Moreover, i) if* $\mathbf{G}$ *is normalized then* $\mathbf{L}(\mathbf{G})$ *is dense ; ii) if* $\mathbf{G}$ *is non-empty-valued then* $\mathbf{L}(\mathbf{G})$ *is co-dense; iii) if* $\mathbf{G}$ *is Ferrers then* $\mathbf{L}(\mathbf{G})$ *is a chain i.e. a linearly ordered set; iv)* $\mathbf{L}(\mathbf{G})$ *is finite whenever either* N *or* X *is finite.*

Remark 2 *While for small finite* N *and* X *computing the concept lattice of a CGF* $\mathbf{G} = (N, X, E)$ *may be easily compued by hand. For larger CGFs some suitable algorithm is needed. A few algorithms for computing concept lattices are presented and discussed in Ganter and Wille (1999), chpt.2.*

Thus, it follows from the foregoing observations and results that a (complete) lattice –the concept lattice–can be effectively attached in a most 'natural' way to each CGF. This fact opens up the opportunity to introduce 'new' classifications of CGFs from a number of interesting perspectives, relying on suitable concept lattice parameters. Of course, those parameters (such as *width, length, size, number of join and/or meet irreducibles*) provide some complexity-evaluation criteria concerning the structure of the underlying CGFs.

I recall here some relevant order- and lattice-theoretic notions. The *width* $w(\mathbf{P})$ of a poset $\mathbf{P} = (P, \geq)$ is the (common) size of its largest antichains (an *antichain* of $\mathbf{P}$ is a set of pairwise $\geq -$incomparable elements). The *length* $l(\mathbf{P})$ of a poset $\mathbf{P} = (P, \geq)$ is the least upper bound of the set of lengths of chains included in $\mathbf{P}$ (a chain is a totally ordered set; the length of a chain of $m + 1$ elements is m).

Thus, the width of the concept lattice of a CGF provides some summary information on the maximum 'degree' of specialization of decision tasks that is allowed by the given CGF. By contrast, the length of the concept lattice of a CGF provides information on the number of layers of decision power induced by the latter.

In particular, the notion of *order-dimension* is made available for CGFs through their concept lattices. Indeed, let $\mathbf{L} = (L, \geq)$ be a lattice. Then, the *order dimension* $d_O(\mathbf{L})$ of $\mathbf{L}$ is given by the minimum positive integer h such that there exist h chains $(L, \geq_1), .., (L, \geq_h)$ with $\geq = \cap_{i=1}^{h} \geq_i$. Therefore, for any CGF $\mathbf{G}$ one may also posit $\dim \mathbf{G} = d_O(\mathbf{L}(\mathbf{G}))$. Moreover, the following fact –which is easily established as an immediate corollary of a well-known result of formal concept analysis (see Ganter,Wille(1999))– entails that the order dimension of any finite CGF $\mathbf{G}$ can be in principle detected by direct inspection of $\mathbf{G}$:

Claim 3 *Let $\mathbf{G} =(N, X, E)$ be a finite CGF. Then its order dimension is given by its so-called* Ferrers dimension *i.e.*

$$\dim(\mathbf{G}) = \min \left\{ \begin{array}{c} k \in \mathbb{Z}_+ : \textit{ there exist} \\ \{E_i : P(N) \to P(P(X)), E_i \textit{ is Ferrers} : i = 1, .., k\} \\ \textit{such that } E(S) = \bigcap_{i=1}^{k} E(S) \textit{ for all } S \subseteq N \end{array} \right\}$$

Summing up, concept-latticial parameters such as *width* and *length* or *order dimension* provide in a most succinct way some basic information on the characteristic degrees of decentralization, specialization and hierarchization of decision tasks among coalitions that are induced by a given game form or decision mechanism. The resulting classifications of CGFs may well turn out to be at odds with more conventional classifications which are typically inspired by behavioural performance of CGFs in certain environments and with respect to certain solution concepts as explained above, or simply by more 'superficial' features of the CGFs themselves. The following example may help clarify this point.

Example 4 *Let $\mathbf{G}_1 = (N, X, E_{W_1})$, $\mathbf{G}_2 = (N, X, E_{W_2})$ be simple CGFs with $\#N \geq 3$, $\#X \geq 3$, $W_1 = \{S \subseteq N : i^* \in S\}$ for some $i^* \in N$, $W_2 = \{S \subseteq N : \#S \geq \lfloor \#N/2 \rfloor + 1\}$. Clearly enough $\mathbf{G}_1$is a dictatorial game form, while $\mathbf{G}_2$ is a simple majority game form. The second one is anonymous, while the first one is definitely not. Moreover, their performance with respect to many typical solution concepts- including the core- is dramatically different. However, it is easily checked that $\mathbf{L}(\mathbf{G}_1) = \mathbf{L}(\mathbf{G}_2) = \mathbf{4}$, the four-element chain.*

One should not conclude, however, that concept-latticial features of a CGF are totally unrelated to their behavioural performance with respect to standard solution concepts. The next section will be devoted to the exploration of one significant link between certain concept-latticial features of a CGF and its core-stability properties.

3 Concept lattices and convexity

As mentioned above in the Introduction, one should like to be able to relate concept-latticial classifications of CGFs to standard *solutions and solution concepts* of the relevant CGF-induced coalitional games. One such link is provided by the next Proposition which relates some features of the concept lattice of a CGF to convexity of the latter which is in turn a well-known sufficient condition for core-stability of a CGF (see e.g. Peleg (1984), Abdou and Keiding (1991)). Moreover, one should like to know whether convexity entails structural restrictions on the Galois lattice of a CGF

A few more definitions are in order here, namely:

Definition 5 *A monotonic CGF* $\mathbf{G}$ =(N, X, E) *is* semi-principal *if for any pair* $\mathbf{S}, \mathbf{A}$ *such that* $\mathbf{S}$ =$g_E(\mathbf{A}), \mathbf{A}$ =$f_E(\mathbf{S})$- *i.e.* $\mathbf{S}, \mathbf{A}$ *are isomorphically related closed families of coalitions and events- it must be the case that at least one of the following conditions holds: i)* $\mathbf{S}$ *is a* principal *order filter of* $(P(N), \supseteq)$*, ii)*$\mathbf{A}$ *is a* principal *order filter of* $(P(X), \supseteq)$.

Definition 6 *A semi-principal CGF* $\mathbf{G}$ =(N, X, E) *is* quasi-principal *if for any pair* $\mathbf{S}, \mathbf{S}'$ *of* K_E*-closed families of coalitions such that* $\mathbf{S}$ *is not a principal order filter and* $\mathbf{S}' \subset \mathbf{S}$ *it must be the case that* $S \cap S' \in \mathbf{S}$ *for any* $S \in \mathbf{S}$ *and any* $S' \in \mathbf{S}'$.

Remark 7 *A semi-principal CGF need not be quasi-principal. To see this consider the CGF* $\mathbf{G}$ =(N, X, E) *defined as follows:* $E(\emptyset) = \emptyset$ *and there exist* $x, y \in X$ *such that for any* $S \subseteq N, A \subseteq X$, $A \in E(S)$ *iff one of the following clauses is satisfied: i)* $S = N$ *and* $A \neq \emptyset$*; ii)* $\#S \geqslant \#N - 1$ *and* $x \in A$*; iii)* $\#S \geqslant \#N - 2$ *and* $\{x, y\} \subseteq A$*; iv)*$S \neq \emptyset$ *and* $A = X$.

By construction $\mathbf{G}$ *is a monotonic EF which is semi-principal (and Ferrers, with concept lattice* $\mathbf{L}(\mathbf{G}) = \mathbf{6}$*) but not quasi-principal. Indeed, posit* $\mathbf{S} = \{S \subseteq N : \#S \geqslant \#N - 2\}$ *and* $\mathbf{S}' = \{S \subseteq N : \#S \geqslant \#N - 1\}$. *It is easily checked that both* $\mathbf{S}$ *and* $\mathbf{S}'$ *are closed sets of the Galois closure operator on* $P(N)$ *induced by* $\mathbf{G}$. *Also, both of them are non-principal order filters of* $(P(N), \supseteq)$ *and clearly* $\mathbf{S}' \subset \mathbf{S}$. *However, for any distinct* $i, j, k \in N$, $N \backslash \{i, j\} \in \mathbf{S}, N \backslash \{k\} \in \mathbf{S}'$ *while* $N \backslash \{i, j\} \cap N \backslash \{k\} \notin \mathbf{S}$.

We are now ready to state the result announced above, namely

Proposition 8 *Let* $\mathbf{G} = (N, X, E)$ *be a CGF which is both Ferrers and quasi-principal. Then* $\mathbf{G}$ *is convex.*

Proof. Let $A \in E(S)$ and $B \in E(T)$.To begin with, notice that for any $(\mathbf{C}, \mathbf{C}^*) \in \mathbf{L}(\mathbf{G})$ there exist $\mathbf{T}, \mathbf{A}$ such that $(\mathbf{C}, \mathbf{C}^*) = (K_E(\mathbf{T}), f_E(\mathbf{T})))$= $(g_E(\mathbf{A}), K_E^*(\mathbf{A}))$ (see e.g. Ganter and Wille (1999)). Then, posit $K_E(\{S\}) = K_E(\mathbf{S})$, $K_E(\{T\}) = K_E(\mathbf{T})$ for some $\mathbf{S}, \mathbf{T} \subseteq P(N)$.If both $K_E(\mathbf{S})$ and $K_E(\mathbf{T})$ are principal then there exist $U, V \subseteq N$ such that $K_E(\mathbf{S})$=$\{S' \subseteq N : S' \supseteq U\}$, and $K_E(\mathbf{T}) = \{S' \subseteq N : S' \supseteq V\}$.But then, since $\mathbf{G}$ is Ferrers, it must be the case that $K_E(\mathbf{S}) \subseteq K_E(\mathbf{T})$ or $K_E(\mathbf{S}) \subseteq K_E(\mathbf{T})$ i.e. by definition $U \supseteq V$ or $V \supseteq U$. Let us suppose w.l.o.g. that $U \supseteq V$.It follows that $S \supseteq U \supseteq V$ whence $S \cap T \in K_E(\mathbf{T})$.Therefore, by definition, $E(S \cap T) \supseteq f_E(\mathbf{T})$. Since -by construction- $B \in f_E(\mathbf{T})$, it also follows that $B \in E(S \cap T)$, hence-by monotonicity- $A \cup B \in E(S \cap T)$. If neither $K_E(\mathbf{S})$ nor $K_E(\mathbf{T})$ are principal then both $f_E(\mathbf{S})$ and $f_E(\mathbf{T})$ are, since $\mathbf{G}$ is semi-principal. Hence, there exist $C, D \subseteq X$ such that $f_E(\mathbf{S}) = \{A' \subseteq X : A' \supseteq C\}$and $f_E(\mathbf{T}) = \{A' \subseteq X : A' \supseteq D\}$.Moreover, since $\mathbf{G}$ is Ferrers it follows that $f_E(\mathbf{S}) \subseteq f_E(\mathbf{T})$ or $f_E(\mathbf{T}) \subseteq f_E(\mathbf{S})$ i.e. $C \supseteq D$ or $D \supseteq C$.Let us then suppose w.l.o.g. that $C \supseteq D$. Since, by construction, $A \in f_E(\mathbf{S})$ and $B \in f_E(\mathbf{T})$, it follows that both $A \supseteq D$ and $B \supseteq D$ i.e. $A \cap B \supseteq D$.Now, $D \in E(T)$, by construction. Therefore, $A \cap B \in E(S \cup T)$, by monotonicity.

Finally, if *either* $K_E(\mathbf{S})$ *or* $K_E(\mathbf{T})$-but not both- are principal, two cases are to be distinguished. First, let us suppose w.l.o.g. that $K_E(\mathbf{T})$ is principal i.e. there exists $V \subseteq N$ such that $K_E(\mathbf{T})=\{S' \subseteq N : S' \supseteq V\}$. Since $\mathbf{G}$ is Ferrers, we know that either i) $K_E(\mathbf{S}) \supset K_E(\mathbf{T})$ or ii)$K_E(\mathbf{T}) \supset K_E(\mathbf{S})$. If $K_E(\mathbf{S}) \supset K_E(\mathbf{T})$, then from quasi-principality of $\mathbf{G}$ it follows that $S \cap T \in K_E(\mathbf{S})$, since by construction $S \in K_E(\mathbf{S})$ and $T \in K_E(\mathbf{T})$.Therefore, $A \in f_E(\mathbf{S}) \subseteq E(S \cap T)$ whence $A \cup B \in E(S \cap T)$, by monotonicity. Conversely, if $K_E(\mathbf{T}) \supset K_E(\mathbf{S})$ holds, then $S \in K_E(\mathbf{T})$ since by construction $S \in K_E(\mathbf{S})$.But then, $S \supseteq V$ whence $S \cap T \supseteq V$ (recall that $T \in K_E(\mathbf{T})$ by construction). Thus, $S \cap T \in K_E(\mathbf{T})$ which entails that $B \in f_E(\mathbf{T}) \subseteq E(S \cap T)$. Hence $A \cup B \in E(S \cap T)$, by monotonicity. ■

Remark 9 *It should be noticed that Proposition 8 is tight. To see this, consider the following examples. First, consider* $\mathbf{G}_2 = (N, X, E_{W_2})$ *with* $\#X \geq 3$ *and* $W_2 = \{S \subseteq N : \#S \geq \lfloor \#N/2 \rfloor + 1\}$ *as defined above (see Example 4). Clearly,* $\mathbf{G}_2$ *is Ferrers but it is neither quasi-principal nor convex. Next, consider* $\mathbf{G} = (N, X, E)$ *with* $N = \{1,2,3,4\}$*,* $X = \{x,y,z,w\}, \#X = 4$*, and such that* $E(N) = P(X)\backslash\{\emptyset\}, E(S) = \{Y \subseteq X : Y \supseteq X \setminus \{z,w\}\}$ *if* $N \neq S \supseteq \{1,2\}, E(S) = \{Y \subseteq X : Y \supseteq X \setminus \{y,w\}\}$ *if* $N \neq S \supseteq \{1,3\}$*,* $E(S) = \{Y \subseteq X : Y \supseteq X \setminus \{x,w\}\}$ *if* $N \neq S \supseteq \{2,3\}$*,* $E(S) = \{X\}$ *if* $\#S \geq 1$*. It can be checked that* $\mathbf{G}$ *is (trivially) quasi-principal but not Ferrers. Moreover,* $\mathbf{G}$ *is not convex: indeed,* $\{x,y\} \in E(\{1,2\})$ *and* $\{x,z\} \in E(\{1,3\})$ *but* $\{x\} \notin E(\{1,2,3\})$ *and* $\{x,y,z\} \notin E(\{1\})$*.On the other hand, quasi-principality and the Ferrers property are mutually consistent. A few examples of Ferrers quasi-principal effectivity functions are presented and studied in Vannucci (2002).*

Conversely, neither the Ferrers property nor quasi-principality are necessary conditions for convexity of a CGF. To see this, just consider the following

Example 10 *Take CGF* $\mathbf{G} = (N = \{1,2\}, X = \{a,b,c,d\}, E\})$ *with* $E(N) = P(X) \setminus \{\emptyset\}, E(\{1\}) = \{Y \subseteq X : Y \supseteq \{a,b\}$ *or* $Y \supseteq \{c,d\}\}$ *,* $E(\{2\}) = \{Y \subseteq X : Y \supseteq \{a,c\}$ *or* $Y \supseteq \{b,d\}\}$ *and* $E(\emptyset) = \emptyset$*. Clearly,* $\mathbf{G}$ *is not Ferrers, by construction. However, it is convex because for any* S, T, Y, Z *such that* $Y \in E(S)$ *and* $Z \in E(T)$ *either* $S \cap T \in \{S, T\}$ *(say, w.l.o.g.* $S \cap T = T$*, whence* $Y \cup Z \in E(S \cap T) = E(T)$*) or* $Y \cap Z \neq \emptyset$ *by construction, which entails that* $Y \cap Z \in E(S \cup T) = E(N)$*.*

The link between Proposition 8, concept-latticial and core-stability properties of a CGF which has been repeatedly alluded to above is made explicit by the following

Corollary 11 *Let* $\mathbf{G} = (N, X, E)$ *be a quasi-principal CGF such that its concept lattice* $\mathbf{L}(\mathbf{G})$ *is a chain. Then* $\mathbf{G}$ *is strongly core-stable on the set of all* N*-profiles of total preorders on* X*.*

Thus, sufficient conditions for (strong) core-stability of a CGF may be established in terms of properties of the concept lattice of the latter. Conversely, such a result sheds some light on the (lack of) constraints that convexity of a CGF imposes upon certain parameters (i.e. lenght and width) of its concept lattice. This is the topic of the following section.

4 Spectral properties: how long and wide can be the concept lattice of a convex CGF?

As mentioned above, concept-latticial parameters such as *width* and *length* or *order dimension* provide in a most succinct way some basic information on the characteristic degrees of decentralization, specialization and hierarchization of decision tasks among coalitions that are induced by a given decision protocol.

I also submit that this last circumstance might be of particular significance for some possible future developments of an *artificial-agent-supported implementation theory* : indeed, suppose one is interested in

a) implementing a certain choice correspondence F (e.g. a cooperative bargaining solution, or any other prescribed social choice rule as defined on a domain of profiles of nonverifiable individual characteristics) via a distributed mechanism, under

b) the additional constraint that the distributed mechanism is to 'faithfully' replicate the allocation of decision power embodied in the choice correspondence itself, and (possibly) with

c) the opportunity to take advantage of suitably designed artificial agents (e.g. artificial 'mediators' or 'arbitrators').

Now, replicating some (standard) effectivity function of choice correspondence F within the similar effectivity function of a mechanism with extra added agents is of course hopeless. Replicating the *concept lattice of the relevant effectivity function of* F , however, is not – and seems indeed to be a sensible and attainable goal for 'artificial-agent-augmented' mechanisms.

Be it as it may, the intuitive meaning of concept latticial parameters of CGFs as outlined above suggests an analysis of the relationship of such parameters to core-stability and related properties of coalitional game forms, which are the focus of a large part of the extant literature on coalitional games. This task is best accomplished by asking– and answering– a few questions concerning *spectral properties of concept lattices of CGFs* , namely questions of the following form:

• what are the possible values of a certain integer parameter t of the concept lattice of a CGF $\mathbf{G}$, when $\mathbf{G}$ is allowed to vary among the CGFs satisfying a given property p?

In view of the well-known fact that convex EFs are strongly core-stable, we address spectral concerning the lengths and widths of their concept lattices. We have the following results, which rely heavily on the classic Sperner's theorem and on the main result of the previous section of this paper, respectively:

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Proposition 12 *Let N, X be finite sets such that $t = \min\{\#N, \#X\}$ is odd, and let $U \in \{N, X\}$ with $\#U = t$. Then,*

for any positive integer $k \leq \#\left\{S \subseteq U : \#S = \frac{1}{2}[(\#U) + 1]\right\}$ there exists a (monotonic) convex CGF $\mathbf{G} = (N, X, E)$ such that $w(\mathbf{L}(\mathbf{G})) = k$.

Proof. Let $t = 2h+1$ for some (positive) integer h, and assume w.l.o.g. that $t = n = \#N$.Now, a well-known extension of the classic Sperner's theorem on antichains that if Y is a finite set of odd cardinality then $\left\{S \subseteq Y : \#S = \frac{1}{2}[\#(Y) + 1]\right\}$ is an antichain of maximum size of $(P(Y), \supseteq)$ (see e.g. Anderson(1987), chpt. 1, Theorem 1.2.2). Thus, take $\mathbf{S} = \left\{S \subseteq N : \#S = \frac{1}{2}(n+1)\right\}$,and let $\#\mathbf{S} = k$, i.e. $\mathbf{S} = \{S_1, .., S_k\}$ .Next, choose a k-subset $\{x_1, .., x_k\}$ of X and define a monotonic EF $\mathbf{G} = (N, X, E)$ by the following rule: $E(\emptyset) = \emptyset$ and for any $S \subseteq N, A \subseteq X$, $A \in E(S)$ iff one of the following clauses is satisfied: i) $S = N$ and $A \neq \emptyset$; ii) $S \supseteq S_i$ and $A \supseteq X \setminus \{x_i\}$; iii) $S \neq \emptyset$ and $A = X$. It is easily checked that indeed $w(\mathbf{L}(\mathbf{G})) = k = \max\left\{w(\mathbf{L}(\mathbf{G}')) : \mathbf{G}' = (N, X, E') \text{ is a CGF with } \min\{\#N, \#X\} = t\ \right\}$.

To check convexity of $\mathbf{G}$, assume $A \in E(S), B \in E(T)$.The following three cases must be distinguished: i) $(S \cap (N \setminus S_i) \neq \emptyset$ for any $S_i \in \mathbf{S}$ or $(T \cap (N \setminus S_i) \neq \emptyset$ for any $S_i \in \mathbf{S}$. In this case, $X \in \{A, B\}$ hence $A \cap B \in \{A, B\}$ which implies-by monotonicity- $A \cap B \in E(S \cup T)$; ii) there exists $S_i \in \mathbf{S}$ such that both $S \supseteq S_i$ and $T \supseteq S_i$.If $A \cap B \in \{A, B\}$ then again $A \cap B \in E(S \cup T)$.Otherwise, by definition of E, $A \cup B = X$: since $S \cap T \supseteq S_i \neq \emptyset$, it follows that $A \cup B \in E(S \cap T)$; iii) there exist $S_i, S_j \in \mathbf{S}$ such that $S_i \subseteq S$ and $S_j \subseteq T$, but $S_h \setminus (S \cap T) \neq \emptyset$ for any $S_h \in \mathbf{S}$. In this case again, $A \cup B = X$, by definition of E.Therefore, $A \cup B \in E(S \cap T)$ since clearly $S \cap T \neq \emptyset$, by definition of $\mathbf{S}$ ■

Proposition 13 *Let N, X be finite sets such that $t = \min\{\#N, \#X\}$. Then, there exists a monotonic convex $\mathbf{G} = (N, X, E)$ such that $l(\mathbf{L}(\mathbf{G})) = 2^t$.*

Proof. It follows at once by Proposition 8 in the previous section, by considering an EF $\mathbf{G} = (N, X, E)$ as defined by the following rule: $E(\emptyset) = \emptyset$ and for any $S \subseteq N, A \subseteq X$, $A \in E(S)$ iff one of the following clauses is satisfied: i) $S = N$ and $A \neq \emptyset$; ii) $S \supseteq \{i_1, .., i_h\}$ and $A \supseteq X \setminus \{x_{j_1}, .., x_{j_s}\}$ where $x_{j_l} \in \{f(i) : i \geqslant i_l\}$, $l = 1, .., h)$, and f is a suitable bijection between two t-subsets of N and X, respectively ; iii) $S \neq \emptyset$ and $A = X$.Indeed, such an EF is both linear and (trivially) quasi-principal, hence -by Proposition 8- convex. ■

Thus, the foregoing propositions establish that the requirement of convexity (hence of strong core stability) on a CGF does not entail by itself any structural constraint on the width or length of the corresponding concept lattice. From the point of view of mechanism design, that proposition amounts of course to an interesting positive result.

5 Concluding remarks

The concept lattice of a coalitional game form, and its parameters, provide useful classificatory tools concerning intrinsic, 'structural' features of the former. We have shown that whenever the concept lattice of a coalitional game form is a chain and the elements of its ground set satisfy suitable closure conditions the relevant coalitional game form is convex hence strongly core-stable on large conventional domains of preference profiles. This is -we submit- a quite remarkable and promising fact, which urges more research on possible further relationships between purely structural features of a coalitional game form and its behavioural performance.

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References

[1] Abdou J., Keiding H. : *Effectivity Functions in Social Choice*. Dordrecht: Kluwer (1991)

[2] Barbut M., Monjardet B.: *Ordre et Classification. Algèbre et Combinatoire. Vol.I,II.* Paris: Hachette (1970)

[3] Ganter B., Wille R. : *Formal Concept Analysis. Mathematical Foundations.* Berlin: Springer (1999)

[4] Otten G.J., Borm P., Storcken T., Tijs S.: Effectivity Functions and Associated Claim Game Correspondences.*Games and Economic Behavior* **9** (1995) 172-190

[5] Peleg B. : *Game Theoretic Analysis of Voting in Committees.* Cambridge: Cambridge University Press (1984)

[6] Vannucci S. : On a Lattice-Theoretic Representation of Coalitional Power in Game Correspondences, in H. de Swart (ed.): *Logic, Game Theory and Social Choice.* Tilburg: Tilburg University Press (1999)

[7] Vannucci S. (2002) : Effectivity Functions and Stable Governance Structures. *Annals of Operations Research* 109, 99-127