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On Multidimensional Inequality in Partitions of Multisets

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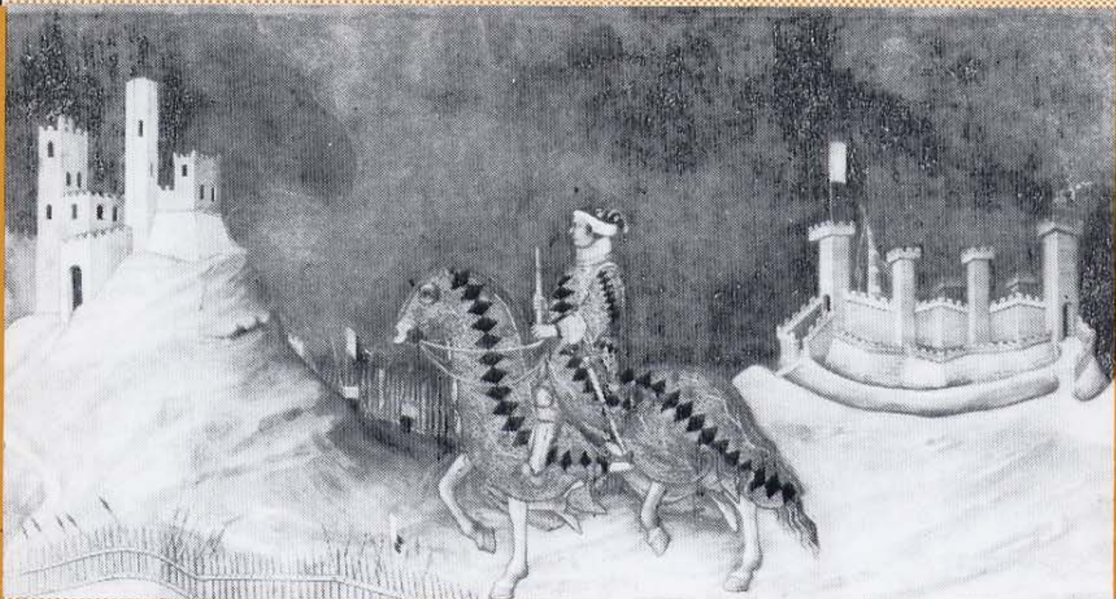

Abstract - We study multidimensional inequality in partitions of finite multisets with thresholds. In such a setting, a Lorenz-like preorder, a family of functions preserving such a preorder, and a counterpart of the Pigou-Dalton transfers are defined, and a version of the celebrated Hardy-Littlewood-Pölya characterization results is provided.

Keywords: Multisets, majorization, Lorenz preorder, Hardy-Littlewood-Polya theorem, transfers
JEL classification: D31; D63; I31

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1. Introduction

The economic literature on inequality measurement is mainly concerned with the comparison of *univariate* indices of well-being, which record the differences in distribution of income (and/or wealth) within and between populations. However, such an approach is considered an inadequate basis for comparing individual disparities because people differ in many aspects besides income. The analysis of different individual attributes is indeed crucial to understand and evaluate inequality among persons. Therefore, a recent research trend is focused on criteria for ranking *multivariate* distributions of individual attributes.[1] Unfortunately, few progress has been made on extending the theory of inequality measurement from univariate to the multivariate case (see e.g. [4], [5], [6], [8], [9] [12]): the works on multidimensional disparity comparisons are indeed rather sparse. In the present paper, we contribute to such a literature by addressing the problem of establishing inequality comparisons among multidimensional distributions of individual endowments of goods. In order to pursue our aim, we extend the notion of majorization (or dually Lorenz) preorder to a multi-attribute framework. Of course, the Lorenz criterion[2] is a fundamental tool for drawing conclusions about inequality of *univariate* (income) distributions, and is amenable to an intuitively appealing interpretation. In fact, according to a celebrated result of Hardy, Littlewood and Polya [HLP, [2]], which is considered a cornerstone of the economic literature on inequality measurement, if a distribution is less unequal than another according to the Lorenz preorder, then the former can be obtained from the latter by means of a finite sequence of Pigou-Dalton transfers[3] (and conversely), and equivalently the inequality associated with the former will be greater than that associated with the latter by the class of *S-concave* real-valued functions[4] (and conversely). Unfortunately, no exact counterparts of the HLP characterizations of Lorenz (majorization) preorder are available in a multidimensional setting. The major difficulty lies in the fact that deriving a more even multivariate distribution from another one by means of the transformation induced by an $n \times n$ bistochastic matrice $\mathcal{B}$ is *not* always equivalent to applying a finite sequence of simple PD transfers, namely transfers where just two coordinates are involved.

Moreover, the very assessment of inequality in resource allocation by means of Lorenz preorder is in fact highly problematic for multivariate distributions. Indeed,

[1] See Savaglio [9] for a survey.

[2] See Marshall and Olkin [7] chapter 1 for the definition of the Lorenz preorder and discussion of the related Lorenz criterion.

[3] A (simple) *Pigou-Dalton (PD) transfer* is a bilateral income transfer, assumed to be inequality diminishing, that takes place from the richer to the poorer, without reversing their relative positions.

[4] A function $f : \mathcal{D} \to \mathbb{R}$, where $\mathcal{D} \subseteq \mathbb{R}^n$, is *S-concave* if $f(\mathcal{B}x) \geq f(x)$ for all $x \in \mathcal{D}$ and all $n \times n$ bistochastic matrices $\mathcal{B}$, i.e. nonnegative square matrices, all row and column sums of which are equal to 1.

in the univariate (income/wealth) case Lorenz-based (or majorization) comparisons of distributions are easily performed, thanks to the *total* ordering of real-valued income levels. On the contrary, in the multidimensional setting individual endowments, that typically admit only *partial* orderings (e.g. dominance orderings), are to be compared. Hence, the problem of building up a majorization preorder, starting from a partial (pre)ordering of individual endowments must be also addressed. In particular, we focus on that issue when individual endowments consist of *private goods*.

Usually, many copies of the same object may appear in an individual endowment. In order to cope with this fact, we introduce the notion of *multisets*, i.e. sets which may include multiple copies of the same element[5]. Therefore, as multiple copies of different goods are distributed among people, we get a partition of multisets, or *multipartitions*. Multipartitions are discrete counterparts of a multivariate distribution and can be represented as a rectangular integer-valued matrix whose generic row i denotes the assignment of the annual vector of goods to the i-th agent. Such an analytical structure, induced by finite multiset-partitions, is exactly the framework used to study multidimensional inequality, i.e. the disparity of a population of N individuals distinguished for several attributes (see e.g. [4], [5], [6], [8], [9]). In such a setting, we first define a multidimensional counterpart of majorization preorder. In order to achieve that, we start from the partial order induced by the *strict dominance* vector order for *multisets*. We enrich such a class of dominance ordering of multisets with a threshold, a sort of multidimensional poverty line, in order to generate a *dominance filtral preorder* (henceforth *DFP*). To put it briefly, we compare distributions of goods on a finite population of N individuals in terms of componentwise (strict) rankings and introduce a minimum requirement (the threshold), such that any bundle of goods which does not reach the required standard is deemed actually equivalent to the empty set. However, in order to define the majorization preorder, we need to extend the DFP, that is in general non-total, to a total preorder. We accomplish the foregoing task by relying on the *height-function* of a poset, defined as the maximum number of steps you may go down a *DFP-discending chain* before reaching the threshold. In words, a height-function assigns to each endowments of goods its height, a non-negative number, therefore providing an objective numerical scale for assessing the relative importance of an individual endowment. Of course, the '*higher than*' binary relation is a total preorder, thus it supports a DFP-based majorization inequality ranking of multiprofiles. In fact, to consider the resulting heights of the profiles of individual endowments allows us to apply the majorization preorder to the set of height profiles we obtain. In such a way, we have achieved a counterpart of the (dual of) Lorenz ranking of unidimensional (income) distributions.

There have been a number of different ways proposed for generalizing the unidimensional transfer principle so that it can be applied when there are multiple individual attributes (see e.g. [4], [7], [9]). In such a setting, the notion of transfer is rather demanding. Indeed, we restrict the class of all possible multidimensional

[5]Multisets have been defined by assuming that for a given set A an element x occurs a finite number of times. They constitute a generalization of the notion of set and as sets, support operations to insert and withdraw items, provide a means to test the membership of a given item, and support the basic set operations of union, intersection, and difference. For example, in the multiset $\{a, a, b, b, b, c\}$, the multiplicities of the members a, b, and c (which could be interpreted as different private goods), are respectively 2, 3, and 1.

transfers to the subset of all bilateral transfers that take place from the richer[6] to the poorer, leaving the set of all total goods in the multivariate distribution unchanged. Moreover, we define as *minimal* a transfer which involves one unit for each component of the personal distribution of goods. In other words, we focus on those transfers such that for each good the endowment of the richer person is diminished by one unit and the corresponding endowment of the poorer one is increased by the same amount (without reversing the respective ranks). We show that the DFP-majorization is equivalent to reachability by means of finite sequence of minimal uniform transfers. This is the first fragment of our extension of the HLP characterization results to a multidimensional framework.

The last part of our analogue to the classic HLP theorem consists in proving that a version of the result due to Schur and Ostrowski on the class of majorization order-preserving functions (see [[7], Chapter 1]) also holds in the setting of finite multipartitions with thresholds. In particular, we focus on the class of real-valued functions which preserve the majorization preorder as defined on the totally ordered space of the height vectors. The importance of defining an order-preserving function in such a setting relies on the one-to-one correspondence between isotone functions, maps that preserve the majorization relation, and the so-called social evaluation functions (SEF)). SEFs are widely used to define inequality indices, which in turn provide the basis for welfare comparisons between and within populations by equity-concerned policy-makers.

In a previous work, we have already addressed the issue of building up a Lorenz preorder starting from a partial ordering induced by set-inclusion as augmented with a minimum 'filtral' threshold (see [10]). In fact in [10], we focus on the extension of the celebrated HLP results to the measurement of opportunity inequality starting from a class of set-inclusion monotonic total preorders of (opportunity) sets different from the cardinality preorder. However, the set-inclusion preorder is only satisfactory when at least some of the relevant resources are public, or at least non-rival, goods. Thus, by addressing the same problem of building up a Lorenz preorder starting from the strict dominance order for *multisets*, the present paper allows a proper treatment of the standard *pure*-private good case.

All things considered, there are therefore at least two good reasons for exploring if a version of the HLP characterization results holds starting from a (dual of) Lorenz-like preorder of partitions of finite multisets: (i) the possibility to extend the unidimensional inequality criteria to a multidimensional setting where several individual characteristics are simultaneusly considered and (ii) it is after all possible to extend the celebrated HLP theorem to the measurement of opportunity inequality even starting from several preorders of opportunity sets which are different from the cardinality preorder even when opportunities to be considered are private goods.

The paper is organized as follows. In Section 2, we introduce and discuss our basic notions. We show the analogy between our approach to inequality rankings of profiles of individual endowments and the standard multidimensional analysis of disparity. We introduce a version of the Pigou–Dalton "principle of transfers" for multipartitions of goods and define an equality-enhancing, rank-preserving transfer of items from a richer to a poorer individual. Finally, a class of multidimensional majorization order-preserving functions is also provided. Section 3 includes the

[6] In terms of the component-wise strict ranking of opportunity endowments.

main results of the present work, namely a partial extension of the classic HLP theorem for individual multiprofiles of goods. Section 4 is devoted to some concluding remarks, while proofs are collected in Section 5.

2. Notation and Definitions

We study a class of rankings of individual endowments which typically arise when all the alternatives are *pure private goods*. Then, let N denote a finite population of agents, with $\#N \geqslant 2$ (where $\#$ denotes cardinality), X the finite set of available goods and assume that $\#X \geqslant 2$ in order to avoid trivial qualifications. As the elements of X represent private goods, the same item may appear more than once. Thus, as people commonly buy many copies of the same (private) good, we allow that the objects in X have duplicates. Hence, in order to cope with this fact, we introduce the notion of *finite multisets* on X, i.e. a function $m : X \to \mathbb{Z}_+$ such that $\sum_{x \in X} m(x) < \infty$. In other words, function m registers how many memberships each item has in a multiset by assigning to this result a natural number. Since copies of different goods are distributed among N people, we get a partition of multiset m, or *multipartion* of m, on population N, which is a profile $\mathbf{m} = \{m_i\}_{i \in N}$ of multisets on X, a multiprofile of individual endowments then, such that for any $x \in X$: $\sum_{i \in N} m_i(x) = m(x)$. We denote by Π_m^N the set of multipartitions of m on population N. In such a manner, an element $\mathbf{m}$ of Π_m^N could be figured out as a distribution matrix (see 2.1), representing a population with N agents among which a set of goods is distributed and where the generic row i denotes the assignment of the goods to the i-th agent.

$$
\begin{array}{cc}
 & \underbrace{\begin{array}{cccc} x & y \ \ldots & w \ \ldots & z \end{array}} \quad \longleftarrow \text{goods} \\
\text{people} & \\
\downarrow & \\
\mathbf{m} = & \begin{bmatrix} m_1(x) & m_1(y) & \ldots & m_1(z) \\ . & . & & . \\ . & . & m_i(w) & . \\ . & . & & . \\ . & . & . & . \\ m_n(x) & \ldots & . & m_n(z) \end{bmatrix}
\end{array}
\tag{2.1}
$$

Hence the following question rapidly arises: "Given two distribution matrices $\mathbf{m}$ and $\mathbf{m}^{'}$, which one contains the lower level of disparity?". To answer the question, we generalize some suitable unidimensional dominance criteria to the multidimensional case. In particular, we generalize, as mentioned above, the notion of majorization preorder to that of majoriaztion preorder of partitions of finite multisets, defined with the reference to a preorder of sets of goods as induced by strict dominance and augmented with a threshold. Then, in order to proceed with our analysis, let M_X be the set of all multisets on X and define the natural componentwise (strict) order $>$ on M_X as follows: for any $m,\ m^{'} \in M_X$, $m > m^{'}$ if and only if $m(x) > m^{'}(x)$ for any $x \in X$. In particular, for any $m^* \in M_X$, we may consider the subposet $\mathcal{M}_{m^*} = (M_{X,m^*},\ >)$ of the poset $\mathcal{M} = (M_X,\ >)$, where $M_{X,m^*} = \{m \in M_X : m > m^* \text{ or } m = m^*\}$. In order to capture the notion of a threshold in this setting, we shall rely on the definition of an *order filter of the posets* $(M_{X,m^*},\ >)$.

Definition 1. *Let* $m^* \in M_X$. *An order filter of* $(M_{X,m^*},\ >)$ *is a set* $F \subseteq M_{X,m^*}$ *such that for any* $m, m^{'} \in M_{X,m^*}$, *if* $m^{'} \in F$ *and* $m > m^{'}$ *then* $m \in F$.

Such an order filter F collects all the elements of the multisets of M_{X,m^*}, which are $>$-larger than *some element* from a specified (finite) list $\mathcal{B}_F = \{b_1, ..., b_l\}$, the *basis* of the filter, where $b_g = (b_g(x))_{x \in X} \in M_{X,m^*}$, $g \in \{1, ..., l\}$ are non-comparable elements, called *generators* of the filter. The elements of the basis, which jointly constitute a threshold (a sort of multidimensional poverty line), correspond to the set of minimally acceptable endowment of goods. In this way, we associate a threshold to each order filter as defined on the individual endowment. In particular, when $\#\mathcal{B}_F = 1$, i.e. $\mathcal{B}_F$ is a singleton, the order-filter F is said to be *principal.* All this is made precise by the following:

Definition 2 (Dominance Filtral Preorder (DFP))**.** *For any (principal) order filter* F *of* $(M_{X,m^*},\ >)$ *the* F*-generated* dominance (principal) filtral preorder (DFP) *is the binary relation* $\succcurlyeq_F$ *on* M_{X,m^*} *defined as follows: for any* $m, m' \in M_{X,m^*}$, $m \succcurlyeq_F m'$ *if and only if* $m > m'$ *or* $m' \notin F$.

Since the main goal of the present paper is to study a DFP-based method of ranking multiprofiles of goods in terms of inequality, which reproduces the HLP characterization for majorization preorder, we need to extend the *partial* order $\succcurlyeq_F$ induced by the F-generated DFP on M_{X,m^*} to a *total* order. As mentioned in the Introduction, the (dual of the) Lorenz criterion is indeed induced by the *total order* of the elements in the univariate distribution (of incomes). On the contrary, our $(M_{X,m^*},\ \succcurlyeq_F)$ is an extremely irregular partial order set. Moreover, we need to find a way to compare individual multivariate distributions of goods in a manner that preserved their relationship under the partial order. In the theory of combinatorial posets, the *rank function* should constitute the natural candidate for extending the DFP induced by the filter F to a total order. But, when the poset is irregular, the rank function (preserving the covering-relation on the elements of the set) violates the so-called Jordan-Dedekind condition and then fails to be well-defined (see e.g. Greene and Kleitman [1]). In absence of a rank function, the notion of *height* is often a useful substitute. In fact, a way to extend a poset to a linear order is to arrange all elements of the poset in a latticial diagram (the so-called Hasse diagram), by height, where 'height' is established by the level in the lattice diagram at which the element, whose height we test, appears. Then, the so-called height-function is a natural-valued function that evaluates the lenght (numer of elements plus one), of the longest *chain*[7] from m to the basis of filter F and is defined as follows:[8]

Definition 3. *Let* F *be an order filter of* $(M_{X,m^*},\ >)$ *and* $\succcurlyeq_F$ *the (principal) DFP induced by F. Then, the* $\succcurlyeq_F$-induced height function

$$h_{\succcurlyeq_F} : M_{X,m^*} \to \mathbb{Z}_+$$

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[7]Notice that a chain of a poset $(X, \succcurlyeq)$ is a subset $Z \subseteq X$ such that $(Z, \succcurlyeq)$ is a totally ordered set.

[8]See [3] for an extensive treatment of the height function.

is defined as follows: for any $m \in M_{X,m^*}$*:*

$$h_{\succcurlyeq_F}(m) = \max \left\{ \begin{array}{c} \#\mathcal{C} : \mathcal{C} \text{ is a } \succcurlyeq_F \text{-chain, such that} \\ m \in \mathcal{C} \text{ and } m \succ_F m' \text{ for any } m' \in \mathcal{C} \backslash \{m\} \end{array} \right\}.$$

In words, the height function assigns to each individual profile of goods $m_i(x)$, for any $x \in X$, with $i = 1, ..., N$, a non-negative number, namely the size of the longest strictly ascending chain having $m_i(x)$ as its maximum. Thus, the height of a vector m_i (i.e. the endowment of goods of the individual i in M_{X,m^*}), counts the number of goods which stand below $m_i(x)$ according to the preorder $\succcurlyeq_F$, when considering the longest path ending in $m_i(x)$. In other words, if we consider the Hasse diagram of the poset $(M_{X,m^*}, \succcurlyeq_F)$, then the application of the height function to M_{X,m^*} consists in counting *how many steps* we go down following the vertical directions before getting the threshold generated by the (principal) filter.[9] In our case, the height function provides a *total extension* of the DFP, i.e. a DFP-method of ranking multiprofiles, by inducing the '*higher than*' binary relation. Moreover, it essentially allows us to replicate, in the more general context of individual endowments of goods, some of the fundamental results of the theory of income inequality measurement as collected in the following:

Theorem 1 (HLP [2])**.** *For any* $\mathbf{x} = (x_1, ..., x_n)$, $\mathbf{y} = (y_1, ..., y_n) \in \mathbb{R}_+^N$*, the following conditions are equivalent:*

i) $\mathbf{y} \succcurlyeq^M \mathbf{x}$*: i.e.* $\sum_{i=1}^k y_i \geq \sum_{i=1}^k x_i$, $i = 1, ..., n-1$, *and* $\sum_{i=1}^n y_i = \sum_{i=1}^n x_i$*, where, the components of* $\mathbf{x}$ *and* $\mathbf{y}$ *are arranged in non-increasing order;*

ii) $\mathbf{x}$ *can be derived from* $\mathbf{y}$ *through a finite sequence of transformations* $\mathbf{z}' = f(\mathbf{z})$ *of the following type:* $z_i' = z_i + \delta$, $z_j' = z_j - \delta$ *with* $j \leq i$ *and* $z_k' = z_k$*, for* $k \neq i, j$ *and* $\delta > 0$*, provided* $\delta \leq (z_j - z_i)/2$*;*

iii) $f(\mathbf{y}) \geq f(\mathbf{x})$ *holds for any* $f : \mathbb{A} \subset \mathbb{R}_+^N \to \mathbb{R}$ *of the following form: for each* $\mathbf{z} \in \mathbb{A}$, $f(\mathbf{z}) = \sum_{i=1}^n \varphi(z_i)$ *where* $\varphi : \mathbb{R} \to \mathbb{R}$ *is a continuous convex function.*[10]

All in all, in what follows, we explore the possibility to export into a multidimensional setting the criteria sub (*i*), (*ii*), (*iii*) of the foregoing theorem. Then, in order to accomplish the foregoing task we first introduce the following:

Claim 1. *Let* $m \in M_{X,m}$*,* F *a principal order filter of* $(M_{X,m}, >)$ *with basis* $\mathcal{B}_F = \{b\}$ *and* $(M_{X,m}, \succcurlyeq_F)$ *the* F*-generated* dominance filtral preorder*, then*

$$h_{\succcurlyeq_F}(m) = \max \left\{ 0, \min_x \{m(x) - b(x)\} + 1 \right\}.$$

Thus, Claim 1 provides us with a handy formula to compute heights, which will prove to be very useful in the ensuing analysis.

Following our approach to the inequality ranking of profiles of individual endowments, we now apply the majorization preorder to the set of the resulting heights for the profiles of personal endowments under consideration. Such a preorder induces a further preorder, which is an inequality ranking of multiprofiles of goods, which is a counterpart of the (*dual of*) Lorenz ranking of (income) distributions.

[9] Note that a strictly ascending chain is said to be maximal when it cannot be extended without changing one of its extrema.

[10] Functions $f(.) = \sum \varphi(.)$ as defined above are indeed Schur convex, i.e. such that $f(\mathbf{y}) \geq f(\mathbf{x})$ whenever $\mathbf{y} \succcurlyeq^M \mathbf{x}$.

Indeed, let us denote by $\mathbf{m} = (m_i(x))_{i \in N}^{x \in X}$ a generic multiprofile, i.e. a multivariate distribution of a finite number of goods among a population of N individuals. In $\mathbf{m}$, the generic row $m_i(x)$ represents the distributions of goods, for any $x \in X$, which are allotted to individual i, for $i \in \{1, ..., N\}$, according to $\mathbf{m}$. Then, a counterpart of the majorization preorder in our framework is defined as follows:

Definition 4. *Let $\mathbf{m}, \mathbf{m}' \in \Pi_m^N$ be two profiles of individual endowments of goods, F an order filter of $(M_{X,m^*}, >)$, $\succcurlyeq_F$ the corresponding filtral preorder on M_{X,m^*}, and $h_{\succcurlyeq_F}$ the $\succcurlyeq_F$-induced height function on M_{X,m^*}. Then, we say that $\mathbf{m}$ majorizes $\mathbf{m}'$, denoted $\mathbf{m} \succcurlyeq_F^{maj} \mathbf{m}'$, if:*

$$h_{\succcurlyeq_F}(\mathbf{m}) = (h_{\succcurlyeq_F}(m_1), ..., h_{\succcurlyeq_F}(m_n)) \succcurlyeq^{maj} (h_{\succcurlyeq_F}(m_1'), ..., h_{\succcurlyeq_F}(m_n')) = h_{\succcurlyeq_F}\left(\mathbf{m}'\right),$$

namely:

$$\begin{aligned} \sum_{i=1}^{k} h_{\succcurlyeq_F}(m_i) &\geqslant \sum_{i=1}^{k} h_{\succcurlyeq_F}(m_i') \qquad k = 1, ..., n-1, \\ \text{and } \sum_{i=1}^{n} h_{\succcurlyeq_F}(m_i) &= \sum_{i=1}^{n} h_{\succcurlyeq_F}(m_i'), \end{aligned}$$

whenever the height vectors are arranged in non-increasing order.

In words, focusing on a domain of multivariate distributions of pure private goods among a population, we map the space of individual profiles of goods $\left(\Pi_m^N, \succcurlyeq_F^{maj}\right)$ into a set of integer points in $\mathbb{Z}_+^N$, i.e. the totally ordered space of the height vectors, on which we apply the majorization preorder. Notice as such a domain will depend on the order filter selected and therefore is denoted as the (*height*) span of $\succcurlyeq_F$, written as $H_{\succcurlyeq_F}$.

Now, in order to extend the basic results of the literature on unidimensional inequality measurement to our multidimensional context of profiles of individual endowments of private goods, we must provide a suitable counterpart of the Pigou-Dalton transfer principle. Thus, let us state the notion of *transfer* with respect to height-extensions of DFPs, by first defining a *transfer operator* as follows:

Definition 5. *A transfer operator on Π_m^N is a nonempty correspondence $\Im : \Pi_m^N \rightrightarrows \Pi_m^N$ such that*

$$\forall (\mathbf{m}, \mathbf{m}') \in \Pi_m^N \times \Pi_m^N, \ \mathbf{m}' \in \Im(\mathbf{m}).$$

Then, a transfer operator is a transformation which leaves the set of all total alternatives/goods in $\mathbf{m}$ and $\mathbf{m}'$ unchanged. Next, we define a notion of *minimal* transfer:

Definition 6. *Let $\mathbf{m}, \mathbf{m}' \in \Pi_m^N$ be two profiles of individual endowments of goods, F a principal order filter of $(M_{X,m^*}, >)$ with basis $\mathcal{B}_F = \{b\}$ and $i, j \in N$ such that $m_i \succ_F m_j$, $h_{\succcurlyeq_F}(m_i) > h_{\succcurlyeq_F}(m_j) + 1$ such that:*

$$\begin{aligned} m_i'(x) &= m_i(x) - 1 \text{ for any } x \in X \\ m_j'(x) &= m_j(x) + 1 \text{ for any } x \in X \\ \text{and } m_l'(x^*) &= m_l(x^*) \text{ for any } l \neq i, j, \text{ and } x^* \in X \end{aligned} \tag{2.2}$$

Then $\mathbf{m}'$ is said to arise from $\mathbf{m}$ through a minimal uniform transfer (MUT) (from richer i to poorer j).

By analogy with the Pigou-Dalton principle, we also require that transfers of goods be not large enough to reverse the relative height-induced positions of the donor and recipient, namely:

Definition 7. *A transfer operator* $\Im$ *is said to be:*

(i): *MUT if and only if for any* $\mathbf{m}, \mathbf{m}' \in \Pi_m^N$, *if* $\mathbf{m}' \in \Im(\mathbf{m})$ *then* $\mathbf{m}'$ *arises from* $\mathbf{m}$ *through a MUT.*

(ii): *weakly rank-monotonic w.r.t* $\succcurlyeq_F$ *if and only if it does not cause height-reversals i.e. for any* $\mathbf{m}, \mathbf{m}' \in \Pi_m^N$ *and any* $i, j \in N$, *if*

$$\mathbf{m}' \in \Im(\mathbf{m}),\ m_i' \neq m_i,\ m_j' \neq m_j$$

and

$$h_{\succcurlyeq_F}(m_i) \geqslant h_{\succcurlyeq_F}(m_j)$$

then

$$h_{\succcurlyeq_F}(m_i') \geqslant h_{\succcurlyeq_F}(m_j').$$

(iii): *weakly progressive w.r.t.* $\succcurlyeq_F$ *if and only if for any* $\mathbf{m}, \mathbf{m}' \in \Pi_m^N$:

$$\mathbf{m}' \in \Im(\mathbf{m}),\ m_i' > m_i \text{ and } m_j > m_j'$$

entails that

$$h_{\succcurlyeq_F}(m_i) \geqslant h_{\succcurlyeq_F}(m_j).$$

(iv): *weakly-equalizing w.r.t.* $\succcurlyeq_F$ *if it is both weakly rank-monotonic w.r.t.* $\succcurlyeq_F$ *and weakly progressive w.r.t.* $\succcurlyeq_F$.

Moreover, in order to pursue our search for a DFP-counterpart of the HLP's celebrated result, we have to focus on the class of real-valued functions which preserve DFP-induced majorization preorders.

Definition 8 (real-valued $\succcurlyeq_F^{maj}$-isotonic functions)**.** *Let* F *be an order filter of* $(M_{X,m^*}, >)$ *and* $\succcurlyeq_F^{maj}$ *the majorization preorder on* Π_m^N *induced by the DFP* $\succcurlyeq_F$ *as defined above. Then a real-valued function*

$$f : \Pi_m^N \longrightarrow \mathbb{R}$$

is isotonic *(wrt* $\succcurlyeq_F^{maj}$*) on domain* $D \subseteq \Pi_m^N$ *if and only if for any* $\mathbf{m}, \mathbf{m}' \in D$

$$f(\mathbf{m}) \geqslant f(\mathbf{m}') \qquad \text{whenever } \mathbf{m} \succ_F^{maj} \mathbf{m}'.$$

Real-valued isotonic functions are simply the DFP-counterparts of so-called Schur-convex functions (see e.g. [7] chapter 3), namely functions $f(\cdot) = \sum \varphi(\cdot)$ such that $f(x) \geqslant f(y)$ whenever $x \succcurlyeq^{maj} y$, with $x, y \in \mathbb{R}^n$.

Finally, the use of the foregoing Definitions is clarified in the following:

Example 1. *Let us suppose that the set of available goods* X *is composed of six copies of good* x *and ten copies of good* y, *(i.e.* $m(x) = 6$ *and* $m(y) = 10$*), distributed over a population of three agents* $\{i, j, l\}$ *in order to get a partition of multiset* m, *namely the multiprofile:*

$$\mathbf{m} = \begin{array}{c c} & \begin{array}{cc} x & y \end{array} \\ \begin{array}{c} i \\ j \\ l \end{array} & \begin{pmatrix} 5 & 6 \\ 1 & 2 \\ 0 & 2 \end{pmatrix} \end{array}$$

If we consider as the basis *of the filter* $\mathcal{B}_F = \{b_1, b_2\}$, *where* $b_1 = (b_1(x)) = 1$ *and* $b_2 = (b_2(y)) = 1$, *then the corresponding filter-induced height function will be* *tantamount to* $h_{\succcurlyeq_F}(\mathbf{m}) = (5,1,0)$. *Thus, suppose that, according to Definition 6, a transfer takes place from richer* i *to poorer* l *in order to get the new multidimensional distribution:*

$$\begin{array}{cc} & \begin{array}{cc} x & y \end{array} \\ \mathbf{m}' = \begin{array}{c} i \\ j \\ l \end{array} & \begin{pmatrix} 4 & 5 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \end{array}$$

and the corresponding $h_{\succcurlyeq_F}\left(\mathbf{m}'\right) = (4,1,1)$. *Hence, it is obvious that* $\mathbf{m} \succcurlyeq_F^{maj} \mathbf{m}'$ *and that* $f(\mathbf{m}) \geqslant f(\mathbf{m}')$ *where* f *is, for example, a function that simply sums the value of the heights of the multipartitions. On the contrary, if* $\mathcal{B}_F = \{b_1, b_2\} = (1,3)$, *and the same transfer takes place in* $\mathbf{m}$, *we now have that* $h_{\succcurlyeq_F}(\mathbf{m}) = (4,0,0)$ *and* $h_{\succcurlyeq_F}\left(\mathbf{m}'\right) = (3,0,0)$, *with corresponding net loss of height mass. It is worth noticing here how a careful check that numbers always vary according to transfers of goods is often required.*

3. HLP Theorem for finite multipartitions

Let us start with a multidimensional counterpart of an important fragment of the theory of inequality measurement, namely the Muirhead Lemma (see e.g Marshall and Olkin [7] chapter 1), which says that is possible to obtain one (univariate) distribution from another one throughout a finite sequence of simple (in the sense of (*ii*) Theorem 1) transfers, which minimally alter the initial distribution if and only if the former is majorized (or dually Lorenz dominated according to the Lorenz criterion) by the latter :

Proposition 1. *Let* F *be a principal order filter of* $(M_{X,m^*}, >)$, *and* $\mathbf{m}, \mathbf{m}' \in \Pi_m^N$ *two multidimensional individual profiles of individual endowments of goods such that* $\left(h_{\succcurlyeq_F}(\mathbf{m}), h_{\succcurlyeq_F}(\mathbf{m}')\right) \in \left(\mathbb{Z}_+ \setminus \{0\}\right)^N$. *Then, the following statements are equivalent:*

(1) $\mathbf{m} \succ_F^{maj} \mathbf{m}'$;
(2) *There exist a weakly-equalizing (wrt* $\succcurlyeq_F$*) MUT operator* $\Im$ *and a positive integer* k *such that* $\mathbf{m}' \in \Im^{(t)}(\mathbf{m})$.[11]

The foregoing Proposition is related to the result on integer majorization due to Muirhead (see e.g. [7]). Nevertheless, it does not reduce to it, because here transfers involves goods and are only indirectly reflected into numbers. In other words, we need to double check that numbers always chance according to transfer of goods by avoiding loss in the total sum of the height-values.

In order to establish a close DFP-analogue of the Theorem 1, let us start with a useful characterization of real-valued $\succcurlyeq_F^M$-isotonic functions which also mimics a well known result on Schur-convex functions (see e.g. Lemma 3.A.2 in Marshall and Olkin [7]).

Lemma 1. *Let* F *be a principal order filter of* $(M_{X,m^*}, >)$, $h_{\succcurlyeq_F} : (M_{X,m^*}, >) \to \mathbb{Z}_+^N$, *the* $\succcurlyeq_F$*-induced height function as defined above, and* $\varphi : \mathbb{Z}_+^N \to \mathbb{R}$. *Then* $\varphi \circ h_{\succcurlyeq_F}$ *is* $\succcurlyeq_F^{maj}$*-isotonic on* $[(H_{\succcurlyeq_F}^+)^{-1}]\downarrow$[12] *if and only if for all* $\mathbf{z} \in \mathbb{Z}_+^N \cap H_{\succcurlyeq_F}^+$ *such*

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[11]Let $\Im$ be a transfer operator, then, for any positive integer t and $\mathbf{m} \in \Pi_m^N$ we define inductively $\Im^{(t)}(\mathbf{m}) = \Im(\Im^{(t-1)}(\mathbf{m}))$.

[12]$[(H_{\succcurlyeq_F}^+)^{-1}]\downarrow$ is the $h_{\succcurlyeq_F}$-counter-image of the positive height span of $(M_{X,m^*}, \succcurlyeq_F)$. Moreover, notice that for any $\mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}^n$, we shall denote $\mathbf{x}\downarrow$ the vector of $\mathbf{x}$'s components arranged in non-increasing order.

that $\mathbf{z} = \mathbf{z}\downarrow$, *and* $k = 2, ..., n$ *the value*

$$\varphi(z_1, \, ..., \, z_{k-1}, \, z_k + \Delta, \, z_{k+1} - \Delta, \, z_{k+2}, \, ..., \, z_n)$$

is non-decreasing in $\Delta \in \mathbb{Z}_+$ *provided that:*

(1) $0 \leqslant \Delta \leqslant \min\{z_{k-1} - z_k, z_{k+1} - z_{k+2}\}, \quad k = 2, ..., n-2$;
(2) $0 \leqslant \Delta \leqslant (z_{k-1} - z_k), \qquad k = n-1, \, n$

Then, a version of the characterization of the majorization preorder in terms of Schur-convex functions is provided by the following:

Proposition 2. *Let* F *be an order filter of* $(M_{X,m^*}, >)$, $h_{\succcurlyeq_F}$ *the* $\succcurlyeq_F$*-induced height function as defined above, and* $\mathbf{m}, \mathbf{m}' \in \Pi_m^N$ *such that*

$$(\varphi \circ h_{\succcurlyeq_F})(\mathbf{m}) \geqslant (\varphi \circ h_{\succcurlyeq_F})(\mathbf{m}')$$

for any $\varphi : \mathbb{Z}_+^N \to \mathbb{R}$ *such that* $\varphi \circ h_{\succcurlyeq_F}$ *is a* $\succcurlyeq_F^{maj}$*-isotonic function on* $[(H^+_{\succcurlyeq_F})^{-1}]\downarrow$*. Then,*

$$\mathbf{m} \succ_F^{maj} \mathbf{m}'.^{13}$$

Altogether, Propositions 1, Lemma 1 and Proposition 2 entail the following:

Theorem 2. *Let* F *be a principal order filter of* $(M_{X,m^*}, >)$, *and* $\mathbf{m}, \mathbf{m}' \in \Pi_m^N$ *two opportunity profiles such that* $h_{\succcurlyeq_F}(\mathbf{m}), h_{\succcurlyeq_F}(\mathbf{m}') \in H^+_{\succcurlyeq_F}$*. Then, the following statements are equivalent:*

(1) $\mathbf{m} \succ_F^{maj} \mathbf{m}'$;
(2) There exist a $\succcurlyeq_F$*-weakly equalizing MUT operator* $\Im$ and a positive integer k such that $\mathbf{m}' \in \Im^{(k)}(\mathbf{m})$
(3)
$$(\varphi \circ h_{\succcurlyeq_F})(\mathbf{m}) \geqslant (\varphi \circ h_{\succcurlyeq_F})(m')$$
for any $\varphi : \mathbb{Z}_+^N \to \mathbb{R}$ such that $\varphi \circ h_{\succcurlyeq_F}$is a $\succcurlyeq_F^{maj}$-isotonic function on $[(H^+_{\succcurlyeq_F})^{-1}]\downarrow$

Thus, Theorem 2 is a multidimensional counterpart for profiles of individual endowments to the HLP theorem on inequality measurement as required.

4. Concluding remarks

The relevance of the foregoing results relies on the fact that the DFP-approach is conducive to a majorization preorder of multiprofiles of goods that extends the classic unidimensional analysis of income inequality to a multivariate context. Since the comparison of multidimensional distributions typically admits only a non-total preorder of individual endowments, we have suggested the possibility to rely on height-based total extensions in order to reproduce some relevant parts of the theory of majorization (or, dually, Lorenz) preorders. Indeed, we have shown that the componentwise strict preorders of vectors, representing the assignment of the goods to the agents, support a multipartition counterpart to the celebreted HLP Theorem. In a broad sense, we answer to the question: "*A lost paradise?*", posed by Trannoy [12] and concerning the *impossibility* of finding again "*the miracle of the HLP theorem*" in the multidimensional context. Of course, this does not come totally for free. We first needed to use a two-steps procedure in order to compare

[13]Notice as Proposition 2 holds for any (as opposed to principal) order filter defined on the domain of partitional profiles.

rectangular matrices, representing the disparity of a population of N individuals distinguished for several attributes, namely multivariate ditributions of goods. Then, we adopted a very restricted version of the Pigou-Dalton principle of transfers to define a distributive profile as less even than another one.

Although our work represents a new fruitful approach to the analysis of multidimensional inequality, much more remains to be discovered, at least on the problem to compare our solution to the issue of building up a Lorenz preorder of multivariate distributions with the main results on matrix majorization existing in economic literature (see e.g. [5], [6], [7], [8], [9] among others), but this task is best left as a possible topic for further research.

5. Appendix: Proofs

Proof of Claim 1. Let us consider two cases, namely (i) there exists $x \in X$ such that $m(x) < b(x)$, then $\min(m(x) - b(x)) < -1$ and therefore $\max\{1, \min_x \{m(x) - b(x)\} + 1\} = 1$. On the other hand, by definition $m \sim_F \varnothing$, hence $h_{\succcurlyeq F}(m) = 1$; (ii) for any $x \in X$, $m(x) \geqslant b(x)$, then $\min(m(x) - b(x)) + 1 \geqslant 1$, i.e. $\max\{1, \min_x \{m(x) - b(x)\} + 1\} = \min(m(x) - b(x)) + 1$. In particular, let us assume $\min(m(x) - b(x)) = k \geqslant 1$, then there exists $x \in X$ such that $m(x) = b(x) + k$ and $m(y) \geqslant b(y) + k$ for any $y \in X$. Thus, we may define $m_1, ..., m_k \in M_{X,m^*}$ such that for any $y \in X$,

$$\begin{aligned} m_1(y) &= b(y) + 1 \\ m_2(y) &= b(y) + 2 \\ &\;\;... \\ m_k(y) &= b(y) + k. \end{aligned}$$

Now, observe that $m_k > ... > m_1 > b$ and either $m > m_k$ or both $m > m_{k-1}$ and m and m_k are not comparable with respect to $>$. In any case, $h_{\succcurlyeq F}(m) \geqslant k + 1$. Now, suppose there exists $m_1^{'}, ..., m_{k+1}^{'} \in M_{X,m^*}$ such that $m_{k+1}^{'} > ... > m_1^{'} > b$ and either $m > m_{k+1}^{'}$ or both $m > m_{k+1}^{'}$ and m and $m_{k+1}^{'}$ are not dominance-comparable with respect to $>$. Then, by definition for any $y \in X$, $m(y) \geqslant b(y) + k + 1$, i.e. $\min\limits_{y \in X} \{m(y) - b(y)\} \geqslant k + 1$, a contradiction.

It follows that $h_{\succcurlyeq F}(m) \leqslant k + 1$, whence $h_{\succcurlyeq F}(m) = k + 1$. $\square$

Proof of Proposition 1. [14]$(1 \Rightarrow 2)$ Since $\sum_{i=1}^{k} h_{\succcurlyeq F}(m_i) - \sum_{i=1}^{k} h_{\succcurlyeq F}(m_i') \geq 0$ for any $k = 1, ..., n-1$, $\sum_{i=1}^{n} h_{\succcurlyeq F}(m_i) - \sum_{i=1}^{n} h_{\succcurlyeq F}(m_i') = 0$ and there exists an $l \in \{1, ..., n-1\}$ such that $\sum_{i=1}^{l} h_{\succcurlyeq F}(m_i) - \sum_{i=1}^{l} h_{\succcurlyeq F}(m_i') > 0$, it follows that there also exist $l' \in \{l+1, ..., n\}$ such that $\sum_{i=l'}^{n} h_{\succcurlyeq F}(m_i) - \sum_{i=l'}^{n} h_{\succcurlyeq F}(m_i') < 0$. Now, let $i^* \in \{1, ..., n-1\}$ be the smallest integer and $j^* \in \{l+1, ..., n\}$ be the largest integer such that $h_{\succcurlyeq F}(m_{i^*}) - h_{\succcurlyeq F}(m_{i^*}') > 0$ and $h_{\succcurlyeq F}(m_{j^*}) - h_{\succcurlyeq F}\left(m_{j^*}'\right) < 0$. Therefore,

$$h_{\succcurlyeq F}(m_{i^*}) > h_{\succcurlyeq F}(m_{i^*}') \geqslant h_{\succcurlyeq F}\left(m_{j^*}'\right) > h_{\succcurlyeq F}(m_{j^*}),$$

whence,

$$h_{\succcurlyeq F}(m_{i^*}) - h_{\succcurlyeq F}(m_{j^*}) > 2.$$

[14]The style of the proof is standard (see e.g. [7]), we report it here for the sake of completeness.

Next, consider a MUT in $\mathbf{m}$ from i^* to j^* and denote by $\mathbf{m}^*$ the resulting profile. Clearly, $\mathbf{m} \succ_F^{maj} \mathbf{m}^*$. Repetition of the same argument entails the thesis.

(1 $\Leftarrow$ 2) We proceed by induction. If $k = 1$, then, by definition of a MUT,

$$\begin{aligned} m_i'(x) &= m_i(x) - 1 \text{ for any } x \in X \\ m_j'(x) &= m_j(x) + 1 \text{ for any } x \in X \\ \text{and } m_l'(x^*) &= m_l(x^*) \text{ for any } l \neq i, j, \text{ and } x^* \in X \end{aligned} \tag{5.1}$$

Hence, by definition of the height function,

$$\begin{aligned} h_{\succcurlyeq_F}\left(m_j'\right) &= h_{\succcurlyeq_F}(m_j) - 1 \text{ and} \\ h_{\succcurlyeq_F}(m_i') &= h_{\succcurlyeq_F}(m_i) + 1. \end{aligned}$$

It follows that $(h_{\succcurlyeq_F}(m_1), ..., h_{\succcurlyeq_F}(m_n)) \succcurlyeq^{maj} (h_{\succcurlyeq_F}(m_1'), ..., h_{\succcurlyeq_F}(m_n'))$, namely $\mathbf{m} \succ_F^{maj} \mathbf{m}'$. A similar argument applies to the inductive step of the proof. $\square$

Proof of Lemma 1. ($\Rightarrow$)Let $\varphi : \mathbb{Z}_+^N \to \mathbb{R}$ be a function such that $\varphi \circ h_{\succcurlyeq_F}$ is $\succcurlyeq_F^{maj}$-isotonic. Then, for any pair of profiles of individual endowments $\mathbf{m}, \mathbf{m}' \in \Pi_m^N$ such that $\mathbf{m} \succ_F^{maj} \mathbf{m}'$, consider $(h_{\succcurlyeq_F}(\mathbf{m})) \downarrow$ and $(h_{\succcurlyeq_F}(\mathbf{m}')) \downarrow$. Then, by definition of $\succcurlyeq_F^{maj}$,

$$h_{\succcurlyeq_F}(\mathbf{m}) \downarrow \succ^{maj} h_{\succcurlyeq_F}(\mathbf{m}') \downarrow$$

and by hypothesis,

$$\varphi(h_{\succcurlyeq_F}(\mathbf{m}) \downarrow) \geqslant \varphi(h_{\succcurlyeq_F}(\mathbf{m}') \downarrow).$$

Now, suppose there exists a vector $\mathbf{z} \in \mathbb{Z}_+^N \cap H_{\succcurlyeq_F}^+$, $\mathbf{z} = \mathbf{z} \downarrow$, $\Delta \in \mathbb{Z}_+$ and $k \in \{2, ..., n\}$ such that:

$$\begin{aligned} \text{either } k &\leqslant n - 1 \text{ and } 0 \leqslant \Delta \leqslant \min\{z_{k-1} - z_k,\, z_{k+1} - z_{k+2}\} \\ \text{or } k &= n \quad \text{and } 0 \leqslant \Delta \leqslant (z_{k-1} - z_k) \end{aligned}$$

with

$$\varphi(z_1, ..., z_{k-1}, z_k + \Delta, z_{k+1} - \Delta, ..., z_n) < \varphi(z_1, ..., z_n) \tag{5.2}$$

i.e. let us suppose that φ is decreasing in Δ.

Then, first, observe that $\mathbf{z} = (z_1, ..., z_n) \in H_{\succcurlyeq_F}^+$ entails:

$$(z_1, ..., z_k + \Delta, z_{k+1} - \Delta, ..., z_n) \in H_{\succcurlyeq_F}^+ \text{ for any } \Delta$$

as defined above. Indeed, suppose w.l.o.g. that $\Delta \geqslant 1$. Then $z_{k+1} \geqslant 1$, whence $z_k \geqslant z_{k+1} \geqslant 1$. Moreover, by hypothesis Δ is such that $z_k + \Delta \geqslant z_{k+1} - \Delta \geqslant 0$. Now, take a profile $\mathbf{m} \in \Pi_m^N$ such that $\mathbf{z} = h_{\succcurlyeq_F}(\mathbf{m})$. Recall that by hypothesis $\mathbf{z} \in H_{\succcurlyeq_F}^+$. By definition of $h_{\succcurlyeq_F}$,

$$h_{\succcurlyeq_F}(m_{k+1}) - \Delta := z_{k+1} - \Delta \geqslant 0$$

entails:

$$\min_x \{m_{k+1}(x) - b(x)\} + 1 \geqslant \Delta.$$

Let $x^* \in \arg\min_X \{m_{k+1}(x) - b(x)\}$, and posit $\mathbf{m}' = \mathbf{m}(k, \Delta)$ such that

$$m_k'(x^*) = m_k(x^*) + \Delta \text{ and } m_k'(x') = m_k(x') \text{ for any } x' \in X \setminus \{x^*\}$$

$$m_{k+1}'(x^*) = m_{k+1}(x^*) - \Delta \text{ and } m_{k+1}'(x') = m_{k+1}(x') \text{ for any } x' \in X \setminus \{x^*\}, \text{ and}$$
$$m_l'(x^*) = m_l(x^*) \text{ for any } l \neq k, k+1, \text{ where } x^* \in \arg\min_X (m_i(x) - b(x)).$$

Hence,

$$\mathbf{m}' = \mathbf{m}(k,\Delta) = (m_1, ..., m_{k-1}, m_k + \Delta, m_{k+1} - \Delta, m_{k+2}, ..., m_n).$$

Then,

$$h_{\succcurlyeq F}(\mathbf{m}') = (z_1, ..., z_{k-1}, z_k + \Delta, z_{k+1} - \Delta, z_{k+2}, ..., z_n)$$

i.e.

$$(z_1, ..., z_{k-1}, z_k + \Delta, z_{k+1} - \Delta, z_{k+2}, ..., z_n) \in H_{\succcurlyeq F}$$

as required.

Next, observe that by definition of the ordinary integer majorization preorder $\succcurlyeq^{maj}$

$$h_{\succcurlyeq F}(\mathbf{m}') \succ^{maj} h_{\succcurlyeq F}(\mathbf{m}).$$

It follows that, as observed above,

$$\varphi(h_{\succcurlyeq F}(\mathbf{m}')) \geqslant \varphi(h_{\succcurlyeq F}(\mathbf{m}))$$

a contradiction in view of 5.2.

($\Leftarrow$)[15] Let φ be non-decreasing in Δ, for any $k = 2, ..., n$ under (1) and (2), but suppose $(\varphi \circ h_{\succcurlyeq F})$ is not $\succcurlyeq_F^{maj}$-isotonic on $[(H^+_{\succcurlyeq F})^{-1}]$, i.e. $\varphi(\mathbf{y}) < \varphi(\mathbf{x})$ whenever $\mathbf{m} \succ_F^{maj} \mathbf{m}'$, with $\mathbf{m}, \mathbf{m}' \in \Pi_m^N$ such that $h_{\succcurlyeq F}(\mathbf{m})$, $h_{\succcurlyeq F}(\mathbf{m}') \in \mathbb{Z}_+^N$, and $h_{\succcurlyeq F}(\mathbf{m}) := \mathbf{y} \succ^{maj} \mathbf{x} := h_{\succcurlyeq F}(\mathbf{m}')$.

Now, change the variables by positing $\mathbf{z}^* = (z_1^*, ..., z_k^*)$ with $z_k^* = \sum_{i=1}^k z_i$, such that $\mathbf{y} \succ^{maj} \mathbf{x}$ if and only if

$$\begin{array}{rcll} y_k^* & \geqslant & x_k^* & \text{for any } k = 1, ..., n-1, \\ \text{and} \quad y_n^* & = & x_n^*, & \end{array}$$

where $\geqslant$ is the componentwise ordering. By definition, $\varphi(z_1, ..., z_n)$ non-decreasing in Δ entails that $\varphi(z_1, ..., z_n) = \varphi(z_1^*, z_2^* - z_1^*, ..., z_n^* - z_{n-1}^*)$ is non-decreasing in z_k^*. Thus, the fact that $\mathbf{y} \succ^{maj} \mathbf{x}$ implies:

$$\varphi(y_1^*, ..., y_k^* - y_{k-1}^*, ..., y_n^* - y_{n-1}^*) \geqslant \varphi(x_1^*, ..., x_k^* - x_{k-1}^*, ..., x_n^* - x_{n-1}^*)$$

i.e.:

$$\varphi(\mathbf{y}) \geqslant \varphi(\mathbf{x})$$

entails a contradiction. □

Proof of Proposition 2. [16] Suppose $(\varphi \circ h_{\succcurlyeq F})(\mathbf{m}) \geqslant (\varphi \circ h_{\succcurlyeq F})(\mathbf{m}')$ is a $\succcurlyeq_F^{maj}$-isotonic function on $[(H^+_{\succcurlyeq F})^{-1}] \downarrow$ and consider the following real-valued non-decreasing function: $\varphi_1(z_1, ..., z_n) =: \sum_{i=1}^n z_i$ and $\varphi_2(z_1, .., z_n) =: -\varphi_1(z_1, ..., z_n)$ defined on

$$\mathbf{D} = \left\{ \mathbf{z} : \mathbf{z} = (z_1, ... z_n) \in \mathbb{Z}_+^N \cap H^+_{\succcurlyeq F} \text{ such that } \mathbf{z} = \mathbf{z} \downarrow \right\}.$$

Thus, in particular, $\varphi_1 \circ h_{\succcurlyeq F}$ and $\varphi_2 \circ h_{\succcurlyeq F}$ are both $\succcurlyeq_F^{maj}$-isotonic in view of our previous lemma. Therefore,

$$(\varphi_i \circ h_{\succcurlyeq F})(\mathbf{m}) \geqslant (\varphi_i \circ h_{\succcurlyeq F})(\mathbf{m}'), \quad i = 1, 2$$

[15] The converse part of the proof is standard. We provide here a sketch of the proof and refer to [7] chapter 3, pages 54-55 and to [10] Lemma 1 for more details.

[16] It is standard, then we provide here a sketch of the proof and refer, for more details, to [7] chapter 4, *Propositon B*.1 and to the proof of *Propositon* 2 in [10].

entails that:

$$\begin{aligned}
(5.3) \qquad \sum_{i=1}^{n} h_{\succcurlyeq_F}(m_i) &\geqslant \sum_{i=1}^{n} h_{\succcurlyeq_F}(m_i') \\
\text{and } -\sum_{i=1}^{n} h_{\succcurlyeq_F}(m_i) &\geqslant -\sum_{i=1}^{n} h_{\succcurlyeq_F}(m_i'), \\
\text{i.e. } \sum_{i=1}^{n} h_{\succcurlyeq_F}(m_i) &= \sum_{i=1}^{n} h_{\succcurlyeq_F}(m_i').
\end{aligned}$$

Next, posit w.l.o.g.:

$$\mathbf{y} = \mathbf{y}\downarrow = (h_{\succcurlyeq_F}(m_1), ..., h_{\succcurlyeq_F}(m_n)) \text{ and } \mathbf{x} = \mathbf{x}\downarrow = (h_{\succcurlyeq_F}(m_1'), ..., h_{\succcurlyeq_F}(m_n')),$$

and for any $i = 1, ..., n$, $k = 1, ..., n-1$ and $\mathbf{w} \in \mathbb{Z}_+^N$ define a function $\varphi_{\mathbf{w}_k}^i$ on $\mathbb{Z}_+^N$ as follows:

$$\varphi_{\mathbf{w}_k}^i(z_1, ..., z_n) = \max\{z_i - w_k, 0\} \qquad \text{for all } \mathbf{z} \in \mathbb{Z}_+^N$$

Now, for any $k = 1, ..., n-1$, $(\sum_{i=1}^n \varphi_{\mathbf{w}_k}^i) \circ h_{\succcurlyeq_F}$ is a *non-negative* (by construction) real-valued $\succcurlyeq_F^{maj}$ -isotonic function, then, for any $k = 1, ..., n-1$

$$\sum_{i=1}^{k} x_i - kw_k \leqslant \sum_{i=1}^{n} \varphi_{\mathbf{y}_k}^i(\mathbf{x}) \leqslant \sum_{i=1}^{n} \varphi_{\mathbf{y}_k}^i(\mathbf{y}) \leqslant \sum_{i=1}^{k} y_i - kw_k$$

i.e.

$$\sum_{i=1}^{k} x_i \leqslant \sum_{i=1}^{k} y_i.$$

From this and from 5.3 follows that $\mathbf{y} \succ^{maj} \mathbf{x}$, or equivalently $\mathbf{m} \succ_F^{maj} \mathbf{m}'$ as required. $\square$

References

[1] Greene, C. and D.J. Kleitman, (1986) - Longest chains in the lattice of integer partitions ordered by majorization - *European Journal of Combinatorics* 7, 1-10.

[2] Hardy, G.H., J.E. Littlewood, and G. Polya, (1934, 1952) - *Inequalities* - Cambridge University Press, London.

[3] Haskins, S. and S. Gudder (1972) - Height on posets and graphs - *Discrete Mathematics* 2, 357-382.

[4] Kolm, S.C. (1977) - Multidimensional egalitarianisms - *Quarterly Journal of Economics* 91, 1-13.

[5] Koshevoy, G. (1995) - Multivariate Lorenz majorization - *Social Choice and Welfare* 12, 93-102.

[6] Koshevoy, G. (1998) - The Lorenz zonotope and multivariate majorizations - *Social Choice and Welfare* 15, 1-14.

[7] Marshall A. W. and I. Olkin (1979) - *Inequalities: Theory of Majorization and its Applications* - Academic Press.

[8] Savaglio, E. (2006) - On multidimensional inequality with variable population size - *Economic Theory* 28, 85-94.

[9] Savaglio, E. (2006) - Three approaches to the analysis of multidimensional inequality - in Farina, F. and E. Savaglio (eds), *Inequality and Economic Integration*, Routledge, London.

[10] Savaglio, E. and S. Vannucci (2007) - Filtral Preorders and Opportunity Inequality - *Journal of Economic Theory.*

[11] Sen, A. K. (1997) - *On Economic Inequality* - Expanded edition with a substantial annex by James E. Foster and Amartya Sen - Clarendon Press: Oxford.

[12] Trannoy, A. (2006) - Multidimensional egalitarianism and the dominance approach: a lost paradise? - in Farina, F. and E. Savaglio (eds), *Inequality and Economic Integration*, Routledge, London.

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