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Tetsuya Shinkai
Kwansei Gakuin University
Satoru Tanaka
Kobe City University of Foreign Studies
and
Makoto Okamura
,iroshima University

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SCHOOL OF ECONOMICS
KWANSEI GAKUIN UNIVERSITY
1-155 Uegahara Ichiban-cho
Nishinomiya 662-8501, Japan

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Tetsuya Shinkai, Kwansei Gakuin University,
Satoru Tanaka, Kobe City University of Foreign Studies
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Makoto Okamura, Hiroshima University

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Abstract

We consider research and development (R&D) investment competition between duopolistic firms that independently invest in two complementary technologies to produce their products. By "*partially complementary technologies*", we mean that each firm *can* produce the goods *without both technologies* but they incur more redundant costs than *with* both technologies. We derive the investment competition equilibria in R&D of the two technologies with and without a licensing system. By comparing R&D investment levels in the two equilibria, we show that the licensing system discourages R&D investment in most cases; however, it encourages R&D investment in some cases when the duopolistic firms can produce the goods using both technologies. We also show that (cross-) licensing increases the expected social surplus at the symmetric equilibrium.

JEL Classification Numbers: D45, L13, O32
Key Words: partially complementary technologies, licensing system, duopoly, R&D investment

Corresponding To: Tetsuya shinkai, ph D, 1-155, Uegahara-ichibancho, Nishinomiya, Hyogo 662-8501, JAPAN

E-mail: shinkai@kwansei.ac.jp, FAX 81-798-51-0944

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1. Introduction

One significant feature of recent technological innovation is that a firm often employs multiple distinct technologies to produce a commodity. Especially in information technology (IT) industries, one product is composed of numerous separable patentable elements. For example, the production of a mobile phone with a digital camera involves about 19,000 (Japanese) patents and/or utility models.[1] In this environment, which has been called "cumulative-systems technologies" (Merges and Nelson (1994)) or "complex technologies" (Cohen, Nelson and Walsh (2000)), many economic agents hold and share the separable patentable elements. The method of coordination among these patent holders affects the interests of each inventor and also affects their R&D incentives. Over the last decade of the previous century, a number of studies have discussed the effects on R&D activities, licensing and the patent systems of the mode of coordination of inventions.

Considering complex technologies, we can identify in principle two types of relationships between inventions. The first type is cumulative or one-way complementary. As Scotchmer (1991) pointed out, many inventors engage in R&D activities based on the outcome of preceding inventions. Here, while an applied technology invention that is based on basic technologies is not possible without the existence of these basic technologies, invention of the basic technologies in themselves is possible without the outcome of the applied technologies. With respect to this relationship, Green and Scotchmer (1995), and Chang (1995) showed that externalities, due to the lack of coordination among the creators of plural distinct inventions, discourage the development of these technologies. In the second type of relationship among inventions, various mutually interdependent inventions are required, without which production of the goods is very difficult. Thus, this relationship among inventions is called two-way complementary.

[1] *Nihon Keizai Shinbun* (August 18, 2003)

In fact, as we have seen typically in the IT industries, technological innovations occur on the basis of plural distinct inventions developed in different systems of technologies. Distinct technologies are complementary to each other as parts of the product produced. An externality problem occurs due to a lack of coordination among the discoverers of plural complementary technologies. Heller and Eisenberg (1998) stated that the existence of such externality results in “the tragedy of anti-commons”. When the intellectual property rights of plural distinct technologies are assigned to different agents (firms), the externality generates excessive exercises of exclusive rights and leads to under-utilization of these technologies, and this under-utilization discourages R&D activities of agents (firms).

When all complementary technologies are necessary to produce a product, licensing has strategic importance. If two firms own each of two distinct inventions with complete complementarity, then the two firms cannot produce a product at all without a cross-licensing contract. The form of this coordination affects R&D activities of the firms. Grindley and Teece (1997) and Hall and Ziedonis (2001) conducted empirical investigations of the appliance and integrated circuit (IC) industries. Their results show the conditions of the firms’ (cross-) licensing of technologies have a significant effect on the incentives for R&D activities in these industries where complementary inventions are indispensable for production. While there are many empirical studies on this subject, few theoretical studies have examined how the conditions of firms’ (cross-) licensing of technologies affect firms’ incentives for R&D activities. Fershtman and Kamien (1992) and Okamura, Shinkai and Tanaka (2002) offer two of the few studies of two firms engaging in R&D activities for two distinct technological inventions with complete complementarity. They both established that the existence of a cross-licensing system reduces the firms’ R&D activities in such a context.

Complementary technological inventions are not always indispensable for production, in which case firms may produce a new product without using any one of two complementary inventions. For example, in IC technologies, a great number of distinct technological inventions with complementarity exist such as software technologies, liquid crystal display (LCD) technologies and so on, all of which are indispensable for producing a mobile phone. Some firms, however, can develop and produce a new and superior mobile phone by using the outcomes of the successful invention with regard to software technologies and LCD technologies. In this environment, the margin created by the cost reduction (e.g. of the mobile phone) depends on the degree of complementarity of the underlying technologies. When the degree of complementarity is large (small), we expect that the cost reduction created by invention of only one element of the underlying technologies is small (large). Such an environment opens the possibility of unilateral licensing for coordinating technological inventions. When firms invest in R&D in two distinct technologies with complete complementarity, both technologies are indispensable for producing products, and the realized pattern of licensing becomes cross-licensing. On the contrary, consider the invention of one element of the underlying technologies that is dispensable for production but also contributes to cost reduction. This invention may be unilaterally licensed. Therefore, we employ a static framework to examine how the degree of complementarity between underlying technologies and the difference between cross-licensing and unilateral licensing changes firms' incentives for R&D activities in a Cournot duopoly. We concentrate on the case where each duopolistic firm can invest in R&D for two distinct technological inventions *with partial complementarity* with each other.

In Section 2, we describe our model. In Section 3, we analyze the problem of R&D in a Cournot duopoly with partially complementary technological innovations without licensing as a benchmark. In Section 4, we examine the conditions under which (cross-) licensing occurs. The appendix presents the conditions under which (cross-) licensing may occur at every state of nature. After extending our analysis to the case of (cross-) licensing in Section 5, we analyze theoretically how the difference between cross-licensing and unilateral licensing affects firms' incentives for R&D activities in a Cournot duopoly in Sections 5 and 6. In

Section 7, we use a numerical example to discuss briefly how the difference between cross-licensing and unilateral licensing affects welfare at the equilibria. In the final section, we present our concluding remarks.

2. The model

We consider a duopolistic market in which two firms with identical production technology, firms *x* and *y,* produce a homogeneous product. At the first stage, each firm simultaneously invests in R&D for the two distinct but partially complementary technologies, *A* and *B*. By "*partially complementary technologies*," we mean that each firm *can* produce the goods *without both two technologies* but it incurs additional costs than with both technologies. Denote by $x_A, x_B (\geq 0)$ and $y_A, y_B (\geq 0)$ the investment levels for the technologies A, B of firm *x* and those for the technologies A, B of firm *y*. If each firm succeeds in the development of at least one of these technologies, it can reduce marginal cost through a process innovation. Assume that each firm has a constant return to scale production technology as follows:

$$
\begin{aligned}
C_i(q_i) = c_i q_i &= (\underline{c}+0)\cdot q_i \text{, if it succeeds in the development of both technologies } A \text{ and } B,\\
&= (\underline{c}+k)\cdot q_i \text{, if it succeeds in the development of technologies } A \text{ or } B, \text{ where } 0 \leq k \leq 1,\\
&= (\underline{c}+1)\cdot q_i \text{, if it fails to develop both technologies } A \text{ and } B\\
i &= x, y, \qquad (1)
\end{aligned}
$$

where $\underline{c}$ is an intrinsic marginal cost and we set $\underline{c}$ = 0 without loss of generality.

This cost function implies that marginal cost decreases by 1, $1-k$ and 0 if firm $i(=x,y)$ succeeds in the development of both, either and none of the two technologies. We say the two technologies *A* and *B* are *less partially complementary*, *even partially complementary* and *more partially complementary*, if $0 \leq k < 1/2$, $k = 1/2$ and $1/2 < k \leq 1$, respectively. Especially, the two technologies are *the least partially complementary* or *the most partially complementary*, if $k = 0$ or $k = 1$.[2]

If each firm succeeds in the development of either technology A or B, it can reduce its marginal cost by $1-k$. If it succeeds in the development of both technologies, the firm can reduce its marginal cost by 1. Suppose that the firm can develop the two technologies sequentially. This implies that the later developed technology decreases marginal cost by k, which is the value of the development of the second technology. Hence, if $1/2 < k \leq 1$, the second technology reduces marginal cost more than does the first. In that case, increasing returns to R&D activity occurs. If $k = 1/2$, the values of both technologies are equivalent. If $0 \leq k < 1/2$, the R&D technologies exhibit decreasing returns. At the end of the first stage, "nature" chooses whether each firm succeeds in developing the technologies or not. Suppose that each firm succeeds in the development of the technology j with probability and assume that $p_j(\cdot)$ are identically and independently distributed. Therefore, we have $p_x(x_j) = 1 - e^{-x_j} = p_y(y_j) = 1 - e^{-y_j} = p(z) = 1 - e^{-z}, j = A, B$. These probability functions are well defined since we have

$$p'(\cdot) > 0, p''(\cdot) < 0, p'(0) \to \infty \text{ and } p(0) = 0, p(\infty) = 1.^{3}$$

The inverse market demand function for the product is given by

$$p = a - Q, \tag{2}$$

---

[2] That is, we distinguish "the *most partially* complementary technologies" from "*completely* complementary technologies"; that is, the latter implies that *no* firm *can* produce the goods *at all without the use of both technologies*. In this paper, "*completely complementary technologies*" is expressed by the case where the marginal cost k is infinitely large; that is $k \to \infty$. Okamura, Shinkai and Tanaka (2002) analyzed the case of completely complementary technologies.

[3] We assume the effect of the R&D activity on a process innovation as static. That is, the successes or failures of the development do not obey a stochastic process. However, these properties of the success probability function are similar to the dynamic "memoryless" or "Poisson" patent race model associated with Reinganum (1982). In her model of the research technology, it is assumed that a firm's probability of making a discovery and obtaining a patent at a point of time depends only on this firm's current R&D investment level and not on its past R&D experience. For illustrations of the dynamic patent race model, see Chapter 10 in Tirole (1989).

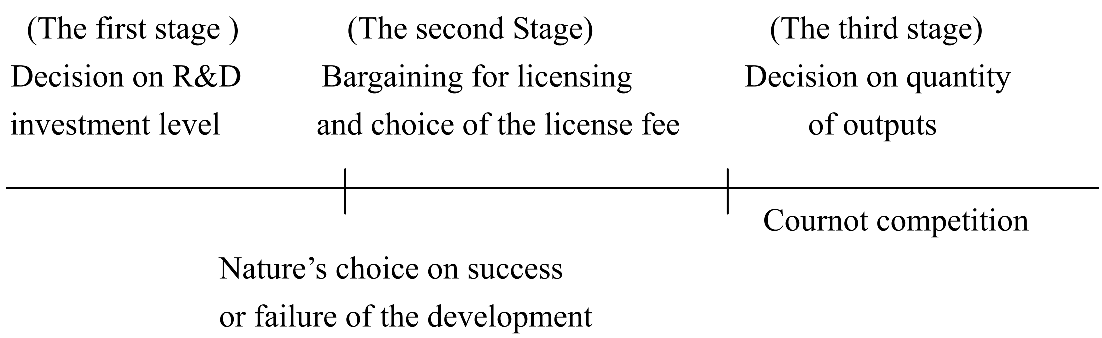
where p is the market price and Q is the aggregate output in the market, that is $Q = q_x + q_y$. We assume that the market is sufficiently large, i.e. $a \geq \frac{39}{4} - \frac{k}{2}$. At the beginning of the second stage, each firm knows all successes or failures of the both firms' developments of the technologies. At the second stage, if a (cross-) licensing system is available, then each firm bargains with its rival and agrees on a (cross-) licensing contract through the Nash bargaining process. The licensing contract describes how both firms divide the total profit. If a (cross-) licensing system is not available, then the game proceeds to the third stage. At the third stage, each firm's marginal cost is realized and it chooses its output simultaneously, that is, Cournot competition occurs.[4] Finally, the profit of each firm is realized and the game is over. The timing of the game is illustrated in Figure 1.

Let us conduct some preliminary work. Denote firm i's profit by $\pi_i(q_i, q_j)$.

$$\pi_i(q_i, q_j) = (a - q_i - q_j - c_i)q_i \quad (i \neq j, i = x, y), \tag{3}$$

where $c_i = c \in \{0, k, 1\}$. Since each firm engages in Cournot competition, the equilibrium output of firm i is given by

$$q_i^*(c_i, c_j) = \frac{(a - 2c_i + c_j)}{3} \quad i, j = x, y, i \neq j. \tag{4}$$



[4] Note that each firm can produce its product by using its own existing technology, even if it fails to develop both technologies, A and B in our model.

Figure 1. Timing of the game

Substituting (4) into (3) yields firm i's Cournot equilibrium profit,

$$\pi_i(c_i, c_j) \equiv \pi_i(q_i^*(c_i, c_j), q_j^*(c_j, c_i)) = \left(\frac{a - 2c_i + c_j}{3}\right)^2. \tag{5}$$

3. R&D investment without (cross-) licensing: A benchmark

In this section, as a benchmark, we analyze the problem of R&D in a Cournot duopoly with partially complementary technological innovation without licensing.[5]

Denote by {*X*,*Y*}, the combination of the states of nature which firms *x* and *y* face: Where $X, Y \in \{AB, A, B, \phi\}$ and "*AB*", "*A*", "*B*" and "ϕ" implies that each firm succeeds in development of technologies *A* and *B*, *A* or *B* and neither A nor B. All possible states of nature are as follows: {*AB*, *AB*}, {*AB*, *A*}, {*AB*, *B*}, {*AB*, ϕ}, {*A*, *AB*}, {*A*, *A*}, {*A*, *B*}, {*A*,ϕ}, {*B*, *AB*}, {*B*, *A*}, {*B*, *B*}, {*B*, ϕ}, {ϕ, *AB*}, {ϕ, *A*}, {ϕ, *B*} and {ϕ, ϕ}.[6]

For these states of nature, the corresponding realized equilibrium firm *x*'s profits are $\pi_x(0,0), \pi_x(0,k), \pi_x(0,k), \pi_x(0,1)$, $\pi_x(k,0), \pi_x(k,k), \pi_x(k,k), \pi_x(k,1)$, $\pi_x(k,0), \pi_x(k,k)$, $\pi_x(k,k), \pi_x(k,1)$, $\pi_x(1,0), \pi_x(1,k), \pi_x(1,k)$ and $\pi_x(1,1)$.

The expected profit of firm *x* without (cross-) licensing is given by

$$\prod\nolimits_x \equiv E\Pi_x(x_A, x_B, y_A, y_B) = (1 - e^{-x_A})(1 - e^{-x_B})H_1 + (1 - e^{-x_A})e^{-x_B}H_2$$

$$+ e^{-x_A}(1 - e^{-x_B})H_3 + e^{-x_A}e^{-x_B}H_4 - x_A - x_B, \tag{6}$$

---

[5] A lemma needed to derive the proposition and all proofs of the lemma and proposition in this section are presented in Appendix 1.

6 In our model, we also allow each firm to utilize the same technology as its rival's for production, if each firm succeeds in the development of a technology by itself. There are two interpretations of the patent breadth that economists have suggested: They have modeled breadth in "product space," defining how "similar" a product must be to infringe a patent, and in "technology space," defining how costly it is to find non-infringing substitutes for the protected market. See Section 2 of Chapter 4 in Scotchmer (2004) for details. We follow the latter interpretation of patent breadth and consider the case where patent breadth in this sense is narrow.

where

$$H_1 = (1-e^{-y_A})(1-e^{-y_B})\pi_x(0,0) + (1-e^{-y_A})e^{-y_B}\ \pi_x(0,k)\ + e^{-y_A}(1-e^{-y_B})\pi_x(0,k)$$

$$+\ e^{-y_A}e^{-y_B}\pi_x(0,1), \tag{7a}$$

$$H_2 = (1-e^{-y_A})(1-e^{-y_B})\pi_x(k,0)\ + (1-e^{-y_A})e^{-y_B}\ \pi_x(k,k)\ + e^{-y_A}(1-e^{-y_B})\pi_x(k,k)$$

$$+\ e^{-y_A}e^{-y_B}\pi_x(k,1), \tag{7b}$$

$$H_3 = (1-e^{-y_A})(1-e^{-y_B})\pi_x(k,0)\ + (1-e^{-y_A})e^{-y_B}\ \pi_x(k,k) + e^{-y_A}(1-e^{-y_B})\pi_x(k,k)$$

$$+\ e^{-y_A}e^{-y_B}\pi_x(k,1) \tag{7c}$$

and

$$H_4 = (1-e^{-y_A})(1-e^{-y_B})\pi_x(1,0) + (1-e^{-y_A})e^{-y_B}\pi_x(1,k) +\ e^{-y_A}(1-e^{-y_B})\pi_x(1,k)$$

$$+\ e^{-y_A}e^{-y_B}\pi_x(1,1). \tag{7d}$$

The first-order condition for expected profit maximization with respect to (w.r.t.) R&D activity of technology A is given by

$$\frac{\partial \Pi_x(x_A, x_B, y_A, y_B)}{\partial x_A} = e^{-x_A}(1-e^{-x_B})H_1 +\ e^{-x_A}e^{-x_B}H_2 - e^{-x_A}(1-e^{-x_B})H_3$$

$$-\ e^{-x_A}e^{-x_B}H_4 - 1 = 0. \tag{8}$$

From (7b) and (7c), we see that $H_2 = H_3$, and obtain

$$e^{-x_A}\{(1-e^{-x_B})H_1 +\ (2e^{-x_B}-1)H_2 -\ e^{-x_B}H_4\} - 1 = 0. \tag{9}$$

Since both firms are identical, we focus on the symmetric equilibrium hereafter. We can denote the probability of failure for the development of each technology that plays a key role in our analysis by $s = e^{-x_A} = e^{-x_B} = e^{-y_A} = e^{-y_B}$.[7]

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[7] The second-order condition at the equilibrium is that $1-\left[\frac{s}{1-s}-\frac{s^2}{1-s}(H_2-H_4)\right]^2 > 0$ holds. If $0 < s \le 1/2$ holds, then we can easily show that this inequality holds.

The first-order condition (9) is expressed by

$$\frac{\partial \Pi_x(x_A, x_B, y_A, y_B)}{\partial x_A} = sV(s) - 1$$
$$= s\{(1-s)(H_1 - H_2) + s(H_2 - H_4)\} - 1 = 0,$$

where $V(s) \equiv (1-s)(H_1 - H_2) + s(H_2 - H_4)$. (10)

We rewrite (10) as

$$\begin{aligned}\phi(s,k,a) &= N_1 s^4 + N_2 s^3 + N_3 s^2 + N^4 s - 1 \\ &= \frac{4}{9}(1-2k)^2 s^4 + \frac{4}{3}k(1-2k)s^3 + \frac{4}{9}(2k-1)(2k-a+1)s^2 + \frac{4}{9}k(a-k)s - 1 = 0.\end{aligned} \tag{11}$$

We assume that

$$\phi(\frac{1}{2},k,a) = -\frac{13}{12} + \frac{1}{18}k + \frac{1}{9}a > 0. \text{ Or,}$$

$$a \geq \frac{39}{4} - \frac{k}{2}.\text{[8]} \tag{12}$$

We examine the properties of $\phi(s,k,a)$. By substituting 0 and 1 into s and rearranging terms we have

$$\phi(0,k,a) = -1 < 0, \tag{13}$$

$$\begin{aligned}\phi(1,k,a) &= N_1 + N_2 + N_3 + N_4 - 1 \\ &= \frac{4}{9}(k-1)(k-a) - 1 \\ &\geq -1 = \phi(0,k,a),\end{aligned} \tag{14}$$

where the last equality holds when k = 1, that is, the two technologies are most partially completely complementary. Setting $f(k) = \frac{4}{9}(k-1)(k-a) - 1$, we see that

$$f(0) = \frac{9}{4}a - 1 > 0, f(1) = -1 < 0, f(\frac{1}{2}) = -\frac{2}{9}(\frac{1}{2} - a) - 1 > 0$$
$$f^{'}(k) = \frac{4}{9}(2k - (1+a)) < 0, \mathit{for} \quad 0 \leq k \leq 1$$
$$\because a \geq \frac{39}{4} - \frac{k}{2} \geq \frac{37}{4}.$$

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[8] This assumption implies that the marginal benefit with respect to x_A is greater than the marginal cost w.r.t. x_A when the development success chance is even. Increase of the investment level, or equivalently, decrease of the probability of failure of the development, is beneficial to the firm if the probability of failure is 1/2.

The smaller roots of $f(k)=0$ is given by

$$k_L^* = \frac{1}{2}\{1+a-\sqrt{a^2-2a+10}\} \equiv k_L^*(a) \quad \text{and we see that}$$

$$k_L^*(\frac{37}{4}) = \frac{41}{8} - \frac{3}{8}\sqrt{137} \cong 0.73574 \quad and \quad \lim_{a\to\infty} k_L^*(a) = 1.$$

Differentiating partially $\phi(s,k,a)$ w.r.t. s yields

$$\begin{aligned}\frac{\partial\phi(s,k,a)}{\partial s} &= \phi_s(s,k,a) = 4N_1s^3 + 3N_2s^2 + 2N_3s + N_4 \\ &= \frac{4}{9}\{4(1-2k)^2s^3 + 9k(1-2k)s^2 + 2(1-2k)(a-1-2k)s + k(a-k)\}.\end{aligned} \tag{15}$$

We have

$$\phi_{s}(0,k,a) = N_4 = \frac{4}{9}k(a-k) \geq 0, \tag{16}$$

$$\begin{aligned}\phi_s(1,k,a)) &= 4N_1 + 3N_2 + 2N_3 + N_4 \\ &= \frac{4}{9}\{5k^2 - (7+3a)k + 2(a+1)\}.\end{aligned}$$

Defining $g(k) = 5k^2 - (7+3a)k + 2(a+1)$, we see that

$$g(0) = 2(a+1) > 0, g(1) = -a < 0, g(\frac{1}{2}) = \frac{1}{2}(a - \frac{1}{2}) > 0$$

$$g'(k) = 10k - (7+3a) < 0.$$

Since the smaller roots of the quadratic equation of k, $g(k)=0$ is given by

$\hat{k} = \frac{1}{10}\{(7+3a) - \sqrt{9a^2+2a+9}\}(> \frac{1}{2})$, we obtain

$$\begin{aligned}\phi_s(1,k,a) &\geq 0, \quad if \quad 0 \leq k \leq \hat{k}, \\ &\leq 0, \quad if \quad \hat{k} < k \leq 1.\end{aligned} \tag{17}$$

We can show that

$$k_L^* > \hat{k} > \frac{1}{2} \quad \text{if} \quad a \geq \frac{39}{4} - \frac{k}{2} \geq \frac{37}{4}. \tag{18}$$

From (15), we have

$$\phi_{ss}(s,k,a) = 2(6N_1s^2 + 3N_2s + N_3) = \frac{8}{9}(1-2k)\{6(1-2k)s^2 + 9ks + (a-1-2k)\} \tag{19}$$

$$\phi_{sss}(s,k,a) = 6(4N_1s + N_2) = \frac{8}{3}(2k-1)(4s(2k-1) - 3k) \tag{20}$$

Now, we present the following proposition on the R&D investment equilibrium in a Cournot duopoly with partially complementary technologies without licensing.[9]

Proposition 1

Suppose that $a \geq \frac{39}{4} - \frac{k}{2} \geq \frac{37}{4}$. Then, there exists at least one positive symmetric equilibrium s^* in our model without (cross-) licensing,

$$0 < s^* < \frac{1}{2}, \text{ and } \frac{\partial s^*}{\partial a} < 0, \qquad \frac{\partial s^*}{\partial k} < 0.$$

The proposition asserts that there exists at least one equilibrium with large R&D investments, if the market is sufficiently large. Since $s^* = e^{-x^*} < 1/2$ is the probability of failure in the development of R&D, the equilibrium R&D investment level is obtained by $x^* = -\ln s^*$. With the failure probability sufficiently small, each firm invests relatively aggressively in R&D technologies. The comparative static results show that the equilibrium investment level increases as the market becomes large or as the complementarity between the two technologies grows strong. These results seem to be plausible. The first result implies that the improvement of the market condition encourages the R&D investments. Now define $e(k) = k/1-k$, which measures the relative economic values of cost reduction if a firm succeeds in developing another technology, given it has already developed one technology. We interpret $e(k)$ as the measure of relative cost efficiency of the first and second developed technologies. The value of $e(k)$ increases from zero to infinitely large as k increases from zero to one. The second result implies that each firm increases R&D investment if relative cost efficiency improves.

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[9] Before deriving the sub game perfect equilibrium strategies, we need a lemma. The lemma is presented in Appendix 1.

4. The conditions under which (cross-) licensing may occur

In this section, we explore the conditions under which each firm engages in (cross-) licensing. We assume that the patent breadth authorized by the patent protection authorities is narrow. That is, if both firms independently succeed in developing versions of technology A (B) that differ slightly from each other, they can acquire a patent for their own outcomes and can utilize the technology. We assume that the terms of the licensing contract entail a fixed licensing fee. We also assume that each firm produces the *Cournot equilibrium quantity of output* given the realized marginal cost under licensing, if it agrees to the licensing contract and it is executed.[10]

The state of nature of each firm depends on success(es) or failure(s) of the development of technologies. All cases where (cross-) licensing occurs are summarized in Table 1.

We classify the conditions under which (cross-) licensing occurs into four cases and derive the corresponding licensing fee in these cases.[11]

(Case Ⅰ) The cross-licensing fee is given by

$$F_{\mathrm{I}} = 0. \tag{21}$$

Cross-licensing occurs where

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[10] Both firms may agree to a licensing contract in which a licensee firm pays half the monopoly profit brought by producing at the lowered marginal cost realized by licensing, as the license fee in compensation for no production. The final gain of each firm after the side payment in this contract is, of course, larger than that where firms compete in a Cournot manner. However, the former is interpreted as an illegal act from the antitrust point of view. See the description in Section 3.2 and Example 4 in the Appendix of Antitrust Guidelines for Collaborations Among Competitors (April 2000) issued by the Federal Trade Commission and the U. S. Department of Justice. They say that 'Agreements of a type that always tends to raise price or reduce output are per se illegal.' The contract in which the licensee firm produces the monopoly output and pays half the monopoly profit as a licensing fee to the licensor firm seems to be per se illegal. We thank Kuninobu Takeda, Associate Professor of Antitrust Law in Osaka University, for this justification of our assumption from the antitrust point of view and for the source of this citation. Hence, we assume that two firms compete in a Cournot manner after the licensing contract.

[11] For the concrete derivation of the licensing conditions and the licensing fees in the case containing the four categories, see Appendix 2.

$$\pi_x(0,0) - \pi_y(k,k) = \frac{k(2a-k)}{9} \geq 0, \tag{22}$$

in which case the inequality holds since $a \geq \frac{39}{4} - \frac{k}{2} \geq \frac{37}{4}$ and $0 \leq k \leq 1$.

(Case Ⅱ) The unilateral licensing fee is given by

$$F_{\text{II}} = \frac{1}{2}[\pi_x(k,1) - \pi_y(1,k)] = \frac{(1-k)(2a-k-1)}{6} > 0. \tag{23}$$

Unilateral licensing occurs where

$$\frac{(-2a-3k+5)(k-1)}{18} \geq 0, \tag{24}$$

Firm y's state Y / Firm x's state X	φ	A	B	AB
φ		UL: $y \to x$ Case II	UL: $y \to x$ Case II	UL: $y \to x$ Case IV
A	UL: $x \to y$ Case II		CL: $x \leftrightarrow y$ Case I	UL: $y \to x$ Case III
B	UL: $x \to y$ Case II	CL: $x \leftrightarrow y$ Case I		UL: $y \to x$ Case III
AB	UL: $x \to y$ Case IV	UL: $x \to y$ Case III	UL: $x \to y$ Case III	

In the table, "UL: $y \to x$" and "CL: $x \leftrightarrow y$" imply "unilateral licensing from Firm y to Firm x," and "cross-licensing between two firms," respectively.

Table 1. Possible Licensing Patterns

in which case the inequality holds since $a \geq \frac{39}{4} - \frac{k}{2} \geq \frac{37}{4}$ and $0 \leq k \leq 1$.

(Case Ⅲ) The unilateral licensing fee is given by

$$F_{\mathrm{III}} = \frac{1}{2}[\pi_x(0,k) - \pi_y(k,0)] = \frac{k(2a-k)}{6} \geq 0. \tag{25}$$

Unilateral licensing occurs where

$$\frac{k(2a-5k)}{18} \geq 0, \tag{26}$$

in which case, again, the inequality holds since $a \geq \frac{39}{4} - \frac{k}{2} \geq \frac{37}{4}$ and $0 \leq k \leq 1$.

(Case Ⅳ) This case consists of two sub-cases in which unilateral licensing occurs. One sub-case is where only one technology is licensed. The other sub-case is where both technologies are licensed. As we show in Appendix 2, the strategy of unilateral licensing of both technologies is more beneficial for the licenser firm than that of licensing only one technology. We analyze this latter type of licensing. The unilateral licensing fee is given by

$$F_{\mathrm{IV}} = \frac{1}{2}[\pi_x(0,1) - \pi_y(1,0)] = \frac{2a-1}{6} > 0. \tag{27}$$

Unilateral licensing occurs where

$$\frac{2a-5}{18} > 0, \tag{28}$$

in which case the inequalities in (27) and (28) hold because $a \geq \frac{39}{4} - \frac{k}{2} \geq \frac{37}{4}$.

Now, we are ready to derive the R&D investment game in duopoly with a (cross-) licensing system.

5. R&D investment with (cross-) licensing

Examining cells in Table 1 where (cross-) licensing occurs, we can express the case by using the realized marginal cost of firm $i(j)$ before (cross-) licensing as (c_i, c_j). Then, we see that all the cases with (cross-) licensing are (*k*, *k*), (0, *k*), (0, 1), (*k*, 1), (*k*, 0), (1, 0) and (1, *k*).

(1) The firm i's profit realized in state (k, k) (in this case, cross-licensing occurs and the licensing fee is zero) is

$$\pi_i^{kk} = ((1-e^{-x_A})e^{-x_B}e^{-y_A}(1-e^{-y_B}) + e^{-x_A}(1-e^{-x_B})(1-e^{-y_A})e^{-y_B})\ \pi_i(0,0)\,. \tag{29}$$

(2) The firm i's profit realized in (0, k) is

$$\pi_i^{0k} = (1-e^{-x_A})(1-e^{-x_B})\{e^{-y_A} + e^{-y_B} - 2e^{-y_A}e^{-y_B}\}\,[\,\pi_i(0,0) + F_{\mathrm{III}}\,]. \tag{30}$$

(3) The firm i's profit realized in (0, 1) is

$$\pi_i^{01} = (1-e^{-x_A})(1-e^{-x_B})e^{-y_A}e^{-y_B}\,[\,\pi_i(0,0) + F_{\mathrm{IV}}\,]. \tag{31}$$

(4) The firm i's profit realized in (k, 1) is

$$\pi_i^{k1} = [\,e^{-x_A} + e^{-x_B} - 2e^{-x_A}e^{-x_B}\,]e^{-y_A}e^{-y_B}\,[\,\pi_i(k,k) + F_{\mathrm{II}}\,]. \tag{32}$$

(5) The firm i's profit realized in (k, 0) is

$$\pi_i^{k0} = [\,e^{-x_A} + e^{-x_B} - 2e^{-x_A}e^{-x_B}\,]\,(1-e^{-y_A})(1-e^{-y_B})\ \ [\,\pi_i(0,0) - F_{\mathrm{III}}\,]. \tag{33}$$

(6) The firm i's profit realized in (1, 0) is

$$\pi_i^{10} = e^{-x_A}e^{-x_B}\ (1-e^{-y_A})(1-e^{-y_B})\ \ [\,\pi_i(0,0) - F_{\mathrm{IV}}\,]. \tag{34}$$

(7) The firm i's profit realized in (1, k) is

$$\pi_i^{1k} = e^{-x_A}e^{-x_B}\ (e^{-y_A} + e^{-y_B} - 2e^{-y_A}e^{-y_B})\ \ [\,\pi_i(k,k) - F_{\mathrm{II}}\,]. \tag{35}$$

Using the profits realized in states above, we express the expected profit of firm i as

$$\begin{aligned}
\widetilde{\Pi}_i &= \Pi_i + \Pi_i^{CL} \\
&+ (1-e^{-x_A})(1-e^{-x_B})\left[A\{\pi_i(0,0) - \pi_i(0,k) + F_{\mathrm{III}}\} + B\{\pi_i(0,0) - \pi_i(0,1) + F_{\mathrm{IV}}\}\right] \\
&+ (e^{-x_A} + e^{-x_B} - 2e^{-x_A}e^{-x_B})\left[B\{\pi_i(k,k) - \pi_i(k,1) + F_{\mathrm{II}}\} + C\{\pi_i(0,0) - \pi_i(k,0) - F_{\mathrm{III}}\}\right] \\
&\quad + e^{-x_A}e^{-x_B}\left[C\{\pi_i(0,0) - \pi_i(1,0) - F_{\mathrm{IV}}\} + A\{\pi_i(k,k) - \pi_i(1,k) - F_{\mathrm{II}}\}\right]
\end{aligned} \tag{36}$$

where $A = e^{-y_A} + e^{-y_B} - 2e^{-y_A}e^{-y_B}$, $B = e^{-y_A}e^{-y_B}$, $C = (1-e^{-y_A})(1-e^{-y_B})$ (37)

The increment to Firm i's expected profit associated only with cross-licensing Π_i^{CL} is given by

$$\Pi_i^{CL} = (1-e^{-x_A})e^{-x_B}(1-e^{-y_B})e^{-u_A}h + e^{-x_A}(1-e^{-x_B})(1-e^{-y_A})e^{-y_B}h, \tag{38}$$

where $h = \pi_i(0,0) - \pi_i(k,k)$.

Deriving the first-order condition and setting $s = e^{-y_A} = e^{-y_B} = e^{-x_A} = e^{-x_B}$, by using the fact that we derive the symmetric equilibrium, yields

$$\Omega(s,k,a) \equiv \left.\frac{\partial\tilde{\Pi}}{\partial x_A}\right|_{s=e^{-x_A}=e^{-x_B}=e^{-y_A}=e^{-y_B}} = \left.\frac{\partial\Pi}{\partial x_A}\right|_{s=e^{-x_A}=e^{-x_B}=e^{-y_A}=e^{-y_B}} + \left.\frac{\partial\Pi^{CL}}{\partial x_A}\right|_{s=e^{-x_A}=e^{-x_B}=e^{-y_A}=e^{-y_B}} +$$

$$s(1-s)[A(n_7+F_{\mathrm{III}})+B(n_8+F_{\mathrm{IV}})]+s(2s-1)[B(n_9+F_{\mathrm{II}})+C(n_1-F_{\mathrm{III}})]$$

$$-s^2[C(n_1+n_4-F_{\mathrm{IV}})+A(n_5-F_{\mathrm{II}})]=0. \tag{39}$$

After tedious calculations, we obtain the profit-maximization condition:[12]

$$\Omega(s,k,a) = \frac{1}{9}(2k^2-2ak+2a-1)s^4 + \frac{1}{18}(4k^2-8ak+6a-3)s^2 + \frac{1}{6}k(2a-k)s - 1 = 0 \tag{40}$$

From the l.h.s. of (40) we see that

$$\Omega(\frac{1}{2},k,a) = -\frac{1}{72}(k^2-2ak-7a+\frac{151}{2}) = (\frac{1}{36}k+\frac{7}{72})a - \frac{1}{72}k^2 - \frac{151}{144}.$$

As in the benchmark case, we assume that

$$\Omega(\frac{1}{2},k,a) = (\frac{1}{36}k+\frac{7}{72})a - \frac{1}{72}k^2 - \frac{151}{144} > 0.$$

This assumption implies that

$$a > \frac{2k^2+151}{4k+14}, 0 \le k \le 1. \tag{41}$$

Now, we need two lemmas to prepare for the result on equilibrium existence.

The two lemmas and their proofs and the proof of the following proposition are presented in Appendix 3.

[12] For derivation of (40), see Appendix 3.

From these two lemmas, we immediately obtain the following equilibrium existence result.

Proposition 2

Suppose that $a > \max\{\frac{39}{4} - \frac{k}{2}, \frac{2k^2+151}{4k+14}\}$. Then there exists a positive symmetric equilibrium $\tilde{s} \in (0, \frac{1}{2})$ in our model with (cross-) licensing.[13]

In any $\tilde{s} \in (0, \frac{1}{2})$, we have $\frac{\partial \tilde{s}}{\partial k} < 0$ and $\frac{\partial \tilde{s}}{\partial a} < 0$.

6. Effects of a (cross-) licensing system on R&D investment

In this section, we compare the equilibrium investment level *without a (cross-) licensing system* with that *with a (cross-) licensing system*. The two lemmas needed for derivation of the main result and their proofs are presented in Appendix 4. We also present the proof of the proposition in Appendix 4.

We establish the following proposition.

Proposition 3

(1) Suppose that the two technologies are not very partially complementary, such that $0 \le k \le k^{**} < \frac{1}{2}$. The licensing system discourages R&D investment, i.e. $0 < s^* < \tilde{s} < \frac{1}{2}$.

(2) Suppose that the two technologies are sufficiently partially complementary, such that

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[13] We can show that $\Omega(1,k,a) > (\le) 0 \Leftrightarrow \forall k \in [0, \quad k^*]([k^*,1])$, where $k^* = a - \frac{1}{5}\sqrt{25a^2 - 50a + 115}$. Then, there exists an $\tilde{\tilde{s}}$ such that $\frac{1}{2} < \tilde{\tilde{s}} < 1$ and $\Omega(\tilde{\tilde{s}},k,a) = 0, \quad$ *for* $k \in [k^*,1]$. We also show, as we do in footnote 6, that $\tilde{\tilde{s}}$ ($\frac{1}{2} < \tilde{\tilde{s}} < 1$) never satisfies the second-order condition. By Kuhn–Tucker conditions, in this case, there exists equilibrium $\tilde{s} = 1$. It implies, however that each firm does not invest at all at the equilibrium. This case is not interesting and is omitted.

$k^{**} < 1/2 < k \le 1$ and there *exist any points* $s^0 \in (0, \frac{1}{2})$ such that $\phi(s^0, k, a) = \Omega(s^0, k, a)$.

(2-a) If there exists a unique $s^0 \in (0, \frac{1}{2})$ such that $\phi(s^0, k, a) = \Omega(s^0, k, a) > 0$, the licensing system discourages R&D investment, i.e. $0 < s^* < \tilde{s} < \frac{1}{2}$.

(2-b) If there exists a unique $s^0 \in (0, \frac{1}{2})$ such that $\phi(s^0, k, a) = \Omega(s^0, k, a) < 0$, the licensing system encourages R&D investment, i.e. $0 < \tilde{s} < s^* < \frac{1}{2}$.

(2-c) If there exists a unique $s^0 \in (0, \frac{1}{2})$ such that $\phi(s^0, k, a) = \Omega(s^0, k, a) = 0$, the licensing system is neutral for R&D investment, i.e. $0 < \tilde{s} = s^* < \frac{1}{2}$.

We give some numerical examples for this proposition. See Figure 2 for (1), Figure 3 for (2-b) and Figure 4 for (2-a).

[Insert Figure 2, Figure 3 and Figure 4 here]

We explain intuitively the discouragement to R&D investment result. From (38) the increment to firm i's expected profit with only cross-licensing Π_i^{CL} is given by

$$\Pi_i^{CL} = (1 - e^{-x_A})e^{-x_B}e^{-y_A}(1 - e^{-y_B})h + e^{-x_A}(1 - e^{-x_B})(1 - e^{-y_A})e^{-y_B}h,$$

$$\text{where } h = \pi_x(0,0) - \pi_y(k,k) = \frac{k(2a-k)}{9} \ge 0 .(\because (22)) \tag{42}$$

The partial derivative term of π_i^{CL} evaluated at the symmetric equilibrium ($s = e^{-x_A} = e^{-x_B} = e^{-y_A} = e^{-y_B}$) is given by the following expression.

$$\left.\frac{\partial \Pi^{CL}}{\partial x_A}\right|_{s=e^{-x_A}=e^{-x_B}=e^{-y_A}=e^{-y_B}} = M(s,k,a) = \frac{1}{9}s^2(2s-1)(1-s)h < 0 \tag{43}$$

However, from Proposition 2, we see that $0 < \tilde{s}(< \frac{1}{2})$ at equilibrium. The expression above shows that the existence of cross-licensing discourages R&D investment. On the other hand, the increment to firm i 's expected profit resulting from unilateral licensing is given by the following formula:

$$\begin{aligned}\Pi^{UL} &= (1-e^{-x_A})(1-e^{-x_B})e^{-y_A}e^{-y_B}\pi_{\mathrm{IV}} + e^{-x_A}e^{-x_B}(1-e^{-y_A})(1-e^{-y_B})\pi_{\mathrm{IV}} \\ &+ (1-e^{-x_A})(1-e^{-x_B})(1-e^{-y_A})e^{-y_B}\pi_{\mathrm{III}} + (1-e^{-x_A})e^{-x_B}(1-e^{-y_A})(1-e^{-y_B})\pi_{\mathrm{III}} \\ &+ (1-e^{-x_A})(1-e^{-x_B})e^{-y_A}(1-e^{-y_B})\pi_{\mathrm{III}} + e^{-x_A}(1-e^{-x_B})(1-e^{-y_A})(1-e^{-y_B})\pi_{\mathrm{III}} \\ &+ (1-e^{-x_A})e^{-x_B}e^{-y_A}e^{-y_B}\pi_{\mathrm{II}} + e^{-x_A}e^{-x_B}(1-e^{-y_A})e^{-y_B}\pi_{\mathrm{II}} \\ &+ e^{-x_A}(1-e^{-x_B})e^{-y_A}e^{-y_B}\pi_{\mathrm{II}} + e^{-x_A}e^{-x_B}e^{-y_A}(1-e^{-y_B})\pi_{\mathrm{II}},\end{aligned}$$

where

$$\begin{aligned}\pi_{\mathrm{IV}} &\equiv \pi_x(0,0) - \pi_x(0,1) + F_{\mathrm{IV}} = \pi_x(0,0) - \pi_x(1,0) - F_{\mathrm{IV}}, \\ \pi_{\mathrm{III}} &\equiv \pi_x(0,0) - \pi_x(0,k) + F_{\mathrm{III}} = \pi_x(0,0) - \pi_x(k,0) - F_{\mathrm{III}}, \\ \pi_{\mathrm{II}} &\equiv \pi_x(k,k) - \pi_x(k,1) + F_{\mathrm{II}} = \pi_x(k,k) - \pi_x(1,k) - F_{\mathrm{II}}.\end{aligned}$$

From the corresponding part of the first-order condition of the symmetric equilibrium ($s = e^{-x_A} = e^{-x_B} = e^{-y_A} = e^{-y_B}$) is given by the following expression.

$$\begin{aligned}\left.\frac{\partial \Pi^{UL}}{\partial x_A}\right|_{\tilde{s}} &= e^{-x_A}(1-e^{-x_B})e^{-y_A}e^{-y_B}\pi_{\mathrm{IV}} - e^{-x_A}e^{-x_B}(1-e^{-y_A})(1-e^{-y_B})\pi_{\mathrm{IV}} \\ &+ e^{-x_A}(1-e^{-x_B})(1-e^{-y_A})e^{-y_B}\pi_{\mathrm{III}} + e^{-x_A}e^{-x_B}(1-e^{-y_A})(1-e^{-y_B})\pi_{\mathrm{III}} \\ &+ e^{-x_A}(1-e^{-x_B})e^{-y_A}(1-e^{-y_B})\pi_{\mathrm{III}} - e^{-x_A}(1-e^{-x_B})(1-e^{-y_A})(1-e^{-y_B})\pi_{\mathrm{III}} \\ &+ e^{-x_A}e^{-x_B}e^{-y_A}e^{-y_B}\pi_{\mathrm{II}} - e^{-x_A}e^{-x_B}(1-e^{-y_A})e^{-y_B}\pi_{\mathrm{II}} \\ &- e^{-x_A}(1-e^{-x_B})e^{-y_A}e^{-y_B}\pi_{\mathrm{II}} - e^{-x_A}e^{-x_B}e^{-y_A}(1-e^{-y_B})\pi_{\mathrm{II}} \\ &= \tilde{s}^2(1-\tilde{s})(2\tilde{s}-1)\pi_{\mathrm{IV}} + 2\tilde{s}^2(1-\tilde{s})^2\pi_{\mathrm{III}} + \tilde{s}(1-\tilde{s})^2(2\tilde{s}-1)\pi_{\mathrm{III}} \\ &+ \tilde{s}^3(2\tilde{s}-1)\pi_{\mathrm{II}} - 2\tilde{s}^3(1-\tilde{s})\pi_{\mathrm{II}} \\ &= \tilde{s}^2(1-\tilde{s})(2\tilde{s}-1)\pi_{\mathrm{IV}} + \tilde{s}(1-\tilde{s})^2(4\tilde{s}-1)\pi_{\mathrm{III}} + \tilde{s}^3(4\tilde{s}-3)\pi_{\mathrm{II}}\end{aligned} \tag{44}$$

From (23), (25), (27) and the definitions of π_{II}, π_{III}, π_{IV} above, we obtain

$$\pi_{\mathrm{II}} = \frac{(1-k)(2a+3k-5)}{18}, \quad \pi_{\mathrm{III}} = \frac{k(2a-5k)}{18}, \quad \pi_{\mathrm{IV}} = \frac{2a-5}{18}. \tag{45}$$

Calculating the partial derivatives of π_{II}, π_{III}, π_{IV} w.r.t. a and k in (45) yields

$$\begin{aligned}&\frac{\partial \pi_{\mathrm{II}}}{\partial a} = \frac{1-k}{9} \geq 0, \frac{\partial \pi_{\mathrm{III}}}{\partial a} = \frac{k}{9} \geq 0, \frac{\partial \pi_{\mathrm{IV}}}{\partial a} = \frac{1}{9} > 0, 0 \leq k \leq 1 \\ &\frac{\partial \pi_{\mathrm{II}}}{\partial k} = \frac{8-6k-2a}{18} < 0, \frac{\partial \pi_{\mathrm{III}}}{\partial k} = \frac{a-5k}{9} \geq 0, \frac{\partial \pi_{\mathrm{IV}}}{\partial k} = 0, 0 \leq k \leq 1, \quad a > \frac{37}{4}.\end{aligned}$$

At the symmetric equilibrium $0 < \tilde{s} (< \frac{1}{2})$, the only two terms in (44) and

$$e^{-x_A}(1-e^{-x_B})(1-e^{-y_A})e^{-y_B}\pi_{\mathrm{III}}+e^{-x_A}e^{-x_B}(1-e^{-y_A})(1-e^{-y_B})\pi_{\mathrm{III}} \\ =2\tilde{s}^2(1-\tilde{s})^2\pi_{\mathrm{III}} \tag{46}$$

work in the same direction to increase R&D investment (shift the function Ω upward). These two terms are the marginal benefit to firm *x* when it slightly increases investment in development of technology *A* in the cases {*AB*, *A*} and {*A*, *AB*}, respectively (which correspond to the two shaded cells in Table 1). In these two cases, the increase of x_A is always beneficial to firm *x* since it succeeds in developing technology *A*. However, all the other eight terms in (44) work to shrink R&D investment (shift the function Ω downward). In these cases, although the increase of x_A brings firm *x* a positive marginal benefit if it succeeds in developing technology *A* (for example, see the corresponding cell to the case $\mathrm{IV}\{AB,\phi\}$ in Table 1), at the same time it brings a negative marginal benefit since the expected benefit as a unilateral licensee of technology *A* from firm *y* decreases in the pair case where firm *x* fails to develop technology *A* (See the UL cells to case $\mathrm{IV}\{\phi, AB\}$ in Table 1). The corresponding part of the first-order condition of the increment to firm i's expected profit with only cross-licensing Π_i^{CL} also works to shrink R&D investment (shift the function Ω downward). The total negative effect dominates in most cases when the underlying market demand for the product a is sufficiently large, since π_{II}, π_{III}, π_{IV} are nondecreasing in *a*. In some cases, however, the positive effects dominate. If the extent of complementarity $k \in [0,1]$ is so large and the underlying demand a is small enough, these cases tend to occur.

To explain these results intuitively, note that we normalize the reduction of the marginal cost of production associated with the R&D development to unity. The underlying demand a is also looked upon as reduced to unity (a is the original market size divided by the amount of marginal cost reduction). Thus, the change in a is incomparably larger than the change of k.

Let us focus on the comparative statics w.r.t. k when the underlying demand a is small enough and the magnitude of the marginal cost reduction is large compared with the underlying demand. From the above definitions, π_{II}, π_{III}, π_{IV} and h represent the ex post profit of each firm under unilateral licensing of only one technology where the licensee has not developed any technology, the ex post profit of the firm under unilateral licensing of one technology by the licensor who has developed both technologies and the licensee has developed only one technology, the ex post profit of the firm under unilateral licensing of two developed technologies where the licensee has no developed technologies, and the incremental profit of each firm under cross-licensing, respectively. As the extent of the complementarity k increases, π_{III} and h increase while π_{IV} does not change and π_{II} decreases. Therefore, when k is large but a is small enough, the positive effect (associated with π_{III}) presented in (46) dominates the total negative effects associated with terms π_{II} in (44) and h (given by (43)).

This proposition shows that a (cross-) licensing system promotes R&D investment in some cases when the duopolistic firms produce goods by using two partially complementary technologies. In these cases, the extent of complementarity k is sufficiently large and the underlying demand a is small enough.

Okamura, Shinkai and Tanaka (2002) established that the existence of a cross-licensing system always discourages firm’s R&D investments, when the duopolistic firms produce a good by using the two completely complementary technologies. In their model, no unilateral licensing can occur since firms require both technologies to produce the good. The existence of a cross-licensing system decreases firms’ incentives for R&D through the chance to exchange their technologies. As we have shown in this paper, however, unilateral licensing may encourage firms’ incentives for R&D through the chance of their receiving (paying) the licensing fee when the extent of complementarity k is sufficiently large and the underlying demand a is small enough. When two not quite completely complementary technological innovations occur, this positive effect of unilateral licensing on firms' incentives for R&D may surpass the negative effect of cross-licensing upon their incentives.

7. Welfare comparison at the equilibria with and without licensing

In this section, we compare economic welfare evaluated at the equilibria, s^* and $\tilde{s}$ with and without (cross-) licensing. Since we see that we cannot derive two equilibrium solutions s^* and $\tilde{s}$ analytically from the discussion in the preceding section, so comparison of economic welfare at s^* and $\tilde{s}$ is conducted for the numerical solutions s^*'s and $\tilde{s}$'s presented in preceding section in three cases in the preceding section, where a = 15, k = 0.3, where a = 9.5, k = 0.95 and where a = 15, k = 0.95.

Set $s^* = e^{-x_A^*} = e^{-x_B^*} = e^{-y_A^*} = e^{-y_B^*}$ $(x_A^* = x_B^* = -\ln s^*)$ in (6) and multiply it by 2, we define the expected producers' surplus at the symmetric equilibrium without licensing as

$$EPS^*(s^*, k, a) \equiv 2(1-s^*)^2 H_1(s^*, k, a) + 4s^*(1-s^*)H_2(s^*, k, a) + 2s^{*2} H_4(s^*, k, a) + 4\ln s^*, \tag{47}$$

where $H_i(s^*, a, k), i = 1,2,4$ implies $H_i, i = 1,2,4$ given by (7a), (7b) and (7d) evaluated at $s^* = e^{-x_A^*} = e^{-x_B^*} = e^{-y_A^*} = e^{-y_B^*}$. That is, we have

$$H_1(s^*, k, a) = (1-s^*)^2 \pi_x(0,0) + 2s^*(1-s^*)\pi_x(0,k) + s^{*2}\pi_x(0,1), \tag{48a}$$

$$H_2(s^*, k, a) = (1-s^*)^2 \pi_x(k,0) + 2s^*(1-s^*)\pi_x(k,k) + s^{*2}\pi_x(k,1), \tag{48b}$$

$$H_4(s^*, k, a) = (1-s^*)^2 \pi_x(1,0) + 2s^*(1-s^*)\pi_x(1,k) + s^{*2}\pi_x(1,1). \tag{48c}$$

We know well that the consumers' surplus in the Cournot equilibrium of our setting is given by $CS(Q(c_x, c_y)) = \frac{1}{2}Q(c_x, c_y)^2$, where $Q(c_x, c_y) = q_x(c_x, c_y) + q_y(c_y + c_x)$.

Replacing $\pi_x(c_x, c_y)$ by $CS(Q(c_x, c_y))$ in (48a), (48b) and (48c), we define the expected consumers' surplus as

$$ECS^*(s^*,k,a) \equiv 2(1-s^*)^2 J_1(s^*,k,a) + 4s^*(1-s^*)J_2(s^*,k,a) + 2s^{*2} J_4(s^*,k,a), \quad (49)$$

where

$$J_1(s^*,k,a) = (1-s^*)^2\, CS(Q(0,0)) + 2s^*(1-s^*)\, CS(Q(0,k)) + s^{*2} CS(Q(0,1))\,, \quad (50a)$$

$$J_2(s^*,k,a) = (1-s^*)^2\, CS(Q(k,0)) + 2s^*(1-s^*)\, CS(Q(k,k)) + s^{*2} CS(Q(k,1))\,, \quad (50b)$$

$$J_4(s^*,k,a) = (1-s^*)^2\, CS(Q(1,0)) + 2s^*(1-s^*)\, CS(Q(1,k)) + s^{*2} CS(Q(1,1))\,. \quad (50c)$$

Accordingly, the expected social surplus at the symmetric equilibrium without licensing is defined as

$$ESS^*(s^*,k,a) = EPS^*(s^*,k,a) + ECS^*(s^*,k,a)\,. \quad (51)$$

From Table 1 and the description following Table 1, the expect profit of firm x at the equilibrium with (cross-) licensing is given by

$$\Pi_x^{WL} \equiv (1-e^{-x_A})(1-e^{-x_B})L_1 + (1-e^{-x_A})e^{-x_B} L_2 + e^{-x_A}(1-e^{-x_B})L_3 + e^{-x_A}e^{-x_A} L_4 - x_A - x_B, \quad (52)$$

where

$$\begin{aligned} L_1 = {} & (1-e^{-y_A})(1-e^{-y_B})\pi_x(0,0) + (1-e^{-y_A})e^{-y_B}[\pi_x(0,0)+F_{\mathrm{III}}] + e^{-y_A}(1-e^{-y_B})[\pi_x(0,0)+F_{\mathrm{III}}] \\ & + e^{-y_A}e^{-y_A}[\pi_x(0,0)+F_{\mathrm{IV}}], \end{aligned} \quad (53a)$$

$$\begin{aligned} L_2 = {} & (1-e^{-y_A})(1-e^{-y_B})[\pi_x(0,0)-F_{\mathrm{III}}] + (1-e^{-y_A})e^{-y_B}\pi_x(k,k) + e^{-y_A}(1-e^{-y_B})\pi_x(0,0) \\ & + e^{-y_A}e^{-y_A}[\pi_x(k,k)+F_{\mathrm{II}}], \end{aligned} \quad (53b)$$

$$\begin{aligned} L_3 = {} & (1-e^{-y_A})(1-e^{-y_B})[\pi_x(0,0)-F_{\mathrm{III}}] + (1-e^{-y_A})e^{-y_B}\pi_x(0,0) + e^{-y_A}(1-e^{-y_B})\pi_x(k,k) \\ & + e^{-y_A}e^{-y_A}[\pi_x(k,k)+F_{\mathrm{II}}], \end{aligned} \quad (53c)$$

$$\begin{aligned} L_4 = {} & (1-e^{-y_A})(1-e^{-y_B})[\pi_x(0,0)-F_{\mathrm{IV}}] + (1-e^{-y_A})e^{-y_B}[\pi_x(k,k)-F_{\mathrm{II}}] \\ & + e^{-y_A}(1-e^{-y_B})[\pi_x(k,k)-F_{\mathrm{II}}] + e^{-y_A}e^{-y_A}\pi_x(1,1). \end{aligned} \quad (53d)$$

Similarly, we can obtain the expect profit of firm *y* at the equilibrium. Setting $\tilde{s} = e^{-\tilde{x}_A} = e^{-\tilde{x}_B} = e^{-\tilde{y}_A} = e^{-\tilde{y}_B}$ $(\tilde{x}_A = \tilde{x}_B = -\ln \tilde{s})$ in these expected profits, and taking into consideration that all license fee terms cancel out at the equilibrium. Then, we can obtain the expected producers' surplus at the symmetric equilibrium with licensing as

$$\begin{aligned} EPS^{WL}(\tilde{s},k,a) &\equiv 2(1-\tilde{s})^2(1+\tilde{s})^2 \cdot \pi_x(0,0) + 4\tilde{s}^2(1-\tilde{s})(1+\tilde{s}) \cdot \pi_x(k,k) + 2\tilde{s}^4 \cdot \pi_x(1,1) + 4\ln\tilde{s} \\ &= 2(1-\tilde{s})^2(1+\tilde{s})^2 \cdot \frac{a^2}{9} + 4\tilde{s}^2(1-\tilde{s})(1+\tilde{s}) \cdot \frac{(a-k)^2}{9} + 2\tilde{s}^4 \cdot \frac{(a-1)^2}{9} + 4\ln\tilde{s} \end{aligned} \tag{54}$$

Similar to $EPS^{WL}(\tilde{s},k,a)$, we also define the expected consumers' surplus as

$$\begin{aligned} ECS^{WL}(\tilde{s},k,a) &\equiv (1-\tilde{s})^2(1+\tilde{s})^2 \cdot CS(Q(0,0)) + 2\tilde{s}^2(1-\tilde{s})(1+\tilde{s}) \cdot CS(Q(k,k)) + \tilde{s}^4 \cdot CS(Q((1,1)) \\ &= (1-\tilde{s})^2(1+\tilde{s})^2 \cdot \frac{2a^2}{9} + 2\tilde{s}^2(1-\tilde{s})(1+\tilde{s}) \cdot \frac{2(a-k)^2}{9} + \tilde{s}^4 \cdot \frac{2(a-1)^2}{9}. \end{aligned} \tag{55}$$

The expected social surplus at the symmetric equilibrium with licensing is defined as

$$ESS^{WL}(\tilde{s},k,a) = EPS^{WL}(\tilde{s},k,a) + ECS^{WL}(\tilde{s},k,a). \tag{56}$$

To compare economic welfare evaluated at the equilibria s^* and $\tilde{s}$ with and without (cross-) licensing, solved numerically in three cases where $a = 15$, $k = 0.3$, where $a = 9.5$, $k = 0.95$ and where $a = 15$, $k = 0.95$), we calculate $EPS^*(s^*,k,a), ECS^*(s^*,k,a), ESS^*(s^*,k,a), EPS^{WL}(\tilde{s},k,a), ECS^{WL}(\tilde{s},k,a)$ and $ESS^{WL}(\tilde{s},k,a)$ in these cases. Denote the variations of the expected producers', consumers' and social surplus by $\Delta PS = EPS^{WL} - EPS^*$, $\Delta CS = ECS^{WL} - ECS^*$ and $\Delta SS = ESS^{WL} - ESS^*$.

(i) Where $a = 15$, $k = 0.3$, the two equilibria are $s^* = 0.35384$ and $\tilde{s} = 0.37943$ (Figure 2). From (47),(49),(51),(54),(56) and (56) we have

$$EPS^*(0.35384,0.3,15) = 44.219 < EPS^{WL}(0.37943,0.3,15) = 45.502, \tag{57a}$$

$$ECS^*(0.35384,0.3,15) = 48.277 < ECS^{WL}(0.37943,0.3,15) = 49.378, \tag{57b}$$

$$ESS^*(0.35384,0.3,15) = 92.496 < ESS^{WL}(0.37943,0.3,15) = 94.88. \tag{57c}$$

$$\Delta PS = 1.283,\quad \Delta CS = 1.101,\quad \Delta SS = 2.384$$

$$\frac{\Delta PS}{\Delta SS} = 0.53817 > \frac{\Delta CS}{\Delta SS} = 0.46183 \tag{58}$$

(ii) Where $a = 9.5$, $k = 0.95$, the two equilibria are $s^* = 0.44885$ and $\tilde{s} = 0.41939$ (Figure 3). From (47), (49), (51), (54), (56) and (56) we have

$$EPS^*(0.44885,0.95,9.5) = 14.335 < EPS^{WL}(0.41939\ ,0.95,9.5) = 15.351\ , \tag{59a}$$

$$ECS^*(0.44885,0.95,9.5) = 17.342 < ECS^{WL}(0.41939,0.95,9.5) = 18.827, \tag{59b}$$

$$ESS^*(0.44885,0.95,9.5) = 31.678 < ESS^{WL}(0.41939\ ,0.95,9.5) = 34.178\,. \tag{59c}$$

$$\Delta PS = 1.016,\quad \Delta CS = 1.485,\quad \Delta SS = 2.5.$$

$$\frac{\Delta PS}{\Delta SS} = 0.4064 < \frac{\Delta CS}{\Delta SS} = 0.594 \tag{60}$$

(iii) Where $a = 15$, $k = 0.95$, the two equilibria are $s^* = 0.20408$ and $\tilde{s} = 0.23493$ (Figure 3). From (47),(49),(51),(54),(56) and (56) we have

$$EPS^*(0.20408,0.95,15) = 41.571 < EPS^{WL}(0.23493,0.95,15) = 43.547, \tag{61a}$$

$$ECS^*(0.20408,0.95,15) = 47.716 < ECS^{WL}(0.23493,0.95,15) = 49.341, \tag{61b}$$

$$ESS^*(0.20408,0.95,15) = 89.287 < ESS^{WL}(0.23493\ ,0.95,15) = 92.888\,. \tag{61c}$$

$$\Delta PS = 1.976,\quad \Delta CS = 1.625,\quad \Delta SS = 3.601$$

$$\frac{\Delta PS}{\Delta SS} = 0.54874 > \frac{\Delta CS}{\Delta SS} = 0.45126 \tag{62}$$

From the above, we see that (cross-) licensing increases the expected producers', consumers' and social surplus at the symmetric equilibrium in all cases. However, the contributions of the improvement of the expected producers' or consumers' surplus to the total improvement of the expected social welfare measured by $\Delta PS/\Delta SS$ or $\Delta CS/\Delta SS$ differ between each case. Remember, here, that (cross-) licensing always occurs in all possible cases in our model. Accordingly, note that (cross-) licensing unties each firm's over-investment in research and development since there is never a spill-over of technologies in our duopoly. If underlying demand is sufficient, large and the two technologies are less or more partially complementary (in cases (i) and (iii)), then licensing discourages technological development and improves producers' welfare more than consumers' (see (58), (62)). Since consumers' surplus increases with (expected) output but output decreases in these cases, it seems to be contradicting the consumers' surplus increase. Note, however, that a cross-licensing system has two opposite effects on outputs. The first is an output-reducing effect associated with a reduction in the probability of success of R&D investment. The second is an output-expanding effect associated with the fact that a cross-licensing contract allows each firm to produce the good if it succeeds in inventing a single different technology. If, however, the underlying demand is so small and the two technologies are nearly completely complementary (in case (ii)), then licensing encourages technological development and improves producers' welfare less than that of consumers (see (60)). These results hold at least within the neighborhood of the numerical solutions s^* and $\tilde{s}$ in each of three cases.

8. Concluding remarks

In this paper, we explored the incentive for R&D investment of duopolistic firms facing technological innovations in nearly completely complementary, partially complementary and less partially complementary technologies by analyzing a simple static innovation model. The first result we obtained was that the effect of cross-licensing of technologies on the incentive for R&D differs from the effect of licensing as a unilateral imposition. The effects due to the difference between cross-licensing and unilateral licensing systems on the incentive for R&D change the relationships among technologies such as the extent of weakly complementarity. Therefore, the above discussion suggests the importance of noticing the relationships among technologies, to analyze how firms determine their R&D under complex technological innovations. The second result is that (cross-) licensing increases the expected producers', consumers' and social surplus at the symmetric equilibria in the three cases where the equilibria are solved numerically. If underlying demand is sufficient, large and two partially complementary technologies are more or less complementary, then licensing discourages the incentive for technological development and improves producers' welfare more than consumers'. If, however, the underlying demand is so small and two technologies are nearly completely complementary, then licensing encourages technological development and improves producers' welfare less than consumers'.

There remain many problems for future research. In this paper, for example, we focus on symmetric equilibrium for tractability. In practice, however, firms *cannot be symmetric* in the industries where complementary technologies are indispensable for the production of goods. In addition, the role of R&D ventures that do not produce products but concentrate on R&D increases its importance in such industries. Second, we cannot derive equilibria analytically in our model setting. Therefore, the analysis of properties of the equilibria investments and welfare are not always sufficient. If we can amend our model and make it more tractable, we can derive clearer results.

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Appendix 1

Lemma 1

For $\forall k \in [0,1/2)$, $\phi_s(s,k,a) > 0, 0 \le s \le 1$, $\phi_{ss}(s,k,a) > 0, 0 \le s \le \frac{1}{2}$.

For $\forall k \in (1/2,\hat{k}]$, $\phi_s(s,k,a) > 0, 0 \le s \le \frac{1}{2}$ and

for $\forall k \in (1/2,1]$ $\phi_{ss}(s,k,a) < 0, 0 \le s \le 1$.

[Proof]

From (21), we easily see that for $k \in [0,1/2)$, $s \le \frac{3k}{4(2k-1)} \le 0 \Leftrightarrow \phi_{sss}(s,k,a) \le 0$ and $\frac{3k}{4(2k-1)} < s \Leftrightarrow \phi_{sss}(s,k,a) > 0$.

We also know form (20) that $\phi_{ss}(0,k,a) = \frac{8}{9}(1-2k)(a-1-2k) > 0$ and $\phi_{ss}(1,k,a) = -\frac{8}{9}(1-2k)(5(k-1)-a) > 0.$ Hence, we have $\phi_{ss}(s,k,a) > 0, 0 \le s \le 1$. Taking this into consideration with that $\phi_s(0,k,a) \ge 0,$ $\phi_s(\frac{1}{2},k,a) > 0$ from (17), (18) and (19), we can conclude that a $\phi_s(s,k,a) > 0$ for $\forall k \in [0,1/2), \forall s \in [0,\frac{1}{2}]$. For $\forall k \in (1/2,1]$, however, from (21) we see that

$$0 \le s \le \frac{3k}{4(2k-1)} \le 1,\ \frac{5}{4} \le k \le 1 \quad or \quad 0 \le s \le 1 < \frac{3k}{4(2k-1)}, \frac{1}{2} < k < \frac{4}{5} \Leftrightarrow \phi_{sss}(s,k,a) \le 0$$ and

$(0 < \frac{3k}{4(2k-1)}) < s \le 1, \frac{4}{5} < k \le 1 \Leftrightarrow \phi_{sss}(s,k,a) > 0.$ So $\phi_{ss}(s,k,a)$ has a minimal value $\phi_{ss}(\frac{3k}{4(2k-1)},k,a) = \frac{1}{18}(5k^2 - 16ak + 8(a-1))$ at $s = \frac{3k}{4(2k-1)}$ for $0 \le s \le 1$. The two real roots of the quadratic equation $5k^2 - 16ak + 8(a-1) = 0$ are given by $\widetilde{k}_1 = \frac{8a - 2\sqrt{2(8a^2 - 5a - 5)}}{5}$ and $\widetilde{k}_2 = \frac{8a + \sqrt{2(8a^2 - 5a - 5)}}{5}$. From the assumption (13), we can show that $0 < \widetilde{k}_1 < \frac{1}{2}$ and $1 < \widetilde{k}_2$. So we see that for $\forall k \in (1/2,1]$,

$\phi_{ss}(\frac{3k}{4(2k-1)},k,a)=\frac{1}{18}(5k^2-16ak+8(a-1))<0$. Furthermore, from (13) and (20) we see that $\phi_{ss}(0,k,a)=\frac{4}{9}(1-2k)(a-1-2k)<0$ and $\phi_{ss}(1,k,a)=-\frac{8}{9}(1-2k)(5(k-1)-a)<0$ for $\forall k\in(1/2,1]$. Hence we have that $\phi_{ss}(s,k,a)<0,\quad 0\le s\le 1$. Finally, from (18), (19), the fact that $\forall k\in(1/2,\hat{k}]\subset(1/2,1]$ and $\phi_{ss}(s,k,a)<0,\quad 0\le s\le 1$, we can conclude that

$$\phi_s(s,k,a)>0 \text{ for } \forall k\in(1/2,1], \forall s\in[0,\frac{1}{2}]. \qquad \text{(Q.E.D.)}$$

[Proof of Proposition 1]

Suppose that $k\neq 1/2$. Divide both sides of the first order condition (12) by N_1, and define

$$\begin{aligned} G(s)\equiv\frac{\phi(s,k,a)}{N_1}&=s^4+\frac{N_2}{N_1}s^3+\frac{N_3}{N_1}s^2+\frac{N_4}{N_1}s-\frac{1}{N_1}\\ &=s^4+\frac{3k}{(1-2k)}s^3-\frac{(2k-a+1)}{(1-2k)}s^2+\frac{k(a-k)}{(1-2k)^2}s-\frac{9}{4(1-2k)^2}=0. \end{aligned} \tag{A1}$$

Define by α,β,γ and $\delta(\alpha<\beta<\gamma<\delta)$, the biquadrate equation $G(s)$ of s, we see that

$$\begin{aligned} (s-\alpha)(s-\beta)(s-\gamma)(s-\delta)&=s^4-(\alpha+\beta+\gamma+\delta)s^3+(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)s^2\\ &\quad-(\alpha\gamma\delta+\beta\gamma\delta+\alpha\beta\gamma+\alpha\beta\delta)s+\alpha\beta\gamma\delta\\ &=0. \end{aligned} \tag{A2}$$

Comparing the coefficients of each terms in (A1) with the correspondence coefficients in (A2), we have

$$\alpha+\beta+\gamma+\delta=-\frac{3k}{(1-2k)} \tag{A3}$$

and

$$\alpha\beta\gamma\delta=-\frac{9}{4(1-2k)^2}<0,\quad k\neq 1/2. \tag{A4}$$

From (A4), if (A2) has four real roots, then three roots have the same sign and the other one has the opposite sign. Furthermore, if (A2) has two real roots and two imaginary roots, then the two real ones have the opposite sign each other form (A4). Threfore, from (A3) and

(A4) all the cases we should examine are

(i) the case where (A2) has four real roots, α, β, γ and δ such that $\alpha < 0 < \beta < \gamma < \delta$,

(ii) the case where (A2) has four real roots, α, β, γ and δ such that $\alpha < \beta < \gamma < 0 < \delta$, and

(iii) the case where (A2) has one positive real root and one negative real root, and two imaginary roots.

From (A3), the case (i) may occur for $\forall k \in (1/2, 1]$. From the monotonicity of $\phi(s,k,a)$ presented in Lemma 1 and the assumption that $a \geq \frac{39}{4} - \frac{k}{2}$, there exists a unique solution $s^* \in [0,1/2)$ of $\phi(s,k,a) = 0$, for $\forall k \in (1/2, \hat{k}]$. For example, see Figure A-1, in which we set $a = 11, k = 0.55 < \hat{k} = 4 - \frac{2}{5}\sqrt{70} \cong 0.65336$. For $\forall k \in (\hat{k}, 1]$, there exists at least one solution $s^* \in [0,1/2)$ of $\phi(s,k,a) = 0$ and at most two solutions in $s \in [0,1]$ from the concavity of $\phi(s,k,a)$ proved in Lemma 1 and the assumption that $a \geq \frac{39}{4} - \frac{k}{2}$. For example, see Figure A-2, in which we set $a = 11, k = 0.8 > \hat{k} = 4 - \frac{2}{5}\sqrt{70} \cong 0.65336$.

From (A3), the case (ii) and (iii) may occur for $\forall k \in [0,1/2)$. From the monotonicity of $\phi(s,k,a)$ in s presented in Lemma 1 and the assumption that $a \geq \frac{39}{4} - \frac{k}{2}$, there exists a unique solution $s^* \in (0,1/2)$ of $\phi(s,k,a) = 0$, for $\forall k \in [0,1/2)$. For example, see Figure A-3, in which we set $a = 11, k = 0.3 < \hat{k} = 4 - \frac{2}{5}\sqrt{70} \cong 0.65336$.

Now, we prove the comparative statistics results on $s^* \in (0,1/2)$.

From Lemma 1 and $s^* \in (0,1/2)$, $\phi_s(s^*,k,a) > 0$. Here, the first order condition (12) can be rewritten

$$\begin{aligned}\phi(s,k,a) &= N_1 s^4 + N_2 s^3 + N_3 s^2 + N^4 s - 1 \\ &= \frac{4}{9}(1-2k)^2 s^4 + \frac{4}{3}k(1-2k)s^3 + \frac{4}{9}(2k-1)(2k-a+1)s^2 + \frac{4}{9}k(a-k)s - 1 \\ &= h(s,k) + i(s,k)a = 0,\end{aligned}$$

where

$$h(s,k)=\frac{4}{9}(1-2k)^2 s^4+\frac{4}{3}k(1-2k)s^3+\frac{4}{9}(4k^2-1)s^2-\frac{4}{9}k^2 s-1 \qquad and$$

$$i(s,k)=\frac{4}{9}(1-2k)s^2+\frac{4}{9}ks$$

So, if $h(s^*,k)<0$, then we have $\phi_a(s^*,k,a)>0$. However, we now that $s^*\in(0,1/2)$ and $0\le k\le 1, k\ne 1/2$. So we can show that

$\frac{4}{9}(1-2k)^2(s^*)^4\le\frac{4}{9}\cdot\frac{1}{16}=\frac{1}{36}$, $\frac{4}{3}(1-2k)(s^*)^3\le\frac{4}{3}\cdot\frac{1}{8}=\frac{1}{6}$, $\frac{4}{9}(4k^2-1)(s^*)^2\le\frac{4}{9}\cdot 3\cdot\frac{1}{4}=\frac{1}{3}$ and $-\frac{4}{9}k^2 s^*\le 0$. So $h(s^*,k)\le\frac{1}{36}+\frac{1}{6}+\frac{1}{3}+0-1=-\frac{17}{36}<0$, and $\phi_a(s^*,k,a)>0$. Hence, by implicite function theorem, we have $\frac{ds^*}{da}=-\frac{\phi_a(s^*,k,a)}{\phi_s(s^*,k,a)}<0$.

Next, we show that $\phi_k(s,k,a)>0, \quad \forall s\in(0,\frac{1}{2}], 0\le k\le 1, k\ne 1/2$. From (12), we have

$$\begin{aligned}\phi_k(s,k,a)&=s\cdot\{\frac{16}{9}(2k-1)s^3-\frac{4}{3}(4k-1)s^2+\frac{8}{9}(4k-a)s+\frac{4}{9}(a-2k)\}\\&\equiv s\cdot\nu(s,k,a).\end{aligned}$$

For $0\le k<\frac{1}{2}$, we show $\nu(s,k,a)>0,\forall s\in[0,\frac{1}{2}]$. One hand, we have

$\nu_s(s,k,a)=\frac{8}{9}\{6(2k-1)s^2-3(4k-1)s+4k-a\}<\frac{8}{9}\{-3(4k-1)s+4k-a\}$, for $0\le k<\frac{1}{2}$. We have $\forall s\in[0,\frac{1}{2}]$

$\frac{8}{9}\{-3(4k-1)s+4k-a\}<\frac{8}{9}\{-3(4k-1)\frac{1}{2}+4k-a\} \quad =-2k+\frac{3}{2}-a<0, a\ge\frac{39}{4}-\frac{k}{2}$ for $0\le k<\frac{1}{4}$, and $\frac{8}{9}\{-3(4k-1)s+4k-a\}<\frac{8}{9}\{4k-a\}<0$ for $\frac{1}{4}\le k<\frac{1}{2}, a\ge\frac{39}{4}-\frac{k}{2}$.

Hence $\nu_s(s,k,a)<\frac{8}{9}\{-3(4k-1)s+4k-a\}<0$, for $0\le k<\frac{1}{2},\forall s\in[0,\frac{1}{2}]$. So, we can conclude that $\nu_s(s,k,a)<0, \quad \forall s\in[0,\frac{1}{2}]$, for $0\le k<1/2$. However, we know that $\nu(0,k,a)=\frac{4}{9}(a-2k)>0, \quad \nu(\frac{1}{2},k,a)=\frac{1}{9}>0 \qquad a\ge\frac{39}{4}-\frac{k}{2}$. Hence, $\nu(s,k,a)>0, \phi_k(s,k,a)=s\cdot\nu(s,k,a)>0 \quad \forall s\in[0,\frac{1}{2}], 0\le k<\frac{1}{2}$.

On the other hand, for $\frac{1}{2} < k \le 1$, we see that $\phi_k(s,k,a)$ is non-increasing in k, since $\phi_{kk}(s,k,a) = \frac{\partial \phi_k}{\partial k} = \frac{8}{9}s(2s-1)(2s^2-2s+1) \le 0, \forall s \in [0,\frac{1}{2}].$ We have

$\phi_k(s,\frac{1}{2},a) = \frac{4}{9}s\{-3s^2+2(2-a)s+(a-1)\}$. The two solutions of the quadratic equation of s, $-3s^2+2(2-a)s+(a-1)=0$ are $s^0(a) = \frac{2-a-\sqrt{a^2-a+1}}{3}, s^1(a) = \frac{2-a+\sqrt{a^2-a+1}}{3}$, and we see that $s^0(\frac{39}{4}) = -\frac{31}{12} - \frac{\sqrt{1381}}{12} \cong -5.6802 < 0, s^1(\frac{39}{4}) = -\frac{31}{12} - \frac{\sqrt{1381}}{12} \cong 0.51348 > 0,$ $\lim_{a\to\infty} s^1(a) = \frac{1}{2}.$ The derivatives of the two solutions w.r.t. a are $\frac{ds^0(a)}{da} = -\frac{2\sqrt{a^2-a+1}-1+2a}{6\sqrt{a^2-a+1}} < 0, \frac{ds^1(a)}{da} = -\frac{2\sqrt{a^2-a+1}+1-2a}{6\sqrt{a^2-a+1}} < 0.$ Hence, we see that $-3s^2+2(2-a)s+(a-1) > 0, \forall s \in [0,\frac{1}{2}], \frac{39}{4}-\frac{k}{2}, \frac{1}{2} < k \le 1$ and

$\phi_k(s,\frac{1}{2},a) = \frac{4}{9}s\{-3s^2+2(2-a)s+(a-1)\} \ge 0, \forall s \in [0,\frac{1}{2}]$ for $0 \le k < 1/2$. Next, we examine the sign of $\phi_k(s,1,a) = s\{\frac{16}{9}s^3 - 4s^2 + \frac{8}{9}(4-a)s + \frac{4}{9}a - \frac{8}{9})\}$. From this formula we have $\phi_{kss}(s,1,a) = \frac{\partial^2 \phi_k}{\partial s^2} = \frac{64}{3}s^2 - 24s + \frac{64}{9} - \frac{16}{9}a$. The two solutions of the quadratic equation of of s $\phi_{kss} = 0$ are $\hat{s}^0(a) = \frac{9}{16} - \frac{1}{48}\sqrt{192a-39}, \hat{s}^1(a) = \frac{9}{16} - \frac{1}{48}\sqrt{192a-39}$, and we see that $\hat{s}^0(\frac{39}{4}) = \frac{9}{16} - \frac{\sqrt{1833}}{48} \cong -0.32945 < 0, \hat{s}^1(\frac{39}{4}) = \frac{9}{16} - \frac{\sqrt{1833}}{48} \cong 1.4544 > 0.$ The derivatives of the two solutions w.r.t. a are $\frac{d\hat{s}^0(a)}{da} = -\frac{2}{\sqrt{192a-39}} < 0, \frac{d\hat{s}^1(a)}{da} = \frac{2}{\sqrt{192a-39}} > 0.$ Hence, we see that $\phi_{kss}(s,1,a) = \frac{\partial^2 \phi_k}{\partial s^2} = \frac{64}{3}s^2 - 24s + \frac{64}{9} - \frac{16}{9}a < 0, \forall s \in [0,\frac{1}{2}] \subset [0,1],$ which implies $\phi_k(s,1,a)$ is concave in s. We also see that $\phi_k(0,1,a) = 0,$ $\phi_k(\frac{1}{2},1,a) = \frac{1}{18} > 0.$ Thus, we show that for $\phi_k(s,1,a) \ge 0, \forall s \in [0,\frac{1}{2}]$ $1/2 < k \le 1$..

However, we know that $v(0,k,a)=\frac{4}{9}(a-2k)>0$, $v(1,k,a)=\frac{4}{9}(a+2k)-\frac{4}{3}>0$, for $a\geq\frac{39}{4}, 0\leq k<1/2$. Consequently, $v(s,k,a)\geq 0$, $\forall s\in[0,1], 0\leq k<1/2$. Hence, we can conclude that $\phi_k(s,k,a)=s\cdot v(s,k,a)>0$, $s\in(0,\frac{1}{2}]$ and $0\leq k\leq 1, k\neq 1/2$. By implicit function theorem, we have $\frac{ds^*}{dk}=-\frac{\phi_k(s^*,k,a)}{\phi_s(s^*,k,a)}<0$, and the result follows.

(Q.E.D)

Appendix 2

In this Appendix, we derive the licensing fees and the conditions for that each licensing occurs in the four types of cases presented in section 4. For in the type Ⅳ cases , we also show that the unilateral licensing strategy of both of the two technologies dominates that of only one technology for the licenser firm.

Ⅰ. In the cases included in this category, the Nash bargaining function B_{I} is given by,

$$B_{\mathrm{I}}=[\pi_x(0,0)+F_{\mathrm{I}}-\pi_x(k,k)][\pi_y(0,0)-F_{\mathrm{I}}-\pi_y(k,k)]. \tag{A2.1}$$

Since the licensing fee F_{I} is determined so as to maximize (A2.1), we have

$$\begin{aligned}\frac{dB_{\mathrm{I}}}{dF_{\mathrm{I}}}&=\pi_y(0,0)-F_{\mathrm{I}}-\pi_y(k,k)-\{\pi_x(0,0)+F_{\mathrm{I}}-\pi_x(k,k)\}\\&=-2F_{\mathrm{I}}=0.\end{aligned} \tag{A2.2}$$

Then we obtain

$$F_{\mathrm{I}}=0. \tag{A2.3}$$

Each firm must have a positive gain from this licensing:

$$\pi_x(0,0)+F_{\mathrm{I}}-\pi_x(k,k)=\frac{a^2}{9}-\frac{(a-k)^2}{9}=\frac{k(2a-k)}{9}\geq 0,$$

which is the condition for this cross-licensing.

Ⅱ. In the cases included in this category, the Nash bargaining function B_{II} is given by,

$$B_{\mathrm{II}} = [\pi_x(k,k) + F_{\mathrm{II}} - \pi_x(k,1)][\pi_y(k,k) - F_{\mathrm{II}} - \pi_y(1,k)]. \qquad \text{(A2.4)}$$

Since the licensing fee F_{II} is determined so as to maximize (A4), we have

$$\frac{dB_{\mathrm{II}}}{dF_{\mathrm{II}}} = 6\{\pi_x(k,k) + F_{\mathrm{II}} - \pi_y(k,1)\}$$

$$= -2F_{\mathrm{II}} + \frac{(a-2k+1)^2}{9} - \frac{(a-2+k)^2}{9} = 0. \qquad \text{(A2.5)}$$

Then we obtain

$$F_{\mathrm{II}} = \frac{1}{2}\left[\frac{(a-2k+1)^2}{9} - \frac{(a-2+k)^2}{9}\right] = \frac{(1-k)(2a-k-1)}{6} \geq 0. \qquad \text{(A2.6)}$$

Each firm must have a positive gain from this licensing:

$$\pi_x(k,k) + F_{\mathrm{II}} - \pi_y(k,1) = \frac{(a-k)^2}{9} - \frac{(a-2k+1)^2}{9} + \frac{(1-k)(2a-k-1)}{6} =$$

$$= \frac{(2a-k-1)(k-1)}{18} \geq 0,$$

which is the condition for this licensing.

Ⅲ. In the cases included in this category, the Nash bargaining function B_{III} is given by,

$$B_{\mathrm{III}} = [\pi_x(0,0) + F_{\mathrm{III}} - \pi_x(0,k)][\pi_y(0,0) - F_{\mathrm{III}} - \pi_y(k,0)]. \qquad \text{(A2.7)}$$

Since the licensing fee F_{III} is determined so as to maximize (A2.7), we have

$$\frac{dB_{\mathrm{III}}}{dF_{\mathrm{III}}} = \{-2F_{\mathrm{III}} + \pi_x(0,k) + F_{\mathrm{III}} - \pi_y(k,0)\}$$

$$= 0. \qquad \text{(A2.8)}$$

Then we obtain

$$F_{\mathrm{III}} = \frac{1}{2}\left[\frac{(a+k)^2}{9} - \frac{(a-2k)^2}{9}\right] = \frac{k(2a-k)}{6} \geq 0. \qquad \text{(A2.9)}$$

Each firm must have a positive gain from this licensing:

$$\pi_x(0,0)+F_{\mathrm{III}}-\pi_y(0,k)=\frac{a^2}{9}-\frac{(a+k)^2}{9}+\frac{k(2a-k)}{6}= \; =\frac{k(2a-5k)}{18}\geq 0,$$

which is the condition for this licensing.

Ⅳ. In the cases included in this category, there are two type of cases in which the unilateral licensing occurs. In one type of the cases, the unilateral licensing of only one technology occurs. In the other type of the cases, the unilateral licensing of both of the two technologies occurs. In the first, we derive the licensing fee in which the unilateral licensing of only one technology occurs.

Ⅳ-1. In the cases included in this category, the Nash bargaining function B^1_{IV} is given by,

$$B^1_{\mathrm{IV}}=[\pi_x(0,k)+F^1_{\mathrm{IV}}-\pi_x(0,1)][\pi_y(k,0)-F^1_{\mathrm{IV}}-\pi_y(1,0)]. \tag{A2.10}$$

Since the licensing fee F^1_{IV} is determined so as to maximize (A10), we have

$$\frac{dB^1_{\mathrm{IV}}}{dF^1_{\mathrm{IV}}}=\{-2F^1_{\mathrm{IV}}-\pi_x(0,k)+F^1_{\mathrm{IV}}+\pi_x(0,1)+\pi_y(k,0)-\pi_x(0,1)-\pi_y(1,0)\}$$

$$=0. \tag{A2.11}$$

Then we obtain

$$F^1_{\mathrm{IV}}=\frac{(k-1)(k-(2a-1))}{6}\geq 0. \tag{A2.12}$$

Each firm must have a positive gain from this licensing:

$$\pi_x(0,k)+F^1_{\mathrm{IV}}-\pi_y(0,1)=\frac{(a+k)^2}{9}-\frac{(a+1)^2}{9}+\frac{(k-1)(k-(2a-1))}{6}=$$

$$=\frac{(k-1)(5k-2a+5)}{18}\geq 0, \tag{A2.13}$$

which is the condition for this licensing. Next, we derive the licensing fee in which the unilateral licensing of both of the two technologies occurs.

Ⅳ-2 .In the cases included in this category, the Nash bargaining function B^2_{IV} is given by,

$$B^2_{\mathrm{IV}} = [\,\pi_x(0,0) + F^2_{\mathrm{IV}} - \pi_x(0,1)\,]\,[\,\pi_y(0,0) - F^2_{\mathrm{IV}} - \pi_y(1,0)\,]. \qquad \text{(A2.14)}$$

Since the licensing fee F^2_{IV} is determined so as to maximize (A10), we have

$$\frac{dB^2_{\mathrm{IV}}}{dF^2_{\mathrm{IV}}} = -2F^2_{\mathrm{IV}} + \pi_x(0,1) - \pi_y(1,0) = 0 \qquad \text{(A2.15)}$$

Then we obtain

$$F^2_{\mathrm{IV}} = \frac{(2a-1)}{6} > 0. \qquad \text{(A2.16)}$$

Each firm must have a positive gain from this licensing:

$$\pi_x(0,0) + F^2_{\mathrm{IV}} - \pi_y(0,1) = \frac{a^2}{9} - \frac{(a+1)^2}{9} + \frac{2a-1}{6} = \frac{2a-5}{18} > 0, \qquad \text{(A2.17)}$$

which is the condition for this licensing.

However, from (A2.13) and (A2.17), we have

$$\pi_x(0,0) + F^2_{\mathrm{IV}} - \pi_y(0,1) - [\pi_x(0,k) + F^1_{\mathrm{IV}} - \pi_y(0,1)]$$

$$= \frac{2a-5}{18} - \frac{(k-1)(5k-2a+5)}{18} = \frac{1}{18}(-5k^2 + 2a) > 0, \because a \geq \frac{39}{4} - \frac{k}{2}, 0 \leq k \leq 1.$$

Consequently, we show in the above that the unilateral licensing strategy of both of the two technologies dominates that of only one technology for the licenser firm.

Appendix 3

In this Appendix, we derive the first order condition for firm i w.r.t. its own R&D level in the case with a (cross) licensing contract at first. Then, we also present two lemmas and the proofs of four lemmas, also present the proof of the proposition in section 5.

Derivation of $\Omega(s,k,a)$

Now from (11) and (12) in section 3, we know that the first term of r.h.s. in (39) is

$$\left.\frac{\partial \Pi}{\partial x_A}\right|_{s=e^{-x_A}=e^{-x_B}=e^{-y_A}=e^{\backslash y_B}} = \phi(s,k,a).$$ From (38), we also see that the second term of r.h.s.

in (39) yields

$$M(s,k,a) \equiv \left.\frac{\partial \Pi^{CL}}{\partial x_A}\right|_{s=e^{-x_A}=e^{-x_B}=e^{-y_A}=e^{\backslash y_B}} = \frac{1}{9}s^2(2s-1)(1-s)h. \qquad \text{(A3.1)}$$

For any $s < 1/2$, we see that $M(s,k,a) < 0$.

We set the last three terms in the l.h.s. of (39) by $N(s,k,a)$,

$$\begin{aligned} N(s,k,a) = {} & s(1-s)[A(n_7 + F_{\mathrm{III}}) + B(n_8 + F_{\mathrm{IV}})] + s(2s-1)[B(n_9 + F_{\mathrm{II}}) + C(n_1 - F_{\mathrm{III}})] \\ & - s^2[C(n_1 + n_4 - F_{\mathrm{IV}}) + A(n_5 - F_{\mathrm{II}})], \end{aligned}$$

(A3.2),

where $n_1 = \pi_i(0,0) - \pi_i(k,0) = \frac{4}{9}k(a-k)$, $\quad n_2 = \pi_i(0,k) - \pi_i(k,k) = \frac{4}{9}ka$,

$n_3 = \pi_i(0,1) - \pi_i(k,1) = \frac{4}{9}k(a-k+1)$, $\quad n_4 = \pi_i(k,0) - \pi_i(1,0) = \frac{4}{9}(1-k)(a-k-1)$,

$n_5 = \pi_i(k,k) - \pi_i(1,k) = \frac{4}{9}(1-k)(a-1)$, $\quad n_6 = \pi_i(k,1) - \pi_i(1,1) = \frac{4}{9}(1-k)(a-k)$

$n_7 = \pi_i(0,0) - \pi_i(0,k) = \frac{-k(2a+k)}{9}$, $\quad n_8 = \pi_i(0,0) - \pi_i(0,1) = \frac{-2a-1}{9}$

$n_9 = \pi_i(k,k) - \pi_i(k,1) = \frac{(k-1)(2a-3k+1)}{9}$.

From (23), (25), (27) and the above n_i s($i = 1,2,\cdots,9$), we see that

$$F_{\mathrm{II}} = \frac{1}{2}(n_5 - n_9), \quad F_{\mathrm{III}} = \frac{1}{2}(n_1 - n_7), \quad F_{\mathrm{IV}} = \frac{1}{2}(n_1 + n_4 - n_8). \qquad \text{(A3.3)}$$

From the symmetry of the equilibrium and (37), we see that

$$A = 2s(1-s), \quad B = s^2, \quad C = (1-s)^2. \qquad \text{(A3.4)}$$

Substituting (A3.3) and (A3.4) into (A3.2) and rearranging yields:

$$\begin{aligned} N(s) = \frac{1}{18}s[& 2(-16k^2 + 16k + 2a - 5)s^3 + 6k(9k - 2(a+2))s^2 + (-30k^2 + 12ak - 2a + 5)s \\ & + k(5k - 2a)] \end{aligned} \qquad \text{(A3.}$$

5)

From (39), (A3.1) and (A3.5), we obtain final form of the profit-maximization condition (40), i.e. $\Omega(s,k,a)=0$.

Lemma 2

Suppose that $a>\max\{\frac{39}{4}-\frac{k}{2},\frac{2k^2+151}{4k+14}\}$. Then we have $\Omega_{ssk}(s,k,a)<0$, for $\forall s\in[0,1],\forall k\in(0,1]$. If $0\le k\le\frac{1}{2}$, then $\Omega_{ss}(s,k,a)>0$, for $s\in[0,\frac{1}{2}]$. If $\frac{1}{2}<k\le 1$, then $\Omega_{sss}(s,k,a)<0$, for $s\in(0,\frac{1}{2}]$. We have $\Omega_{s}(s,k,a)>0\ \forall s\in(0,\frac{1}{2}],\forall k\in(0,1]$.

[Proof]

At first, we show that

$\Omega_{ssk}(s,k,a)<0, 0\le s\le 1, 0\le k\le 1$ for $a>\max\{\frac{39}{4}-\frac{k}{2},\frac{2k^2+151}{4k+14}\}$.

At first, define $f(k)\equiv\frac{39}{4}-\frac{k}{2}$ and $g(k)=\frac{2k^2+151}{4k+14}$. Then we can show that $f(0)=\frac{39}{4}<g(0)=\frac{151}{14}$, $f(1)=\frac{37}{4}>g(1)=\frac{153}{18}=\frac{17}{2}=8.5$,

$f'(k)=-\frac{1}{2}<0$ and $g'(k)=\frac{2k^2+14k-151}{(2k+7)^2}<0$ for $\forall k\ni[0,1]$. Solving $\frac{39}{4}-\frac{k}{2}=\frac{2k^2+151}{4k+14}$ w.r.t. k, we obtain $k=4-\frac{3}{4}\sqrt{22},4+\frac{3}{4}\sqrt{22}$. From this, we see that

$$g(k)=\frac{2k^2+151}{4k+14}\ge(<)f(k)=\frac{39}{4}-\frac{k}{2}\Leftrightarrow\quad k\le(>)4-\frac{3}{4}\sqrt{22}\cong 0.48219.$$

From (40) and the above discussion, for $a>\max\{\frac{39}{4}-\frac{k}{2},\frac{2k^2+151}{4k+14}\}>\frac{37}{4}$ we have,

$$\Omega_{ssk}(s,k,a)=\frac{\partial}{\partial k}(\frac{\partial^2\Omega}{\partial s^2})=\frac{8}{9}(6s^2+1)(k-a)<0, 0\le s\le 1, 0\le k\le 1. \quad (A3.6)$$

From (40), we also have

$$\Omega_{ss}(s,k,a)=\frac{4}{3}(2k^2-4ak+2a-1)s^2+\frac{1}{9}(4k^2-8ak+6a-3). \qquad \text{(A3.7)}$$

Substituting $k=1/2$ into (A.3.7), we can show that for $a>\max\{\frac{39}{4}-\frac{1}{4},\frac{1/2+151}{1/2+14}\}=\frac{1/2+151}{1/2+14}=\frac{303}{29}\cong 10.448$

$$\Omega_{ss}(s,\frac{1}{2},a)=\frac{2}{9}(a-3s^2+1)>0, \quad \forall s\in[0,1]. \qquad \text{(A3.8)}$$

Setting $k=0$ in the assumption (41), we have $a>\max\{\frac{39}{4},\frac{151}{14}\}=\frac{151}{14}$.

Then, we can show that

$$\Omega_{ss}(s,0,a)=\frac{1}{3}(2a-1)(4s^2+1)>0, \quad \forall s\in[0,1].$$

Considering together (A3.6), (A3.8) and $\Omega_{ss}(s,0,a)>0, \forall s\in[0,1]$, we can conclude that $\Omega_{ss}(s,k,a)>0, \forall s\in[0,\frac{1}{2}], \forall k\in[0,\frac{1}{2}]$. Next we show that if $\frac{1}{2}<k\leq 1$, then for $s\in[0,\frac{1}{2})$, $\Omega_{ss}(s,k,a)>0$. Next let us show that $\Omega_{sss}(s,k,a)<0, \forall k\in(\frac{1}{2},1], \forall s\in(0,1]$. Differentiating (A3.7) partially by s, we obtain

$$\Omega_{sss}(s,k,a)=\frac{8}{3}(2k^2-4ak+2a-1)s. \qquad \text{(A3.9)}$$

Since we can express the part within the parentheses coefficient of s in (A3.9) as

$$(2k^2-4ak+2a-1)=(k-(a-1/2\sqrt{2(2a^2-2a+1)})(k-(a+1/2\sqrt{2(2a^2-2a+1)}),$$

we can see that

$$\Omega_{sss}(s,k,a)<0, \quad a-1/2\sqrt{2(2a^2-2a+1)}, \quad a+1/2\sqrt{2(2a^2-2a+1)}.$$

Remember that

$\frac{2k^2+151}{4k+14}<\frac{39}{4}-\frac{k}{2} \Leftrightarrow 1\geq k\geq\frac{1}{2}(>4-\frac{3}{4}\sqrt{22}\cong 0.48219)$ and

$g(k)=\dfrac{2k^2+151}{4k+14}$ is decreasing in k. So we evaluate $a-1/2\sqrt{2(2a^2-2a+1)}$, $a+1/2\sqrt{2(2a^2-2a+1)}$ at $a=\dfrac{2+151}{4+14}=\dfrac{17}{2}$ yields

$$\frac{17-\sqrt{257}}{2}\cong 0.48439,\frac{17+\sqrt{257}}{2}\cong 16.516\text{, respectively and}$$

$$\frac{d}{da}(a-1/2\sqrt{2(2a^2-2a+1)})=\frac{\sqrt{2}(1-2a)+2\sqrt{2a^2-2a+1}}{2\sqrt{2a^2-2a+1}}>0,$$

$$\lim_{a\to\infty}(a-1/2\sqrt{2(2a^2-2a+1)})=\frac{1}{2}\text{ and}$$

$$\frac{d}{da}(a+1/2\sqrt{2(2a^2-2a+1)})=\frac{\sqrt{2}(2a-1)+2\sqrt{2a^2-2a+1}}{2\sqrt{2a^2-2a+1}}>0.$$

Since we see that

$$0.48439<a-1/2\sqrt{2('2a^2-2a+1)}<\frac{1}{2}<k<1<a+1/2\sqrt{2('2a^2-2a+1)},$$

we can conclude that

$$2k^2-4ak+2a-1<0 \text{ and } \Omega_{sss}(s,k,a)<0,\forall k\in(\frac{1}{2},1],\forall s\in(0,1]. \quad \text{(A3.10)}$$

Finally, we show that $\Omega_s(s,k,a)>0,\forall s\in[0,\frac{1}{2}],\forall k\in(0,1]$. From (40), we have

$$\Omega_s(s,k,a)=\frac{4}{9}(2k^2-4ak+2a-4)s^3+\frac{1}{9}(4k^2-8ak+6a-3)s++\frac{1}{6}k(2a-k) \quad \text{(A3.11)}$$

From (A3.11), we have

$\Omega_s(0,k,a)=\frac{1}{6}k(2a-k)>0,\forall k\in(0,\frac{1}{2}]$. From this and the fact that $\Omega_{ss}(s,k,a)>0,\forall s\in[0,\frac{1}{2}],\forall k\in[0,\frac{1}{2}]$, $\Omega_s(s,k,a)>0,\forall s\in[0,\frac{1}{2}],\forall k\in(0,\frac{1}{2}]$.

At one hand, for $\forall k\in(\frac{1}{2},1]$, $\Omega_{sss}(s,k,a)<0,\forall k\in(\frac{1}{2},1],\forall s\in(0,1]$ from (A3.10). Hence $\Omega_s(s,k,a)$ is concave in $s(\in(0,1].)$ $\forall k\in(\frac{1}{2},1]$, On the other hand, we have

$$\Omega_s(\frac{1}{2},k,a)=\frac{1}{18}(3k^2-6ak+8a-4)$$

The two solutions of the quadratic equation of *s*, $3k^2-6ak+8a-4=0$ are given by

$$a-1/3\sqrt{3(3a^2-8a+4)},\quad a+1/3\sqrt{3(3a^2-8a+4)},$$

where $3a^2-8a+4=(3a-2)(a-2)>0, a\geq(2+151)/(4+14)=17/2$.

Evaluating at $a=17/2$

$$a-1/3\sqrt{3('3a^2-8a+4)},\quad a+1/3\sqrt{3('3a^2-8a+4)}$$

yields

$$\frac{51-\sqrt{1833}}{6}\cong 1.3644, \frac{51+\sqrt{1833}}{6}\cong 15.636.$$

We see that $\frac{d}{da}(a-1/3\sqrt{3(3a^2-8a+4)})=\frac{3\sqrt{3a^2-8a+4}-\sqrt{3}(3a-4)}{3\sqrt{3a^2-8a+4}}<0,$

$$\lim_{a\to\infty}(a-3\sqrt{3(3a^2-8a+4)})=\frac{4}{3}\text{ and}$$

$$\frac{d}{da}(a+1/3\sqrt{3(3a^2-8a+4)})=\frac{3\sqrt{3a^2-8a+4}+\sqrt{3}(3a-4)}{3\sqrt{3a^2-8a+4}}>0.$$

Hence,

$3k^2-6ak+8a-4>0, \forall k\in[0,1], a\geq 8.5,$

$\Omega_s(\frac{1}{2},k,a)=\frac{1}{18}(3k^2-6ak+8a-4)>0, \forall k\in[0,1], a\geq 8.5$. From this

$\Omega_s(0,k,a)=\frac{1}{6}k(2a-k)>0, \forall k\in[0,1],$ and concavity of $\Omega_s(s,k,a)$ w.r.t. *s*, $\Omega_s(s,k,a)>0, \forall s\in[0,\frac{1}{2}], \forall k\in(\frac{1}{2},1]$. In consequent, we can conclude that $\Omega_s(s,k,a)>0, \forall s\in(0,\frac{1}{2}], \forall k\in[0,1]. \Omega_s(s,k,a)$. (A3.12)

(Q.E.D)

Lemma 3

$$\frac{\partial\Omega(s,k,a)}{\partial k}>0,\ \frac{\partial\Omega(s,k,a)}{\partial a}>0,\quad s\in[0,\frac{1}{2}).$$

[Proof]

Since $0<\widetilde{s}<\frac{1}{2}$ must satisfy the first order condition (40),

$$\begin{aligned}
\Omega(\widetilde{s},k,a) &= \frac{1}{18}[\{2(2k^2-4ak+2a-1)\}\widetilde{s}^4+(4k^2-8ak+6a-3)\widetilde{s}^2+3k(2a-k)\widetilde{s}]-1\\
&= \frac{1}{9}(2k^2-1)\widetilde{s}^4+\frac{1}{18}(4k^2-3)\widetilde{s}^2-\frac{1}{6}k^2\widetilde{s}-1+\{\frac{2}{9}(1-2k)\widetilde{s}^4+\frac{1}{9}(3-4k)\widetilde{s}^2+\frac{1}{3}k\widetilde{s}\}a\\
&= \alpha(\widetilde{s},k)+\beta(\widetilde{s},k)a=0,
\end{aligned}$$

where $\alpha(\widetilde{s},k)=\frac{1}{9}(2k^2-1)\widetilde{s}^4+\frac{1}{18}(4k^2-3)\widetilde{s}^2-\frac{1}{6}k^2\widetilde{s}-1$ and

$\beta(\widetilde{s},k)=\frac{2}{9}(1-2k)\widetilde{s}^4+\frac{1}{9}(3-4k)\widetilde{s}^2+\frac{1}{3}k\widetilde{s}$. Then we can show that

$$\begin{aligned}
\alpha(\widetilde{s},k) &= \frac{1}{9}(2k^2-1)\widetilde{s}^4+\frac{1}{18}(4k^2-3)\widetilde{s}^2-\frac{1}{6}k^2\widetilde{s}-1\\
&< \frac{1}{9}(2\cdot1^2-1)\widetilde{s}^4+\frac{1}{18}(4\cdot1^2-3)\widetilde{s}^2-\frac{1}{6}\cdot1^2\widetilde{s}-1\\
&= \frac{2}{9}\widetilde{s}^4+\frac{1}{18}\widetilde{s}^2-\frac{1}{6}\widetilde{s}-1\\
&< \frac{2}{9}\widetilde{s}^4+\frac{1}{18}\widetilde{s}^2-1<\frac{2}{9}\cdot\frac{1}{2}^4+\frac{1}{18}\cdot\frac{1}{2}^2-1=-\frac{47}{48}<0.
\end{aligned}$$

Since $\alpha(\widetilde{s},k)+\beta(\widetilde{s},k)a=0$ and $a\geq\frac{17}{2}>0$ however, it follows that

$\Omega_a(\widetilde{s},k,a)=\beta(\widetilde{s},k)>0$. Next we show that $\Omega_k(s,k,a)>0$ for $0<s<\frac{1}{2}$.

From Young theorem and partial differentiability and the continuity of Ω, $\Omega_{kss}(\widetilde{s},k,a)=\Omega_{ssk}(\widetilde{s},k,a)$. In the proof of lemma 2, we have already shown that

$$\Omega_{ssk}(s,k,a)<0, 0\leq s\leq1, 0\leq k\leq1,\quad a\geq\max\{\frac{39}{4}-\frac{k}{2},\frac{2k^2+151}{4k+14}\}>\frac{37}{4}\qquad\text{(A3.6)}.$$

Hence we also see that

$$\Omega_{kss}(s,k,a)<0, 0\leq s\leq1, 0\leq k\leq1,\quad a>\max\{\frac{39}{4}-\frac{k}{2},\frac{2k^2+151}{4k+14}\}.$$

So we see that $\Omega_k(\widetilde{s},k,a)$ is concave in $\widetilde{s}$. From (40) we have

$\Omega_k(s,k,a)=\frac{1}{3}s(k-a)(\frac{4}{3}s(s^2+1)-1)$. Therefore, we have for

$\forall k \in [0,1], a \geq \max\{\frac{39}{4} - \frac{k}{2}, \frac{2k^2+151}{4k+14}\},$

$$\Omega_k(0,k,a) = 0,\ \Omega_k(\frac{1}{2},k,a) = \frac{1}{3}\cdot\frac{1}{2}(k-a)(\frac{4}{3}\cdot\frac{1}{2}(\frac{1}{2}^2+1)-1) = \frac{1}{36}(a-k) > 0.$$

Thus we have shown that

$\Omega_k(s,k,a) > 0$ for $0 < s < \frac{1}{2}$. Thus, the lemma holds.

(Q.E.D)

[Proof of Proposition 2]

For the first existence result of the proposition, combining the fact that $\Omega(0,k,a) = -1 < 0$, the assumption that $\Omega(\frac{1}{2},k,a) > 0$ and that for $\frac{1}{2} < k \leq 1$, $s \in [0, \frac{1}{2})$, $\Omega_s(0,k,a) > 0$ and $\Omega_{ss}(s,k,a) > 0$, therefore $\Omega_s(s,k,a) > 0$ from Lemma 2, the result follows. For the last part of the proposition, we know that for any $s \in [0, \frac{1}{2})$, $\Omega_s(s,k,a) > 0$ from the above. Also we know that $\Omega_a(\tilde{s},k,a) > 0$ and $\Omega_k(s,k,a) > 0$ for $0 < s < \frac{1}{2}$ from Lemma 3. By the implicit function theorem and the fact that $\Omega_s(s,k,a) > 0$ for any $s \in [0, \frac{1}{2})$, $\frac{\partial \tilde{s}}{\partial a} = -\frac{\Omega_a(\tilde{s},k,a)}{\Omega_s(\tilde{s},k,a)} > 0$ and $\frac{\partial \tilde{s}}{\partial k} = -\frac{\Omega_k(\tilde{s},k,a)}{\Omega_s(\tilde{s},k,a)} > 0, 0 < \tilde{s} < \frac{1}{2}$.

(Q.E.D)

Appendix 4

In this Appendix, we present the two lemmas and their proofs, also present the proof of the proposition in section 6.

Lemma 4

If $k \in (k^{**}, 1] (k \in [0, \quad k^{**}])$, then

$$\Omega(\frac{1}{2},k,a) > (\leq) \phi(\frac{1}{2},k,a),$$

$$\text{where } k^{**} = 4 - \frac{3}{4}\sqrt{22} \cong 0.482188.$$

[Proof]

We assume that $\phi(\frac{1}{2},k,a) > 0$ and $\Omega(\frac{1}{2},k,a) > 0$. The former implies that $a > \frac{39}{4} - \frac{k}{2}$ and the latter implies that $a > \frac{2k^2+151}{4k+14}$, respectively.

Solving $\frac{39}{4} - \frac{k}{2} = \frac{2k^2+151}{4k+14}$ for k, we obtain $k = 4 \mp \frac{3}{4}\sqrt{22}$. Obviously we see that $k = 4 + \frac{3}{4}\sqrt{22} > 1$. Let $k^{**} = 4 - \frac{3}{4}\sqrt{22} \cong 0.482188$. We can easily show that for $k \in [0, \quad k^{**}](k \in (k^{**},1]$), $\frac{39}{4} - \frac{k}{2} \le \frac{2k^2+151}{4k+14}\left(\frac{39}{4} - \frac{k}{2} > \frac{2k^2+151}{4k+14}\right)$. So for $k \in [0, \quad k^{**}](k \in (k^{**},1])$,

$$\Omega(\frac{1}{2},k,a) > 0 \Rightarrow \phi(\frac{1}{2},k,a) > 0 \quad (\phi(\frac{1}{2},k,a) > 0 \Rightarrow \Omega(\frac{1}{2},k,a) > 0).$$

$\Omega(\frac{1}{2},k,a) > 0 \Rightarrow \phi(\frac{1}{2},k,a) > 0 \quad (\phi(\frac{1}{2},k,a) > 0 \Rightarrow \Omega(\frac{1}{2},k,a) > 0)$ imply that $0 < \Omega(\frac{1}{2},k,a) \le \phi(\frac{1}{2},k,a) \quad (\phi(\frac{1}{2},k,a) \ge \Omega(\frac{1}{2},k,a) > 0)$, and the result follows. (Q.E.D)

Lemma 5

If $\max\{\frac{39}{4} - \frac{1}{2}k, \frac{2k^2+151}{4k+14}\} \le a, \quad \forall k \in [0,1]$, then

$$\phi_s(0,k,a) > \Omega_s(0,k,a) > 0.$$

[Proof]

From (A3.6) in the proof of Lemma 2, note that

$$\Omega_s(0,k,a) = \frac{1}{6}k(2a-k) > 0, 0 < k \le 1, a \ge \max\{\frac{39}{4} - \frac{k}{2}, \frac{2k^2+151}{4k+14}\}.$$

On the other hand, from (12), we have

$$\phi_s(s,k,a)=\frac{16}{9}(4k^2-4k+1)s^3+4k(1-2k)s^2+\frac{8}{9}(4k^2-2ak+a-1)s+\frac{4}{9}k(a-k).$$

$$\phi_s(0,k,a)=\frac{4}{9}k(a-k)>0,\ \frac{1}{2}<k\le 1,\quad a>\frac{37}{4}. \tag{A4.2}$$

Subtracting (A4.1) from (A4.2) yields

$$\phi_s(0,k,a)-\Omega_s(0,k,a)=\frac{1}{18}k(2a-5k)>0,\forall k\in(0,1],a\ge\max\{\frac{39}{4}-\frac{k}{2},\frac{2k^2+151}{4k+14}\}.$$

(Q.E.D)

[Proof of Proposition 3]

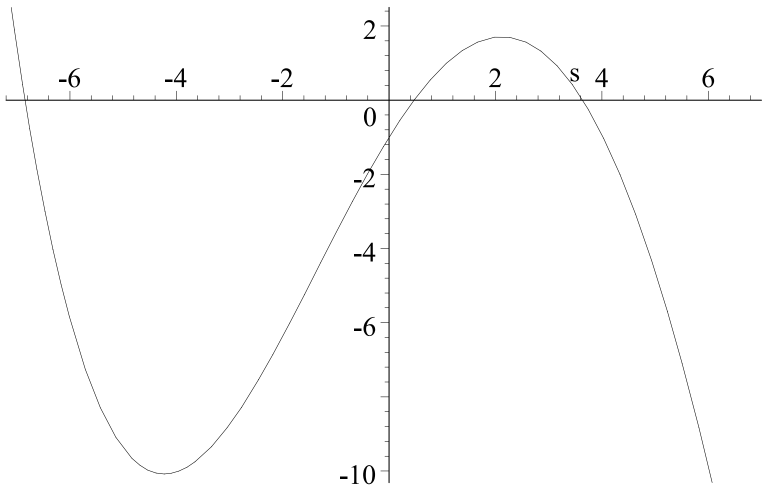
At first, we know that $\phi(\bullet,k,a)$ is convex (concave) in $s\in[0,\frac{1}{2})$, $0\le k<\frac{1}{2},\left(\frac{1}{2}<k\le 1\right)$ from Lemma 1 and $\Omega(\bullet,k,a)$ is convex in $s\in[0,\frac{1}{2})$, $0\le k\le 1$. From (12) and (46), $\phi(0,k,a)=\Omega(0,k,a)=-1$ holds, and by the assumptions we have that $\phi(\frac{1}{2},k,a)>0,\Omega(\frac{1}{2},k,a)>0.$ We also see that $\phi_s(0,k,a)>\Omega_s(0,k,a)>0$ from Lemma 5. Therefore, if $0\le k\le k^{**}<\frac{1}{2}$, then there never exist any intersect points s^0 of $\phi(s,k,a)$ and $\Omega(s,k,a)$ in $s\in[0,\frac{1}{2})$ such that $\phi(s^0,k,a)=\Omega(s^0,k,a)$ since $\phi(\frac{1}{2},k,a)\ge\Omega(\frac{1}{2},k,a)>0$ from Lemma 4. Thus, we can conclude that $0<s^*<\tilde{s}<\frac{1}{2}$, if $0\le k\le k^{**}<\frac{1}{2}$. (See Figure 2, in which $\phi(s,0.3,15)$ and $\Omega(s,0.3,15)$ are depicted.)

While, if $\frac{1}{2}<k^{**}<k\le 1$, then there exist a unique intersect point s^0 of $\phi(s,k,a)$ and $\Omega(s,k,a)$ in $s\in[0,\frac{1}{2})$ such that $\phi(s^0,k,a)=\Omega(s^0,k,a)$ since $0<\phi(\frac{1}{2},k,a)<\Omega(\frac{1}{2},k,a)$ from Lemma 4. At one hand, if $\phi(s^0,k,a)=\Omega(s^0,k,a)<0$ at the $s^0\in(0,\frac{1}{2})$, then we see that

$0 < \tilde{s} < s^* < \dfrac{1}{2}$ under the above conditions stated above. (See Figure 3, in which $\phi(s,0.95,9.5)$ and $\Omega(s,0.95,9.5)$ are depicted.) On the other hand, if $\phi(s^0,k,a) = \Omega(s^0,k,a) > 0$ at the $s^0 \in (0,\dfrac{1}{2})$, then we see that $0 < s^* < \tilde{s} < \dfrac{1}{2}$ under the above conditions stated above. (See Figure 4, in which $\phi(s,0.95,15)$ and $\Omega(s,0.95,15)$ are depicted.)

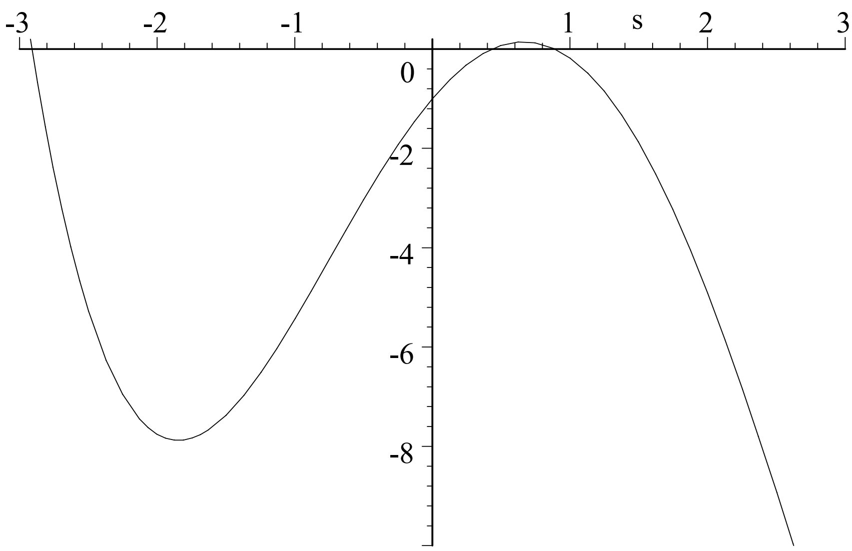
(Q.E.D.)

Figures



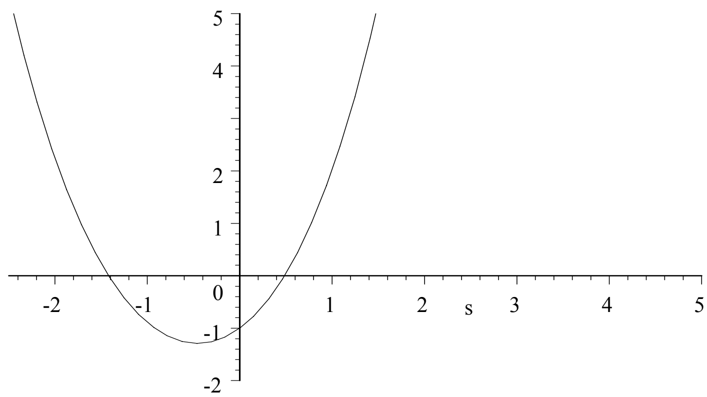
ϕ(s,0.55,11)=0, Solutions : {s=-6. 8435},{s=0.46962}, {s=3. 6399},{s=19. 234}

Figure A-1 ϕ(s,0.55,11)



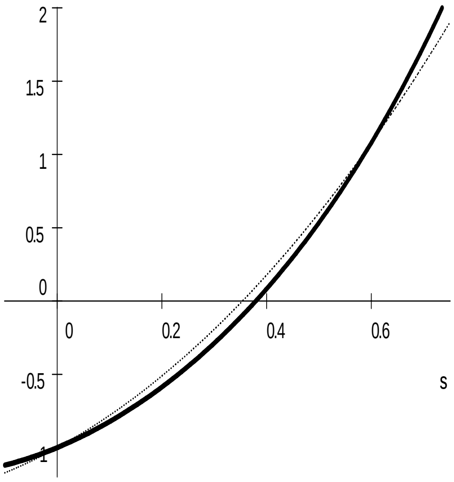
ϕ(s,0.8,11)=0, Solutions: {s=-2. 9091},{s=0.43294}, {s=0.88802},{s=5. 5882}

Figure A-2 ϕ(s,0.8,11)



ϕ(s,0.3,11)=0, solutions: {s=-1. 4180}, {s=-0.65838-4. 4749i}, {s=-0.65838+4. 4749i}, {s=0.48475}

Figure A-3 ϕ(s,0.3,11)



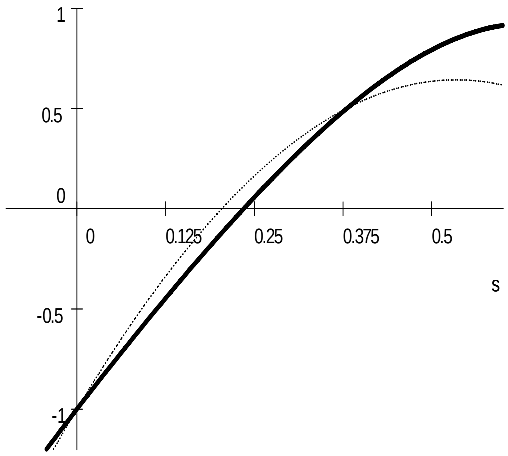
$\phi(s,0.3,15)$: a dotted thin line

$\Omega(s,0.3,15)$: a solid thick line

$\phi(s,0.3,15) = 0$, Numerical Solution: $s^* = 0.35384$

$\Omega(s,0.3,15) = 0$,Numerical Solution: $\tilde{s} = 0.37943$

Figure 2 $\phi(s,0.3,15)$, $\Omega(s,0.3,15)$



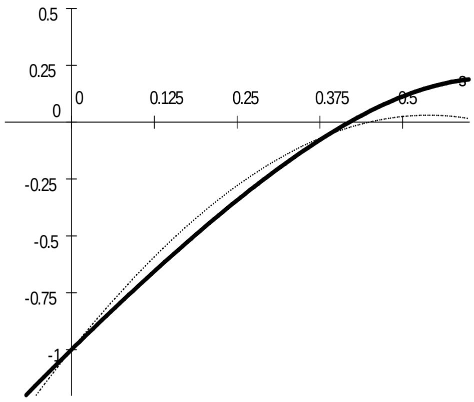
$\phi(s,0.95,15)$: a dotted thin line

$\Omega(s,0.95,15)$: a solid thick line

$\phi(s,0.95,15) = 0$, Numerical Solution: $s^* = 0.20408$

$\Omega(s,0.95,15) = 0$, Numerical Solution: $\tilde{s} = 0.23493$

Figure 3 $\phi(s,0.95,15)$, $\Omega(s,0.95,15)$



$\phi(s, 0.95, 9.5)$: a dotted thin line

$\Omega(s, 0.95, 9.5)$: a solid thick line

$\phi(s, 0.95, 9.5) = 0$, Numerical Solution: $s^* = 0.44885$

$\Omega(s, 0.95, 9.5) = 0$, Numerical Solution: $\tilde{s} = 0.41939$

Figure 4 $\phi(s, 0.95, 9.5)$, $\Omega(s, 0.95, 9.5)$