No arbitrage condition and existence of equilibrium in infinite or finite dimension with expected risk averse utilities*

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October 25, 2008

Abstract

We consider a general equilibrium model in asset markets with a countable set of states and expected risk averse utilities. The agents do not have the same beliefs. We use the methods in Le Van - Truong Xuan (JME, 2001) but one of their assumption which is crucial for obtaining their result cannot be accepted in our model when the number of states is countable. We give a proof of existence of equilibrium when the number of states is infinite or finite.

Keywords: No-arbitrage Conditions, the two-period wealth model, No Unbouded Arbitrage, Weak No Market Arbitrage.

JEL Classification: C62, D50, D81,D84,G1.

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1 Introduction

Expected utility with additive probability theories, e.g, Savage's (1954) and Anscombe and Aumann's (1963) are known as standard formulation of decision under uncertainty. Since the seminal paper of Hart (1974), the question of existence of equilibrium in the unbounded securities exchange model has been a subject of much development. In finite dimension economies, one of a crucial assumption interpreted as a no-arbitrage-condition be used to prove the compactness of the individually rational utility (see,e.g, Werner 1987, Nielsen 1989, Page and Wooders 1996, Allouch et al. 2002). This assumption together with other standard assumptions are sufficient condition for the existence of equilibrium. However, in infinite dimension economies, the no-arbitrage condition are not sufficient to ensure the compactness of the utility set. Therefore, to find the conditions for which the compactness of utility set holds is interested by many authors. (e.g,Cheng 1991, Dana et al, (1999), Dana and LeVan 2000). Recently, Le Van and Truong Xuan (2001) have proved the compactness of utility set (and hence the existence of equilibrium followed), in asset market with consumption set equal to L^p, separable utilities and the continuum states which belong to [0,1]. Following this direction, we consider a general equilibrium model in asset markets with a countable set of states and expected risk-averse utilities. The agents do not have the same beliefs. We use the methods in Le Van - Truong Xuan (2001) but one of their assumption which is crucial for obtaining their result cannot be accepted in our model when the number of states is countable. Moreover, by assuming the existence of a common marginal utility price, the proof we give is more natural and simple than the one given in Le Van and Truong Xuan (2001). The existence of a quasi-equilibrium in L^1 can be also derived.

The paper is organized as follows. In Section 2, we give a proof of existence of equilibrium in a model with expected risk-averse utilities the number of states is infinitely countable. Section 3 we consider the case of continuum states as in Le Van and Truong Xuan (2001) but we relax one of their crucial assumption. Section 4 prove the existence of equilibrium in the case of finite number of states by exploiting the similarity of NUBA and WNMA.

2 The model with infinitely countable states

First, we consider the case where the set of states possible is countable. There are m agents indexed by $1, \ldots, m$. Each agent has a probability $(\pi_s^i)_{s=1}^{\infty}$ in the set $\Delta := \{\pi \in \mathbb{R}^{\infty} : \sum_{s=1}^{+\infty} \pi_s = 1\}$. Let us denote the probability $\pi = \frac{1}{m} \sum_{i=1}^{m} \pi^i$, a consumption set $X^i = L^p(\pi)$ with $1 \le p \le \infty$ and an endowment $e^i \in L^p(\pi)$. We assume that for each agent i, there exists a concave, strictly increasing function u^i from $\mathbb{R}$ to $\mathbb{R}$ and consumer i choose a portfolio $x^i = (x^i_s)_{s=1}^{\infty} \in L^p(\pi)$ to solve the problem

$$\max U^i(x^i) = \max \sum_{s=1}^{\infty} \pi^i_s u^i(x^i_s)$$

We recall the notion set of individually rational attainable allocations A is defined by

$$A = \{(x^i) \in (L^p)^m \mid \sum_{i=1}^{m} x^i = \sum_{i=1}^{m} e^i \text{ and } U^i(x^i) \geq U^i(e^i) \text{ for all } i.\}$$

The individually rational utility set U is defined by

$$U = \{(v_1, v_2, ..., v_m) \in \mathbb{R}^m \mid \exists x \in A \text{ s.t } U^i(e^i) \leq v^i \leq U^i(x^i) \text{ for all } i.\}$$

Let us denote, for each agent i, $a^i := \inf u^{i\prime}(z)$, $b^i := \sup u^{i\prime}(z)$.

Assumption 1 *$\exists\, p \in (L^p)^*$, $\exists (\lambda_i) \in \mathbb{R}^m_{++}$, $\exists (x^1, \ldots, x^m) \in A$ such that: $\forall\, i, s$, $p_s = \lambda_i \pi^i_s u^{i\prime}(\overline{x}^i_s)$ and*

$$\inf_s u^{i\prime}(\overline{x}^i_s) = m^i > a^i$$
$$\sup_s u^{i\prime}(\overline{x}^i_s) = M^i < b^i$$

Remark From the assumption 1, we know that all the probabilities π^i are equivalences and hence equivalences with π.

Assumption 2 *For all $i = 1, 2, ..., m$, $b^i = +\infty$.*

Proposition 1 *With the assumption 1 there exists $C > 0$ such that for all $(x^1, \ldots, x^m) \in A$, we have:*

$$\sum_{s=1}^{+\infty} p_s |x^i_s| < C$$

for all i.

Proof: From the condition $\forall\, i$ $a^i < m^i = \inf_s u^{i\prime}(\overline{x}^i_s) \leq \sup_s u^{i\prime}(\overline{x}^i_s) = M^i < b^i$, there exist $\eta > 0$ such that

$$a^i < u^{i\prime}(\overline{x}^i_s)(1+\eta) < b^i \tag{1}$$

for all i.

Then we define the price q such that, $\forall i, j$,

$$\begin{aligned} q_s &= \lambda_i \pi^i_s u^{i\prime}(\overline{x}^i_s)(1+\eta) \\ &= \lambda_j \pi^j_s u^{j\prime}(\overline{x}^j_s)(1+\eta). \end{aligned}$$

It follows from (1) that, for each i, there exist $\overline{z}^i \in L^\infty$ such that $\forall s$, $q_s = \lambda_i u^{i\prime}(\overline{z}^i_s)$. Note that

$$\forall s,\ p_s < q_s.$$

Denote

$$\begin{aligned} x^{i+} &:= \begin{cases} x^i & \text{if } x^i > 0 \\ 0 & \text{if } x \leq 0 \end{cases} \\ x^{i-} &:= \begin{cases} -x^i & \text{if } x^i < 0 \\ 0 & \text{if } x^i \geq 0 \end{cases} \end{aligned}$$

From the concavity of the utility function u^i we have

$$\begin{aligned} \lambda_i \sum_{s=1}^{\infty} \pi^i_s u^i(\overline{x}^i_s) - \lambda_i \sum_{s=1}^{\infty} \pi^i_s u^i(x^{i+}_s) &\geq \lambda_i \sum_{s=1}^{\infty} \pi^i_s u^{i\prime}(\overline{x}^i_s)(\overline{x}^i_s - x^{i+}_s), \\ \lambda_i \sum_{s=1}^{\infty} \pi^i_s u^i(\overline{z}^i_s) - \lambda_i \sum_{s=1}^{\infty} \pi^i_s u^i(-x^{i-}_s) &\geq \lambda_i \sum_{s=1}^{\infty} \pi^i_s u^{i\prime}(\overline{z}^i_s)(\overline{z}^i_s + x^{i-}_s) \end{aligned}$$

Therefore,

$$\begin{aligned} \lambda_i \sum_{s=1}^{\infty} \pi^i_s u^{i\prime}(\overline{z}^i_s) x^{i-} &\leq \lambda_i \sum_{s=1}^{\infty} \pi^i_s [u^i(\overline{z}^i_s) + u^i(\overline{x}^i_s) - u^i(x^{i+}_s) - u^i(x^{i-}_s)] \\ &\quad - \lambda_i \sum_{s=1}^{\infty} \pi^i_s u^{i\prime}(\overline{z}^i_s)\overline{z}^i_s + \lambda_i \sum_{s=1}^{\infty} \pi^i_s u^{i\prime}(\overline{x}^i_s) x^{i+}_s. \end{aligned}$$

which implies

$$\begin{aligned} \sum_{s=1}^{\infty} q_s x^{i-}_s &\leq \lambda_i [U^i(\overline{z}^i) + U^i(\overline{x}^i) - U^i(x^i)] - \sum_{s=1}^{\infty} q_s \overline{z}^i_s + \sum_{s=1}^{\infty} p_s x^{i+}_s \\ &\leq \lambda_i [U^i(\overline{z}^i) + U^i(\overline{x}^i) - U^i(e^i)] - \sum_{s=1}^{+\infty} q_s \overline{z}^i_s + \sum_{s=1}^{\infty} p_s x^{i+}_s. \end{aligned}$$

Hence, $\forall i$,

$$\sum_{s=1}^{+\infty} (q_s - p_s) x^{i-}_s \leq C^i + \sum_{s=1}^{+\infty} p_s x^i_s$$

where

$$C^i = \lambda_i [U^i(\overline{z}^i) + U^i(\overline{x}^i) - U^i(e^i)].$$

Thus we have

$$\begin{aligned} \sum_{i=1}^{m} \sum_{s=1}^{\infty} (q_s - p_s) x^{i-}_s &\leq \sum_{i=1}^{m} C^i + \sum_{i=1}^{m} \sum_{s=1}^{\infty} p_s x^i_s \\ &= \sum_{i=1}^{m} C^i + \sum_{s=1}^{\infty} p_s \overline{e}^i_s = C_1. \end{aligned}$$

Since $x_s^{i-} \geq 0, \forall i, s$, we get

$$\sum_{s=1}^{\infty}(q_s - p_s)x_s^{i-} \leq C_1$$

for all i. We also have

$$\sum_{i=1}^{m}\sum_{s=1}^{\infty}(q_s - p_q)(x_s^{i+} - x_s^{i-}) = \sum_{s=1}^{\infty}(q_s - p_s)\overline{e}_s$$

which implies

$$\begin{aligned}\sum_{i=1}^{m}\sum_{s=1}^{\infty}(q_s - p_s)x_s^{i+} &= \sum_{s=1}^{\infty}(q_s - p_s)\overline{e}_s - \sum_{i=1}^{m}\sum_{s=1}^{\infty}(q_s - p_s)x_s^{i-} \\ &\leq C_2\end{aligned}$$

Thus

$$\sum_{s=1}^{\infty}(q_s - p_s)|x_s^i| \leq C_1 + C_2 =: C$$

which implies

$$\eta\sum_{s=1}^{\infty}p_s|x_s^i| \leq C.$$

■

Remark *1. The condition* $(\overline{x}^i) \in L^\infty$ *is not sufficient for the existence of Assumption 1, because the utility function can be linear by pieces.*
2. In the proof of boundedness above, we used the property $U^i(x^i) \geq U^i(e^i)$*. However, We can use a weaker assumption that there exists a constant* a *such that* $U^i(x^i) \geq a$ *for all* i*.*
Note that $p_s = \lambda_i \pi_s^i u^{i\prime}(\overline{x}_s^i) \geq \lambda_i \pi_s^i m^i$, so there exist the constant $C > 0$ such that:

$$\sum_{s=1}^{\infty}\pi_s^i|x_s^i| \leq C$$

with all $(x^1, x^2, \ldots, x^m) \in A$. From this property and by using Jensen's inequality, we have the following Lemma

Lemma 1 *There exists* $C > 0$ *such that for all* $(x^1, x^2, \ldots, x^m) \in A$*,* $U^i(x^i) < C$

Thus we get the following Theorem which will be used later.

Theorem 1 *With the assumptions 1 and 2, in the case* $X^i = L^\infty(\pi)$*, for all* $\epsilon > 0$*, there exists* $N > 0$ *such that*

$$\sum_{s=N}^{\infty}\pi_s^i|x_s^i| < \epsilon$$

for all $(x^1, x^2, ..., x^m) \in A$*, for all* i*.*

Proof: Assume the contrary, there exists a sequence $(x^1(n), x^2(n), ..., x^m(n)) \in A$ and a constant $c > 0$ such that

$$\sum_{s=n}^{\infty} |x_s^i(n)| > c$$

Without loosing of the generality, we can suppose that $\sum_{s=n}^{\infty} |x_s^i(n)| \to c > 0$.
We can assume that there exists i $\lim_{n\to\infty} \sum_{s=n}^{\infty} \pi_s^i |x_s^i| = c > 0 \Rightarrow \lim_{n\to\infty} \sum_{s=n}^{\infty} \pi_s^i x_s^{i+}(n) - \lim_{n\to\infty} \sum_{s=n}^{\infty} \pi_s^i x_s^{i-}(n) = c$. For every s, $\exists j$ such that $x_s^j(n) < -\frac{x_s^i - |\bar{e}_s|}{m-1}$. There is an finite $j \neq i$, then for the simplicity, we can assume that there exists j fixed such that i and j satisfies the properties:
1. $\exists\ E_n^i \subset \mathbb{N} \cap \{s \geq n\}$, $x_s^i > 0$ for all s and

$$\lim_{n\to\infty} \sum_{s \in E_n^i} \pi_s^i x_s^i = c^i > 0$$

2. For all $s \in E_n^i$

$$x_s^j(n) \leq -\frac{x_s^i(n) - |\bar{e}_s|}{m-1}$$

With each $M > 0$, define the set $S_n^i \subset E_n^i$ the states s satisfies:

$$\frac{x_s^i(n) - |\bar{e}_s|}{m-1} > M$$

and for all $s \in S_n^i$ we have:

$$x_s^j(n) \leq \frac{|\bar{e}_s| - x_s^i(n)}{m-1} < -M$$

We can see that $\lim_{n\to+\infty} \sum_{s \in S_n^i} \pi_s^i x_s^i = c^i$. The two probabilities π^i and π^j are equivalent, then we have $\lim_{n\to\infty} \sum_{s \in S_n^i} \pi_s^j x_s^i = c^j > 0$. Now consider the sequence $(y^1(n), y^2(n), \ldots, y^m(n))$ defined by:

$$y_s^i(n) := x_s^i(n) - \frac{x_s^i - |\bar{e}_s|}{m-1} + M \text{ with } s \in S_n^i$$

$$y_s^j(n) := x_s^j(n) + \frac{x_s^i - |\bar{e}_s|}{m-1} - M \text{ with } s \in S_n^i$$

$y_s^k = x_s^k$ with every $k \neq i, j$ or $s \notin S_n^i$.
Remarks that $y_s^i(n) \leq x_s^i(n)$ and $y_s^j(n) \geq x_s^j(n)$ for all s. We will prove that $(U^l(y^l(n)))_{l=1,m}$ is bounded below, but is not bounded above, that leads us to

a contradiction with the Lemma 1.

$$\begin{aligned}
U^i(y^i(n)) - U^i(x^i(n)) &= \sum_{s\in S_n^i} \pi_s^i (u^i(y_s^i(n)) - u^i(x_s^i(n))) \\
&\geq \sum_{s\in S_n^i} \pi_s^i u^{i\prime}(x_s^i(n) - \frac{x_s^i - |\bar{e}_s|}{m-1} + M)(-\frac{x_s^i - |\bar{e}_s|}{m-1} + M) \\
&\geq \sum_{s\in S_n^i} \pi_s^i u^{i\prime}(M)(-\frac{x_s^i(n)}{m-1}) + u^{i\prime}(M)(\frac{|\bar{e}_s|}{m-1} + M) \sum_{s\in S_n^i} \pi_s^i \\
&\geq -\frac{u^{i\prime}(M)}{m-1} \sum_{s\in S_n^i} \pi_s^i x_s^i(n) + u^{i\prime}(M)(\frac{|\bar{e}_s|}{m-1} + M) \sum_{s\in S_n^i} \pi_s^i
\end{aligned}$$

Let $n \to +\infty$ we have:

$$\liminf_{n\to+\infty} U^i(y^i(n)) \geq v^i - \frac{u^{i\prime}(M)c^i}{m-1}$$

so for great n, $U^i(y^i(n))$ is bounded below. Now we will see $U^j(y^j(n))$.

$$\begin{aligned}
U^j(y^j(n)) - U^j(x^j(n)) &= \sum_{s\in S_n^i} \pi_s^j (u^j(y_s^j(n)) - u^j(x_s^j(n))) \\
&\geq \sum_{s\in S_n^i} \pi_s^j u^{j\prime}(x_s^j(n) + \frac{x_s^i(n) - |\bar{e}_s|}{m-1} - M)(\frac{x_s^i(n) - |\bar{e}_s|}{m-1} - M)
\end{aligned}$$

$$\begin{aligned}
U^j(y^j(n)) - U^j(x^j(n)) &\geq \sum_{s\in S_n^i} \pi_s^j b_j(\frac{x_s^i(n) - |\bar{e}_s|}{m-1} - M) \\
&\geq \frac{u^{j\prime}(-M)}{m-1} \sum_{s\in S_n^i} \pi_s^j x_s^i(n) - M \sum_{s\in S_N^{i,j}} \pi_s^i
\end{aligned}$$

So we have the limit

$$\liminf_{n\to+\infty} U^j(y^j(n)) \geq v^j + \frac{u^{j\prime}(-M)c^j}{m-1}$$

So if $b^i = +\infty$, we can choose M very large, and the limit of $U^j(y^j(n))$ is unbounded above: a contradiction. ■

The next Lemma show that the sum is bounded uniformly.

Lemma 2 *If $p = \infty$, for all $\epsilon > 0$, there exist $N > 0$ such that for all $(x^1, x^2, \ldots, x^m) \in A$, for all i,*

$$\sum_{s=N}^{\infty} \pi_s^i u^i(x_s^i) < \epsilon$$

Proof: Fixe $a \in \mathbb{R}$ arbitrarily, we have $u^i(a) - u^i(x_s^i) \geq u^{i\prime}(a)(a - x_s^i)$, so with every $N > 0$,

$$\sum_{s=N}^{\infty} \pi_s^i u^i(x_s^i) \leq [u^i(a) - u^{i\prime}(a)a] \sum_{s=N}^{\infty} \pi_s^i + u^{i\prime}(a) \sum_{s=N}^{\infty} \pi_s^i x_s^i$$

From the Lemma above, in choosing N large sufficiently, we have $\sum_{s \geq N} \pi_s^i u^i(x_s^i) < \epsilon$ with every $(x^1, x^2, ..., x^m) \in A$. ■

We know that U is bounded. Suppose that there exists a sequence in U $(v^1(n), v^2(n), ..., v^m(n)) \to (v^1, v^2, ..., v^n)$. We have to find that if $(v^1, v^2, ..., v^m) \in A$. Denote the sequence $(x^1(n), x^2(n), ..., x^m(n)) \in A$ such that $U^i(x^i(n)) = v^i(n)$ for all i.

Theorem 2 *Under the Assumptions 1 and 2, U is closed for every p.*

Proof: Note that $L^\infty \subset L^p$ for every $1 \leq p \leq \infty$. We have two cases:
I There exists $M > 0$,$N > 0$ such that for all $n > N$, for all i, s: $|x_s^i(n)| < M$.
II For all $M > 0$, there exists n, i, s such that $|x_s^i(n)| > M$.
Consider the first case. From the Theorem 1, we know that A is the subset of a compact set of the product topology, then we can assume that $x^i(n) \to y^i$ in this topology for all i. For all s, $\lim_{n\to\infty} x_s^i(n) = y_s^i$. $|y_s^i| \leq M$ for all i, s, then $y^i \in L^\infty$ for all i. For all $\epsilon > 0$, choose $N > 0$ in the theorem 1 and the lemma 2, such that the sum $\sum_{s=N}^{\infty} u^i(x_s^i(n)) < \epsilon$, we have:

$$\begin{aligned}\sum_{s=1}^{\infty} u^i(x_s^i(n)) &= \sum_{s=1}^{N} \pi_s^i u^i(x_s^i(n)) + \sum_{s=N+1}^{\infty} \pi_s^i u^i(x_s^i(n)) \\ &\leq \sum_{s=1}^{N} \pi_s^i u^i(x_s^i(n)) + \epsilon\end{aligned}$$

$\Rightarrow$

$$\lim_{n\to\infty} \sum_{s=1}^{\infty} \pi_s^i u^i(x_s^i(n)) \leq \sum_{s=1}^{N} u^i(y_s^i) + \epsilon$$

for all N sufficiently large, we have:

$$\lim_{n\to\infty} \sum_{s=1}^{\infty} \pi_s^i u^i(x_s^i(n)) \leq \sum_{s=1}^{\infty} \pi_s^i u^i(y_s^i)$$

$\Rightarrow$ for all i, $U^i(y^i) \geq v^i \Rightarrow (v^1, v^2, ..., v^m) \in U$.

Then we consider the second case. Suppose that for every $M > 0$ there exists i and an infinite n such that $x_s^i(n) > M$ with an s, without losing the generality, we can assume that is true for all n and for an i fixed. Choose $\underline{M}$ sufficiently large such that for all i, denote $T_n^i = \{s : |x_s^i| < \underline{M} - 1\}$, we have $\sum_{s \in T_n^i} \pi_s^i > \frac{1}{2}$ With each $\overline{M} > 0$, denote the sets E_n^i, S_n^i as above. Choose $\overline{M}$ sufficiently large such that $\overline{M} > \underline{M}$ and $S_n^i \cap T_n^i = \emptyset$. π^i and π^j are equivalents, so there exist $h > 0$ such that:

$$h\pi_s^i \leq \pi_s^j \leq \frac{1}{h}\pi_s^i$$

We can choose $\overline{M}$ such that:

$$u^{j\prime}(-\overline{M}) > \frac{u^{j\prime}(-\underline{M})}{h^2}$$

We consider two cases:
1. $\exists\, t \in S_n^i$ for an infinite n.
2. $\lim_{n\to\infty} \min S_n^i = \infty$.
Consider the first case. Then $\liminf \sum_{s \in S_n^i} \pi_s^i > 0$. Without lost of generality, we can suppose that there exist

$$a = \lim_{n\to\infty} \frac{\sum_{s \in S_n^i} \pi_s^i x_s^i}{(m-1)\sum_{s \in T_n^i} \pi_s^i}$$

Remarks that $0 < a < \frac{1}{2}$. Define the sequence $(y^i(n))$:

$$\begin{aligned}
y_s^i(n) &= x_s^i(n) - \frac{x_s^i(n) - |\overline{e}_s|}{m-1} + \overline{M} \text{ if } s \in S_n^i \\
y_s^i(n) &= x_s^i(n) + a \text{ if } s \in T_n^i \\
y_s^i(n) &= x_s^i(n) \text{ others cases}
\end{aligned}$$

$$\begin{aligned}
y_s^j(n) &= x_s^j(n) + \frac{x_s^j(n) - |\overline{e}_s|}{m-1} - \overline{M} \text{ if } s \in S_n^i \\
y_s^j(n) &= x_s^j(n) - a \text{ if } s \in T_n^i \\
y_s^j(n) &= x_s^j(n) \text{ others cases}
\end{aligned}$$

And $y^k(n) = x^k(n)$ for all $k \neq i, j$. Easily, we see that $\sum_{k=1}^m y^k(n) = \overline{e}$. We are estimating $U^i(y^i(n))$ and $U^j(x^j(n))$.

$$\begin{aligned}
U^i(y^i(n)) - U^i(x^i(n)) &= \sum_{s \in T_n^i} \pi_s^i[u^i(y_s^i(n)) - u^i(x_s^i(n))] + \sum_{s \in S_n^i} \pi_s^i[u^i(y_s^i(n)) - u^i(x_s^i(n))] \\
&\geq \sum_{s \in T_n^i} \pi_s^i u^{i\prime}(x_s^i(n) + a) - \sum_{s \in S_n^i} \pi_s^i u^{i\prime}(x_s^i(n) - \frac{x_s^i(n) - |\overline{e}_s|}{m-1} + \overline{M})[\frac{x_s^i(n) - |\overline{e}_s|}{m-1} - \overline{M}] \\
&\geq a u^{i\prime}(\underline{M}) \sum_{s \in T_n^i} \pi_s^i - u^{i\prime}(\overline{M}) \sum_{s \in S_n^i} \pi_s^i[\frac{x_s^i(n) - |\overline{e}_s|}{m-1} - \overline{M}] \\
&\geq a u^{i\prime}(\underline{M}) \sum_{s \in T_n^i} \pi_s^i - u^{i\prime}(\overline{M}) \sum_{s \in S_n^i} \pi_s^i \frac{x_s^i}{m-1} + u^{i\prime}(\overline{M}) \sum_{s \in S_n^i} \pi_s^i[\frac{|\overline{e}_s|}{m-1} + \overline{M}] > 0
\end{aligned}$$

with n sufficiently great. So we have the result that $\liminf_{n\to\infty} U^i(y^i(n) > \liminf_{n\to\infty} U^i(n) = v^i$.

$$U^j(y^j(n)) - U^j(x^j(n)) = \sum_{s\in T_n^i} \pi_s^j[u^j(y_s^j(n)) - u^j(x_s^j(n))] + \sum_{s\in S_n^i} \pi_s^j[u^j(y_s^j(n)) - u^j(x_s^j(n))]$$

$$\begin{aligned}
&\geq -a\sum_{s\in T_n^i} \pi_s^j u^{j\prime}(x_s^j(n) - a) + \sum_{s\in S_n^i} \pi_s^i u^{j\prime}(x_s^j(n) + \frac{x_s^i(n) - |\overline{e}_s|}{m-1} - \overline{M})[\frac{x_s^i(n) - |\overline{e}_s|}{m-1} - \overline{M}] \\
&\geq -au^{j\prime}(-\underline{M}) \sum_{s\in T_n^i} \pi_s^j + u^{j\prime}(-\overline{M}) \sum_{s\in S_n^i} \pi_s^j \frac{x_s^i}{m-1} \\
&\geq -au^{j\prime}(-\underline{M}) \sum_{s\in T_n^i} \pi_s^j + u^{j\prime}(-\overline{M}) \sum_{s\in S_n^i} \pi_s^j \frac{x_s^i}{m-1} \\
&\geq -\frac{a}{h}u^{j\prime}(-\underline{M}) \sum_{s\in T_n^i} \pi_s^i + hu^{j\prime}(-\overline{M}) \sum_{s\in S_n^i} \pi_s^i \frac{x_s^i}{m-1}
\end{aligned}$$

We know that $u^{j\prime}(-\overline{M}) > u^{j\prime}(-\underline{M})/h^2$. Then $\liminf_{n\to\infty} U^j(y^j(n)) > \liminf_{n\to} U^j(x^j(n)) = v^j$ too.

Now we will show that we can construct a sequence $(z^k(n))$ such that $\lim_{n\to\infty} U^k(z^k(n)) > v^k$. Choose $k \neq i, j$ above. Choose $\epsilon > 0$ very small such that $\liminf_{n\to\infty} U^i(y^i(n)) - \epsilon u^{i\prime}(-M) > v^i$. Choose $t \in T_n^i$ arbitrarily, we define the new sequence $(z^l(n))$ as:

$$\begin{aligned}
z_t^i(n) &= y_t^i(n) - \epsilon \\
z_t^k(n) &= y_t^k(n) + \epsilon \\
z_s^l(n) &= y_s^l(n) \text{ in others cases}
\end{aligned}$$

$$\begin{aligned}
U^i(z^i(n)) - U^i(y^i(n)) &= \pi_t^i[u^i(y_t^i(n)) - u^i(z_t^i(n))] \\
&\geq -\pi_t^i u^{i\prime}(y_t^i(n) - \epsilon)\epsilon \\
&> -u^{i\prime}(-M)\epsilon
\end{aligned}$$

and then $\liminf_{n\to\infty} u^i(z^i(n)) > v^i$.

$$\begin{aligned}
U^k(z^k(n)) - U^k(y^k(n)) &= \pi_t^k[u^k(y_t^k(n)) - u^k(z_t^k(n))] \\
&\geq \pi_t^k u^{k\prime}(y_t^k(n) + \epsilon)\epsilon \\
&> \pi_t^k u^{k\prime}(M)\epsilon
\end{aligned}$$

then $\liminf_{n\to} U^k(z^k(n)) > v^k$.

By the induction, we can construct the sequence $(z^i(n))$ such that $\sum_{i=1}^m z^i(n) = \overline{e}$ and $\liminf_{n\to\infty} U^i(z^i(n)) > v^i$ for all i. Then there exist n such that $U^i(z^i(n)) > v^i$ for all $i = \overline{1,m} \Rightarrow (v^1, v^2, ..., v^m) \in U$.

Now we return to the case $\lim_{n\to\infty} \inf S_n^i = +\infty$. In this case, we will construct a sequence satisfy the properties: $\liminf U^i(y^i(n)) = v^i$ and $\sup_n \sup_s |y^i_s(n)| < +\infty$. If those properties are true for a sequence $(x^i(n))$, we have nothing to do, in the converse case, there exist i such that for all M, there exist an infinite n and s s.t $x^i_s(n) > M$. Define i, $\overline{M}$, S^i_n, as above, remarks that $\sum_{j\neq i} x^j_s(n) = \overline{e}_s - x^i_s(n) < 0$. Then we have $0 \leq \sum_{j\neq i} x^{j+}_s(n) < \sum_{j\neq i} x^{j-}_s(n)$. Then there exists a sequence $0 \leq z^i_s(n) \leq x^{j-}_s(n)$ such that $\sum_{j\neq i} z^j_s(n) = \sum_{j\neq i} x^{j+}_s(n)$. We define the sequence $(y^i(n))$:

$$\begin{aligned} y^i_s(n) &= \overline{e}_s \text{ if } s \in S^i_n \\ y^i_s(n) &= x^i_s(n) \text{ if } s \notin S^i_n \\ y^j_s(n) &= x^j_s(n) + z^j_s(n) \text{ if } s \in S^i_n \\ y^j_s(n) &= x^j_s(n) \text{ if } s \notin S^i_n \end{aligned}$$

We can check that $\sum_{k=1}^m y^k(n) = \overline{e}$. We have $\inf S^i_n \rightarrow +\infty$, so from the Lemma 2

$$|U^i(y^i(n)) - U^i(x^i(n))| \leq \sum_{s\geq \inf S^i_n} \pi^i_s |u^i(y^i(n)) - u^i(x^i(n))| \rightarrow 0$$

and

$$\begin{aligned} U^j(y^j(n)) - U^j(x^j(n)) &= \sum_{s\in S^i_n} \pi^i_s [u^j(y^j_s(n)) - u^j(x^j_s(n))] \\ &\geq \sum_{s\in S^i_n} \pi^j_s u^{j\prime}(x^j_s(n) + z^j_s(n)) z^j_s(n) \\ &\geq u^{j\prime}(0) \sum_{s\in S^i_n} \pi^j_s z^j_s(n) \geq 0 \end{aligned}$$

So $\lim_{n\rightarrow\infty} U^i(y^i(n)) = v^i$ and for n great enough, for all s, we have $|y^i_s(n)| \leq M(m-1)|\overline{e}_s|$. By induction, in applying the same method, we can construct our sequence with the properties desired. We have the sequence $(y^i(n)) \in A$ satisfy:

$$\begin{gathered} \lim_{n\rightarrow\infty} U^i(y^i(n)) = v^i \\ \exists M > 0 \text{ such that} \|y^i(n)\|_\infty < M \end{gathered}$$

From Proposition 1, we can suppose that $\lim y^i(n) = y^i$ in the L^1. $\|y^i(n)\|_\infty < M \Rightarrow y^i \in L^\infty$ for all i. Then we have $(y^1, y^2, ..., y^m) \in A$ with $U^i(y^i) \geq v^i$, then $(v^1, v^2, ..., v^m) \in U$. U is closed and bounded in L^p, so U is compact. ■

Now we will drop the condition of Assumption 2, $b^i = +\infty$ for all i, we will prove that with only Assumption 1, there is an quasi-equilibrium in L^1.

Theorem 3 *With Assumption 1, there is an quasi-equilibrium in L^1.*

Proof: We construct the sequence of utilities concave functions $u^i_N : \mathbb{R} \to \mathbb{R}$ such that $u^i_N(x) = u^i(x)$ with $x \in [-N, +\infty)$, for all N, $u^{i\prime}_N(-\infty) = \infty$ and $u^i_N \leq u^i_{N+1}$. Remark that $\forall x$, $\lim_{N\to\infty} u^i_N(x) = u^i(x)$.
From the Theorem 2 and [2] we know that for N sufficiently large such that $\overline{x}^i_s, e^i_s \in [-N, +\infty)$ $\forall i, s$, there exists an equilibrium general $(p^*(N), x^{i*}(N))$. From the Theorem 1, we know that $(x^{i*}(N))$ is in a compact set of the topology L^1, and for all $\epsilon > 0$, there exist $N_0 > 0$ such that for all N we have:

$$\sum_{s\geq N_0} \pi^i_s |x^{i*}_s(N)| < \epsilon$$

$\Rightarrow$ in L^1, the sequence $(x^{i*}(N))$ converge to (x^{i*}).
And the price sequence $p^*(N)$ converge to p^*.
Suppose that there exist $x^i \in L^1$ such that $U^i(x^i) > U^i(x^{i*})$. Choose $0 < \epsilon < U^i(x^i) - U^i(x^{i*})$. There exist $M > 0$ such that

$$\begin{aligned} \sum_{s=1}^{M} \pi^i_s u^i(x^i_s) &> U^i(x^{i*}) + \epsilon \\ \Rightarrow \lim_{N\to\infty} \sum_{s=1}^{M} \pi^i_s u^i_N(x^i_s) &> U^i(x^{i*}) + \epsilon \end{aligned}$$

so for N sufficiently large we have

$$\sum_{s=1}^{M} \pi^i_s u^i_N(x^i_s) > U^i(x^{i*}) + \epsilon$$

We can choose M very large such that $\sum_{s>M} \pi^i_s |x^{i*}_s(N)| < \epsilon$ for every N. Construct the sequence $(x^i(N))$ satisfy: $x^i_s(N) = x^i_s$ for $s \leq M$, $x^i_s(N) = x^{i*}_s(N)$ if $s > M$, then we have $U^i(x^i(N)) > U^i(x^{i*}) \Rightarrow p^*(N).x^i(N) > p^*(N).x^{i*}$ for every N sufficiently large. Let M and N tend to infinity, we have $p^*.x^i \geq p^*.x^{i*}$.
■

3 The model with continuum states

In this section, we will give a proof with a similar result as the section above. In using a utility function less general than [4], we can have the result without the assumption H4 in their paper. The set of states we use here as Le-Van and Truong-Xuan, the set $[0, 1]$, the consumption set is $L^p([0, 1])$, $1 \leq p \leq \infty$, each agent i has an endowment $e^i(s)$, utility function under the form

$$U^i(x^i) := \int_0^1 u^i(x^i(s))h^i(s)ds$$

We define A and U as in the section above.

Assumption 3 *For all* i, j, $a^i < b^j$.

Assumption 4 $0 < m <\leq \inf_{[0,1]} h^i(s) \leq \sup_{[0,1]} h^i(s) \leq M < +\infty$

Assumption 5 *For all* i, u^i *is concave and* $u^{i\prime}(-\infty) = +\infty$.

Theorem 4 *Under the Assumption 4 and the Assumption 5, there exists equilibrium.*

Lemma 3 *Assume that* $x^i \in L^p$, $i = \overline{1,m}$ *s.t* $\sum_{i=1}^m x^i(s) = \sum_{i=1}^m e^i(s)$ *for all* s, $U^i(x^i) \geq U^i(e^i)$ *for all* i, *then there exist* $C > 0$ *such that for all* i:

$$\int_0^1 |x^i(s)| h^i(s) ds < C$$

Proof: We will using the same method as the section 1. Note $d^i = \int_0^1 h^i(s)ds$. Choose $a > b$ such that $c_1 = \max_i u^{i\prime}(a) < c_2 = \min_i u^{i\prime}(b)$. We have:

$$\begin{aligned}
\int_0^1 u^i(a)h^i(s)ds - \int_0^1 u^i(x^{i+}(s))h^i(s)ds &\geq u^{i\prime}(a) \int_0^1 (a - x^{i+}(s))h^i(s)ds \\
\int_0^1 u^i(b)h^i(s)ds - \int_0^1 u^i(-x^{i-}(s))h^i(s)ds &\geq u^{i\prime}(b) \int_0^1 (b + x^{i-}(s))h^i(s)ds
\end{aligned}$$

$\Rightarrow$

$$u^{i\prime}(b) \int_0^1 x^{i-}(s)h^i(s)ds \leq [u^i(a) - au^{i\prime}(a) - bu^{i\prime}(b)]d^i - U^i(e^i) + \max_j u^{j\prime}(a) \int_0^1 x^{i+}(s)h^i(s)ds$$

$\Rightarrow$

$$[\min_j u^{j\prime}(b) - \max_j u^{j\prime}(a)] \int_0^1 x^{i-}(s)h^i(s)ds \leq C_i + \max_j u^{j\prime}(a) \int_0^1 x^i(s)h^i(s)ds$$

$\Rightarrow$

$$\begin{aligned}
\int_0^1 x^{i-}(s)h^i(s)ds &< \frac{C_i}{\min_j u^{j\prime}(b) - \max_j u^{j\prime}(a)} + \frac{\max_j u^{j\prime}(a)}{\min_j u^{j\prime}(b) - \max_j u^{j\prime}(a)} \int_0^1 x^i(s)h^i(s)ds \\
&< C^1 + C^2 \int_0^1 x^i(s)h^i(s)ds
\end{aligned}$$

$\Rightarrow$

$$\sum_{i=1}^m \int_0^1 x^{i-}(s)h^i(s)ds < mC^1 + C^2 \int_0^1 \overline{e}^i(s)h^i(s)ds =: X$$

So we have for all i,

$$\int_0^1 x^{i-}(s)h^i(s)ds < mC^1 + C^2 \int_0^1 \overline{e}^i(s)h^i(s)ds =: X$$

$$\sum_{i=1}^{m}\int_0^1 x^{i+}(s)h^i(s)ds = \int_0^1 \overline{e}^i(s)h^i(s)ds + \sum_{i=1}^{m}\int_0^1 x^{i-}(s)h^i(s)ds < Y$$

$\Rightarrow$

$$\int_0^1 x^{i+}(s)h^i(s)ds = \int_0^1 \overline{e}^i(s)h^i(s)ds + \sum_{i=1}^{m}\int_0^1 x^{i-}(s)h^i(s)ds < Y$$

Then we have

$$\int_0^1 |x^i(s)|h^i(s)ds = \int_0^1 x^{i+}(s)h^i(s)ds + \int_0^1 x^{i-}(s)h^i(s)ds < C$$

■

Lemma 4 *U is bounded.*

Proof: U is bounded below, from the definition of U. We will prove that U is bounded above. Suppose $(x^1, ..., x^m) \in A$, u^i is concave, increasing, so we have:

$$\int_0^1 u^i(x^i(s))h^i(s)ds < d^i u^i(\int_0^1 x^i(s)\frac{h^i(s)}{d^i}ds) < d^i u^i(\frac{C}{d^i})$$

■

Theorem 5 *U is closed.*

Proof: Suppose that there exists a sequence $(x_n^1, x_n^2, ..., x_n^m) \in A$, $\lim_{n\to\infty} U^i(x_n^i) = v^i$, we have to prove that $(v^1, v^2, ..., v^m) \in U$.
Firstly, we show that (x_n^i) is weakly compact in $\sigma(L^1, L^\infty)$. Suppose the converse, then there exists a sequence $X_n \subset [0,1]$ with the Lebegue measure $\mu(X_n) \to 0$ and $\liminf_{n\to} \int_{X_n} |x_n^i(s)|h^i(s) > 0$ for some i. With each s, there exists j such that $x_n^j(s) \leq -\frac{x_n^i(s)-\overline{e}(s)}{m-1}$. Without loosing the generality, we can fixe i, and suppose that on the sequence E_n, $x_n^i(s) > 0$ and $x_n^j(s) \leq -\frac{x_n^i(s)-\overline{e}(s)}{m-1}$. Then we can fixe i, j and a subset E_n such that:

$$x_n^j(s) \leq -\frac{x_n^i(s) - \overline{e}(s)}{m-1} \text{ for all } n \text{ and all } s \in E_n$$

$$\lim_{n\to\infty}\int_{E_n} x_n^i(s)h^i(s)ds = c^i > 0$$

With each M very large, we define the set $S_n \subset E_n = \{s : \frac{x_n^i(s)-\overline{e}(s)}{m-1} > M\}$. Note that $\lim_{n\to\infty}\int_{S_n} x_n^i(s)h^i(s)ds = c^i$. Define the new sequence $(y_n^k(s))$ as below:

$$\begin{aligned} y_n^i(s) &= x_n^i(s) - \frac{x_n^i(s) - \overline{e}(s)}{m-1} + M \text{ on } S_n \\ y_n^j(s) &= x_n^j(s) + \frac{x_n^j(s) + \overline{e}(s)}{m-1} - M \text{ on } S_n \\ y_n^k(s) &= x_n^k(s) \text{ with other } k \text{ or } s \end{aligned}$$

Note that $\sum_k y^k = \overline{e}$. As in the section above, we will estimating $U^k(y^k))$ with $k = i, j$.

$$\begin{aligned} U^i(y_n^i) - U^i(x_n^i) &\geq \int_{S_n} u^{i\prime}(y_n^i(s))(y_n^i(s) - x_n^i(s))h^i(s)ds \\ &\geq -\int_{S_n} u^{i\prime}(M)(\frac{x_n^i(s) - \overline{e}(s)}{m-1})h^i(s)ds \end{aligned}$$

$\Rightarrow$

$$\liminf_{n\to\infty} U^i(y_n^i) \geq v^i - \frac{u^{i\prime}(M)c^i}{m-1}$$

$$\begin{aligned} U^j(y_n^j) - U^j(x_n^j) &\geq \int_{S_n} u^{j\prime}(y_n^j(s))(y_n^j(s) - x_n^j(s))h^j(s)ds \\ &\geq \int_{S_n} u^{j\prime}(-M)(\frac{x_n^i(s) - \overline{e}(s)}{m-1})h^j(s)ds \end{aligned}$$

$\Rightarrow$

$$\liminf_{n\to\infty} U^j(y_n^j) \geq v^j + \frac{v^{j\prime}(-M)c^j}{m-1}$$

and $\lim_{n\to\infty} U^k(y_n^k) = v^k$ for others k.
So we have constructed the sequence (y_n^k) with $U^k(y_n^k)$ is bounded below, and if we let $M \to \infty$, our sequence is unbounded above because $u^j(-\infty) = \infty$, that leads us to a contradiction.
Then the sequence $(x_n^1, x_n^2, ..., x_n^m)$ is $\sigma(L^1, L^\infty)$ compact.
With each $\underline{M}$, denote the set $T_n = \{s : |x_n^i(s)| < \underline{M}$ for all $i\}$. We can choose $\underline{M}$ sufficiently large such that Lebesgue measure $\mu(T_n) > \frac{1}{2}$. Choose $\overline{M}$ very large such that for all i, $u^i(-\underline{M})h_2 < h_1 u^i(-\overline{M})$. Define $E_n^i = \{s : \frac{|x_n^i(s)| - \overline{e}(s)}{m-1} > \overline{M}\}$.
Firstly, we consider the case that there exists i, $\liminf_{n\to\infty} \mu(E_n^i) > 0$. Suppose not, then we can find i such that $\lim_{n\to} \mu(E_n^i) = c^i > 0$. Without loosing the generality, we can assume $x_n^i(s) > 0$ on E_n^i. Using the same argument as above, we assume that there exist j and $S_n \subset E_n^i$ satisfy:

$$\begin{aligned} &x_n^j(s) \leq \frac{x_n^i(s) - \overline{e}(s)}{m-1} \text{ for all } s \in S_n \\ &\lim_{n\to\infty} \mu(S_n) = c > 0 \end{aligned}$$

Construct the sequence (y_n^k) as:

$$\begin{aligned} y_n^i(s) &= x_n^i(s) + \alpha \text{ on } T_n \\ y_n^i(s) &= x_n^i(s) - \frac{x_n^i(s) - \overline{e}(s)}{m-1} + \overline{M} \text{ on } S_n \\ y_n^j(s) &= x_n^j(s) - \alpha \text{ on } T_n \\ y_n^j(s) &= x_n^j(s) + \frac{x_n^i(s) - \overline{e}(s)}{m-1} - \overline{M} \text{ on } S_n \\ y_n^k(s) &= x_n^k(s) \text{ for others } k \text{ or } s \end{aligned}$$

Now we estimate $U^i(y^i_n)$ and $U^j(y^j_n)$:

$$U^i(y^i_n) - U^i(x^i_n) \geq \alpha \int_{T_n} u^{i\prime}(x^i_n(s)+\alpha)h^i(s)ds -$$
$$\int_{S_n} u^{i\prime}(x^i_n(s) - \frac{x^i_n(s)-\bar{e}(s)}{m-1} + \overline{M})(\frac{x^i_n(s)-\bar{e}(s)}{m-1} + \overline{M})$$
$$\geq \alpha u^{i\prime}(\underline{M}) \int_{S_n} h^i(s)ds - u^{i\prime}(\overline{M}) \int_{S_n} (\frac{x^i_n(s)-\bar{e}_s}{m-1} - \overline{M})$$

then we have $\liminf_{n\to\infty} U^i(y^i_n) > \lim_{n\to\infty} U^i(x^i_n) = v^i$.

$$U^j(y^j_n) - U^j(x^j_n) \geq -\alpha \int_{T_n} u^{j\prime}(x^j_n(s)-\alpha)h^i(s)ds +$$
$$\int_{S_n} u^{j\prime}(x^j_n(s) + \frac{x^i_n(s)-\bar{e}(s)}{m-1} - \overline{M})(\frac{x^i_n(s)-\bar{e}(s)}{m-1} - \overline{M})$$
$$\geq -\alpha \int_{T_n} u^{j\prime}(-\underline{M})h_2 ds + \int_{S_n} h_1 u^{j\prime}(-\overline{M})ds$$

We have $u^{j\prime}(-\overline{M}) > h_2/h_1 u^{j\prime}(-\underline{M})$, then we have $\liminf U^j(y^j_n) > v^j$. We have constructed the sequence (y^k_n) such that $\sum_k y^k_n = \bar{e}$, $\liminf U^k(y^k_n) \geq v^k$ with the strict inequality when $k = i, j$. Choose $\epsilon > 0$ such that $\liminf_{n\to\infty} U^i(y^i_n) - \epsilon u^{i\prime}(-\underline{M})h_2 > v^i$. Fix $k \neq i$, define a new sequence (z^l_n) as:

$$\begin{aligned} z^i_n(s) &= y^i_n(s) - \epsilon \\ z^k_n(s) &= y^k_n(s) + \epsilon \\ z^l_n(s) &= y^l_n(s) \text{ in others cases} \end{aligned}$$

With the sequence (z^l_n), we have:

$$\begin{aligned} U^i(z^i_n) - U^i(y^i_n) &\geq -\epsilon \int_{T_n} u^{i\prime}(y^i_n(s)-\epsilon)h^i(s)ds \\ &\geq -\epsilon u^{i\prime}(-\underline{M})h_2 \end{aligned}$$

$\Rightarrow \liminf_{n\to\infty} U^i(z^i(n)) \geq \liminf_{n\to\infty} U^i(y^i_n) - \epsilon u^{i\prime}(-\underline{M})h_2 > v^i$.

$$\begin{aligned} U^k(z^k_n) - U^k(y^k_n) &\geq \epsilon \int_{T_n} u^{k\prime}(y^k_n(s)+\epsilon)h^k(s)ds \\ &\geq \frac{1}{2}\epsilon u^{k\prime}(\underline{M})h_1 > 0 \end{aligned}$$

$\Rightarrow \liminf_{n\to\infty} U^k(z^k_n) > v^k$.
By induction, we can construct the sequence (z^k_n) such that for all k, $\liminf_{n\to\infty} U^k(z^k_n) > v^k \Rightarrow$ there exists n such that for all k, $U^k(z^k_n) > v^k \Rightarrow (v^1, v^2, ..., v^m) \in U$.
Now we consider the case for all i, $\lim_{n\to\infty} \mu(E^i_n) = 0$. In this case, we will construct a sequence satisfy the properties: $\liminf_{n\to\infty} U^i(y^i_n) = v^i$ and $\sup_n \sup_s |y^i_n(s)| < +\infty$. If those properties are true for a sequence (x^i_n), we

have nothing to do, in the converse case, there exist i such that for all M, there exist an infinite n with $s \in E_n^i$, $\mu(E_n^i) > 0$, s.t $x_n^i(s) > M$. Define i, $\overline{M}$, S_n, as above, remarks that $\sum_{j\neq i} x_n^j(s) = \overline{e}_s - x_n^i(s) < 0$. Then we have $0 \leq \sum_{j\neq i} x_n^{j+}(s) < \sum_{j\neq i} x_n^{j-}(s)$. Then there exists a sequence $0 \leq z_n^i(s) \leq x_n^{j-}(s)$ such that $\sum_{j\neq i} z_n^j(s) = \sum_{j\neq i} x_n^{j+}(s)$. We define the sequence (y_n^i):

$$
\begin{aligned}
y_n^i(s) &= \overline{e}(s) \text{ if } s \in S_n^i \\
y_n^i(s) &= x_n^i(s) \text{ if } s \notin S_n^i \\
y_n^j(s) &= x_n^j(s) + z_s^j(n) \text{ if } s \in S_n^i \\
y_n^j(s) &= x_n^j(s) \text{ if } s \notin S_n^i
\end{aligned}
$$

We can check that $\sum_{k=1}^m y_n^k = \overline{e}$. We have $\mu(S_n^i) \to +\infty$, so from the Lemma 2

$$
|U^i(y_n^i) - U^i(x_n^i)| \leq \int_{S_n^i} |u^i(y_n^i) - u^i(x_n^i)| h^i(s) ds \to 0
$$

and

$$
\begin{aligned}
U^j(y_n^j) - U^j(x_n^j) &= \int_{S_n^i} [u^j(y_n^j(s)) - u^j(x_n^j(s))] h^i(s) ds \\
&\geq \int_{S_n^i} u^{j\prime}(x_n^j(s) + z_n^j(s)) z_n^j(s) h^j(s) ds \\
&\geq u^{j\prime}(0) \int_{S_n^i} z_n^j(s) h^j(s) ds \geq 0
\end{aligned}
$$

So $\lim_{n\to\infty} U^i(y^i(n)) = v^i$ and for n great enough, for all s, we have $|y_n^i(s)| \leq M(m-1)|\overline{e}(s)|$. By induction, in applying the same method, we can construct our sequence with the properties desired. We have the sequence $(y^i(n)) \in A$ satisfy:

$$
\lim_{n\to\infty} U^i(y_n^i) = v^i
$$
$$
\exists M > 0 \text{ such that} \|y_n^i\|_\infty < M
$$

Then we have the sequence (y_n^i) is $\sigma(L^\infty, L^1)$ compact. We can suppose that $y_n^i \to y^i \in L^\infty$. And $U^i(y^i) \geq v^i$ for all $i \Rightarrow (v^1, v^2, ..., v^m) \in U$. ■

Theorem 6 *U is compact.*

Proof: From Lemma 4 and Theorem 5. ■

4 The case of finite countable states

There are m agents indexed by $1, \ldots, m$, each agent has a consumption set $X_i \subset \mathbb{R}^k$, a vector of endowment e_i and a continuous concave utility function $u^i : \mathbb{R}^k \to \mathbb{R}$. We first recall some standard concepts of general equilibrium theory.

The set of individually rational attainable allocations A is defined by

$$A = \{(x^i) \in (\mathbb{R}^k)^m \mid \sum_{i=1}^{m} x_i = \sum_{i=1}^{m} e_i \text{ and } u^i(x_i) \geq u^i(e_i) \text{ for all } i.\}$$

Definition 1 *A pair $((x_i^*)_{i=1}^m, p^*) \in A \times \mathbb{R}^k$ is a contingent Arrow - Debreu equilibrium if*

1. *for each agent i and $x^i \in \mathbb{R}^k$, $u^i(x_i) > u^i(x_i^*)$ implies $p^* \cdot x_i > p^* \cdot x_i^*$,*
2. *for each agent i, $p^* \cdot x^{i*} = p^* \cdot e^i$.*

For $x \in \mathbb{R}^k$, let

$$\widehat{P}^i(x) = \{y \in \mathbb{R}^s \mid u^i(y) \geq u^i(x)\}$$

and let R^i be its recession cone. R^i is called the set of *useful vectors* for i and is defined as

$$R^i = \{w \in \mathbb{R}^S \mid u^i(x + \lambda w) \geq u^i(x), \text{ for all } \lambda \geq 0\}$$

The lineality space of i is defined by

$$L^i = \{w \in \mathbb{R}^l \mid u^i(x + \lambda w) \geq u^i(x), \text{ for all } \lambda \in \mathbb{R}\} = R^i \cap -R^i$$

Elements in L^i will be called *useless vectors* .

The *no unbounded arbitrage* condition denoted from now on by NUBA is introduced by Page (1987).

Definition 2 *The economy satisfies the NUBA condition if $\sum_{i=1}^m w^i = 0$ and $w^i \in R^i$ for all i implies $w^i = 0$ for all i.*

There exists a weaker condition, called the *weak no market arbitrage* condition (WNMA), introduced by Hart[1974].

Definition 3 *The economy satisfies the WNMA condition if $\sum_{i=1}^m w^i = 0$ and $w^i \in R^i$ for all i implies $w^i \in L^i$ for all i.*

We will prove the propositions that give us the similarity under the NUBA condition and the WNMA condition. Choose θ sufficiently large such that $\|\hat{e}_i\| \leq \theta$ for all i. Define $T_i^\theta := \{t \in L_i \mid \|t\| \leq \theta\}$. We define the new economy $\tilde{\mathcal{E}}^\theta = (\tilde{X}_i^\theta, \tilde{u}_i, e_i)$ such that $\tilde{X}_i^\theta := L_i^\perp \cap T_i^\theta$, $\tilde{u}^i : \mathbb{R}^k \to \mathbb{R}$ defined as the restriction of u^i on $\tilde{X}_i^\theta$. Evidently, we have $e_i \in \tilde{X}_i^\theta$ for all i.

Proposition 2 *If $((\tilde{x}_i^*)_{i=1}^m, \tilde{p}^*)$ is an equilibrium of $\tilde{\mathcal{E}}$ then $((\tilde{x}_i^*)_{i=1}^m, \tilde{p}^*)$ is equilibrium of $\mathcal{E}$.*

Proof: We first prove that $p^* \in \bigcap_{i=1}^m L_i^\perp$. For each i, there exist ϵ_i such that $u^i(\tilde{x}_i^* + \epsilon_i) > u^i(\tilde{x}_i^*)$. $\forall\, y_i \in T_i$, $u^i(\tilde{x}_i^* + \epsilon_i + y_i) > u^i(\tilde{x}_i) \Rightarrow \tilde{p}^*.(\tilde{x}_i^* + \epsilon_i + y_i) > \tilde{p}^*.\tilde{x}_i^*$. Let $\epsilon_i \to 0$, we have $\tilde{p}^*.y_i \geq 0$. With the similar argument, we found that $\tilde{p}^*.(-y_i) \geq 0 \Rightarrow \tilde{p}^*.y_i = 0 \;\forall\, y_i \in T_i^\theta \Rightarrow \tilde{p}^* \in L_i^\perp \;\forall\, i$.
Observe that $((\tilde{x}_i^*)_{i=1}^m, \tilde{p}^*)$ is equilibrium of $\tilde{\mathcal{E}} \Rightarrow \sum_i \tilde{x}_i^* = \sum_i e_i$. Now let $u^i(x^i) > u^i(\tilde{x}_i^*) \Rightarrow u^i(x_i^\perp) > u^i(\tilde{x}_i^*) \Rightarrow \tilde{p}^*.x_i^\perp > \tilde{p}^*.\tilde{x}_i^* \Rightarrow \tilde{p}^*.(x_i^\perp + \hat{x}_i) > \tilde{p}^*.\tilde{x}_i^*$. So $((\tilde{x}_i^*)_{i=1}^m, \tilde{p}^*)$ is equilibrium of $\mathcal{E}$. ■

Proposition 3 *If $((x_i^*)_{i=1}^m, p^*)$ is an equilibrium of $\mathcal{E}$, then there exists $\theta > 0$ such that $((x_i^*)_{i=1}^m, p^*)$ is equilibrium of $\tilde{\mathcal{E}}^\theta$.*

Proof: Choose $\theta \geq \max\{\|x_i^*\|, \|\hat{e}_i\|\}$. ■

Proposition 4 *The economy $\mathcal{E}$ satisfies Weak No Market Arbitrage condition if and only if $\tilde{\mathcal{E}}$ satisfies No Unbounded Arbitrage condition.*

Proof: Firstly, suppose that $\mathcal{E}$ satisfies WNMA condition. In the economy $\tilde{\mathcal{E}}$, $L_i^\theta = \{0\}$, so $R_i^\theta = R_i \cap L^\perp \;\forall\, i$. Suppose that $w_i \in R_i^\theta$ such that $\sum_i w_i = 0 \Rightarrow w_i \in L_i$ for all i, $\Rightarrow w_i \in L_i^\perp \cap L_i \Rightarrow w_i = 0 \;\forall\, i$.
Suppose that $\tilde{\mathcal{E}}$ satisfies NUBA condition. If $w_i \in R_i$ such that $\sum_i w_i = 0$, then we have $\sum_i w_i^\perp = 0 \Rightarrow w_i^\perp = 0$ for all i from the NUBA properties $\Rightarrow w_i \in L_i$ $\forall i$. ■

Now we define the notion of no-arbitrage price as in Allouch, Le Van, Page (2002) and the NAPS notion:

Definition 4 $S_i = \left\{ \begin{matrix} \{p \in L_i^\perp \mid p.w > 0, \forall\; w \in (R_i \cap L_i^\perp)\backslash\{0\} \; \textit{if } R_i \backslash L_i \neq \emptyset\} \\ L_i^\perp \; \textit{if } R_i = L_i \end{matrix} \right\}$

Definition 5 *The economy $\mathcal{E}$ satisfies the NAPS condition if $\cap_i S_i \neq \emptyset$.*

Proposition 5 *(Page and Wooders, 1996) Assume $L_i = \{0\}$, $\forall\ i$, then NUBA $\Rightarrow$ NAPS.*

Proof: In [5] ∎

Proposition 6 *(Allouch, LeVan and Page (2002))*
WNMA $\Rightarrow \cap_i S_i \neq \emptyset$.

Proof: In [1] ∎

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