Uncertainty, Learning and Ambiguity in Economic Models on Climate Policy: Some Classical Results and New Directions*

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Abstract: We present how uncertainty and learning are classically studied in economic models. Specifically, we study a standard expected utility model with two sequential decisions, and consider two particular cases of this model to illustrate how uncertainty and learning may affect climate policy. While uncertainty has generally a negative effect on welfare, learning has always a positive, and thus opposite, effect. The effects of both uncertainty and learning on decisions are less clear. Neither uncertainty nor learning can be used as a general argument to increase or reduce emissions today without studying the specific intertemporal costs and benefits explicitly. Considering limits in applying the expected utility framework to climate change problems, we then consider a more recent framework with ambiguity-aversion which accounts for situations of imprecise or multiple probability distributions. We discuss both the impact of ambiguity-aversion on decisions and difficulties in applying such a non-expected utility framework to a dynamic context.

Keywords: uncertainty, learning, ambiguity-aversion, risk-aversion, climate change.

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1. INTRODUCTION

Climate policy decisions today have to be made under substantial uncertainty: the impact of accumulating greenhouse gases in the atmosphere is not perfectly known, the future economic and social consequences of climate change, in particular the valuation of possible damages, are uncertain. However, learning will change the basis of making future decisions on abatement policies.

The issues of uncertainty and learning are often presented in a colloquial sense. Two opposing effects are typically put forward: First, uncertainty about future climate damage, which is often associated with the possibility of a catastrophic scenario is said to give a premium to slow down global warming and therefore to increase abatement efforts today. Second, learning opportunities will reduce scientific uncertainty about climate damage over time. This is often used as an argument to postpone abatement efforts until new information is received. The effects of uncertainty and learning on the optimal design of current climate policy are still much debated both in the academic and the political arena.

In this paper, we present how uncertainty and learning are classically studied in economics. The characterization of uncertainty and learning in economics relates to early concepts introduced in mathematics and statistics. We believe that there is an interest in introducing these formal concepts to an interdisciplinary audience. Indeed what we present is now a common and broadly accepted approach in economics to formally study the effects of uncertainty and learning. Moreover, we illustrate how one can apply this approach to give insights into the climate change problem.

We proceed as follows. We first define the concepts of uncertainty and learning within the classical framework of economic decision theory, namely the (Bayesian) expected utility framework. We consider a two-decision model that encompasses most existing microeconomics models that have analyzed the effects of uncertainty and learning. For the sake of illustration, we introduce two particular examples of this model. One example is a “climate change” model, and the other example is a “resource depletion” model. We show that the attitude of a decision-maker towards risk and the type of payoff function are instrumental to the sign of the effect of uncertainty and learning on optimal emissions reductions. Specifically, our results indicate that, compared to the reduction of emissions under certainty, uncertainty and learning generally cannot provide a clear argument for stricter abatement of emissions today or their postponement.

While standard economic decision theory relies on an expected utility framework, empirical and experimental data have long suggested that this framework fails to explain observed individuals choices. In particular, the preferences of individuals in situations of imprecise or multiple probabilities are often not consistent with a single (objective or subjective) probability distribution as usually assumed by the theory of expected utility. Climate change policy is a classical example involving imprecise probabilities. Predictions are derived from different models whose results are often presented as a range of probabilities for a single event (IPCC, 2005).

We thus consider an alternative to the expected utility framework that accounts for this type of uncertainty over probabilities, or "ambiguity". We indeed consider that the study of the effect of ambiguity is a promising direction of research. We thus generalize the previous framework to illustrate some immediate implications of ambiguity for climate policy. We show that ambiguity typically leads to stricter abatement policies today. We also point out difficulties in applying such a non-expected utility theory to a dynamic framework where beliefs should be updated frequently to account for new information. We conclude the paper with a word of caution: the optimal response of climate policy to uncertainty and learning is sensitive to which decision theoretical framework is used. Furthermore, alternatives to the standard expected utility framework may have better descriptive power but also can generate unappealing normative effects.

2. THE BASIC EXPECTED UTILITY FRAMEWORK

We first consider a basic decision theoretic framework to study the effects of uncertainty and learning. We concentrate on a model where decisions have to be made at two different points in time. Let x_t denote the decision in period $t \in \{1, 2\}$.[1] The ex post utility derived from decisions x_t is denoted by $v(x_1, x_2, \theta)$ where the parameter θ captures uncertainty, i.e. its realization might not be known when decisions are made.

[1] Throughout the paper, and in line with much of the literature, we assume that x_t is a member of $\mathbb{R}$.

The properties of the function $v(.)$ capture the decision-maker (DM)'s preferences (e.g., the attitude towards risk) as well as characteristics of the economic environment, as we will see. Throughout the paper, we will consider two simple examples.

Example 1. *A two-period "climate change model":*

$$v(x_1, x_2, \theta) = u(x_1) + u(x_2) - (1-\theta)d(x_1 + x_2) \tag{1}$$

> *in which $u(.)$ is the instantaneous utility derived from emissions in each period and $d(.)$ is the damage from climate change occurring in period 2. The extent of the (unknown) future damages is captured by the parameter $\theta \in [0,1]$. The function $u(.)$ is assumed to be increasing and concave as, e.g., emission-intensive production can increase consumption and thereby utility albeit at decreasing marginal returns. The function $d(.)$ is assumed to be increasing and convex, reflecting environmental damages due to climate change which are the more sensitive to increases in emissions, the larger the stock of greenhouse gases in the atmosphere already is. Both functions are assumed to be continuously differentiable.*

Example 2. *A three-period "resource depletion model":*

$$v(x_1, x_2, \theta) = u(x_1) + u(x_2) + u(\theta - x_1 - x_2) \tag{2}$$

> *in which $u(.)$ is the utility derived from resource consumption in each period and θ is the total (unknown) stock size of the resource. The utility function $u(.)$ is assumed to be increasing and concave, and continuously differentiable. This example can be interpreted in terms of climate policy as reflecting the intergenerational problem of "consuming the atmosphere": the more climate change limits future production, i.e. the smaller θ, and the more emission-intensive products we consume today, the smaller are future consumption opportunities.*

In both examples we neither discount the flow of future utilities nor consider a limited atmospheric lifetime of greenhouse gases, i.e. we only consider the sum of period utilities and let the sum of emissions determine the future damages. While discount and atmospheric decay factors play a key role in the analysis of intertemporal climate policy decisions, their qualitative role in understanding the effect of uncertainty and learning is usually small and will be neglected here.

In the following, we first report classical findings under the assumption that the DM maximizes expected utility. Namely, we assume that the DM has beliefs over θ, that can be captured by a probability distribution, and maximizes the expected value of the utility function $v(x_1, x_2, \theta)$ based on using this probability distribution. Formally, the two-decision model is represented by the following optimization program:

$$V_U = \max_{x_1} \max_{x_2} E_{\tilde{\theta}} v(x_1, x_2, \tilde{\theta}) \tag{3}$$

where $E_{\tilde{\theta}}$ denotes the expectation operator over the distribution of $\tilde{\theta}$. Notice that expected utility was first axiomatized by von Neumann and Morgenstern (1944) and is the common framework that is used in economic decision theory. We will, however, consider an alternative framework in section 4. In the main text of the paper, we provide the basic ideas and a summary of findings of economic literature. A more formal analysis can be found in the appendix.

3. THE EFFECT OF UNCERTAINTY AND LEARNING

3.1 The effect of uncertainty

Our first objective in this section is to present the effect of uncertainty. To do so, it is natural to compare the case of uncertainty represented in (3) to a case of certainty. This (hypothetical) case of certainty is usually constructed by replacing an uncertain model parameter by its expected value. In our case, we hereby assume that the random variable $\tilde{\theta}$ can only take one value and therefore is replaced by its mean $E_{\tilde{\theta}}\tilde{\theta}$. The optimization problem in the certainty case is therefore given as follows:

$$V_C = \max_{x_1} \max_{x_2} v(x_1, x_2, E_{\tilde{\theta}}\tilde{\theta}) \tag{4}$$

We first study the effect of uncertainty on the value of the program, i.e. on “welfare”, and then on optimal decisions.

To study the effect of uncertainty on welfare, we must compare V_U and V_C. It is easy to see that V_C is always larger than V_U if and only if $v(x_1, x_2, \theta)$ is concave in θ.[2] This immediately implies that $V_C = V_U$ in Example 1, i.e. there is no effect of uncertainty since $v(.)$ is linear in θ.[3] In Example 2, we have $V_C \geq V_U$ by the concavity of $u(.)$. The concavity of the instantaneous utility is usually interpreted as risk-aversion. Risk-aversion in Example 2 means that the DM prefers that the size of the stock of the resource is equal to $E_{\tilde{\theta}}\tilde{\theta}$ rather than random and distributed as $\tilde{\theta}$. Equivalently, it means that the DM would be willing to pay an insurance premium to convert the uncertain resource stock $\tilde{\theta}$ into a certain one $E_{\tilde{\theta}}\tilde{\theta}$. Risk-aversion is obviously a fundamental concept in risk theory (Pratt, 1964; Arrow; 1971; Mas-Collel, Whinston and Green, 1995; Eeckhoudt, Gollier and Schlesinger, 2005).

We now study the effect of uncertainty on decisions, in particular on x_1. In Example 1, the linearity of the utility function in θ immediately implies that uncertainty has no effect on x_1. In Example 2, uncertainty may have an effect on the initial consumption x_1. It is in fact the case unless the utility function is quadratic. The general result is that uncertainty decreases initial consumption if and only if marginal utility $u'(.)$ is convex, or $u'''(.) \geq 0$. This condition on the utility function is sufficient to induce a precautionary savings motive in standard microeconomic models of consumption and savings decisions (see, e.g., Leland 1968), to which Example 2 could be seen as the simplest illustration. Kimball (1990) coined the term "prudence" to refer to this condition. Notice that in Example 2 the effect of uncertainty on welfare depends on the second derivative of the utility function, while the effect of uncertainty on decisions depends on the third derivative of the utility function. This illustrates that, in expected utility theory, the various derivatives of the utility function capture fundamental but different economic aspects of risk preferences. Specifically the second derivative measures the intensity to which the DM wants to "escape" uncertainty, while the third derivative usually measures the direction and the intensity of the DM's response to uncertainty.

We emphasize here that the climate change model presented Example 1 does not capture some of the aspects related to risk preferences usually considered in the economic literature.

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[2] This immediately follows from Jensen's inequality which states that $E_{\tilde{\theta}} f(\tilde{\theta}) \leq f(E_{\tilde{\theta}}\tilde{\theta})$ if and only if f is concave.

[3] See the appendix for a formal analysis.

Indeed the DM's utility function displays linearity in θ; in other words, the DM is risk-neutral. Still, we will see in the next section that learning may have an effect on the DM welfare and decisions even under risk-neutrality.

Notice finally that we have considered extreme comparisons, i.e. studied the case of uncertainty and certainty. There exist more general notions of partial uncertainty and of increasing uncertainty (Rothschild and Stiglitz 1970, 1971) that we briefly present in the appendix.

3.2 The effect of learning

Our next objective is to study the effect of learning. To do so, it is standard to compare the case of uncertainty represented in (3) to the case of learning in which the DM is informed about the value of $\tilde{\theta}$ before making the decision in period 2. That is, the DM learns the realization of $\tilde{\theta}$ before the period 2 decision, x_2, but after the period 1 decision, x_1.[4] Formally, in the learning case, the optimization program is thus given by:

$$V_L = \max\nolimits_{x_1} E_{\tilde{\theta}} \max\nolimits_{x_2} v(x_1, x_2, \tilde{\theta}) \tag{5}$$

As before, we analyze the effect of learning on welfare first and then on decisions. Comparing V_L to V_U, it is obvious that[5]

$$E_{\tilde{\theta}} \max\nolimits_{x_2} v(x_1, x_2, \tilde{\theta}) \geq \max\nolimits_{x_2} E_{\tilde{\theta}} v(x_1, x_2, \tilde{\theta}) \tag{6}$$

This implies that the DM always prefers the situation in which there is learning, that is, the situation in which he can optimally adjust x_2 to the realized value of the random variable θ.

Consequently, by taking the maximand over x_1 of each expression in (6) we indeed get $V_L \geq V_U$. In other words, the value of information is always positive. This general result holds in fact as long as the DM is an expected utility maximizer (see, e.g., Marschak and Miyasawa 1968).[6] It should be noted that for the case of perfect information not only the ex ante

[4] We here only consider the case in which new information is acquired exogenously over time, e.g. by independent scientific progress. We do not consider active experimentation, that is, a situation where the learning rate is influenced by the decisions in the first period.
[5] Note that the max operator is convex and we can again use Jensen's inequality.
[6] Note that we only consider the case of a single decision maker. As an example of an analysis of interaction of multiple decision-makers, see Kolstad and Ulph (this issue).

expected utility, but also the ex post utility are increased compared to no-learning. This is generally different if the information does not perfectly reveal the parameter θ, as we discuss in the online material. In such a case, information can be misleading: for example, the new pieces of information suggest a smaller probability of severe climate damages, therefore leading to less strict climate policy, while in the end damages turn out to be immense.[7] Oppenheimer et al. (this issue) refer to instances in which information is misleading in this manner as “negative learning”.

While the value of information is always positive, the impact of learning on decisions is generally ambiguous. That is, the optimal x_1 in (3) may be larger or smaller than the optimal x_1 in (5), depending on the value of the parameters of the model.

Indeed in Example 1, the effect of learning is ambiguous in general, even under risk-neutrality. It can easily be shown, however, that for quadratic utility and for quadratic damage functions perfect learning leads to an increase, and not a decrease, of first period emissions compared to the case of uncertainty (see Ulph and Ulph, 1997). Hence, this gives an example in which the prospect of learning over time provides a rationale for emitting more, and not less, pollution today. The intuition is that, under learning, there is an incentive to delay emissions reduction efforts. This allows future reduction efforts to be adapted to the severity of the climate risk that will be known in the future due to learning. It should be noted, however, that this result is not general, namely the positive effect of learning on emissions is not robust to other (non quadratic) functional forms.

In Example 2, it can be shown that learning increases resource depletion compared to the case of uncertainty if the third derivative of u is positive, that is under the condition of prudence (Eeckhoudt, Gollier and Treich, 2005). The intuition is that learning in the future will allow perfect consumption smoothing which operates as a reduction of the future uncertainty. Therefore it makes sense that the condition of prudence is instrumental here as well; indeed we know from the previous section that the effect of uncertainty on early consumption depends on the condition of prudence. Hence for “imprudent” consumers the result is reversed, in the sense that learning decreases early consumption.

[7] There are few other general results on the determinants and the magnitude of the value of information however (Hirshleifer and Riley, 1992). But the value of information has been routinely computed in specific numerical climate-economy models, e.g. Manne and Richels (1992) and Nordhaus (1994).

In summary, we can conclude that, even if perfect learning has always a positive effect on welfare, the qualitative impact on today's decisions is generally ambiguous, and may or may not go in the opposite direction compared to the effect of uncertainty. Moreover observe that we only study the effect of perfect learning, that is the effect of having perfect information about the parameter θ. In the appendix, we present a formal definition for better information that does not require information to be perfect. It is obvious, however, that one cannot expect less ambiguous results on decisions for the case when learning only imperfectly resolves the uncertainty since the results are ambiguous even under perfect learning.

Finally, it is important to indicate that the literature on the effect of learning on early decisions often refers to the "irreversibility effect". There is indeed a general result that learning always favors less irreversible decisions (Arrow and Fisher, 1974, Henry, 1974). We notice that the framework used for analyzing the irreversibility effect is slightly different from the one that we have considered so far. Indeed the analysis of the irreversibility focuses on the effect of the current decision on the future set of choice. We formally discuss the relationship with literature on the irreversibility effect in the appendix.

4. THE EFFECT OF AMBIGUITY

4.1 Introduction to the concept of ambiguity

In our analysis so far we have assumed that the DM behaves as an expected utility maximizer. That is, his beliefs upon the uncertainty parameter $\tilde{\theta}$ are represented by an additive probability measure $\pi \in \Pi$ which is potentially updated according to Bayes rule when new information is received. Preferences are linear in the probabilities. However, one might question the applicability of the expected utility framework to the problem of climate change where no objective probability assessment exists (e.g., IPCC, 1995; Morgan and Keith, 1995). Furthermore, empirical evidence indicates that, when facing uncertainty, people quite often do not maximize expected utility.[8] Such violations have led to the development of several alternatives.

In this section, we report some findings from non-expected utility theories applied to the climate change problem. We focus on theories that deviate from the assumption of *a single*

[8] Prominent examples based on experiments are given by Allais (1953), Ellsberg (1964) and Kahneman and Tversky (1979).

underlying (objective or subjective) probability distribution and instead allow for *multiple priors*. Useful published references on these theories include Gilboa (1987), Schmeidler (1986), Gilboa and Schmeidler (1989), Camerer and Weber (1992), Klibanoff (2001) and Klibanoff et al. (2005). These theories explicitly allow for ambiguity-aversion as opposed to risk-aversion, as we will see. In climate policy, such multiple priors $\pi \in \Pi$ of probability distributions over climate damage θ arise naturally from the use of different models: Predictions from each scientific model are usually given in confidence intervals, i.e. each models generates a probability distribution over outcomes.[9] Similarly, ambiguity naturally arises if experts disagree in their predictions of the future climatic damage. Decision makers therefore have to aggregate findings from these competing models, or from different experts, i.e. they have to deal with a whole set of probability distributions. Under expected utility, decision makers behave as if they still base decisions on a single probability distribution, e.g. by using the average. However observed choice data have often indicated that individuals behave differently: they seem to place excessive weight on the most pessimistic probability distribution (Ellsberg, 1964). Consistent with these observations, Gilboa and Schmeidler (1989) axiomatize a framework in which decisions are derived from the minimal expected utility obtained from a probability distribution in the set of priors,, which represents this focus on the worst case. Ambiguity is then modeled by the size of the set of probability distributions.

Klibanoff, Marinacci and Mukerji (hereafter KMM) (2005) consider a more general ambiguity model than the one of Gilboa and Schmeidler. They model second-order probability distributions, or equivalently "smooth ambiguity". Formally, a DM chooses the action which maximizes $E_{\mu(\pi)}\phi(E_{\pi(\tilde{\theta})}v(x_1, x_2, \tilde{\theta}))$ where μ is a subjective probability over the set $\pi \in \Pi$ of probability distributions over θ and ϕ is an increasing transformation.

Compared to the expected utility framework, the novelty in the KMM framework is therefore the introduction of this function ϕ by which the expected utility measures stemming from the respective probability distributions are evaluated. ϕ thereby captures the attitude towards ambiguity over probability distributions, i.e. towards differences in expected utility measures

[9] Compare, for example, with the policy of the IPCC to account for uncertainties by classifying uncertainties as ranges of probabilities (IPCC, 2005).

implied by the different probability distributions.[10] In particular, a concave ϕ ($\phi'' < 0$) models ambiguity-aversion. Indeed, it can easily be seen that when the probability distributions become “more dispersed” then welfare decreases if and only if ϕ is concave. The qualitative effects of ambiguity-aversion can most easily be seen when relying on a special functional form $\phi(v) = -(1/\alpha)\exp(-\alpha v)$: on the one hand, the limit of ambiguity-neutrality ($\phi(v) = v$) and therefore the standard expected utility framework results for $\alpha \to 0$. On the other hand, KMM’s model yields Gilboa and Schmeidler’s MaxiMin model as a limiting case for infinitely ambiguity-averse DM ($\alpha \to \infty$).

4.2 The effect of ambiguity-aversion

We use the model of smooth ambiguity-aversion (KMM, 2005) to demonstrate how decisions under uncertainty can change when decision makers are ambiguity-averse and thereby deviate from expected utility maximization. We hereby concentrate on optimization programs under uncertainty, analogously to (3):

$$V_U = \max_{x_1} \max_{x_2} E_{\mu(\pi)} \phi(E_{\pi(\tilde{\theta})} v(x_1, x_2, \tilde{\theta})) \tag{7}$$

where ϕ is increasing and concave.

We show in the appendix how the solution to problem (7) depends on the attitude of the DM towards ambiguity. For Example 1, increased ambiguity-aversion is shown to decrease the emission level implemented in period 1 in the case of uncertainty. Since for expected utility maximizers, uncertainty on the damage parameter had no effect on decisions in Example 1 (see section 2.2), ambiguity-aversion can therefore explain why (risk-neutral) decision makers might react to uncertainties regarding future damages by reducing emissions, i.e. increasing abatement efforts.

In Example 2, however, the impact of ambiguity is less clear. Qualitatively, ambiguity-aversion can change the predictions of the effect of uncertainty. We can take quadratic utility functions as an example. For such functions, we have $u'''(.) = 0$, and therefore our results in section 3.1 showed that uncertainty had no effect on decisions if the DM maximizes expected utility, i.e. is ambiguity-neutral. Under ambiguity-aversion, however, uncertainty has an

[10] Note that risk-aversion is still captured by the concavity of v in θ.

ambiguous effect (see the appendix). In particular, one can construct examples where an increased ambiguity-aversion implies more consumption in both periods.

We have thus introduced a more general framework that allows welfare to decrease when probability distributions become different, provided that the DM displays ambiguity-aversion. However, the decisions of an ambiguity-averse DM compared to those of an ambiguity-neutral DM (that is, an expected utility maximizer) can be higher or lower. Nevertheless, Example 1 illustrates a situation in which ambiguity-aversion always leads to less emissions.

4.3 Discussion of further applications of ambiguity-aversion

The analysis above considered the effects of ambiguity-aversion only for the case in which decision makers cannot expect new information before making the decisions in the second period. A complete discussion of perfect and partial learning under ambiguity is beyond the scope of this paper. We therefore only shortly discuss some issues which arise when new information is explicitly considered in such an ambiguity-averse model.

First, the model outlined above which was based on KMM (2005) only applies to evaluating expected utility measures generated from different probability distributions at one particular point in time. This clearly does not cover most applications to the climate change problem, where new information may change the underlying probability distributions and therefore require a reevaluation in future periods. An intertemporal version of ambiguity-aversion model is proposed by KMM (2006). In general, the impact of ambiguity-aversion on decisions under learning again depends on the specific functional forms, i.e. how exactly today's decisions affect current and future utility.

While we will not extensively discuss this framework here in order to limit the technicalities, we want to point out some difficulties which arise when incorporating ambiguity-aversion into an intertemporal decision making context (see Machina, 1989, Epstein and le Breton, 1993, Karni and Schmeidler, 1991). First, it is no longer obvious how to incorporate new information to update beliefs. While in an expected utility framework, prior beliefs are transformed via Bayesian updating, a number of different updating rules exists to deal with a situation under ambiguity, i.e. with multiple priors.[11] Moreover, applying these updated rules in a dynamic framework can induce decisions to be "dynamically inconsistent".[12]

Several ways of dealing with these inconsistencies have been proposed: (i) preferences can be defined in a recursive way, that is ex ante preferences are based on ex post preferences and cannot be reduced to a simple form (e.g., KMM, 2006; Epstein and Schneider, 2003). (ii) preferences can be non-consequentalist, that is future preferences can depend on (non-materialized) branches of the decision tree. Finally, (iii) decisions can be limited to "behaviorally consistent" choices. That is, one accepts the fact that preferences can be dynamically inconsistent but focuses only on decisions which will be carried out given future information. Here, in a decision tree, future choice nodes are replaced by future choices (Karni and Safra, 1989).

The implications of these extensions of ambiguity models in our two-decision framework are largely unexplored and remain subject to future research. A first illustration of the effects for Example 1 is given in Lange (2003) who relies on a behaviorally consistent approach. Lange's results are illustrative of the subtleties of an analysis of multi-period decision problems under ambiguity: dynamic inconsistencies in a behaviorally consistent approach can yield a negative value of learning (see, more generally,Wakker, 1988). That is, for instance in Example 1, ambiguity-averse people could choose not to receive new information on prospective climate damages.

5. SOME IMPLICATIONS FOR CLIMATE POLICY

In this paper, we derived some theoretic results on the effect of uncertainty and learning. We based our analysis on a single individual decision maker, i.e. we did not address strategic interactions between multiple decision-makers. Our two-decision model encompasses most previous microeconomics models of climate change, such as those of Kolstad (1996), Ulph and Ulph (1997) and Gollier, Jullien and Treich (2000) and illustrates the basic determinants

[11] As one starts with multiple probability distributions (e.g. Gilboa and Schmeidler 1989), one could update each prior distribution individually and use all distributions for future decision making, or, alternatively, only keep those probability distributions which gave the observed event the maximal probability. Remember that the probability distributions could stem from sensitivity analyses of different, competing climate models. Now, after observing new events, the question could be to what extent one shifts the decision weights towards those models which were best in predicting the observed event.

[12] That is, viewed from the start of period 1, an optimal policy to implement in period 2 may not be optimal anymore when this policy will be reconsidered at the start of period 2.

of numerical results such as those in the climate economies of Nordhaus (1994) and Ha Duong (1998).

Uncertainty in our framework is captured by an uncertain climate damage parameter. Uncertainty generally has a negative effect on welfare in climate change models in which the climate damages are monetarized and the decision-maker is risk-averse in wealth. Hence, there is an incentive to avoid uncertain situations. In contrast, we showed that the effect of uncertainty on decisions is ambiguous, and depends on how intertemporal costs and benefits are specified (Example 1 vs. Example 2).

When dealing with long-term decision making processes it is, however, important to realize that active research or experimentation as well as passive waiting can yield new information. From an economic perspective, this information is only valuable if it can lead to a change in future decisions. In our model, we therefore explicitly considered the effects of learning, i.e. the ability of decision-makers to gain new information on the uncertain model parameters (e.g., climate damages or future consumption possibilities) before the second period decisions have to be made. Such learning was shown to always have a positive effect on welfare compared the situation of no learning. While the value of information is therefore always positive ex ante, it should be noted that future information can clearly also be misleading, e.g. it could suggest a smaller probability of severe climate damages while in the end damages turn out to be immense. The ex ante positive value of information must therefore be differentiated from the positive or negative ex post consequences of adjusting the climate policy after receiving new information.

Future information affects, however, not only future decisions. Instead, the expectation of receiving new information in the future can already change today’s decisions, e.g. on climate policy. The qualitative effect is less clear: the microeconomics literature has not given a definitive answer to how climate policy decisions should respond to learning. In particular, there is no general support for the argument to delay abatement of emissions if learning is expected.

While these results depend on the assumption of a (Bayesian) expected utility framework, there is some doubt about its applicability to deriving predictions for climate policy: on the one hand, it is well-known that expected utility often fails to explain observed individual decisions under uncertainty. On the other hand, the expected utility framework does not capture the existing ambiguity over the probability distribution to use in climate change models. We therefore discussed a more general model which allows for ambiguity-aversion and in which welfare is reduced when initial priors are more dispersed. Such ambiguity-aversion leads to reduced emissions when utility is linear in the damage parameter while no definite results can be obtained for more general utility specifications.

In conclusion, there is little theoretic support for any claim that uncertainty and learning should affect climate policy in a specific direction. That is, the simple fact of the existence of uncertainty and the potential for learning does not support any strong position either to reduce or to delay emissions abatement. The effect depends not only on the functional forms of the damage and utility functions, but also on the specific modeling approach. From this perspective, the scientific debate on the impact of uncertainty and learning on climate policy is, we believe, mostly an empirical matter. This should, at least, be a word of caution to policy-makers, and to some extent to some environmentalists and politicians.

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APPENDIX

A1. The effect of uncertainty, learning and ambiguity in the "climate change model"

In this section, we give some formal insights on the sign of the effect of uncertainty, learning and ambiguity in the Example 1, that is when the DM's primitive utility equals $v(x_1, x_2, \theta) = u(x_1) + u(x_2) - (1-\theta)d(x_1 + x_2)$.

The first order conditions for problem (3) characterizing optimal emissions under uncertainty, denoted x_1^U, are the same as those for problem (4) and are given by

$$u'(x_1^U) = u'(x_2^U) = (1 - E_{\tilde{\theta}}\tilde{\theta})d'(x_1^U + x_2^U) \qquad \text{(A1)}$$

Consequently, emissions under uncertainty are not different from emissions under certainty.

We now study the effect of learning. The first order conditions for problem (5) are given by

$$\begin{aligned} u'(x_1^L) &= E_{\tilde{\theta}}(1-\tilde{\theta})d'(x_1^L + x_2(\tilde{\theta})) \\ u'(x_2(\theta)) &= (1-\theta)d'(x_1^L + x_2(\theta)) \end{aligned} \qquad \text{(A2)}$$

where x_1^L and $x_2(\theta)$ denote optimal emissions in the learning case. Comparing (A1) and (A2) implies that $x_1^L > x_1^U$ if and only if $(1-\theta)d'(x_1 + x_2(\theta))$ with $x_2(\theta)$ defined by $u'(x_2(\theta)) = (1-\theta)d'(x_1 + x_2(\theta))$ is concave (resp. convex) in θ. While this is ambiguous in general, it can easily be shown that for quadratic utility and damage functions that $(1-\theta)d'(x_1 + x_2(\theta))$ is concave in θ. Therefore, perfect learning leads to increase, and not decrease, first period emissions for quadratic utility functions, as stated in the text. For a complete demonstration and intuition of this result, see Ulph and Ulph (1997).

In the context of the smooth ambiguity-aversion model, the general first order conditions for first and second period decisions based on problem (7) are given by:

$$0 = E_{\mu(\pi)}\phi'(E_{\pi(\tilde{\theta})}v(x_1, x_2, \tilde{\theta}))[E_{\pi(\tilde{\theta})}v_{x_i}(x_1, x_2, \tilde{\theta})] \text{ for both } i = 1,2 \qquad \text{(A.3)}$$

To determine the impact of ambiguity-aversion on the decision x_1, that is comparing (3) and (7), it is essential to study the ϕ'-weights attached to the marginal expected utility based on a probability distribution π in the first order condition (A.3).

Let us define $\hat{\mu}(\pi) = \phi'(E_{\pi(\tilde{\theta})} v(x_1, x_2, \tilde{\theta})) / E_{\mu(\pi)} \phi'(E_{\pi(\tilde{\theta})} v(x_1, x_2, \tilde{\theta}))$. We can then rewrite condition (A.3) as $0 = E_{\hat{\mu}(\pi)}[E_{\pi(\tilde{\theta})} v_{x_i}(x_1, x_2, \tilde{\theta}))]$, that is, the ambiguity-averse DM acts as if his beliefs over the set of probability distributions are distorted from μ to $\hat{\mu}$. Note, however, that these new weights $\hat{\mu}$ depend on expected utility and therefore on the optimal decisions x_i. Hence ambiguity-aversion operates as follows: compared to μ, probability distributions which give a small expected utility are overweighted, while those π with comparably large expected utility receive a small $\hat{\mu}(\pi)$.

The effect of ambiguity-aversion therefore depends on whether – when comparing two probability distributions π and $\hat{\pi}$ – a larger marginal expected utility is associated with a larger or a smaller expected utility.

For Example 1, it is obvious that

$$\begin{aligned} & E_{\pi(\tilde{\theta})}(1-\tilde{\theta})d'(x_1+x_2) > E_{\hat{\pi}(\tilde{\theta})}(1-\tilde{\theta})d'(x_1+x_2) \\ \Leftrightarrow\ & E_{\pi(\tilde{\theta})} v(x_1, x_2, \tilde{\theta}) < E_{\hat{\pi}(\tilde{\theta})} v(x_1, x_2, \tilde{\theta}) \\ \Leftrightarrow\ & \phi'(E_{\pi(\tilde{\theta})} v(x_1, x_2, \tilde{\theta})) > \phi'(E_{\hat{\pi}(\tilde{\theta})} v(x_1, x_2, \tilde{\theta})) \end{aligned}$$

That is, the larger the weight which is implicitly attached to a probability distribution due to ambiguity-aversion, the larger the expected damages are. In this model, this is equivalent to larger expected marginal damages. This implies that increased ambiguity-aversion leads to smaller emission levels in period 1.

A2. The effect of uncertainty, learning and ambiguity in the "resource depletion model"

In this section, we give some formal insights to sign the effect of uncertainty, learning and ambiguity in the Example 2, that is when the DM's primitive utility equals $v(x_1, x_2, \theta) = u(x_1) + u(x_2) + u(\theta - x_1 - x_2)$.

The first order conditions for problem (3) characterizing optimal resource consumption under uncertainty x_1^U in Example 2 are given by

$$u'(x_1^U) = u'(x_2^U) = E_{\tilde{\theta}} u'(\tilde{\theta} - x_1^U - x_2^U) \qquad \text{(A.4)}$$

Notice that, in contrast, under certainty, the first order conditions are given by $u'(x_1) = u'(x_2) = u'(E_{\tilde{\theta}}\tilde{\theta} - x_1 - x_2)$. Comparing the previous first order conditions, it is not difficult to show that uncertainty decreases consumption if and only if $E_{\tilde{\theta}}u'(\tilde{\theta} - x_1 - x_2)$ is larger than $u'(E_{\tilde{\theta}}\tilde{\theta} - x_1 - x_2)$ for any x_1 and x_2. Using again the Jensen's inequality, this is true if and only if marginal utility $u'(.)$ is convex, or $u'''(.) \geq 0$, as stated in the text (Leland, 1968; Kimball, 1990). For a simple demonstration and intuition on this result, see for example Eeckhoudt, Gollier and Schlesinger (2005).

We now examine the effect of learning. In period 2, that is when the value of θ is known to the DM, it is optimal to perfectly smooth consumption. Optimal consumption thus equals $x_2(\theta) = 0.5(\theta - x_1)$. The initial problem thus reduces to maximize $u(x_1) + 2E_{\tilde{\theta}}u(0.5(\tilde{\theta} - x_1))$ over x_1. (Under weak assumptions on the utility function, it is never optimal to run the risk of consuming all the resource in the initial period which insures interiority). The first order condition for initial consumption under learning is then given by

$$u'(x_1^{\,L}) = E_{\tilde{\theta}}u'(0.5(\tilde{\theta} - x_1^{\,L})) \qquad \text{(A.5)}$$

Comparing conditions (A.4) and (A.5), it is not difficult to show that learning increases initial consumption if and only if $E_{\tilde{\theta}}u'(\tilde{\theta} - x_1 - x_2)$ is larger than $E_{\tilde{\theta}}u'(0.5(\tilde{\theta} - x_1))$ for any x_1 and x_2 characterized by the equality $u'(x_2) = E_{\tilde{\theta}}u'(\tilde{\theta} - x_1 - x_2)$. But then notice that we can write $E_{\tilde{\theta}}u'(\tilde{\theta} - x_1 - x_2) = 0.5u'(x_2) + 0.5E_{\tilde{\theta}}u'(\tilde{\theta} - x_1 - x_2)$ which, under $u'''(.) \geq 0$ and the Jensen's inequality, is larger than $E_{\tilde{\theta}}u'(0.5(\tilde{\theta} - x_1))$, which is the condition that we precisely look for. For a complete demonstration and intuition of this result, see Eeckhoudt, Gollier and Treich (2005).

To show the effects of ambiguity-aversion on decision, we concentrate on quadratic utility functions $u(x) = \alpha x - \beta x^2 / 2$ (with $\alpha \leq \beta / x$ and $\beta > 0$). We obtain

$$E_{\pi(\tilde{\theta})}v_{x_i}(x_1, x_2, \tilde{\theta})) = \beta[E_{\pi(\tilde{\theta})}\tilde{\theta} - x_1 - x_2 - x_i]$$

$$E_{\pi(\tilde{\theta})}v(x_1, x_2, \tilde{\theta})) = u(x_1) + u(x_2) + \alpha[E_{\pi(\tilde{\theta})}\tilde{\theta} - x_1 - x_2] - (\beta / 2)[E_{\pi(\tilde{\theta})}(\tilde{\theta} - x_1 - x_2)^2]$$

which implies that expected marginal utility only depends on the expected value $E_{\pi(\tilde{\theta})}\tilde{\theta}$ while expected utility additionally depends on the variance of $\tilde{\theta}$. Considering condition (A.3) and by linking a larger mean to a larger variance of $\tilde{\theta}$, one can therefore construct examples where a larger expected marginal expected utility is linked to a smaller expected utility (and the other way around). Hence, in general, ambiguity-aversion has an ambiguous impact on decisions under uncertainty in Example 2 as claimed in the text.

A3. Partial uncertainty and partial learning

In the paper, we have considered extreme comparisons, that is, certainty vs. uncertainty, and learning vs. no learning. We now introduce the more general concepts of partial uncertainty and partial learning.

Let us start with the notion of partial uncertainty. This notion builds on the literature on stochastic dominance which dates back to the mathematicians Hardy, Littlewood and Polya (1934). The dominant concept used in economics is that of a mean-preserving increase in risk by Rothschild and Stiglitz (1970). The random variable $\tilde{\theta}'$ is a mean-preserving increase in risk of $\tilde{\theta}$ if and only if:

$$\text{for any convex function } f,\ E_{\tilde{\theta}'} f(\tilde{\theta}') \geq E_{\tilde{\theta}} f(\tilde{\theta}) \qquad \text{(A.6)}$$

This is a particular case of second order stochastic dominance in which the two random variables must have the same mean. Any risk-averse agent dislikes mean-preserving increase in risk. See Pratt (1964) and Arrow (1971) for a thorough analysis of the notion risk-aversion, and see Samuelson (1967) for a general proof that risk-averse agents like diversification. The first systematic analysis of the effect of mean-preserving increase in risk on decisions was developed in Rothschild and Stiglitz (1971). For an overview on stochastic dominance and its effects on decisions, see for instance Eeckhoudt, Gollier and Schlesinger (2005).

The general notion of partial learning relies on that of a better information structure. It dates back to the mathematicians Bohnenblust, Shapley, and Sherman (1949), and especially to Blackwell (1951). A convenient definition is introduced by Marschak and Miyasawa (1968). Let $\tilde{y}$ (resp. $\tilde{y}'$) an information structure correlated with $\tilde{\theta}$, and π_y (resp. π'_y) the vector of posterior probabilities of $\tilde{\theta}$ after observing y. Let also define S the set of probability distributions. Then $\tilde{y}$ is a better information structure than $\tilde{y}'$ if and only if:

$$\text{for any convex function } \rho \text{ on } S,\ E_{\tilde{y}}\rho(\pi_{\tilde{y}}) \geq E_{\tilde{y}'}\rho(\pi'_{\tilde{y}'}) \tag{A.7}$$

Thus a better information structure induces a mean-preserving spread in posterior beliefs. Notice that the function $\rho(\pi_y) = \max_{x_2} E_{\tilde{\theta}/y} v(x_1, x_2, \tilde{\theta})$ is always convex in posterior beliefs π_y since it is the maximum of linear functions of π_y. Hence any better information structure increases ex ante expected utility.

Using Marschak and Miyasawa (1968)'s definition, Epstein (1980) derives a general theorem which permits investigation of the effect of learning on decisions under some differentiability assumptions. Jones and Ostroy (1984) generalize Epstein's theorem to non-differentiable problems. Gollier, Jullien and Treich (2000) show that the theorem does not usually yield unambiguous restrictions on the primitives of the model, i.e. on $v(x_1, x_2, \theta)$.

A4. The "irreversibility effect"

In some situations, decisions at one point in time may affect the set of possible decisions that can be made later in the future. Specifically, a current decision may be irreversible in the sense that it prevents the DM from selecting a future decision. This so-called irreversibility constraint should be taken into account at the initial stage, i.e. when the current decision is made. In the paper, we did not consider irreversibility constraints. We show now how to incorporate irreversibility constraints in our framework. In doing so, we also demonstrate the "irreversibility effect", a general effect that states that learning always favors less irreversible decisions.

The essence of how an irreversibility constraint impacts the current decision is best captured in a classical example of an irreversible investment problem. Let $v(x_1, x_2, \theta) = x_1 + x_2\theta$ with the decision set of $x_1 \in D_1 = \{0,1\}$ and $x_2 \in D(x_1) = \{x_1, 1\}$. Here the choice at date 2 is explicitly restricted by the date 1 decision: the project is irreversible in the sense that once it is developed it cannot be stopped ($x_2 = 1$ if $x_1 = 1$). Under uncertainty, program (3) becomes

$$\max_{x_1 \in \{0,1\}, x_2 \in \{x_1, 1\}} E_{\tilde{\theta}}(x_1 + x_2\tilde{\theta}) = \max(1 + E_{\tilde{\theta}}\tilde{\theta}, 0) \tag{A.8}$$

The project is implemented today if its expected present value is positive, that is if $1+E_{\tilde{\theta}}\tilde{\theta}\geq 0$, or it is never implemented. Consider alternatively the case of perfect learning. Program (4) becomes

$$\max_{x_1\in\{0,1\}} E_{\tilde{\theta}} \max_{x_2\in\{x_1,1\}}(x_1+x_2\tilde{\theta}) = \max(1+E_{\tilde{\theta}}\tilde{\theta}, E_{\tilde{\theta}}\max(0,\tilde{\theta})) \tag{A.9}$$

The present value of postponing the decision to develop the project equals $V=E_{\tilde{\theta}}\max(0,\tilde{\theta})$. The project will be initiated today only if it yields a larger present value than that obtained if the decision is postponed to the future: $1+E_{\tilde{\theta}}\tilde{\theta}\geq V$. The quantity V has been coined the (quasi-) option value (Arrow and Fisher, 1974). The prospect of receiving perfect information in the future thus increases the cost of choosing the irreversible decision today from 0 to V. This irreversible decision would indeed prevent the DM from taking advantage of information in the future. This is a general result coined the 'irreversibility effect' (Henry, 1974). See Ha Duong (1998) for a numerical application to the climate change problem.

Epstein (1980) generalizes the irreversibility effect to partial learning; but he has shown that this effect does not usually hold for a general non-separable payoff function, that is when $v(x_1,x_2,\theta)$ is not of the form $v_1(x_1)+v_2(x_2,\theta)$. In other words, when the payoff function is non-separable one cannot be sure that better forthcoming information biases current decision in favor of less irreversibility (that is, inducing a larger set of future choices). Non-separability actually holds in the Examples 1 and 2. The interested reader is referred to Kolstad (1996), Ulph and Ulph (1997) and Gollier, Jullien and Treich (2000) for a discussion of the non-separability involved in a climate change model when combined with an irreversibility constraint. Importantly, Narain, Hanemann and Fisher (2007) provide a more general definition of irreversibility that can be used to study the effect of learning in some models with a non-separable payoff function.