The Alberta Dilemma: Optimal Sharing of a Water Resource by an Agricultural and an Oil Sector

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Abstract

The purpose of this paper is to characterize the optimal time paths of production and water usage by an agricultural and an oil sector that have to share a limited water resource. We show that for any given water stock, if the oil stock is sufficiently large, it will become optimal to have a phase during which the agricultural sector is inactive. This may mean having an initial phase during which the two sectors are active, then a phase during which the water is reserved for the oil sector and the agricultural sector is inactive, followed by a phase during which both sectors are active again. The agricultural sector will always be active in the end as the oil stock is depleted and the demand for water from the oil sector decreases. In the case where agriculture is not constrained by the given natural inflow of water once there is no more oil, we show that oil extraction will always end with a phase during which oil production follows a pure Hotelling path, with the implicit price of oil net of extraction cost growing at the rate of interest. If the natural inflow of water does constitute a constraint for agriculture, then oil production never follows a pure Hotelling path, because its full marginal cost must always reflect not only the imputed rent on the finite oil stock, but also the positive opportunity cost of water. The case of oil and agriculture sharing a water resource fixes ideas, but it constitutes just one example where a nonrenewable resource sector must compete with another sector of the economy for the use of some scarce input. Our analysis provides a framework to generalize the Hotelling rule of nonrenewable resource depletion to the case where the marginal opportunity cost of extracting the resource depends on the endogenous activity of some other sector of the economy.

Key Words: Nonrenewable natural resources, renewable natural resources, optimal order of use, Hotelling rule, oil, water, agriculture.

1 Introduction

Several years of drought have recently exacerbated a dilemma faced by the province of Alberta (Canada) concerning the sustainability of water usage by the various sectors of its economy. The dilemma comes from the choices that must be made between conflicting uses of a limited common water resource by important sectors of its economy. This is particularly true of the agricultural and oil sectors, two of the mainstays of the Alberta economy and two large water users.[1] Water is an essential input for the agricultural sector, for irrigation and other purposes. Water is also used intensively by Alberta's important and growing oil sector in order to enhance oil recovery.[2] The optimal allocation of the scarce water resource between those alternative uses poses a problem of intertemporal choice, given that both water and oil are subject to dynamic constraints.

This Alberta situation is but one example where a nonrenewable resource sector, such as oil, must compete with another sector of the economy for the use of some scarce input. There are many instances where the exploitation of a nonrenewable resource will impact on some scarce resource which is also a valuable input to other sectors of the economy. Mining operations may use or pollute water or may be otherwise detrimental to the natural environment, thus constraining the activities of other sectors that also depend on this water or this natural environment; the common resource may be the absorption capacity of the environment, being shared by two polluting industries, one of which exploits a nonrenewable resource; economic development may irreversibly deplete the environmental base on which depends the exploitation of a renewable resource, such as a fishery.[3]

This paper can be viewed as a generalization of the Hotelling rule (Hotelling, 1931) to

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[1]See Griffiths and Woynillowicz (2003) for an overview of the consequences of the demand for water by Alberta's oil industry on the management of the province's water resources.

[2]For a description of the different ways in which the use of water enters the oil recovery processes in Alberta and for some summary data on water use by that industry, see Canadian Association of Petroleum Producers (2002) and Alberta Environment (2004).

[3]Swallow (1990) provides an excellent example of such an interaction between a nonrenewable and a renewable sector. He analyzes the case of the development of a coastal area which irreversibly changes the character of a watershed on which also relies a fishery.

cases where the nonrenewable resource sector shares a common constraint with other sectors of the economy. As a result of this interdependence, the true opportunity cost of exploiting the nonrenewable resource in question depends on the *endogenous* level of activity of those other sectors. In some cases, this common constraint may be strictly static in nature, in the sense that it applies to the flow of some common input. In other cases it may be dynamic in nature, in the sense that that it applies to the stock of some renewable resource in addition to the rate of renewal (or inflow) of this resource. This is specifically the case retained in this paper. The framework used encompasses both types of constraints and can thus be easily adapted to the analysis of either type of situation. The nonrenewable resource being subject to eventual exhaustion, the question arises as to what is the optimal order in which the two sectors should access this scarce input.

In order to fix ideas we will hereafter call the two sectors agriculture and oil and they will share a water resource, as in the Alberta situation described above. Our purpose is to characterize the optimal time paths of production and water usage of the two sectors. We show that for any given initial water stock, these time paths will take different configurations depending on the size of the initial oil stock and on whether or not the natural water recharge imposes a long-run constraint on the agricultural sector. We are able to identify critical values of the oil stock that determine the specific phases of the optimal paths. *Ceteris paribus*, the larger the oil stock, the greater the pressure on the scarce water resource. We show that for sufficiently large oil stocks, it will become optimal to have a phase during which the agricultural sector is inactive. This may mean having a first phase during which the two sectors are active, then a phase during which the water is reserved for the oil sector and the agricultural sector is inactive, followed by a phase during which both sectors are active again. The agricultural sector will always be active in the end as the oil stock is depleted and the demand for water from the oil sector decreases. Agriculture becomes the only water user once the oil stock is exhausted. It then may or may not be constrained by the natural inflow of water. In the case where it is not, we show that oil extraction will always end with

a phase during which the oil production path follows a pure Hotelling path, with the implicit price of oil net of extraction cost growing at the rate of interest. Otherwise the oil production path never follows a pure Hotelling path, because its full marginal cost must always reflect not only the imputed rent on the finite oil stock, but also the positive opportunity cost of water.

The problem analyzed here concerns the optimal order of use of the common water resource as an input by a renewable and a nonrenewable sector. In this respect, it is related to the literature on the optimal order of use over time of multiple pools of a natural resource to serve a single market (Herfindahl (1967), Kemp and Long (1980), Lewis (1982), Kemp and Long (1984), Hartwick, Kemp and Long (1986), Amigues *et al.* (1998), Favard (2002), Holland (2003)). One particularity however is that the decision concerns the order of use of a single common resource pool by multiple sectors of the economy, rather than multiple resource pools by a single user. As such it is more closely related to Gaudet, Moreaux and Salant (2001), who analyze the optimal order of use of many nonrenewable resource pools to serve multiple markets, and to Chakravorty and Krulce (1994), Chakravorty, Roumasset and Kinping (1997) and Chakravorty, Krulce and Roumasset (2005), where the analysis concerns the optimal order of use of many differentiated resources for different purposes. However none of those analyses can be applied directly to the problem studied in this paper, since another one of its particularities is that the common resource is renewable and one of the sectors using it as an input exploits a nonrenewable resource.

In the next section we present the model and derive some general propositions concerning the rates of production of the two sectors. The optimal paths for the case where the natural inflow of water constitutes a long-run constraint on agriculture are derived in Section 3. In Section 4 we show how these paths are modified when the agricultural sector is not constrained by the natural inflow of water. We then briefly conclude in Section 5.

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2 The model

Consider an economy that produces an agricultural product and oil, both of which use water as an input, drawn from a common source. The agricultural product can be produced indefinitely, as long as the essential water input is available. Oil is a nonrenewable resource, whose initial stock is fixed and therefore subject to exhaustion.

Let $y_a(t)$ denote agricultural production and $y_m(t)$ oil production at time t. The unit cost of production in sector i, $i = a, m$, is $c_i > 0$, excluding any imputed rents on water and oil stocks. The gross social benefit derived from the production of sector i is $u_i(y_i)$, which is assumed to satisfy:

$$u_i'(y_i) > 0, \quad u_i''(y_i) < 0 \quad \forall y_i \geq 0 \quad \text{and} \quad u_i(0) = 0, \quad c_i < u_i'(0) < +\infty, \quad u_i'(\infty) < c_i. \quad (1)$$

We further assume that

$$u_m'(0) > c_m + \frac{k_m}{k_a}[u_a'(0) - c_a] \quad (2)$$

where the right-hand side represents the marginal opportunity cost, excluding any imputed rents, of producing the first unit of oil when no agriculture is being produced. The purpose of these assumptions will become clear in due course.[4]

Sector i consumes net k_i units of water per unit of production.[5] Total net consumption of water by sector i is therefore $k_i y_i$. The total stock of water available at time t is $X(t) \geq 0$ and the given initial stock is $X_0 > 0$. The stock of water is recharged by a natural inflow $\bar{x}$. The dynamics of the water stock, after withdrawal, is therefore given by:

$$\dot{X}(t) = \bar{x} - k_a y_a(t) - k_m y_m(t). \quad (3)$$

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[4]An essential property of the net benefit function of each sector i is its strict concavity in y_i. For ease of exposition, we choose to write it as $u_i(y_i) - c_i y_i$, that is with strictly concave gross benefit and linear costs, as one way of neatly distinguishing benefits and costs. All of our results go through just as well with, for instance, linear gross benefit and convex costs or, for that matter, any strictly concave net benefit function that exhibits a unique interior maximum.

[5]The net consumption of water by a sector may differ from the gross consumption to the extent that a fraction of the water used is returned to the water cycle. So if gross withdrawal per unit of output is h_i and a fraction α_i is returned to the cycle, then $k_i = (1 - \alpha_i)h_i$. Typically α_m is relatively low and $\alpha_a > \alpha_m$ (See Griffiths and Woynillowicz (2003)).

The oil stock to which the oil sector has access at time t is $S(t)$ and its fixed initial stock is $S_0 > 0$. The oil stock dynamics is given by:

$$\dot{S}(t) = -y_m(t). \tag{4}$$

When the water stock is drawn down to zero, the aggregate water consumption is constrained by the natural water inflow : $k_a y_a + k_m y_m \leq \bar{x}$. Each sector then faces an upper bound to its production, given by $\bar{y}_i = \bar{x}/k_i$, which is the maximum output that can be achieved in that situation when the other sector is inactive.

Denote by $\hat{y}_i$ the level of output that would maximize the net benefit generated by sector i if both water and oil were abundant, thus not justifying any scarcity rent. It is given by $u_i'(\hat{y}_i) = c_i$. The assumptions on $u_i(y_i)$ in (1) imply that $\hat{y}_i > 0$ and that it is well defined and unique.

The planner's problem can be formulated as that of choosing the time paths of $y_a(t)$ and $y_m(t)$, for all $t \geq 0$, so as to maximize:

$$\int_0^\infty e^{-rt}[u_a(y_a(t)) - c_a y_a(t) + u_m(y_m(t)) - c_m y_m(t)]dt$$

subject to

$$\dot{X}(t) = \bar{x} - k_a y_a(t) - k_m y_m(t), \ \ X(t) \geq 0, \ \ X(0) = X_0, \text{given} \tag{5}$$

$$\dot{S}(t) = -y_m(t), \ \ \lim_{t \to \infty} S(t) \geq 0, \ \ S(0) = S_0, \text{given} \tag{6}$$

$$y_a(t) \geq 0, \ \ y_m(t) \geq 0. \tag{7}$$

where r is the rate of discount. Notice that contrary to the stock of oil, the stock of water may be replenished by withdrawing less than the constant natural inflow. This explains why it is necessary to impose explicitly that $X(t) \geq 0$ for all $t > 0$ and not only at $t = \infty$, as for $S(t)$.

In order to take into account the pure state constraint $X(t) \geq 0$, define the Lagrangian function:

$$L(X, S, y_a, y_m, \lambda_m, \lambda_w, \mu, t) = H(X, S, y_a, y_m, \lambda_m, \lambda_w, t) + \mu(t)X(t)$$

where the Hamiltonian is given by:

$$H(X, S, y_a, y_m, \lambda_m, \lambda_w, t) = e^{-rt}[u_a(y_a) - c_a y_a + u_m(y_m) - c_m y_m] - \lambda_m y_m + \lambda_w[\bar{x} - k_a y_a - k_m y_m].$$

Then the following conditions, along with (5), (6) and (7), are necessary:[6]

$$u'_a(y_a(t)) \begin{cases} = c_a + e^{rt}\lambda_w(t)k_a & \text{if } y_a(t) > 0 \\ \\ \leq c_a + e^{rt}\lambda_w(t)k_a & \text{otherwise.} \end{cases} \tag{8}$$

$$u'_m(y_m(t)) \begin{cases} = c_m + e^{rt}[\lambda_m(t) + \lambda_w(t)k_m] & \text{if } y_m(t) > 0 \\ \\ \leq c_m + e^{rt}[\lambda_m(t) + \lambda_w(t)k_m] & \text{otherwise.} \end{cases} \tag{9}$$

$$\dot{\lambda}_w(t) \begin{cases} = 0 & \text{if } X(t) > 0 \\ \\ = -\mu(t) \leq 0 & \text{otherwise.} \end{cases} \tag{10}$$

$$\dot{\lambda}_m(t) = 0 \tag{11}$$

$$\lim_{t\to\infty} \lambda_w(t) \geq 0, \quad \lim_{t\to\infty} \lambda_w(t)X(t) = 0, \lim_{t\to\infty} X(t) \geq 0 \tag{12}$$

$$\lim_{t\to\infty} \lambda_m(t) \geq 0, \quad \lim_{t\to\infty} \lambda_m(t)S(t) = 0, \lim_{t\to\infty} S(t) \geq 0 \tag{13}$$

The interpretation of these conditions is straightforward. The co-state variables λ_m and λ_w are the present value shadow prices of the oil stock and the water stock respectively. The right-hand sides of (8) and (9) represent the full marginal cost of agricultural production and oil production respectively. Hence, condition (8) says that if the agricultural sector is active, then gross marginal benefit from agriculture must be equal to the full marginal cost of agricultural production as measured by the sum of c_a and the current marginal shadow costs of the water required. Similarly, condition (9) says that when the oil sector is active its gross marginal benefit must equal its full marginal cost as measured by the sum of c_m and the current marginal shadow cost of the oil being depleted and the water being used.

From condition (11), we know that $\lambda_m(t)$, the present value shadow price of oil, is constant over time. This means that its current value must be growing at the rate of interest. Henceforth we will simply write λ_m without the time argument to signify this. As for $\lambda_w(t)$,

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[6]See Seierstad and Sydsaeter (1987), Theorem 16, page 244, on the necessity of the transversality conditions.

the shadow value of water, we know from condition (10) that it must be constant while the stock of water is positive and decreasing over time while the stock of water is zero. Henceforth, we will denote it simply λ_w over intervals of time where the stock of water is known to be positive and explicitly as $\lambda_w(t)$ otherwise. Note also that $\lambda_w(t)$ must be continuous at the point where $X(t)$ becomes zero. If there was to be a jump in $\lambda_w(t)$, as can occur in problems subject to pure state constraints, it would have to be downward in the case at hand.[7] However this is excluded in our problem since, from (8) and (9), it would imply an upward discontinuity in either $y_a(t)$ or $y_m(t)$ or both. This in turn would lead to a negative water stock, an impossibility.

As can be seen from conditions (8) and (9), the full marginal opportunity cost of oil extraction when both sectors are producing is $c_m + \frac{k_m}{k_a}[u_a'(y_a(t)) - c_m] + e^{rt}\lambda_m$. In other words, the full marginal cost of extracting oil must take into account the fact that one must sacrifice some water usage in the agricultural sector in order to do so. This is reflected in the second term, a term which is absent in the usual Hotelling type nonrenewable resource extraction problem. This means that unlike in the usual Hotelling type problem of resource extraction, the full marginal opportunity cost of extracting the resource now depends on the endogenous level of activity of another sector of the economy. Because of this we need to reestablish the standard result that, under our assumptions, the resource will be extracted at a positive rate until some finite date T_m, at which point its stock becomes fully depleted. We do this in the first of the following three propositions.

Proposition 1 *When assumption (2) is satisfied, (i) the oil stock will be fully depleted in finite time; (ii) over any interval of time such that $S(t) > 0$, we will have $y_m(t) > 0$.*

Proof. We first show by contradiction that $\lambda_m > 0$. Suppose that $\lambda_m = 0$. Then $y_m(t) > 0$ for all $t \in [0, \infty)$. To see this assume $y_m(t) = 0$ for some $t_1 \in [0, \infty]$. Then, by condition (9), $u_m'(0) - c_m \leq \lambda_w(t_1)e^{rt_1}k_m$ and hence, by condition (8) and assumption (2), $y_a(t_1) = 0$. It cannot be optimal to have $y_m(t) = 0$ for all $t \in [t_1, \infty)$, for then also $y_a(t) = 0$

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[7]See Léonard and Long (1992), Theorem 10.3.1, page 334-335.

for all $t \in [t_1, \infty)$. So there must be a $t_2 \geq t_1$ such that $y_m(t) = 0$ for $t \in [t_1, t_2)$ and $y_m(t_2) > 0$. But since neither sector would be producing along $[t_1, t_2)$, the water stock will be increasing and we will necessarily have $X(t) > 0$ and hence, by condition (10), $\dot{\lambda}_w(t) = 0$. In view of the continuity of λ_w, this means that we cannot have $y_m(t_2) > 0$. Therefore $y_m(t) > 0$ for all $t \in [0, \infty)$ if $\lambda_m = 0$. Since by assumption $u_m''(y_m) < 0$, at any date t for which $X(t) > 0$ we will have $\dot{y}_m(t) < 0$. If on the other hand $X(t) = 0$, then either $y_m(t) = \hat{y}_m$ (if $\lambda_w(t) = 0$) or $k_a y_a(t) + k_m y_m(t) = \bar{x}$, in which case $y_m(t) > 0$ from (8), (9) and assumption (2). In any case, $y_m(t)$ is bounded away from zero. But this contradicts the fact that S_0 is finite. Therefore $\lambda_m > 0$. This immediately implies that $y_m(t) = 0$ for all t large enough, which proves part (i) of the proposition.

There remains to prove part (ii). Since the oil stock must be fully depleted, $y_m(t)$ will necessarily become positive at some point in time. Suppose $y_m(t) = 0$ for $t \in [t_1, t_2)$ and $y_m(t_2) > 0$. From (9) and the assumption that $u_m''(y_m) < 0$ for all $y_m > 0$, it follows that $\lambda_w(t_2) < \lambda_w(t_1)$. Hence, by condition (10), there must be a nondegenerate subinterval $[\theta, t_2)$ of $[t_1, t_2)$ along which $X(t) = 0$. But then the initial conditions at t_2 are the same as at θ, since $S(t_2) = S(\theta)$ and $X(t_2) = X(\theta) = 0$. Therefore, if $y_m(\theta) = 0$ was optimal, so must be $y_m(t_2) = 0$, a contradiction. ∎

Notice that we must also have $y_m(T_m) = 0$. This is because the full marginal cost of oil extraction $(c_m + e^{rt}[\lambda_m(t) + \lambda_w(t)k_m])$ must reach the "choke price" $(u_m'(0))$ at the exact moment of exhaustion of the oil stock. Otherwise there would be an upward jump in the implicit price of oil and it would always pay to delay exhaustion in order to benefit from that jump.

We have thus established that whether there is water left in stock or not, as long as there is oil left, oil production will not be interrupted. This means that condition (9) will be satisfied with equality as long as the stock of oil is positive. However, whenever the shadow value of water is positive the rate of extraction of oil will not be following a pure Hotelling path, since the full marginal cost of oil must depend not only on λ_m, the shadow value of oil,

but also on $k_m \lambda_w(t)$, the shadow value of the water required to extract the oil. As already pointed out, the evolution of the shadow value of water will depend on whether the stock of water is positive or zero (condition (10)). Therefore, whether the stock of water is positive or not will also be crucial in determining the evolution over time of the rate of production of oil and of agriculture, which leads to the next two propositions.

Proposition 2 *Over any interval of time where $X(t) > 0$ and $S(t) > 0$, (i) if both sectors are active, then $\dot{y}_a(t) < 0$ and $\dot{y}_m(t) < 0$ over that interval; (ii) if only the oil sector is active, then $\dot{y}_a(t) = 0$ and $\dot{y}_m(t) < 0$.*

Proof. When $X(t) > 0$ and $S(t) > 0$, from (10) and (11), $\dot{\lambda}_w(t) = \dot{\lambda}_m(t) = 0$ and hence, differentiating (8) and (9) with respect to time, we get:

$$\dot{y}_a(t) = \frac{re^{rt}k_a\lambda_w}{u_a''(y_a(t))} < 0 \tag{14}$$

$$\dot{y}_m(t) = \frac{re^{rt}[\lambda_m + k_m\lambda_w]}{u_m''(y_m(t))} < 0, \tag{15}$$

which proves part (i) of the proposition. Part (ii) follows immediately from (15) and the fact that if $y_a(t) = 0$ over the interval in question, then $\dot{y}_a(t) = 0$ over that interval. ∎

The explanation of this result is straightforward. The fact that the discounted shadow prices of water and of oil are both constant when both stocks are positive means that the full marginal cost of production will necessarily be increasing over time in both sectors. As a consequence, if the sector is active — it will necessarily be the case of oil, as proven in Proposition 1 — its rate of production must be decreasing over time since gross marginal benefit is a decreasing function of production in both sectors. It is a different matter however when the stock of water is zero, since then the full marginal costs of production will not remain constant over time anymore.

Proposition 3 *Over any interval of time where $X(t) = 0$ and $S(t) > 0$, (i) if both sectors are active, then $\dot{y}_a(t) > 0$ and $\dot{y}_m(t) < 0$; (ii) if only the oil sector is active, then $\dot{y}_a(t) = 0$ and $\dot{y}_m(t) = 0$.*

Proof. If $X(t) = 0$ over some interval of time, then $\dot{X}(t) = 0$ over that interval. This means that $k_a y_a(t) + k_m y_m(t) = \bar{x}$ and therefore:

$$k_a \dot{y}_a(t) + k_m \dot{y}_m(t) = 0. \tag{16}$$

Differentiating (8) and (9) with respect to time and using (11), we find that:

$$\dot{y}_a(t) = \frac{e^{rt} k_a [r\lambda_w(t) + \dot{\lambda}_w(t)]}{u_a''(y_a(t))} \tag{17}$$

$$\dot{y}_m(t) = \frac{e^{rt} \{r\lambda_m + k_m [r\lambda_w(t) + \dot{\lambda}_w(t)]\}}{u_m''(y_m(t))}. \tag{18}$$

Substituting into (16), we find:

$$r\lambda_w(t) + \dot{\lambda}_w(t) = -\frac{\dfrac{-rk_m \lambda_m}{u_m''(y_m(t))}}{\dfrac{k_a^2}{u_a''(y_a(t))} + \dfrac{k_m^2}{u_m''(y_m(t))}} < 0. \tag{19}$$

Therefore $\dot{y}_a(t) > 0$, from (17), and $\dot{y}_m(t) < 0$, from (16), which proves part (i) of the proposition. The proof of part (ii) follows immediately from the fact that if $y_a(t) = 0$ over the interval in question, then $y_m(t) = \bar{y}_m$ over that interval. ∎

Thus when there is some oil left and both sectors depend strictly on the natural inflow of water, oil production must be decreasing and agricultural production increasing. As a corollary we get from (19) the rate of decrease of $\lambda_w(t)$. It can be seen that it must be decreasing at a rate less than the rate of interest, since $r\lambda_w(t) + \dot{\lambda}_w(t) < 0$. This must be the case, since otherwise both oil and agricultural production would be decreasing over time as is evident from (17) and (18), which is incompatible with water usage being constant at $\bar{x}$. The intuition as to why the present value of the shadow price of water must be decreasing to begin with when the stock of water is zero can be seen by imagining the case where only the agricultural sector would be active. In that case agricultural production would be equal to either $\hat{y}_a$ or $\bar{y}_a$ and would thus be constant over time. This means that the current value of the shadow price of water, namely $e^{rt}\lambda_w(t)$, would also have to be constant and hence its present value, $\lambda_w(t)$, would have to be decreasing exactly at the rate of interest if

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$y_a(t) = \bar{y}_a < \hat{y}_a$, or would be equal to zero in the case where $y_a(t) = \hat{y}_a \leq \bar{y}_a$. When both the agricultural and oil sectors are active, we must in addition take into account the fact that water requirement from the oil sector will be decreasing over time.

It will be useful to distinguish henceforth between the case where $\hat{y}_a > \bar{y}_a$ and that where $\hat{y}_a < \bar{y}_a$. In the first case, discussed in the next section, water availability poses a long-run constraint on agriculture. This is because, even in the absence of the oil sector, a water usage of $k_a \hat{y}_a$ would require more than the natural inflow of water and hence cannot be sustained in the long run. In the second case, discussed in Section 4, a water usage of $k_a \hat{y}_a$ can be sustained indefinitely after the stock of oil has been depleted.

3 The natural water inflow poses a long-run constraint on agriculture

Let us now consider the case where $\hat{y}_a > \bar{y}_a$. It is useful to first characterize the two extreme situations where there is only either an agricultural or an oil sector in operation. After having done this, we turn to the analysis of the situation where the two sectors coexist. We treat the initial oil stock as a pivotal parameter and define a number of critical values of this stock that are important in determining the shapes of the optimal paths. These critical values are then used to fully characterize the optimal paths.

3.1 Only the agricultural sector is active

If $\hat{y}_a > \bar{y}_a$ and there is no oil sector, the optimal path can be sketched as follows. The water stock will be exhausted in finite time T_w, since if $X(t) > 0$ for all $t \geq 0$ we would have λ_w constant, in view of condition (10), and the full marginal cost of agriculture would be increasing forever. This means that we would eventually have $y_a(t) = 0$, since $u'_a(0)$ is finite. But since $u'_a(0) > c_a$, this cannot be optimal. Moreover, the water stock will remain zero after T_w, since producing less than $\bar{y}_a < \hat{y}_a$ could never be optimal in the absence of the oil sector. Hence two phases can be distinguished. During the first phase, which ends at T_w, the water stock is being exhausted. During that phase, the water stock is positive, so that

λ_w is a constant, and condition (8) is satisfied with equality, meaning that:

$$u'_a(y_a(t)) = c_a + e^{rt}\lambda_w k_a.$$

Agricultural production exceeds $\bar{y}_a$ and is decreasing towards $\bar{y}_a$ since $u''_a(y_a(t)) < 0$ and λ_w is constant. Because of continuity of production over time and exhaustion of the water stock, the values of T_w and λ_w are obtained from:

$$u'_a(\bar{y}_a) = c_a + e^{rT_w}\lambda_w k_a \quad \text{and} \quad \int_0^{T_w}(k_a y_a(t) - \bar{x})dt = X_0.$$

The second phase begins at T_w and has $y_a(t) = \bar{y}_a$ and $X(t) = 0$ for all $t > T_w$.

Once the existence of the oil sector is taken into account, these two phases will characterize the agricultural production path after the oil stock is exhausted, provided it is exhausted before the water stock. If the water stock is exhausted before the oil stock, then agricultural production enters the second phase as soon as the oil stock is exhausted. Since the oil stock is always exhausted in finite time, it follows that if $\hat{y}_a > \bar{y}_a$, the optimal path always ends with a final phase during which $y_a = \bar{y}_a$ and $X(t) = 0.$[8]

3.2 Only the oil sector is active

Assume now there is no agricultural sector. Then two cases need to be distinguished, according to whether the initial stock of water is abundant relative to the initial stock of oil or not. In the first case, the stock of oil is exhausted before the stock of water and therefore $\lambda_w = 0$, since by assumption there is no other use for water. We would therefore have a pure Hotelling-type path, with the rate of extraction given by condition (9) satisfied with equality, so that:

$$u'_m(y_m(t)) = c_m + e^{rt}\lambda_m, \tag{20}$$

with λ_m a constant from condition (11). Oil extraction decreases towards zero, with $\dot{y}_m(t)$ given by (15). The date of exhaustion of the oil stock, T_m, and λ_m are determined by:

$$u'_m(0) = c_m + e^{rT_m}\lambda_m \quad \text{and} \quad \int_0^{T_m} y_m(t)dt = S_0.$$

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[8]This assures that the transversality condition (12) is satisfied.

This first case occurs if, for the values of T_m and λ_m just determined and $y_m(t)$ given by (20), we have:

$$\int_0^{T_m} (k_m y_m(t) - \bar{x})dt \leq X_0, \tag{21}$$

meaning that the water stock poses no constraint.

If (21) is not satisfied, so that water does pose a constraint, we have the second case, which is characterized by three phases. In a first phase, the water stock is being exhausted and, from condition (9):

$$u'_m(y_m(t)) = c_m + e^{rt}[\lambda_m + \lambda_w k_m], \tag{22}$$

where λ_m and λ_w are both positive constants, by (11) and (10). Since $u''_m(y_m(t) < 0$, the rate of oil extraction is decreasing towards $\bar{y}_m$ until the exhaustion of the water stock at T_w. Then follows a second phase during which the oil extraction rate is constrained by the natural inflow to $\bar{y}_m$. This phase ends at some date $\widetilde{T} \geq T_w$ defined by:

$$u'_m(\bar{y}_m) = c_m + e^{r\widetilde{T}}\lambda_m.$$

From that date on, there follows a Hotelling-type path like the one just described in the first case. Notice that if (21) happened to be just satisfied with equality, then we are left with the Hotelling-type path of the first case: the second phase collapses, since then $\lambda_w = 0$, and $T_m = T_w$.

3.3 Both sectors are active

Consider now the situation where both sectors are present from the outset. We have already established that since the oil stock will always be exhausted in finite time, the optimum will be characterized by a final phase during which the water stock is zero since $\hat{y}_a > \bar{y}_a$. The proposition that follows further establishes that once the water stock is exhausted, it will *never* become positive again when $\hat{y}_a > \bar{y}_a$.

Proposition 4 *If $\hat{y}_a > \bar{y}_a$, then once the stock of water is exhausted, it will never be replenished.*

Proof. As just shown above, if $\hat{y}_a > \bar{y}_a$, the optimal path always ends with a phase during which $X(t) = 0$. Therefore, if an interval of time during which $X(t) = 0$ is followed by an interval of time during which $X(t) > 0$, there must follow a third interval of time during which $X(t) = 0$. Suppose this were the case. Then it must be that $S(t) > 0$ at the beginning of the second interval, for otherwise it is optimal to keep $X(t) = 0$ forever. By Proposition 2, neither $y_a(t)$ nor $y_m(t)$ can be increasing during an interval where $S(t) > 0$ and $X(t) > 0$. But the assumed sequence of intervals necessitates that $\dot{X}(t)$ be at first positive and then negative during the second interval, which means that total water usage must increase from a level lower than $\bar{x}$ to eventually a level higher than $\bar{x}$. Therefore the assumed sequence cannot be optimal. ∎

In order to complete the characterization of the optimal paths for the case where both sectors are present, it will now be useful to define a number of threshold levels on S_0, the initial stock of oil. These critical values of S_0 will determine whether, for any given initial water stock, X_0:

   i. the water stock is exhausted before the oil stock or not;

   ii. there is a period of inactivity of the agricultural sector or not;

   iii. there is initially a period of inactivity of the agricultural sector or not.

We will denote these threshold values of S_0 by $\widehat{S}_0(X_0)$, $\widetilde{S}_0(X_0)$ and $\overline{S}_0(X_0)$ respectively. We now define, in order, each of those critical values.

3.3.1 The determination of $\widehat{S}_0(X_0)$

An important consideration for the characterization of the overall optimal paths is the identification of cases where the stock of water is exhausted before the stock of oil and vice versa. To do this, we consider a hypothetical situation where both the stock of water and the stock of oil are exhausted at exactly the same instant of time. This allows us to determine the properties that must satisfy S_0, for any given X_0, in order for this to be the case and thus define $\widehat{S}_0(X_0)$.

If the stock of water and the stock of oil were to be exhausted at exactly the same instant of time, then $T_w = T_m$ and:

$$\int_0^{T_m} y_m(t)dt = S_0 \tag{23}$$

$$\int_0^{T_m} [k_a y_a(t) - \bar{x}]dt + k_m S_0 = X_0. \tag{24}$$

From (10) and (11), we know that λ_m and λ_w are constant for all $t \in [0, T_m]$ and from conditions (8) and (9), we must have:

$$u_a'(y_a(t)) = c_a + e^{rt}\lambda_w k_a, \quad t \in [0, T_m] \tag{25}$$

$$u_m'(y_m(t)) = c_m + e^{rt}[\lambda_m + \lambda_w k_m], \quad t \in [0, T_m]. \tag{26}$$

Furthermore, $y_a(t) = \bar{y}_a$ for all $t \in [T_m, \infty)$, as demonstrated above, and $y_m(T_m) = 0$, since the oil price $(c_a + e^{rt}[\lambda_m + \lambda_w k_m])$ must reach the choke price $(u_m'(0))$ at the moment of exhaustion of the oil stock. This means that:

$$u_a'(\bar{y}_a) = c_a + e^{rT_m}\lambda_w k_a \tag{27}$$

$$u_m'(0) = c_m + e^{rT_m}[\lambda_m + \lambda_w k_m]. \tag{28}$$

From (23), (26) and (28) we can uniquely determine T_m, $\lambda_m + \lambda_w k_m$ and the entire path of $y_m(t)$. Then λ_w and the path of $y_a(t)$ for $t \in [0, T_m]$ follow from (25) and (27). Finally (24) determines, for any X_0, the level of S_0 such that the simultaneous activity of both sectors just solved for exactly exhausts X_0 at T_m. This defines the threshold level $\widehat{S}_0(X_0)$. It is monotonically increasing in X_0 and it must go through the origin, since otherwise we could not have $T_m = T_w$ at $X_0 = 0$.

For $\widehat{S}_0(X_0)$ thus defined, we may now state the following:

i. If $S_0 < (>) \widehat{S}_0(X_0)$, we will have $T_w > (<) T_m$.

ii. If $S_0 \leq \widehat{S}_0(X_0)$ then $y_a(t) > 0$ for all $t \geq 0$ and $y_m(t) > 0$ for all $t < T_m$.

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3.3.2 The determination of $\widetilde{S}_0(X_0)$

Assume now $S_0 > \widehat{S}_0(X_0)$ and hence $T_w < T_m$. From Propositions 1, 2 and 3, we know that $y_m(t)$ will be positive and decreasing for all $t \in [0, T_m)$. From Propositions 2 and 3, we also know that when $y_a(t)$ is positive it will be decreasing while $X(t)$ is positive, and increasing while $X(t)$ is zero. This is consistent with the following two possibilities: either $y_a(t)$ is positive for all $t \in [0, T_m)$, switching from decreasing to increasing exactly at T_w, or $y_a(t)$ is zero over some interval of time before it begins increasing. In order to fully characterize the optimal paths, we need to identify the conditions on the initial stocks under which each of those cases holds.

To do this, consider a scenario where the agricultural sector is active throughout the interval of time over which the water stock is being exhausted, just becomes inactive at the exact moment that the water stock is exhausted and immediately becomes active again. Hence $y_a(t) > 0$ for $t \in [0, T_w)$, $y_a(T_w) = 0$ (with condition (8) just satisfied with equality) and $y_a(t) > 0$ for $t \in (T_w, \infty)$.

If $y_a(T_w) = 0$, with (8) just satisfied with equality at $t = T_w$, and $y_a(t) > 0$ for all $t > T_w$, it is optimal to have $X(t) = 0$ for all $t \geq T_w$ (Proposition 4). From (3), we therefore have:

$$k_a y_a(t) + k_m y_m(t) = \bar{x}, \quad t \in [T_w, \infty), \tag{29}$$

from which it follows that $y_m(T_w) = \bar{y}_m$. We also know that $y_m(T_m) = 0$ and $y_m(t) > 0$ for $t \in [T_w, T_m)$ (Proposition 1). The solution being interior in both sectors, we must therefore have, from (8), (9) and (29):

$$u_a'(\bar{x} - \frac{k_m}{k_a} y_m(t)) - c_a = e^{rt} \lambda_w(t) k_a, \quad t \in [T_w, T_m] \tag{30}$$

$$u_m'(y_m(t)) - c_m = e^{rt}[\lambda_m + \lambda_w(t) k_m], \quad t \in [T_w, T_m], \tag{31}$$

and hence, after eliminating $\lambda_w(t)$:

$$u_m'(y_m(t)) - c_m - \frac{k_m}{k_a}[u_a'(\bar{x} - \frac{k_m}{k_a} y_m(t)) - c_a] = e^{rt} \lambda_m, \quad t \in [T_w, T_m]. \tag{32}$$

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Knowing that $y_a(T_w) = 0$, $y_m(T_w) = \bar{y}_m$ and $y_m(T_m) = 0$, we therefore have:

$$u'_m(\bar{y}_m) - c_m - \frac{k_m}{k_a}[u'_a(0) - c_a] = e^{rT_w}\lambda_m \tag{33}$$

$$u'_m(0) - c_m - \frac{k_m}{k_a}[u'_a(\bar{y}_a) - c_a] = e^{rT_m}\lambda_m \tag{34}$$

For a given T_w, conditions (33) and (34) determine λ_m and T_m. The entire path of $y_m(t)$ after T_w then follows from (32) and that of $y_a(t)$ from (29). Over the interval $[T_w, T_m]$ the resulting cumulative oil extraction, which we denote $\tilde{S}$, is:

$$\int_{T_w}^{T_m} y_m(t)dt = \tilde{S}.$$

The paths thus derived are optimal if and only if $S(T_w) = \tilde{S}$.

This determines the stock of oil that should be left at the date the stock of water becomes exhausted if this scenario is to be optimal for T_w and beyond. It remains to be determined what must happen before T_w.

In order for the oil stock to be exhausted over the interval $[0, T_m]$ and for the water stock to be exhausted over the interval $[0, T_w]$ it is necessary that:

$$\int_0^{T_w} y_m(t)dt + \tilde{S} = S_0 \tag{35}$$

and

$$\int_0^{T_w} k_a y_a(t)dt + k_m[S_0 - \bar{x}T_w - \tilde{S}] = X_0. \tag{36}$$

Over the interval $[0, T_w]$, λ_m and λ_w are constant and the solution for both sectors is interior, so that:

$$u'_a(y_a(t)) - c_a = e^{rt}\lambda_w k_a, \quad t \in [0, T_w] \tag{37}$$

$$u'_m(y_m(t)) - c_m = e^{rt}[\lambda_m + \lambda_w k_m], \quad t \in [0, T_w]. \tag{38}$$

At $t = T_w$, we must have $y_m(T_w) = \bar{y}_m$, since $y_a(T_w) = 0$ by assumption. Hence:

$$u'_a(0) - c_a = e^{rT_w}\lambda_w k_a \tag{39}$$

$$u'_m(\bar{y}_m) - c_m = e^{rT_w}[\lambda_m + \lambda_w k_m]. \tag{40}$$

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Conditions (35), (38) and (40) uniquely determine T_w, $\lambda_m + \lambda_w k_m$ and the path of $y_m(t)$ over the interval of time $[0, T_w]$. Then λ_w and the path of $y_a(t)$ over the same interval are determined from (39) and (37).

Finally, in order for this to constitute an optimal solution, the constraint (36) must also be satisfied. This determines, for any X_0, the level of S_0 that will exactly exhaust the water stock at T_w, determined above, and hence defines $\widetilde{S}_0(X_0)$. $\widetilde{S}_0(X_0)$ is monotonically increasing in X_0, with $\widetilde{S}_0(0) = \widetilde{S}$. The scenario posited at the outset, namely $y_a(t) > 0$ for $t \in [0, T_w)$, $y_a(T_w) = 0$ and $y_a(t) > 0$ for $t \in (T_w, \infty)$, will therefore be optimal if and only $S_0 = \widetilde{S}_0(X_0)$.

If $S_0 < \widetilde{S}_0(X_0)$, then there is relatively less pressure on water demand from the oil sector than with $S_0 = \widetilde{S}_0(X_0)$ and the left-hand side of (39) exceeds the right-hand side: $u'_a(0) > c_a + e^{rT_w} \lambda_w k_a$. Optimality then requires $y_a(T_w) > 0$. If on the other hand $S_0 > \widetilde{S}_0(X_0)$, then the demand for water from the oil sector pushes the shadow value of water up to a level such that the nonnegativity constraint on $y_a(t)$ becomes strictly binding at T_w and $u'_a(0) < c_a + e^{rT_w} \lambda_w k_a$. The agricultural sector will therefore be inactive over a positive interval of time instead of just at $t = T_w$.

Note that this interval must end at some date $\widetilde{T} > T_w$ such that $S(\widetilde{T}) = \widetilde{S}$. The reason for this follows from the definition of $\widetilde{S}$. When $X(t) = 0$ and $S(t) = \widetilde{S}$ we must have $y_a(t) = 0$ for the path to be optimal from that date on. But since oil extraction is positive, then $S(t) < \widetilde{S}$ beyond $\widetilde{T}$ and hence $y_a(t) > 0$. If we now denote by $\tau \in [0, T_w)$ the date at which this interval begins, then for $\widetilde{S}_0(X_0)$ as just defined, we may state:

i. If $\widetilde{S}_0(X_0) > S_0 > \widehat{S}_0(X_0)$ then $T_w < T_m$ and $y_a(t) > 0$ for all $t \in [0, \infty)$.

ii. If $S_0 > \widetilde{S}(X_0)$ then $T_w < T_m$ and there exist an interval $[\tau, \widetilde{T}]$ such that $y_a(t) = 0$ for all $t \in [\tau, \widetilde{T}]$, with $0 \leq \tau < T_w < \widetilde{T} < T_m$.

3.3.3 The determination of $\overline{S}_0(X_0)$

In view of the results just established we still need to distinguish between the case where $\tau = 0$ and that where $\tau > 0$. In order to do this, consider the following hypothetical situation:

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$S_0 > \widetilde{S}_0(X_0)$, $y_a(t) = 0$ for $t \in [0, \widetilde{T})$, with the first-order condition (8) just satisfied with equality at $t = 0$, and $y_a(t) > 0$ for $t \in [\widetilde{T}, \infty)$. So $\tau = 0$. Then, since $T_m > \widetilde{T} > T_w$, the water stock will be exhausted by the oil sector alone and hence:

$$k_m \int_0^{T_w} y_m(t)dt = X_0 + \bar{x}T_w \tag{41}$$

In addition, the following must hold:

$$u_a'(0) = c_a + \lambda_w k_a \tag{42}$$

$$u_a'(0) = c_a + e^{r\widetilde{T}}\lambda_w(\widetilde{T})k_a \tag{43}$$

$$u_a'(\bar{y}_a) = c_a + e^{rT_m}\lambda_w(T_m)k_a \tag{44}$$

and

$$u_m'(\bar{y}_m) = c_m + e^{rT_w}[\lambda_m + \lambda_w(T_w)k_m] \tag{45}$$

$$u_m'(\bar{y}_m) = c_m + e^{r\widetilde{T}}[\lambda_m + \lambda_w(\widetilde{T})k_m] \tag{46}$$

$$u_m'(0) = c_m + e^{rT_m}[\lambda_m + \lambda_w(T_m)k_m] \tag{47}$$

$$u'(y_m(t)) = c_m + e^{rt}[\lambda_m + \lambda_w(t)k_m], \quad t \in [0, T_w]. \tag{48}$$

Furthermore, since $\dot{\lambda}_w(t) = 0$ for $t \in [0, T_w)$ and $\lambda_w(t)$ is continuous, we must have:

$$\lambda_w(T_w) = \lambda_w. \tag{49}$$

Substituting in (45) to (48) for λ_w, $e^{r\widetilde{T}}\lambda_w(\widetilde{T})$ and $e^{rT_m}\lambda_w(T_m)$ obtained from (42), (43) and (44), we get:

$$u_m'(\bar{y}_m) = c_m + e^{rT_w}\lambda_m + e^{rT_w}\frac{k_m}{k_a}[u_a'(0) - c_a] \tag{50}$$

$$u_m'(\bar{y}_m) = c_m + e^{r\widetilde{T}}\lambda_m + \frac{k_m}{k_a}[u_a'(0) - c_a] \tag{51}$$

$$u_m'(0) = c_m + e^{rT_m}\lambda_m + \frac{k_m}{k_a}[u_a'(\bar{y}_a) - c_a] \tag{52}$$

$$u'(y_m(t)) = c_m + e^{rt}[\lambda_m + \frac{k_m}{k_a}[u_a'(0) - c_a]], \quad t \in [0, T_w]. \tag{53}$$

Those four equations, together with (41) determine T_w, $\widetilde{T}$, T_m, λ_m and the path of $y_m(t)$ for $t \in [0, T_w]$. Knowing T_w and T_m, the path of $y_m(t)$ for $t \in (T_w, T_m]$ then follows from (9).

This scenario will constitute and optimum if and only if the total extraction over the interval $[\widetilde{T}, T_m]$ equals $\widetilde{S}$, the total extraction over the interval $(T_w, \widetilde{T})$ equals $(\widetilde{T} - T_w)\bar{y}_m$, and the total extraction over the interval $[0, T_m]$ equals S_0. Hence we must have in addition:

$$\frac{X_0 + \bar{x}T_w}{k_m} + (\widetilde{T} - T_w)\bar{y}_m + \widetilde{S} = S_0. \tag{54}$$

This defines $\overline{S}_0(X_0)$. The function $\overline{S}_0(X_0)$ is monotonically increasing in X_0, with $\overline{S}_0(0) = \overline{S} = \widetilde{T}\bar{y}_m + \widetilde{S}$.

If $\widetilde{S}_0(X_0) < S_0 < \overline{S}_0(X_0)$, the water demand from the oil sector puts relatively less pressure on the value of water than when $S_0 = \overline{S}_0(X_0)$. As a result $u_a'(0) > c_a + \lambda_w k_a$. It therefore becomes optimal for the agricultural sector to be active during a positive interval of time $[0, \tau]$, where $\tau < \widetilde{T}$ denotes the time at which

$$u_a'(0) = c_a + e^{r\tau} \lambda_w k_a. \tag{55}$$

The agricultural sector then becomes inactive and remains so until the oil stock reaches $\widetilde{S}$, at time $\widetilde{T}$. On the other hand, if $S_0 > \overline{S}_0(X_0)$, then $u_a'(0) < c_a + \lambda_w k_a$ and the agricultural sector is inactive from the start and remains so until $\widetilde{T}$.

We may therefore state:

i If $\overline{S}(X_0) > S_0 > \widetilde{S}(X_0)$ then $T_w < T_m$ and there exist an interval $[\tau, \widetilde{T}]$ such that $y_a(t) = 0$ for all $t \in [\tau, \widetilde{T}]$, with $0 < \tau < T_w < \widetilde{T} < T_m$.

ii If $S_0 > \overline{S}(X_0)$, then $T_w < T_m$ and there exist an interval $[0, \widetilde{T}]$ such that $y_a(t) = 0$ for all $t \in [0, \widetilde{T}]$, with $0 < T_w < \widetilde{T} < T_m$.

3.4 The optimal paths

The threshold values $\widehat{S}_0(X_0)$, $\widetilde{S}_0(X_0)$, $\widetilde{S}$, $\overline{S}_0(X_0)$ and $\overline{S}$ just defined now allow us to fully characterize the optimal paths in $(X(t), S(t))$-space. For any given $X_0 > 0$, the optimal

paths of the agricultural sector and of the oil sector have the following properties, where $y_a^*(t)$ and $y_m^*(t)$ denote the interior solution to (8) and (9) respectively:

If $S_0 \geq \overline{S}_0(X_0)$:

$$y_a(t) = \begin{cases} 0 & \text{for } t \in [0, \widetilde{T}]; \\ \bar{y}_a - \frac{k_m}{k_a} y_m(t) > 0 & \text{for } t \in (\widetilde{T}, T_m); \\ \bar{y}_a & \text{for } t \in [T_m, \infty). \end{cases}$$

$$y_m(t) = \begin{cases} y_m^*(t) > 0 & \text{for } t \in [0, T_w); \\ \bar{y}_m & \text{for } t \in [T_w, \widetilde{T}]; \\ y_m^*(t) > 0 & \text{for } t \in (\widetilde{T}, T_m); \\ 0 & \text{for } t \in [T_m, \infty). \end{cases}$$

If $\overline{S}_0(X_0) > S_0 > \widetilde{S}_0(X_0)$:

$$y_a(t) = \begin{cases} y_a^*(t) > 0 & \text{for } t \in [0, \tau); \\ 0 & \text{for } t \in [\tau, \widetilde{T}]; \\ \bar{y}_a - \frac{k_m}{k_a} y_m(t) > 0 & \text{for } t \in (\widetilde{T}, T_m); \\ \bar{y}_a & \text{for } t \in [T_m, \infty). \end{cases}$$

$$y_m(t) = \begin{cases} y_m^*(t) > 0 & \text{for } t \in [0, T_w); \\ \bar{y}_m & \text{for } t \in [T_w, \widetilde{T}]; \\ y_m^*(t) > 0 & \text{for } t \in (\widetilde{T}, T_m); \\ 0 & \text{for } t \in [T_m, \infty). \end{cases}$$

If $S_0 = \widetilde{S}_0(X_0)$:

$$y_a(t) = \begin{cases} y_a^*(t) > 0 & \text{for } t \in [0, T_w); \\ 0 & \text{for } t = T_w; \\ \bar{y}_a - \frac{k_m}{k_a} y_m(t) > 0 & \text{for } t \in (T_w, T_m); \\ \bar{y}_a & \text{for } t \in [T_m, \infty). \end{cases}$$

$$y_m(t) = \begin{cases} y_m^*(t) > 0 & \text{for } t \in [0, T_m); \\ 0 & \text{for } t \in [T_m, \infty). \end{cases}$$

If $\widetilde{S}_0(X_0) > S_0 > \widehat{S}_0(X_0)$:

$$y_a(t) = \begin{cases} y_a^*(t) > 0 & \text{for } t \in [0, T_w); \\ \bar{y}_a - \frac{k_m}{k_a} y_m(t) > 0 & \text{for } t \in (T_w, T_m); \\ \bar{y}_a & \text{for } t \in [T_m, \infty). \end{cases}$$

$$y_m(t) = \begin{cases} y_m^*(t) > 0 & \text{for } t \in [0, T_m); \\ 0 & \text{for } t \in [T_m, \infty). \end{cases}$$

If $\widehat{S}_0(X_0) \geq S_0$:

$$y_a(t) = \begin{cases} y_a^*(t) > 0 & \text{for } t \in [0, T_w); \\ \bar{y}_a & \text{for } t \in [T_w, \infty). \end{cases}$$

$$y_m(t) = \begin{cases} y_m^*(t) > 0 & \text{for } t \in [0, T_m); \\ 0 & \text{for } t \in [T_m, \infty). \end{cases}$$

Figure 1 illustrates the optimal paths in $(X(t), S(t))$-space for different values of S_0 and a given X_0.

The case of $\overline{S}_0(X_0) > S_0 > \widetilde{S}_0(X_0)$ offers a rather rich structure and is well suited to illustrate the time paths of the different implicit current value prices. In that case, there are five distinct phases, as depicted in Figure 2.

In the first phase, during the interval $[0, \tau)$, the stock of water is positive and both sectors are active, with $(y_a(t), y_m(t)) = (y_a^*(t), y_m^*(t))$. During this phase, the full marginal cost of production of the agricultural sector, $c_a + k_a e^{rt} \lambda_w$, is increasing. It reaches the agricultural choke price, $u_a'(0)$, at $t = \tau$, at which time the agricultural sector stops producing.

Then begins the second phase, which lasts throughout the interval $[\tau, T_w)$. Since the full marginal cost of agriculture continues to increase over that interval, the agricultural sector remains inactive and we have $(y_a(t), y_m(t)) = (0, y_m^*(t))$.

At time T_w, the water stock is exhausted. From that point on, the water stock will remain at zero (Proposition 4) and total water consumption becomes constrained by $\bar{x}$, the natural water inflow. Although the shadow value of water then begins decreasing, the full marginal cost of agricultural production is higher than the choke price and will remain so for some time.

We therefore have a third phase, over the interval $[T_w, \widetilde{T}]$, during which $(y_a(t), y_m(t)) = (0, \bar{y}_m)$. The implicit price of oil remains constant over that interval, at $u_m'(\bar{y}_m) = c_m + e^{rt}[\lambda_m + \lambda_w(t)k_m]$, since water consumption is constrained to $\bar{x}$ and hence oil production is constrained to $\bar{y}_m$. Note that since $y_a(\tau) = y_a(\widetilde{T}) = 0$, it must be the case that $\lambda_w(\widetilde{T}) = e^{-r(\widetilde{T}-\tau)}\lambda_w(\tau)$, with $\lambda_w(\tau) = \lambda_w$, the constant discounted shadow value of water over the interval $[0, T_w]$. The new shadow value of water is decreasing during that third phase,
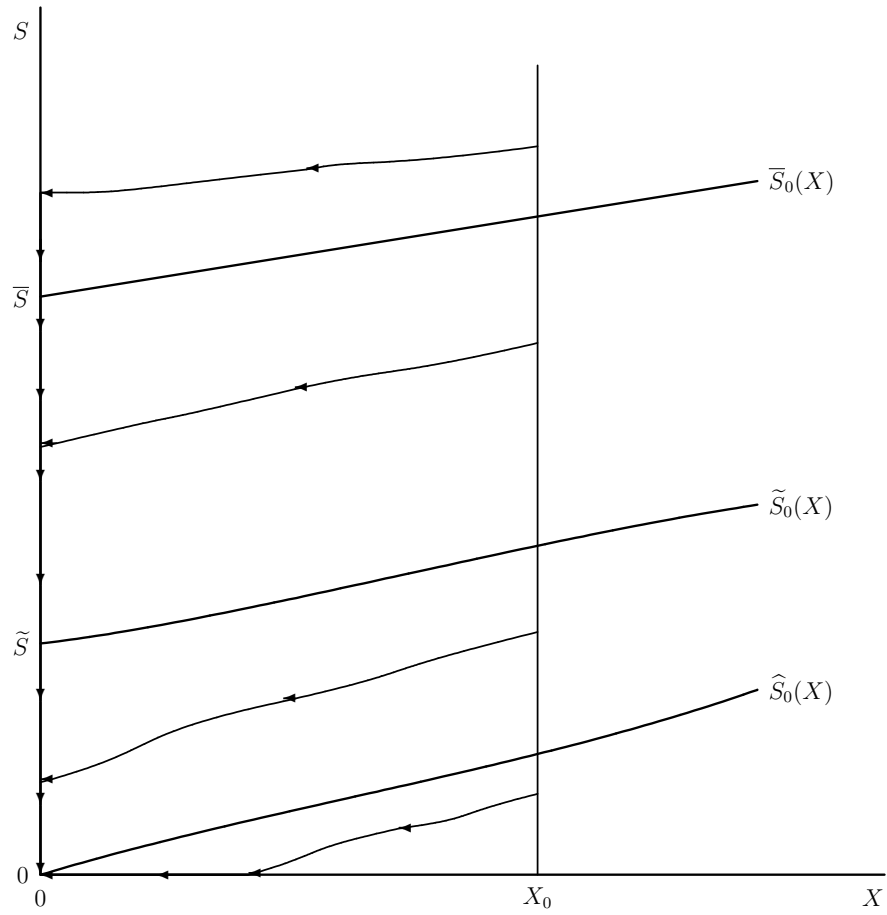
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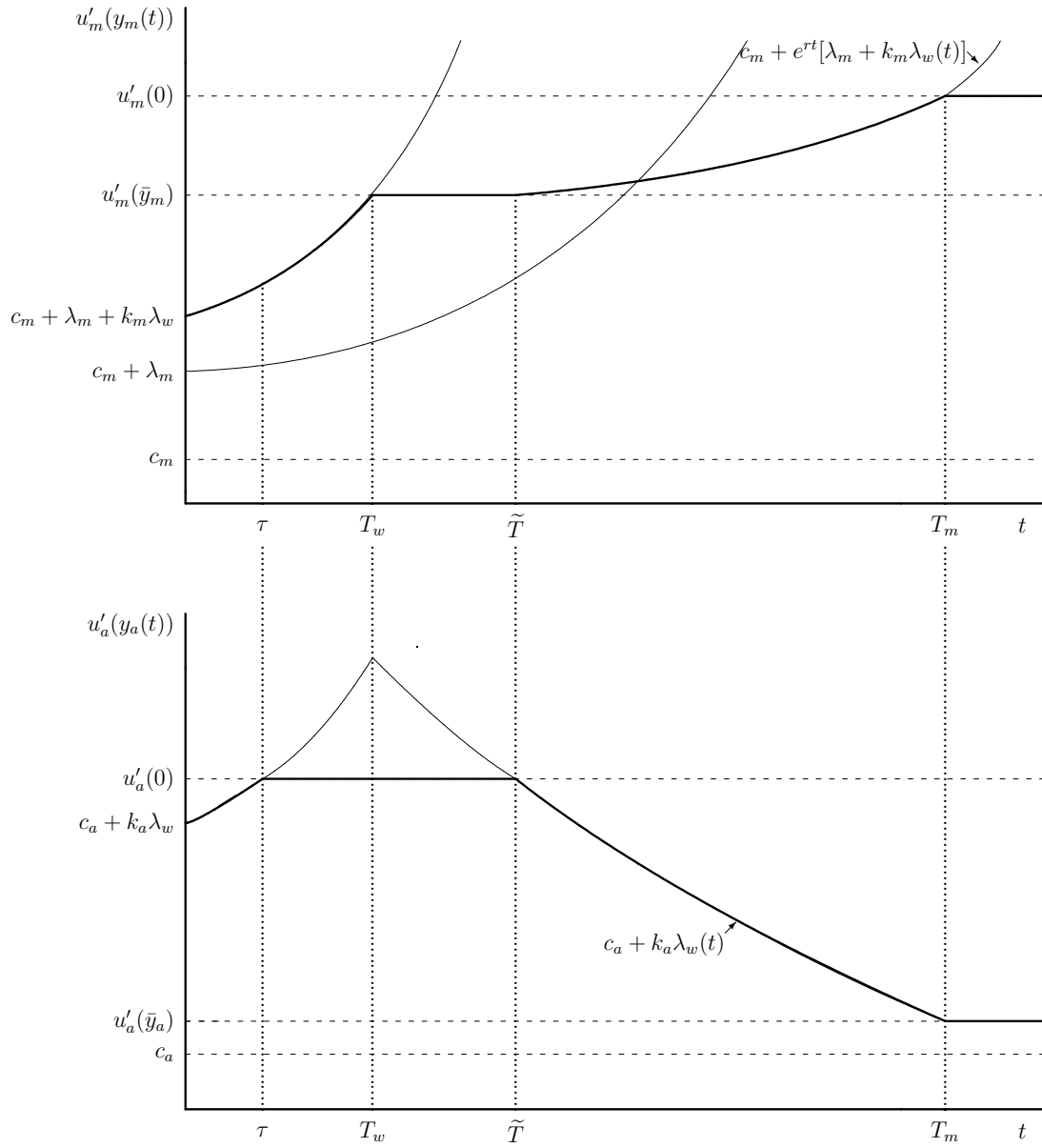Figure 1 : The optimal paths in (X, S)-space

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Figure 2 : $\hat{y}_a > \bar{y}_a$ and $\overline{S}_0(X_0) > S_0 > \widetilde{S}_0(X_0)$

because, as the oil stock decreases, so does the pressure on water demand. At time $\widetilde{T}$, the full marginal cost of agriculture becomes just low enough for agricultural production to resume.

Then begins a fourth phase, during which $(y_a(t), y_m(t)) = (\bar{y}_a - \frac{k_m}{k_a} y_m^*(t), y_m^*(t))$ until the oil stock is exhausted, at T_m. Over the interval $(\widetilde{T}, T_m)$, the full marginal cost of oil production is increasing and eventually reaches the choke price for oil at T_m, when $u_m'(0) = c_m + e^{rT_m}[\lambda_m + \lambda_w(T_m)k_m]$. The full marginal cost of agriculture is decreasing during this phase, until at T_m we have $u_a'(\bar{y}_a) = c_a + e^{rT_m}\lambda_w(T_m)k_a$.

In the final phase there is no more oil, so there remains only the agricultural sector. Therefore $(y_a(t), y_m(t)) = (\bar{y}_a, 0)$ for all $t \in [T_m, \infty)$ and the implicit price of agriculture is constant at $u_a'(\bar{y}_a)$.

The other cases are now easily characterized. If $S_0 \geq \overline{S}_0(X_0)$, the price paths have exactly the same configuration as in Figure 2. Only now the pressure on water demand from the oil sector is so high that $\tau = 0$ and the first phase collapses: the agricultural sector is inactive from the beginning and remains inactive until time $\widetilde{T}$.

If $S_0 = \widetilde{S}_0(X_0)$, then $\tau = T_w = \widetilde{T}$, which means that the second and third phases collapse. The agricultural sector is active throughout except for an instant, at T_w. We therefore have a phase ending at T_w during which the water stock is being exhausted, with both sectors active and the full marginal cost of production increasing in both sectors. This is followed by a phase ending at T_m during which the remaining oil stock is being exhausted, still with both sectors active, but now with the full marginal cost of agriculture decreasing and that of oil still increasing, although at a slower rate due to the fact that $\lambda_w(t)$ is now decreasing. The final phase has the agricultural sector producing indefinitely at the full capacity permitted by the natural water inflow and the price of agriculture constant. This case is a borderline case. It separates the cases where, given the initial water stock, the size of the initial oil stock dictates that the agricultural sector should remain inactive during some period of time, from those cases where it does not.

When $S_0 < \widetilde{S}_0(X_0)$, then the initial oil stock is not sufficiently large, relative to the water stock, for it to be optimal to interrupt agricultural production in order to favor oil production. Therefore the agricultural sector will always be active, $\tau = T_w = \widetilde{T}$, and there are only three phases, as in the case when $S_0 = \widetilde{S}_0(X_0)$.

Two subcases of $S_0 < \widetilde{S}_0(X_0)$ need to be distinguished. If $\widetilde{S}_0(X_0) > S_0 > \widehat{S}_0(X_0)$, then the water stock will be exhausted before the oil stock. The three phases are characterized, on the production side, by: $(y_a(t), y_m(t)) = (y_a^*(t), y_m^*(t))$ during the interval $[0, T_w)$; $(y_a(t), y_m(t)) = (\bar{y}_a - \frac{k_m}{k_a} y_m^*(t), y_m^*(t))$ during the interval $[T_w, T_m)$; $(y_a(t), y_m(t)) = (\bar{y}_a, 0)$ during the interval $[T_m, \infty)$. As for the implicit price paths, both are increasing during the interval $[0, T_w)$, while the water stock is being depleted, but decreasing for agriculture and increasing for oil during the interval $[T_w, T_m)$, at which point begins the final phase, with the implicit price of agriculture given by $u_a'(\bar{y})$ for all $t \geq T_m$.

On the other hand, if $S_0 < \widehat{S}_0(X_0)$, the initial oil stock is small enough that it is optimal to exhaust it before the water stock. Then the three phases are characterized on the production side by: $(y_a(t), y_m(t)) = (y_a^*(t), y_m^*(t))$ during the interval $[0, T_m)$; $(y_a(t), y_m(t)) = (y_a^*(t), 0)$ during the interval $[T_m, T_w)$; $(y_a(t), y_m(t)) = (\bar{y}_a, 0)$ during the interval $[T_w, \infty)$. During the first of those phases, the full marginal costs and hence the implicit prices are increasing in both sectors, until there is no more oil. Since the water stock is still positive at that point, the shadow value of water remains constant at λ_w and therefore the implicit price of agriculture keeps increasing, until the water stock is exhausted. This occurs at T_w, when $u_a'(\bar{y}_a) = c_a + e^{rT_w} \lambda_w k_a$. Then follows the usual final phase, with the price of agriculture constant at $u_a'(\bar{y}_a)$ for all $t \geq T_w$.

4 The natural water inflow poses no constraint on agriculture

Consider now the case where $\hat{y}_a < \bar{y}_a$. In this case water availability poses no constraint on the agricultural sector and, if there were no oil sector, the shadow value of water would be zero. From condition (8) we then have $u_a'(y_a(t)) = c_a$ and hence $y_a(t) = \hat{y}_a$ for all $t \geq 0$.

This will obviously be the case for all $t \geq T_m$, once the existence of an oil sector is taken into account.[9]

If there were no agricultural sector, then exactly the same two cases as in Section 3.2 need to be distinguished. In one case, water is abundant, $\lambda_w = 0$, and we have a pure Hotelling-type path for the oil sector. In the other case, water is scarce and the optimal path would be characterized by the same three phases derived in Section 3.2.

Now let the two sectors be present from the outset. All the threshold levels introduced in Section 3.3 remain pertinent and can be similarly defined. Clearly, if $S_0 < \widehat{S}_0(X_0)$, so that $T_w > T_m$, then water availability is never a constraint for either sector and $\lambda_w = 0$ for all $t > 0$. We then have $y_a(t) = \hat{y}_a$ and oil production follows the same Hotelling-type path as if there were no agricultural sector.

It is not necessary however that $T_m < T_w$ in order for water to have no value. Indeed, assume $S_0 > \widehat{S}_0(X_0)$, so that $T_m > T_w$, and consider a hypothetical situation where $y_a(t) = \hat{y}_a$ for all $t \in [0, T_m]$ and where λ_m, T_m and $y_m^*(t)$ solve:

$$u'(y_m(t)) = c_m + e^{rt}\lambda_m, \quad t \in [0, T_m],$$

$$u'(0) = c_m + e^{rT_m}\lambda_m,$$

and

$$\int_0^{T_m} y_m(t)dt = S_0.$$

For this to constitute the optimal solution, S_0 must be such that it also satisfies:

$$T_m[k_a\hat{y}_a - \bar{x}] + k_m S_0 = X_0. \tag{56}$$

Denote the level of S_0 required to satisfy (56) by $S_0^H(X_0)$. Then for any initial oil stock $S_0 \leq S_0^H(X_0)$, $\lambda_w = 0$, the optimal oil production path is a pure Hotelling-type path and $y_a(t) = \hat{y}_a$ for all $t > 0$. On the other hand, if $S_0 > S_0^H(X_0)$, then water is scarce and $\lambda_w > 0$.

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[9]Since $\hat{y}_a < \bar{y}_a$, this means that the water stock will be replenished once the oil stock is exhausted. It would be natural to impose an upper bound on the stock of water. We have chosen to ignore this issue here, since, if any excess can simply be wasted or freely disposed of, the existence of this upper bound will have no impact on the nature of the optimal paths. Note that in this case, since the stock of water is positive in the end, the transversality condition (12) will be satisfied with the shadow value of water becoming zero.

Since the oil stock is continuously decreasing over the interval $[0, T_m)$ (Proposition 1) and $y_m(T_m) = 0$, for any $S_0 > S_0^H(X_0)$, the stock of oil must eventually reach $S_0^H(X_0)$ at some date $T_H < T_m$. When the oil stock reaches $S_0^H(X_0)$, water becomes abundant and $\lambda_w(t)$ becomes zero and remains at zero for all $t \geq T_H$. This means that the final phase, during which agriculture is the only active sector, with $y_a(t) = \hat{y}_a$ for all $t \in [T_m, \infty)$, is necessarily preceded by a phase during which $y_a(t) = \hat{y}_a$ and oil production follows a pure Hotelling-type path, with $y_m(t) = y_m^*(t) < \bar{x} - \frac{k_a}{k_m}\hat{y}_a$.

Figure 3 depicts the implicit price paths for the case where $\overline{S}_0(X_0) > S_0 > \widetilde{S}_0(X_0)$. The first three phases are exactly the same as in Section 3. The first phase, for $t \in [0, \tau)$, has $(y_a(t), y_m(t)) = (y_a^*(t), y_m^*(t))$, with the full marginal cost of both oil and agricultural production increasing. At $t = \tau$, the full marginal cost of agricultural production reaches the choke price from below and the agricultural sector ceases to produce. The second phase, for $t \in [\tau, T_w)$, has $(y_a(t), y_m(t)) = (0, y_m^*(t))$. Oil production becomes constrained by the natural inflow of water just as the water stock becomes exhausted, $t = T_w$. The third phase, for $t \in [T_w, \widetilde{T})$, has $(y_a(t), y_m(t)) = (0, \bar{y}_m)$. The full marginal cost of water is decreasing during that phase and reaches the agricultural choke price from above at $t = \widetilde{T}$, after which point agricultural production resumes.

During the fourth phase, for $t \in (\widetilde{T}, T_m)$, both sectors are active. This phase can now be divided into two sub-phases. The first sub-phase occurs during the interval $(\widetilde{T}, T_H)$, when the natural water inflow constitutes a binding constraint on total water consumption. The optimal production paths are $(y_a(t), y_m(t)) = (y_a^*(t), \bar{y}_m - \frac{k_a}{k_m}y_a^*(t))$. By Proposition 3, oil production is decreasing and agricultural production is increasing towards $\hat{y}_a$. The second sub-phase occurs during the interval $[T_H, T_m)$. Total water consumption is not constrained by the natural water inflow, $\lambda_w(t) = 0$ and the optimal production paths are given by $(y_a(t), y_m(t)) = (\hat{y}_a, y_m^*(t))$, with $y_m^*(t) < \bar{y}_m - \frac{k_a}{k_m}\hat{y}_a$. Thus oil production follows a pure Hotelling-type path during that sub-phase. The fifth phase is the final phase, with $y_a(t) = \hat{y}$ for all $t \in [T_m, \infty)$.
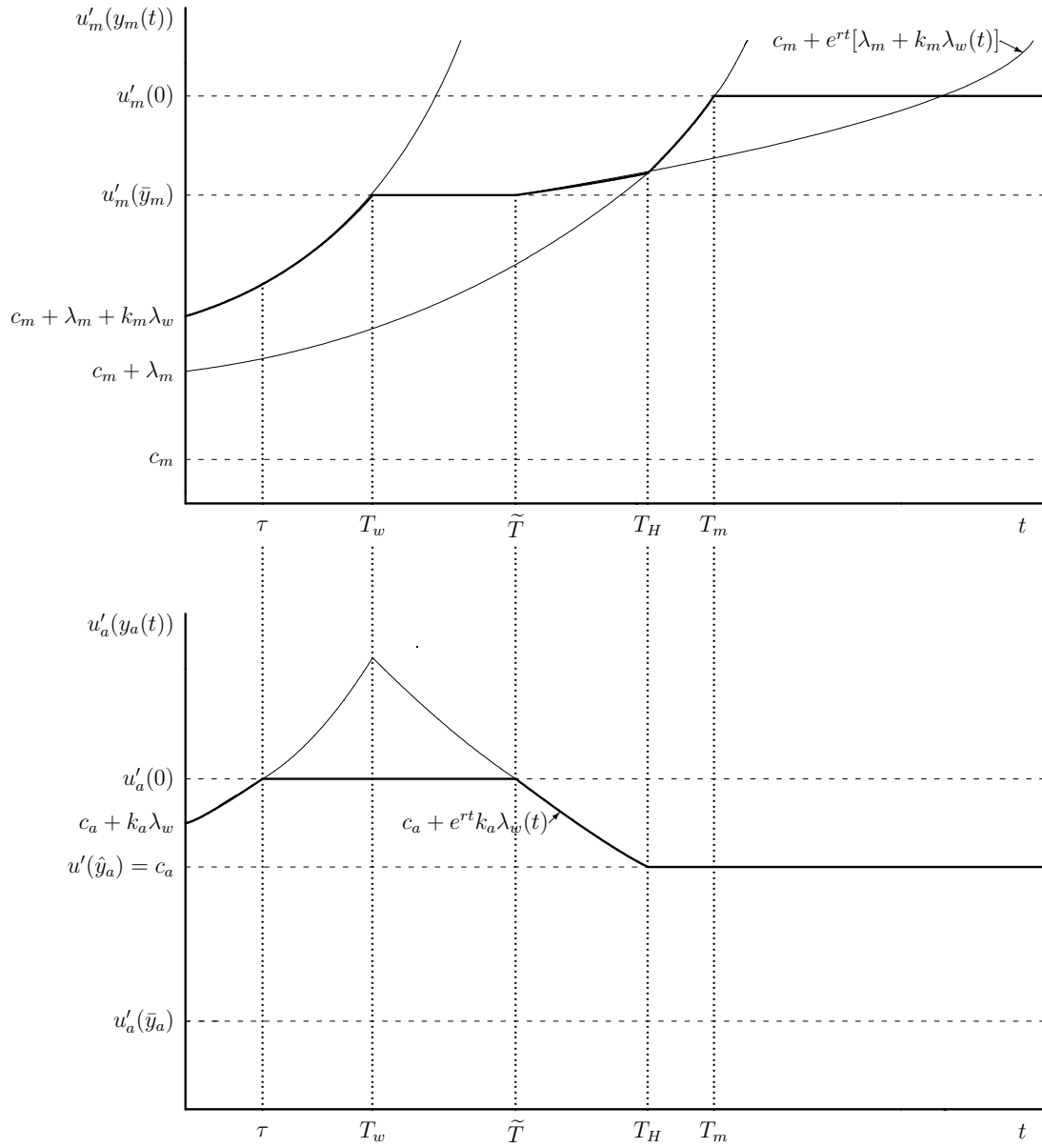
Figure 3 : $\hat{y}_a < \bar{y}_a$ and $\overline{S}_0(X_0) > S_0 > \widetilde{S}_0(X_0)$

As with the paths depicted in Figure 2 of Section 3, for any given X_0 the paths depicted in Figure 3 contain all the other possible path configurations as special cases, depending on S_0. If $S_0 > \overline{S}_0(X_0)$, then $\tau = 0$ and the agricultural sector is inactive from the beginning and remains inactive until $t = \widetilde{T}$. If $S_0 < \widetilde{S}_0(X_0)$, the five phases corresponding to the case where $\overline{S}_0(X_0) > S_0 > \widetilde{S}_0(X_0)$ described in Figure 3 collapse into three phases, since then $\tau = T_w = \widetilde{T}$ and the agricultural sector is always active. The optimal paths during those three phases are exactly as in the case where $\hat{y}_a > \bar{y}_a$, except for the fact that now the next to last phase will always be composed of the two sub-phases described above. The second of those two sub-phases is always characterized by a pure Hotelling-type path, due to the fact that water availability does not constitute a constraint beyond T_H when $\hat{y}_a < \bar{y}_a$.

5   Conclusion

We have analyzed the problem faced by an economy in which a nonrenewable resource sector, such as oil, and a reproducible good sector, such as agriculture, must share as an essential input some renewable resource, such as water. The optimal allocation over time of the scarce resource between the two sectors poses a dynamic optimization problem involving two state variables: the stock of oil and the stock of water. We have been able to fully characterize the solution to this problem in order to show how, for a given initial stock of water, the production paths and the water usage of the two sectors depend on the size of the initial stock of oil and on whether or not the natural inflow of water constitutes a constraint on the agricultural sector in the long run, when there is no more oil left.

A striking result is that the optimal paths may involve abandoning agriculture after some time, in order to reserve the water for the oil sector during an interval of time, at the end of which agricultural activity resumes. This can occur whether the water resource constitutes a long-run constraint on agriculture or not. It will occur when the demand pressure on the value of water from the oil sector is such that the full marginal cost of agriculture reaches the agricultural choke price from below before the water stock is exhausted. We have identified,

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for any given initial stock of water, the critical range inside which the initial oil stock must fall in order for this to be a characteristic of the optimal paths. If the initial oil stock is above that critical range, then the full marginal cost of agriculture is initially higher than the agricultural choke price and the agricultural sector is inactive from the outset. If the initial oil stock is below that critical range, then both sectors are always active, as long as the oil is not fully depleted. Once the oil stock is depleted, the agricultural sector produces indefinitely at the level that equates gross marginal benefit to marginal cost of production, as in a static equilibrium, unless its production is constrained by the natural inflow of water.

Another feature of the solution is that the optimal path of the oil sector does not generally follow a pure Hotelling-type path, with the implicit price of oil net of extraction cost growing at the rate of interest. This is because the full marginal opportunity cost of oil production must account not only for the rent imputed on the finite oil stock but also that imputed on the stock of water, which in turn depends on the level of activity of the agricultural sector. Our model thus provides a framework for generalizing the Hotelling rule to cases where the full marginal cost of extracting the nonrenewable resource depends on the endogenous level of activity of another sector of the economy that shares a common availability constraint on an essential input. In the particular problem analyzed in this paper, only in the case where the natural inflow of water does not pose a long-run constraint on agricultural production will there be a phase during which oil production follows a pure Hotelling path. In that case, this will occur once the oil stock falls below a certain critical value, beyond which water becomes abundant, being a constraint neither for the oil nor for the agricultural sector. But even then, the whole path will always be characterized by other phases where it does not follow a pure Hotelling rule.

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