Existence of competitive equilibrium in an optimal growth model with heterogeneous agents and endogenous leisure*

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Abstract

This paper proves the existence of competitive equilibrium in a single-sector dynamic economy with heterogeneous agents and elastic labor supply. The method of proof relies on some recent results concerning the existence of Lagrange multipliers in infinite dimensional spaces and their representation as a summable sequence and a direct application of the Brouwer fixed point theorem.

Keywords: Optimal growth model, Lagrange multipliers, Competitive equilibrium, Elastic labor supply.
JEL Classification: *C61, D51, E13, O41*

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1 Introduction

Since the seminal work of Ramsey [1928], optimal growth models have played a central role in modern macroeconomics. Classical growth theory relies on the assumption that labor is supplied in fixed amounts, although the original paper of Ramsey did include the disutility of labor as an argument in consumers' utility functions. Subsequent research in applied macroeconomics (theories of business cycles fluctuations) have reassessed the role of labor-leisure choice in the process of growth. Nowadays, intertemporal models with elastic labor continue to be the standard setting used to model many issues in applied macroeconomics.

Lagrange multiplier techniques have facilitated considerably the analysis of constrained optimization problems. The applications of those techniques in the analysis of intertemporal models inherits most of the tractability found in a finite setting. However, the passage to an infinite dimensional setting raises additional questions. These questions concern both the extension of the Lagrangean in an infinite dimensional setting as well as the representation of the Lagrange multipliers as a summable sequence.

Our purpose is to prove existence of competitive equilibrium for the basic neoclassical model with elastic labor using some recent results (see Le Van and Saglam [2004]) concerning the existence of Lagrange multipliers in infinite dimensional spaces and their representation as a summable sequence.

Previous work addressing existence of competitive equilibrium issues in intertemporal models attacks the problem of existence from an abstract point of view. Following the early work of Peleg and Yaari [1970], this approach is based on separation arguments applied to arbitrary vector spaces (see see Bewley [1972], Bewley [1982], Aliprantis et al. [1997], Dana and Le Van [1991]). The advantage of this approach is that it yields general results capable of application in a wide variety of specific models but they require a high level of abstraction.

Dynamic optimizing models where infinitely-lived heterogeneous agents maximize their lifetime utilities in perfect foresight equilibrium settings have, to date, concentrated on perfectly inelastic labor supply cases. In a complete market model, Le Van and Vailakis [2003] have studied the so-called reduced form associated with the welfare maximization problem in a single-sector model with inelastic labor supply. Many difficulties arise when they prove convergence of the optimal path due to the fact that the Pareto-optimum problem is non-stationary. Their arguments exploit the fact that the stationary problem involving only the agents with a discount factor equal to the maximum one has a unique stable state k^s. This property enable them to prove the consumption paths of all agents with a discount factor equal to the maximum one converge to strictly positive limit points. Subsequently, by using this limit points, they define a sequence of prices as the marginal utility and prove that the sequence of prices belong to l^1_+ (see Lemma 9 in Le Van and Vailakis [2003]). This method, with some additional assumptions[1], also has been used in Le Van and Vailakis [2004] to prove the existence of competitive equilibrium in a model with one representative agent and elastic labor supply.

Recently, C. Le Van, M.H. Nguyen and Y. Vailakis [2007] extend the canonical representative Ramsey model to include heterogeneous agents and elastic labor supply where supermodularity is used to establish the convergence of optimal paths. The novelty in their works is that relatively impatient consumers have their consumption and leisure converging to zero as time tends towards infinity. However, they did not prove the existence of competitive equilibrium of the economy and the method used in Le Van and Vailakis [2003] could not apply due to the presence of leisure. The purpose of this paper is to complete this important point. Our approach is based on the result of existence of Lagrange multipliers of the Pareto problem and their representation as a summable sequence to define the sequence of prices and wage as these multipliers rather than marginal utility as usual. Following the Negishi approach, our strategy for tackling the question of existence relies on exploiting the link between Pareto-optima and competitive equilibria. We show that there exists a Lagrange multiplier as a price system such that together with the Pareto-optimal solution they constitute *a price equilibrium with transfers*. These transfers depend on the individual weights involved in the social welfare function. An equilibrium exists provided that there is a set of welfare weights such that the corresponding transfers equal zero.

The organization of the paper is as follows. In section 2, we present the model and provide sufficient conditions on the objective function and the constraint functions so that Lagrangean multipliers can be presented by an l^1_+ sequence. We characterize some dynamic properties of the Pareto optimal paths of capital and of consumption-leisure. In particular, we prove that the optimal consumption and leisure paths of the most impatient agents will converge to zero in the long run, with a very elementary proof compared to the one in C. Le Van, M-H. Nguyen and Y. Vailakis [2007] which uses supermodularity for lattice programming. In section 3, we prove the existence of competitive equilibrium by using the Negishi approach and the Brouwer fixed point theorem.

[1] They used assumptions $\frac{u(\epsilon,\epsilon)}{\epsilon} \to +\infty$ as $\epsilon \to 0$ for the proof $c_t > 0, l_t > 0$ and $\frac{u_{cc}}{u_c} \leq \frac{u_{cl}}{u_l}$ for the proof $k_t > 0$ for all t.

2 The model

We consider an intertemporal model with $m \geq 1$ consumers and one firm. The preferences of each consumer take additively form: $\sum_{t=0}^{\infty} \beta_i^t u^i(c_t^i, l_t^i)$ where $\beta_i \in (0,1)$ is the discount factor. At date t, agent i consumes the quantity c_t^i, spends a quantity of leisure l_t^i and supplies a quantity of labor L_t^i. Production possibilities are presented by the gross production function F and a physical depreciation $\delta \in (0,1)$. Denote $F(k_t, \sum_{i=1}^{m}(1-l_t^i)) + (1-\delta)k_t = f(k_t, \sum_{i=1}^{m}(1-l_t^i))$.

We next specify a set of restrictions imposed on preferences and production technology.[2] The assumptions on period utility function $u^i : \mathbb{R}_+^2 \to \mathbb{R}$ are as follows:

Assumption U1: *u^i is continuous, concave, increasing on $\mathbb{R}_+^2$ and strictly increasing, strictly concave on $\mathbb{R}_{++}^2$.*

Assumption U2: $u^i(0,0) = 0$.

Assumption U3: *u^i is twice continuously differentiable on $\mathbb{R}_{++}^2$ with partial derivatives satisfying the Inada conditions:* $\lim_{c\to 0} u_c^i(c,l) = +\infty$, $\forall l > 0$ and $\lim_{l\to 0} u_l^i(c,l) = +\infty$, $\forall c > 0$.

The assumptions on the production function $F : \mathbb{R}_+^2 \to \mathbb{R}_+$ are as follows:

Assumption F1: *F is continuous, concave, increasing on $\mathbb{R}_+^2$ and strictly increasing, strictly concave on $\mathbb{R}_{++}^2$.*

Assumption F2: $F(0,0) = 0$.

Assumption F3: *F is twice continuously differentiable on $\mathbb{R}_{++}^2$ with partial derivatives satisfying the Inada conditions:* $\lim_{k\to 0} F_k(k,1) = +\infty$, $\lim_{k\to +\infty} F_k(k,m) < \delta$ and $\lim_{L\to 0} F_L(k,L) = +\infty$, $\forall k > 0$.

For any initial condition $k_0 \geq 0$, when a sequence $\mathbf{k} = (k_0, k_1, k_2, ..., k_t, ...)$ such that $0 \leq k_{t+1} \leq f(k_t, m)$ for all t, we say it is feasible from k_0 and we denote the class of feasible capital paths by $\Pi(k_0)$. Let $c_t = (c_t^1, c_t^2, ... c_t^m)$ denote the m-vector of consumptions and $l_t = (l_t^1, l_t^2, ... l_t^m)$ denote m-vector of leisure of all agents at date t. A pair of consumption-leisure sequences $(\mathbf{c}, \mathbf{l}) = ((c_0, l_0),(c_1, l_1), ...)$ is feasible from $k_0 \geq 0$ if there exists a sequence $\mathbf{k} \in \Pi(k_0)$ that satisfies

$$\sum_{i=1}^{m} c_t^i + k_{t+1} \leq f(k_t, \sum_{i=1}^{m}(1-l_t^i)) \text{ and } 0 \leq l_t^i \leq 1 \forall t.$$

The set of feasible from k_0 consumption-leisure is denoted by $\sum(k_0)$. Assumption **F3** implies that

[2] We relaxed some important assumptions in the literature. For example, the convex cone of zero of the production set (Bewley [1972]) or the strictly positiveness of derivatives of utility functions on $\mathbb{R}_+^L$ (Bewley [1982]). The utility functions in our model may not differentiable in $\mathbb{R}_+^2$ from which many difficulties arise when we deal with boundary points.

$$
\begin{aligned}
f_k(+\infty, m) &= F_k(+\infty, m) + (1-\delta) < 1, \\
f_k(0, m) &= F_k(0, m) + (1-\delta) > 1.
\end{aligned}
$$

From above, it follows that there exists $\overline{k} > 0$ such that: (i) $f(\overline{k}, m) = \overline{k}$, (ii) $k > \overline{k}$ implies $f(k, m) < k$, (iii) $k < \overline{k}$ implies $f(k, m) > k$. Therefore for any $\mathbf{k} \in \Pi(k_0)$, we have $0 \le k_t \le \max(k_0, \overline{k})$. Thus, a feasible sequence $\mathbf{k} \in l_+^\infty$ which in turn implies a feasible sequence $(\mathbf{c}, \mathbf{l}) \in l_+^\infty \times [0,1]^\infty$.

In what follows, we first study the Pareto optimum problem from which we obtain the Lagrange multipliers in l_+^1. Then these multipliers will be used to define prices and wages systems for the equilibrium.

Let $\Delta = \{\eta_1, \eta_2, ..., \eta_m | \eta_i \ge 0 \text{ and } \sum_{i=0}^m \eta_i = 1\}$. Given a vector of welfare weights $\eta \in \Delta$, define the Pareto problem

$$
\begin{aligned}
\max \quad & \sum_{i=1}^m \eta_i \sum_{t=0}^\infty \beta_i^t u_i(c_t^i, l_t^i) \\
s.t. \quad & \sum_{i=1}^m c_t^i + k_{t+1} \le f(k_t, \sum_{i=1}^m (1 - l_t^i)), \forall t \\
& c_t^i \ge 0,\ l_t^i \ge 0,\ l_t^i \le 1,\ \forall i, \forall t \\
& k_t \ge 0,\ \forall t \text{ and } k_0 \text{ given.}
\end{aligned}
$$

It is well known that any Pareto-efficient allocation can be represented as the solution to Pareto optimum problem. By varying the welfare weights it is possible to trace the economy's utility possibility frontier. Following the Negishi approach, this procedure can also be used to prove the existence of a price system that supports Pareto-optima and characterize competitive equilibria as a set of welfare weights such that the associated transfer payments are zero. Note that, for all $k_0 \ge 0$, $0 \le k_t \le \max(k_0, \overline{k})$, then $0 \le c_t^i \le f(\max(k_0, \overline{k}), m)$ $\forall t$, $\forall i = 1...m$. Therefore, the sequence $(u_i^n)_n = \sum_{i=1}^n \beta_i^t u_i(c_t^i, l_t^i)$ is increasing and bounded, it will converge. Thus we can write

$$
\sum_{i=1}^m \eta_i \sum_{t=0}^\infty \beta_i^t u_i(c_t^i, l_t^i) = \sum_{t=0}^\infty \sum_{i=1}^m \eta_i \beta_i^t u_i(c_t^i, l_t^i).
$$

Let denote

$$
\begin{aligned}
\mathbf{c} &= (\mathbf{c}^1, \mathbf{c}^2, ..., \mathbf{c}^i, ..., \mathbf{c}^m) \text{ where } \mathbf{c}^i = (c_0^i, c_1^i, ...c_t^i, ...), \\
\mathbf{l} &= (\mathbf{l}^1, \mathbf{l}^2, ..., \mathbf{l}^i, ..., \mathbf{l}^m) \text{ where } \mathbf{l}^i = (l_0^i, l_1^i, ...l_t^i, ...), \\
\mathbf{x} &= (\mathbf{c}, \mathbf{k}, \mathbf{l}) \in (l_+^\infty)^m \times l_+^\infty \times (l_+^\infty)^m.
\end{aligned}
$$

Define

$$
\begin{aligned}
\mathcal{F}(\mathbf{x}) &= -\sum_{t=0}^{\infty}\sum_{i=1}^{m}\eta_i\beta_i^t u_i(c_t^i, l_t^i) \\
\Phi_t^1(\mathbf{x}) &= \sum_{i=1}^{m} c_t^i + k_{t+1} - f(k_t, \sum_{i=1}^{m}(1-l_t^i))\ \forall t \\
\Phi_t^{2i}(\mathbf{x}) &= -c_t^i, \forall t, \forall i = 1...m \\
\Phi_t^3(\mathbf{x}) &= -k_t,\ \forall t \\
\Phi_t^{4i}(\mathbf{x}) &= -l_t^i, \forall t, \forall i = 1...m \\
\Phi_t^{5i}(\mathbf{x}) &= l_t^i - 1, \forall t, \forall i = 1...m \\
\Phi_t &= (\Phi_t^1, \Phi_t^{2i}, \Phi_{t+1}^3, \Phi_t^{4i}, \Phi_t^{5i}),\ \forall t, \forall i = 1...m
\end{aligned}
$$

The Pareto problem can be written as:

$$
\begin{aligned}
&\min \mathcal{F}(\mathbf{x}) \\
\text{s.t.}\Phi(\mathbf{x}) &\leq \mathbf{0}, \mathbf{x} \in (l_+^\infty)^m \times l_+^\infty \times (l_+^\infty)^m
\end{aligned}
$$

where:

$$
\begin{aligned}
\mathcal{F} &: (l_+^\infty)^m \times l_+^\infty \times (l_+^\infty)^m \to \mathbb{R} \cup \{+\infty\} \\
\Phi &= (\Phi_t)_{t=0...\infty} : (l_+^\infty)^m \times l_+^\infty \times (l_+^\infty)^m \to \mathbb{R} \cup \{+\infty\} \\
\text{Let } C &= dom(\mathcal{F}) = \{\mathbf{x} \in (l_+^\infty)^m \times l_+^\infty \times (l_+^\infty)^m | \mathcal{F}(\mathbf{x}) < +\infty\} \\
\Gamma &= dom(\Phi) = \{\mathbf{x} \in (l_+^\infty)^m \times l_+^\infty \times (l_+^\infty)^m | \Phi_t(\mathbf{x}) < +\infty,\ \forall t\}.
\end{aligned}
$$

The following theorem follows from Theorem1 and Theorem2 in Le Van and Saglam [2004].

Theorem 1 *Let* $\mathbf{x}$, $\mathbf{y} \in (l_+^\infty)^m \times l_+^\infty \times (l_+^\infty)^m$, $T \in N$.

Define $x_t^T(\mathbf{x}, \mathbf{y}) = \begin{cases} x_t & if \quad t \leq T \\ y_t & if \quad t > T \end{cases}$

Suppose that two following assumptions are satisfied:

T1*: If* $\mathbf{x} \in C$, $\mathbf{y} \in (l_+^\infty)^m \times l_+^\infty \times (l_+^\infty)^m$ *satisfy* $\forall T \geq T_0$, $\mathbf{x}^T(\mathbf{x}, \mathbf{y}) \in C$ *then* $\mathcal{F}(\mathbf{x}^T(\mathbf{x}, \mathbf{y})) \to \mathcal{F}(\mathbf{x})$ *when* $T \to \infty$.

T2*: If* $\mathbf{x} \in \Gamma$, $\mathbf{y} \in \Gamma$ *and* $\mathbf{x}^T(\mathbf{x}, \mathbf{y}) \in \Gamma$, $\forall T \geq T_0$, *then*

$$
\begin{aligned}
&a)\ \Phi_t(\mathbf{x}^T(\mathbf{x}, \mathbf{y})) \to \Phi_t(\mathbf{x}) as\ T \to \infty \\
&b)\ \exists M s.t. \forall T \geq T_0, \|\Phi_t(\mathbf{x}^T(\mathbf{x}, \mathbf{y}))\| \leq M \\
&c)\ \forall N \geq T_0, \lim_{t\to\infty}[\Phi_t(\mathbf{x}^T(\mathbf{x}, \mathbf{y})) - \Phi_t(\mathbf{y})] = 0
\end{aligned}
$$

Let $\mathbf{x}^*$ *be a solution to* ***(P)*** *and* $\bar{\mathbf{x}} \in C$ *satisfy the Slater condition:*

$$
\sup_t \Phi_t(\bar{\mathbf{x}}) < 0.
$$

Suppose $\mathbf{x}^T(\mathbf{x}^*,\bar{\mathbf{x}}) \in C \cap \Gamma$. *Then, there exists* $\mathbf{\Lambda} \in l_+^1 \backslash \{0\}$ *such that*

$$\mathcal{F}(\mathbf{x}) + \mathbf{\Lambda}\Phi(\mathbf{x}) \geq \mathcal{F}(\mathbf{x}^*) + \Lambda\Phi(\mathbf{x}^*),\ \forall \mathbf{x} \in (C \cap \Gamma)$$

and $\Lambda\Phi(\mathbf{x}^*) = 0$.

Obviously, for any $\eta \in \Delta$, an optimal path will depend on η. In what follows, we will suppress η and denote by $(\mathbf{c}^{i*}, \mathbf{k}^*, \mathbf{L}^{i*}, \mathbf{l}^{i*})$ any optimal path for each agent i if possible. The following proposition characterize the Lagrange multipliers of the Pareto problem.

Proposition 1 *If* $\mathbf{x}^* = (\mathbf{c}^{i*}, \mathbf{k}^*, \mathbf{l}^{i*})$ *is a solution to the Pareto problem:*

$$\max \sum_{t=0}^{\infty} \sum_{i=1}^{m} \eta_i \beta_i^t u_i(c_t^i, l_t^i) \tag{Q}$$

$$\begin{aligned} s.t. \sum_{i=1}^{m} c_t^i + k_{t+1} &\leq f(k_t, \sum_{i=1}^{m}(1 - l_t^i)) \forall t \geq 0 \\ c_t^i &\geq 0,\ l_t^i \geq 0,\ l_t^i \leq 1,\ \forall i, \forall t \\ k_t &\geq 0,\ \forall t \textit{ and } k_0 \textit{ given.} \end{aligned}$$

Then there exists, $\forall i = 1...m$, $\lambda = (\lambda^1, \lambda^{2i}, \lambda^3, \lambda^{4i}, \lambda^{5i}) \in l_+^1 \times\ (l_+^1)^m \times l_+^1 \times (l_+^1)^m \times (l_+^1)^m$ $\lambda \neq \mathbf{0}$ *such that*

$$\begin{aligned} &\sum_{t=0}^{\infty} \sum_{i=1}^{m} \eta_i \beta_i^t u_i(c_t^{i*}, l_t^{i*}) - \sum_{t=0}^{\infty} \lambda_t^1 (\sum_{i=1}^{m} c_t^{i*} + k_{t+1}^* - f(k_t^*, \sum_{i=1}^{m}(1 - l_t^{i*}))) \\ &-(1-\delta)k_t^*) + \sum_{t=0}^{\infty} \lambda_t^{2i} c_t^{i*} + \sum_{t=0}^{\infty} \lambda_t^3 k_t^* + \sum_{t=0}^{\infty} \lambda_t^{4i} l_t^{i*} + \sum_{t=0}^{\infty} \lambda_t^{5i}(1 - l_t^{i*}) \end{aligned}$$

$$\begin{aligned} \geq\ &\sum_{t=0}^{\infty} \sum_{i=1}^{m} \eta_i \beta_i^t u_i(c_t^i, l_t^i) - \sum_{t=0}^{\infty} \lambda_t^1 (\sum_{i=1}^{m} c_t^i + k_{t+1} - f(k_t, \sum_{i=1}^{m}(1 - l_t^i))) \\ &-(1-\delta)k_t) + \sum_{t=0}^{\infty} \lambda_t^{2i} c_t^i + \sum_{t=0}^{\infty} \lambda_t^3 k_t + \sum_{t=0}^{\infty} \lambda_t^{4i} l_t^i + \sum_{t=0}^{\infty} \lambda_t^{5i}(1 - l_t^i) \end{aligned} \tag{1}$$

$$\lambda_t^1 [\sum_{i=1}^{m} c_t^{i*} + k_{t+1}^* - f(k_t^*, \sum_{i=1}^{m} L_t^{i*})] = 0 \tag{2}$$

$$\lambda_t^{2i} c_t^{i*} = 0, \forall i = 1...m \tag{3}$$

$$\lambda_t^3 k_t^* = 0 \tag{4}$$

$$\lambda_t^{4i} l_t^{i*} = 0, \forall i = 1...m \tag{5}$$

$$\lambda_t^{5i}(1 - l_t^{i*}) = 0, \forall i = 1...m \tag{6}$$

$$0 \in \eta_i \beta_i^t \partial_1 u^i(c_t^{i*}, l_t^{i*}) - \{\lambda_t^1\} + \{\lambda_t^{2i}\}, \forall i = 1...m \tag{7}$$

$$0 \in \eta_i \beta_i^t \partial_2 u^i(c_t^{i*}, l_t^{i*}) - \lambda_t^1 \partial_2 f(k_t^*, L_t^*) + \{\lambda_t^{4i}\} - \{\lambda_t^{5i}\}, \forall i = 1...m \tag{8}$$

$$0 \in \lambda_t^1 \partial_1 f(k_t^*, L_t^*) + \{\lambda_t^3\} - \{\lambda_{t-1}^1\} \tag{9}$$

where, $L_t^* = \sum_{i=1}^m L_t^{i*} = \sum_{i=1}^m (1 - l_t^{i*})$, $\partial_j u(c_t^{i*}, l_t^{i*}), \partial_j f(k_t^*, L_t^*)$ *respectively denote the projection on the* j^{th} *component of the subdifferential of function* u *at* (c_t^{i*}, l_t^{i*}) *and the function* f *at* (k_t^*, L_t^*)[3]

Proof: We show that the Slater condition holds. Since $f_k(0, m) > 1$,[4] as in the Theorem2 in Le Van - Saglam [2004]. then for all $k_0 > 0$, there exists some $0 < \widehat{k} < k_0$ such that: $0 < \widehat{k} < f(\widehat{k}, m)$ and $0 < \widehat{k} < f(k_0, m)$.Thus, there exists two small positive numbers ε, ε_1 such that:

$$0 < \widehat{k} + \varepsilon < f(\widehat{k}, m - \varepsilon_1) \text{ and } 0 < \widehat{k} + \varepsilon < f(k_0, m - \varepsilon_1).$$

Denote $\bar{\mathbf{x}} = (\bar{\mathbf{c}}, \bar{\mathbf{k}}, \bar{\mathbf{l}})$ such that $\bar{\mathbf{c}} = (\bar{\mathbf{c}}^1, \bar{\mathbf{c}}^2, ..., \bar{\mathbf{c}}^i, ..., \bar{\mathbf{c}}^m)$, where

$$\bar{\mathbf{c}}^i = (\bar{c_t}^i)_{t=0,...\infty} = (\frac{\varepsilon}{m}, \frac{\varepsilon}{m}, \frac{\varepsilon}{m}, ...)$$

$\bar{\mathbf{l}} = (\bar{\mathbf{l}}^1, \bar{\mathbf{l}}^2, ..., \bar{\mathbf{l}}^i, ..., \bar{\mathbf{l}}^m)$, where

$$\bar{\mathbf{l}}^i = (\bar{l_t}^i)_{t=0,...\infty} = (\frac{\varepsilon_1}{m}, \frac{\varepsilon_1}{m}, \frac{\varepsilon_1}{m}, ...).$$

and $\bar{\mathbf{k}} = (k_0, \widehat{k}, \widehat{k}, ...)$. We have

$$\begin{aligned}
\Phi_0^1(\bar{\mathbf{x}}) &= \sum_{i=0}^m c_0^i + k_1 \leq f(k_0, \sum_{i=1}^m (1 - l_0^i)) \\
&= \varepsilon + \widehat{k} - f(k_0, m - \varepsilon_1) < 0 \\
\Phi_1^1(\bar{\mathbf{x}}) &= \sum_{i=0}^m c_1^i + k_2 \leq f(k_1, \sum_{i=1}^m (1 - l_1^i)) \\
&= \varepsilon + \widehat{k} - f(\widehat{k}, m - \varepsilon_1) < 0 \\
\Phi_t^1(\bar{\mathbf{x}}) &= \varepsilon + \widehat{k} - f(\widehat{k}, m - \varepsilon_1) < 0, \ \forall t \geq 2
\end{aligned}$$

$$\Phi_t^{2i}(\bar{\mathbf{x}}) = -\bar{c_t}^i = -\frac{\varepsilon}{m} < 0, \ \forall t \geq 0, \forall i = 1...m$$

[3]For a concave function f defined on $\mathbb{R}^n, \partial f(x)$ denotes the subdifferential of f at x. We have to write the first-order conditions by the subgradient set since at the point $(0, 0)$, the functions u^i and f are not assumed to be differentiable.

[4]As the Remark 6.1.1 in LeVan and Dana [2003], assumption $f_k(0, 1) > 1$ is equivalent to the Adequacy Assumption in Bewley (1972) and this assumption is crucial to have equilibrium prices in l_+^1 since it implies that the production set has an interior point. Subsequenctly, it allows using a separation theorem in the infinite dimensional space to derive Lagrange multipliers.

$$\begin{aligned}\Phi_0^3(\bar{\mathbf{x}}) &= -k_0 < 0;\\ \Phi_t^3(\bar{\mathbf{x}}) &= -\widehat{k} < 0 \ \ \forall t \geq 1.\end{aligned}$$

$$\Phi_t^{4i}(\bar{\mathbf{x}}) = -\frac{\varepsilon_1}{m} < 0,\ \forall t \geq 0, \forall i = 1...m$$

$$\Phi_t^{5i}(\bar{\mathbf{x}}) = \frac{\varepsilon_1}{m} - 1 < 0,\ \forall t \geq 0, \forall i = 1...m$$

Therefore the Slater condition is satisfied.

It is obvious that, $\forall T$, $\mathbf{x}^T(\mathbf{x}^*,\bar{\mathbf{x}})$ belongs to $(l_+^\infty)^m \times l_+^\infty \times (l_+^\infty)^m$.

As in Le Van-Saglam 2004, Assumption **T2** is satisfied. We now check Assumption **T1**.

For any $\widetilde{\mathbf{x}} \in C, \widetilde{\widetilde{\mathbf{x}}} \in (l_+^\infty)^m \times l_+^\infty \times (l_+^\infty)^m$ such that for any T, $\mathbf{x}^T(\widetilde{\mathbf{x}}, \widetilde{\widetilde{\mathbf{x}}}) \in C$ we have

$$\mathcal{F}(\mathbf{x}^T(\widetilde{\mathbf{x}}, \widetilde{\widetilde{\mathbf{x}}})) = -\sum_{t=0}^{T}\sum_{i=1}^{m}\eta_i\beta_i^t u_i(\widetilde{c}_t^i, \widetilde{l}_t^i) - \sum_{t=T+1}^{\infty}\sum_{i=1}^{m}\eta_i\beta_i^t u_i(\widetilde{\widetilde{c}}_t^i, \widetilde{\widetilde{l}}_t^i).$$

As $\widetilde{\widetilde{\mathbf{x}}} \in (l_+^\infty)^m \times l_+^\infty \times (l_+^\infty)^m$, $\sup_t |\widetilde{\widetilde{c}}_t| < +\infty$, there exists $a > 0, \forall t, |\widetilde{\widetilde{c}}_t| \leq a$. Since $\beta \in (0,1)$,as $T \to \infty$ we have

$$0 \leq |\sum_{t=T+1}^{\infty}\sum_{i=1}^{m}\eta_i\beta_i^t u_i(\widetilde{\widetilde{c}}_t^i, \widetilde{\widetilde{l}}_t^i)| \leq u(a,1)\sum_{t=T+1}^{\infty}\sum_{i=1}^{m}\eta_i\beta_i^t = \sum_{i=1}^{m}\sum_{t=T+1}^{\infty}\eta_i\beta_i^t \to 0\ .$$

Hence, $\mathcal{F}(\mathbf{x}^T(\widetilde{\mathbf{x}}, \widetilde{\widetilde{\mathbf{x}}})) \to \mathcal{F}(\widetilde{\mathbf{x}})$ when $T \to \infty$.Taking account of the Theorem 1, we get (1) - (6).

Obviously, $\cap_{i=1}^m ri(dom(u_i)) \neq \emptyset$ where $ri(dom(u_i))$ is the relative interior of $dom(u_i)$. It follows from the Proposition 6.5.5 in Florenzano and Le Van (2001), we have

$$\partial\sum_{i=1}^{m}\eta_i\beta_i^t u_i(c_t^{i*}, l_t^{i*}) = \eta_i\beta_i^t\sum_{i=1}^{m}\partial u_i(c_t^{i*}, l_t^{i*})$$

We then get (7) - (9) as the Kuhn-Tucker first-order conditions. ■

Let us denote $I = \{i\ |\eta_i > 0\}$, $\beta = \max\{\beta_i | i \in I\}$, $I_1 = \{i \in I \mid \beta_i = \beta\}$ and $I_2 = \{i \in I \mid \beta_i < \beta\}$.

In the following proposition, we will prove the positiveness of the optimal capital, consumption and leisure paths which will be used later.

Proposition 2 *i) If $k_0 > 0$, the capital optimal paths $k_t^* > 0, \forall t$.*
ii) If $\eta_i > 0$ then $c_t^{i*} > 0$, $l_t^{i*} > 0\ \forall t$.

The proof is given in the Appendix. C. Le Van, M.H. Nguyen and Y. Vailakis [2007] did not prove the positiveness of consumption and leisure paths. For the capital path, by choosing only one agent in an alternative path, our proof is simpler since it does not require to consider two separated cases of one agent and more than one agents.

We now show that the consumption and leisure paths of all agents with a discount factor less than the maximum one converge to zero. The proof is very simple compared to the one in C. Le Van, M.H. Nguyen and Y. Vailakis [2007] which uses the supermodular structure inspired by lattice programming.

Proposition 3 *If* $(\mathbf{k}^*, \mathbf{c}^{i*}, \mathbf{l}^{i*})$ *denotes the optimal path starting from* k_0, *then* $\forall i \in I_2$, $c_t^{i*} \longrightarrow 0$ *and* $l_t^{i*} \longrightarrow 0$.

Proof: Let

$$\begin{aligned} V(k_t, k_{t+1}) &= \max \sum_{i=1}^{m} \eta_i \beta_i^t u_i(c_t^i, l_t^i) \\ \text{s.t. } \sum_{i=1}^{m} c_t^i + k_{t+1} &\leq F(k_t, \sum_{i=1}^{m}(1 - l_t^i)) + (1-\delta)k_t \ . \end{aligned}$$

It is easy to see that problem (Q) is equivalent to

$$\begin{aligned} &\max \sum_{t=0}^{\infty} V(k_t, k_{t+1}) \\ \text{s.t. } 0 \ &\leq \ k_{t+1} \leq F(k_t, m) + (1-\delta)k_t \\ &\quad\ k_0 \text{ is given.} \end{aligned}$$

Assume that there exist $i_2 \in I_2$ and a subsequence $(c_\tau^{*i_2})$ which converges to $\overline{c}^{i_2} > 0$ when $\tau \to \infty$. Let a small $\varepsilon > 0$ and $i_1 \in I_1$. At the time $t = \tau$, consider a feasible path $((\mathbf{c}^i, \mathbf{l}^i)_i, \mathbf{k})$ defined as follows:

$$\begin{aligned} &\text{i)} && c_\tau^{i_1} = c_\tau^{*i_1} + \varepsilon, \\ &\text{ii)} && c_\tau^{i_2} = c_\tau^{*i_2} - \varepsilon, c_\tau^i = c_\tau^{*i}, \forall i \in I \backslash \{i_1, i_2\} \\ &\text{iii)} && l_\tau^i = l_\tau^{*i}, \ \forall i \in I, \\ &\text{iv)} && k_t = k_t^*, \ \forall t. \end{aligned}$$

Define

$$\begin{aligned} \Delta_\tau(\varepsilon) &= \sum_{i=1}^{m} \eta_i \beta_i^t u_i(c_t^i, l_t^i) - \sum_{i=1}^{m} \eta_i \beta_i^t u_i(c_t^{*i}, l_t^{*i}) \\ &= \eta_{i_1} \beta^\tau \left[u_{i_1}(c_\tau^{i_1}, l_\tau^{i_1}) - u_{i_1}(c_\tau^{*i_1}, l_\tau^{*i_1}) \right] \\ &+ \eta_{i_2} \beta_{i_2}^\tau \left[u_{i_2}(c_\tau^{i_2}, l_\tau^{i_2}) - u_{i_2}(c_\tau^{*i_2}, l_\tau^{*i_2}) \right] \\ &= \beta^\tau [\eta_{i_1} u_{i_1}(c_\tau^{i_1}, l_\tau^{i_1}) - \eta_{i_1} u_{i_1}(c_\tau^{*i_1}, l_\tau^{*i_1}) \\ &\quad + [\eta_{i_2} u_{i_2}(c_\tau^{i_2}, l_\tau^{i_2}) - \eta_{i_2} u_{i_2}(c_\tau^{*i_2}, l_\tau^{*i_2})](\frac{\beta_{i_2}}{\beta})^\tau] \end{aligned}$$

The concavity of u_{i_1}, u_{i_2} imply that

$$\Delta_\tau(\varepsilon) \geq \beta^\tau \left[\eta_{i_1} u_c^{i_1}(c_\tau^{i_1}, l_\tau^{*i_1}) - \eta_{i_2} u_c^{i_2}(c_\tau^{i_2}, l_\tau^{*i_2}) (\frac{\beta_{i_2}}{\beta})^\tau \right] \varepsilon.$$

Since $(c_\tau^{*i_2}) \to \overline{c}^{i_2} > 0, \eta_{i_2} u_c^{i_2}(c_\tau^{i_2}, l_\tau^{*i_2})(\frac{\beta_{i_2}}{\beta})^\tau \to 0$ when $\tau \to \infty$, $c_\tau^{i_1}$ is bounded, $\Delta_\tau(\varepsilon) > 0$ when $\tau \to \infty$. We get a contradiction. Hence $\forall i \in I_2$, $c_t^{i*} \longrightarrow 0$.

Similarly, we can prove that $l_t^{i*} \longrightarrow 0 \ \forall i \in I_2$. ■

3 Existence of competitive equilibrium

Let us now give the characterization of equilibrium. For each consumer i, let denote:

A sequence of prices $(p_0, p_{1,...)} \in l_+^1 \backslash \{0\}$,a price $r > 0$ for the initial capital stock.

A consumption allocation $\mathbf{c}^i = (c_0^i, c_1^i, ...c_t^i, ...)$ where c_t^i denote the quantity which agent i consumes at date t.

A sequence of capital stocks $\mathbf{k} = (k_0, k_1, ...k_t, ...)$ where k_0 is the initial endowment of capital. Denote $\alpha^i > 0$ be the share the profit of the firm owned by consumer i, $\sum_{i=1}^m \alpha^i = 1$, $\vartheta^i > 0$ be the share of initial endowment owned by consumer i, $\sum_{i=1}^m \vartheta^i = 1$ and $\vartheta^i \ k_0$ be the endowment of consumer i. Let denote $\mathbf{l}^i = (l_0^i, l_1^i, ..., l_t^i, ...)$, $\mathbf{L}^i = (L_0^i, L_1^i, ..., L_t^i, ...), \mathbf{w} = (w_0, w_1, ..., w_t, ...)$ be the sequences of leisure, labor supply and wage, respectively.

Definition 1 *A competitive equilibrium for this model is defined as follows. With an allocation* $\{\mathbf{c}^{i*}, \mathbf{k}^*, \mathbf{l}^{i*}, \mathbf{L}^{i*}\}$, *one can associate a price sequence* $\mathbf{p}^*$ *for consumption good, a wage sequence* $\mathbf{w}^*$ *for labor and a price* r *for the initial capital stock* k_0 *such that*

i)

$$\begin{aligned} \mathbf{c}^* &\in l_+^\infty, \mathbf{l}^{i*} \in l_+^\infty, \mathbf{L}^{i*} \in l_+^\infty, \mathbf{k}^* \in l_+^\infty, \\ \mathbf{p}^* &\in l_1^+ \backslash \{0\}, \mathbf{w}^* \in l_1^+ \backslash \{0\}, r > 0. \end{aligned}$$

ii)For every i, $(\mathbf{c}^{i*}, \mathbf{l}^{i*})$ *is a solution to the problem*

$$\begin{aligned} &\max \textstyle\sum_{t=0}^\infty \beta_i^t u_i(c_t^i, l_t^i) \\ s.t \quad &\textstyle\sum_{t=0}^\infty p_t^* c_t^i + \sum_{t=0}^\infty w_t^* l_t^i \ \leq \sum_{t=0}^\infty w_t^* + \vartheta^i r k_0 + \alpha^i \pi^* \end{aligned}$$

where π^* *is the maximum profit of the single firm.*

iii) $(\mathbf{k}^*, \mathbf{L}^*)$ *is a solution to the firm's problem:*

iv)Markets clear: $\forall t$,

$$\sum_{t=1}^{m} c_t^{i*} + k_{t+1}^* = f(k_t^*, \sum_{t=1}^{m} L_t^{i*}),$$
$$l_t^{i*} + L_t^{i*} = 1, L_t^* = \sum_{t=1}^{m} L_t^{i^*} \text{ and } k_0^* = k_0.$$

With the optimal path $(\mathbf{c}^*, \mathbf{k}^*, \mathbf{l}^*, \mathbf{L}^*)$ we have proved that there exists the Lagrange multipliers

$$\begin{aligned} \lambda(\eta) &= (\lambda^{\mathbf{1}}(\eta), \lambda^{\mathbf{2i}}(\eta), \lambda^{\mathbf{3}}(\eta), \lambda^{\mathbf{4i}}(\eta), \lambda^{\mathbf{5i}}(\eta)) \\ &\in l_+^1 \times (l_+^1)^m \times l_+^1 \times (l_+^1)^m \times (l_+^1)^m, i = 1...m, \end{aligned}$$

for the Pareto problem. In what follow, we want to prove, with the optimal path $(\mathbf{c}^*, \mathbf{k}^*, \mathbf{l}^*, \mathbf{L}^*)$, one can associate a sequence of prices p_t^*, a sequence of wages w_t^* defined as

$$\begin{aligned} p_t^* &= \lambda_t^1 \ \forall t \\ w_t^* &= \lambda_t^1 f_L(k_t^*, L_t^*) \ \forall t \end{aligned}$$

where $f_L(k_t^*, L_t^*) \in \partial_2 f(k_t^*, L_t^*)$, and a price $r > 0$ for the initial capital stock k_0 such that ($\mathbf{c}^*, \mathbf{k}^*, \mathbf{l}^*, \mathbf{L}^*, \mathbf{p}^*, \mathbf{w}^*, r$) is a price equilibrium with transfers. We next show that, there exists a set of welfare weights such that these transfers equal to zero.

Lemma 1 *Let $k_0 > 0$. The sequence of prices p_t^*, the sequence of wages w_t^* defined as*

$$\begin{aligned} p_t^* &= \lambda_t^1, \\ w_t^* &= \lambda_t^1 f_L(k_t^*, L_t^*) \ \forall t \text{ where } f_L(k_t^*, L_t^*) \in \partial_2 f(k_t^*, L_t^*) \end{aligned}$$

belong to $l_1^+ \backslash \{0\}$.

Proof: See in the Appendix. ■

Now we define a price equilibrium with transfers

Definition 2 *A given allocation $\{\mathbf{c}^{i*}, \mathbf{k}^*, \mathbf{l}^{i*}, \mathbf{L}^{i*}\}$, together with a price sequence $\mathbf{p}^*$ for consumption good, a wage sequence $\mathbf{w}^*$ for labor and a price r for the initial capital stock k_0 which constitute a price equilibrium with transfers if*

i)

$$\begin{aligned} \mathbf{c}^* &\in (l_+^\infty)^m, \mathbf{l}^* \in (l_+^\infty)^m, \mathbf{L}^* \in (l_+^\infty)^m, \mathbf{k}^* \in l_+^\infty, \\ \mathbf{p}^* &\in l_1^+ \backslash \{0\}, \mathbf{w}^* \in l_1^+ \backslash \{0\}, \ r > 0 \end{aligned}$$

ii) For every $i = 1...m$, $(\mathbf{c}^{i*}, \mathbf{l}^{i*})$ is a solution to the problem

$$\begin{aligned} &\max \sum_{t=0}^{\infty} \beta_i^t u_i(c_t^i, l_t^i) \\ st \quad &\sum_{t=0}^{\infty} p_t^* c_t^i + \sum_{t=0}^{\infty} w_t^* l_t^i \leq \sum_{t=0}^{\infty} p_t^* c_t^{i*} + \sum_{t=0}^{\infty} w_t^* l_t^{i*} \end{aligned}$$

iii) $(\mathbf{k}^*, \mathbf{L}^*)$ is a solution to the firm's problem:

$$\begin{aligned} \pi^* &= \max \sum_{t=0}^{\infty} p_t^*[f(k_t, L_t) - k_{t+1}] - \sum_{t=0}^{\infty} w_t^* L_t - rk_0 \\ st \quad 0 &\leq k_{t+1} \leq f(k_t, L_t), 0 \leq L_t, \forall t \end{aligned}$$

iv)Markets clear

$$\begin{aligned} \forall t, \ \textstyle\sum_{i=1}^m c_t^{i*} + k_{t+1}^* = & \quad f(k_t^*, \sum_{i=1}^m L_t^{i*}), \\ L_t^* = \textstyle\sum_{i=1}^m L_t^{i*}, \ l_t^{i*} = 1 - L_t^{i*} & \quad \text{and } k_0^* = k_0 \end{aligned}$$

Theorem 2 *Let $(\mathbf{k}^*, \mathbf{c}^*, \mathbf{L}^*, \mathbf{l}^*)$ solve Problem (Q). Take*

$$\begin{aligned} p_t^* &= \lambda_t^1, \ w_t^* = \lambda_t^1 f_L(k_t^*, L_t^*) \text{ for any } t \\ \text{and } r &= \lambda_0^1[F_k(k_0, 0) + 1 - \delta]. \end{aligned}$$

Then $\{\mathbf{c}^*, \mathbf{k}^*, \mathbf{L}^*, \mathbf{p}^*, \mathbf{w}^*, r\}$ is a price equilibrium with transfers .

Proof: See in the Appendix. ■

The appropriate transfer to each consumer is the amount that just allows the consumer to afford the consumption stream allocated by the social optimization problem. Thus, for given weight $\eta \in \Delta$, the required transfers are:

$$\phi_i(\eta) = \sum_{t=0}^{\infty} p_t^*(\eta) c_t^{i*}(\eta) + \sum_{t=0}^{\infty} w_t^*(\eta) l_t^{i*}(\eta) - \sum_{t=0}^{\infty} w_t^*(\eta) - \vartheta^i rk_0 - \alpha^i \pi^*(\eta)$$

where

$$\pi^*(\eta) = \sum_{t=0}^{\infty} p_t^*(\eta)[f(k_t^*(\eta), L_t^*(\eta)) - k_{t+1}^*(\eta)] - \sum_{t=0}^{\infty} w_t^*(\eta) L_t^*(\eta) - rk_0.$$

A competitive equilibrium for this economy corresponds to a set of welfare weights $\eta \in \Delta$ such that these transfers equal to zero.

Proposition 4 *i) Let $k_0 > 0$. Then for any $\eta \in \Delta$, $\pi^*(\eta) \geq 0$.*

ii) If $\eta_i = 0$ then $\forall t$, $c_t^{i*} = 0$, $l_t^{i*} = 0$ and $\phi_i(\eta) < 0$.

Proof: i) Let $(k_0, 0, 0, ...) \in \Pi(k_0)$. Then

$$\begin{aligned}
\pi^*(\eta) &\geq \lambda_0^1(\eta)[F(k_0, 0) + (1-\delta)k_0] - rk_0 \\
&= \lambda_0^1(\eta)[F(k_0, 0) + (1-\delta)k_0] - \lambda_0^1(\eta)[F_k(k_0, 0) + 1 - \delta]k_0 \\
&\geq 0.
\end{aligned}$$

ii) Let $\eta_i = 0$. Suppose for simplicity that $c_0^{i*} > 0$.

Let j satisfies $\eta_j > 0$. Define $c_0^{i**} = 0$, $c_0^{j**} = c_0^{j*} + c_0^{i*}$. We have

$$\eta_i u_i(c_0^{i**}, l_0^{i*}) = \eta_i u_i(c_0^{i*}, l_0^{i*}) = 0, \eta_j u_j(c_0^{j**}, l_0^{j*}) > \eta_j u_j(c_0^{j*}, l_0^{j*}).$$

Hence we get new utility is greater than the optimum which leads to contradiction. Now, assume that $l_0^{i*} > 0$. Let j satisfies $\eta_j > 0$. Define

$$\begin{aligned}
c_0^{j**} &= F(k_0, m - \sum_{k \neq i} l_0^k) + (1-\delta)k_0 - k_1 - \sum_{k \neq j} c_0^{k*} \\
l_0^{i**} &= 0
\end{aligned}$$

We have $c_0^{j**} > c_0^{j*}$ and

$$\eta_i u_i(c_0^{i*}, l_0^{i**}) = \eta_i u_i(c_0^{i*}, l_0^{i*}) = 0, \ \ \eta_j u_j(c_0^{j**}, l_0^{j*}) > \eta_j u_j(c_0^{j*}, l_0^{j*}).$$

that also leads to contradiction. Thus, $c_t^{i*} = 0$, $l_t^{i*} = 0$ $\forall t$. Now, we have

$$\begin{aligned}
\phi_i(\eta) = \sum_{t=0}^{\infty} p_t^*(\eta) c_t^{i*}(\eta) &+ \sum_{t=0}^{\infty} w_t^*(\eta) l_t^{i*}(\eta) \\
-\sum_{t=0}^{\infty} w_t^*(\eta) &- \vartheta^i r k_0 - \alpha^i \pi^*(\eta) \\
= \ -\sum_{t=0}^{\infty} w_t^*(\eta) &- \vartheta^i r k_0 - \alpha^i \pi^*(\eta) < 0.
\end{aligned}$$

■

For given η, properties in the Proposition 2 , $k_t^* > 0$, $c_t^{i*} > 0$, $l_t^{i*} > 0$ $\forall t, \forall i \in I$, and Inada condition on function F implies the uniqueness of Lagrange multipliers $\lambda_t^1 = p_t^*(\eta) = \eta_i \beta_i^t u_c^i(c_t^{i*}, l_t^{i*})$, $w_t^*(\eta) = \lambda_t^1 f_L(k_t^*, L_t^*)$. Thus, $\phi_i(.)$ is a function of η. The following theorem is a direct application of Brouwer's fixed point theorem.

Theorem 3 *Let $k_0 > 0$. Then there exists $\bar{\eta} \in \Delta, \bar{\eta} >> 0$, such that $\phi_i(\bar{\eta}) = 0, \forall i$. That means there exists an equilibrium.*

Proof: First, we prove that $\phi_i(.)$ is a continuous function of η for every i. Let $\eta^n \in \Delta$ and $\eta^n \to \eta \in \Delta$. We shall prove that $\phi_i(\eta^n) \to \phi_i(\eta)$. It is easy to check that, for given $\eta \in \Delta$,

$$U(\eta, \mathbf{k}, \mathbf{c}, \mathbf{l}) = \sum_{t=0}^{\infty} \sum_{i=1}^{m} \eta_i \beta_i^t u_i(c_t^i, l_t^i)$$

is continuous over $\Delta \times \Pi(k_0)\times \sum(k_0)$, $\Pi(k_0)\times \sum(k_0)$ is compact, it follows from Berge's Theorem that $c_t^{i*}(\eta), k_t^*(\eta), l_t^{i*}(\eta)$ are continuous functions of η for the product topology. Let us recall that $\phi_i(\eta) =$

$$\sum_{t=0}^{\infty} p_t^*(\eta)c_t^{i*}(\eta) + \sum_{t=0}^{\infty} w_t^*(\eta)l_t^{i*}(\eta) - \sum_{t=0}^{\infty} w_t^*(\eta) - \vartheta^i r k_0 - \alpha^i \pi^*(\eta).$$

Firstly, we consider the first three terms of $\phi_i(\eta)$. Boundedness of $c_t^{i*}(\eta), l_t^{i*}(\eta)$, concavity of u_i together with conditions (3),(5),(6),(10),(11) imply that, $\forall i = 1, .., m$,

$$\begin{aligned} &\eta_i\beta_i^t u_i(c_t^{i*}(\eta), l_t^{i*}(\eta)) - \eta_i\beta_i^t u_i(0,0) \\ &\geq \eta_i\beta_i^t u_c^i(c_t^{i*}(\eta), l_t^{i*}(\eta))c_t^{i*}(\eta) \\ &= \left[\lambda_t^1(\eta) - \lambda_t^{2i}(\eta)\right] c_t^{i*}(\eta) = \lambda_t^1(\eta)c_t^{i*}(\eta). \end{aligned}$$

$$\begin{aligned} & \eta_i\beta_i^t u_i(c_t^{i*}(\eta), l_t^{i*}(\eta)) - \eta_i\beta_i^t u_i(0,0) \\ \geq \; & \eta_i\beta_i^t u_l^i(c_t^{i*}(\eta), l_t^{i*}(\eta))l_t^{i*}(\eta) \\ = \; & [w_t^*(\eta) - \lambda_t^{4i}(\eta) + \lambda_t^{5i}(\eta)]l_t^{i*}(\eta) \\ = \; & [w_t^*(\eta) + \lambda_t^{5i}(\eta)]l_t^{i*}(\eta) \geq w_t^*(\eta)l_t^{i*}(\eta). \end{aligned}$$

(Note that if $i \notin I$ then $c_t^{i*}(\eta) = 0, l_t^{i*}(\eta) = 0$).

Thus, $\forall \varepsilon > 0$, there exist T and a real number $M > 0$ such that

$$\sum_{t=T}^{\infty} p_t^*(\eta)c_t^{i*}(\eta) = \sum_{t=T}^{\infty} \lambda_t^1(\eta^n)c_t^{i*}(\eta) \leq \sum_{t=T}^{\infty} M\beta_i^t < \frac{\varepsilon}{9}.$$

$$\sum_{t=T}^{\infty} w_t^*(\eta)l_t^{i*}(\eta) \leq \sum_{t=T}^{\infty} M\beta_i^t < \frac{\varepsilon}{9}.$$

Moreover, since $\sum_{t=0}^{\infty} w_t^*(\eta) < +\infty$, we also can write

$$\sum_{t=T}^{\infty} w_t^*(\eta) < \frac{\varepsilon}{9}.$$

For some $i \in I$, we have

$$\begin{aligned} p_t^*(\eta^n) &\rightarrow \eta_i\beta_i^t u_c^i(c_t^{i*}, l_t^{i*}) = p_t^*(\eta), \\ w_t^*(\eta^n) &\rightarrow \lambda_t^1 f_L(k_t^*, L_t^*) = w_t^*(\eta) \end{aligned}$$

since $c_t^{i*}(\eta^n) \rightarrow c_t^{i*}(\eta) > 0$, $l_t^{i*}(\eta^n) \rightarrow l_t^{i*}(\eta) > 0$. Thus,

$$\begin{aligned} &|\sum_{t=0}^{\infty} p_t^*(\eta^n)c_t^{i*}(\eta^n) + \sum_{t=0}^{\infty} w_t^*(\eta^n)l_t^{i*}(\eta^n) - \sum_{t=T}^{\infty} w_t^*(\eta^n) \\ &- \sum_{t=0}^{\infty} p_t^*(\eta)c_t^{i*}(\eta) - \sum_{t=0}^{\infty} w_t^*(\eta)l_t^{i*}(\eta) + \sum_{t=T}^{\infty} w_t^*(\eta)| \end{aligned}$$

$$\leq |\sum_{t=0}^{T} p_t^*(\eta^n) c_t^{i*}(\eta^n) - \sum_{t=0}^{T} p_t^*(\eta) c_t^{i*}(\eta)| + \tag{i}$$

$$|\sum_{t=0}^{T} w_t^*(\eta^n) l_t^{i*}(\eta^n) - \sum_{t=0}^{T} w_t^*(\eta) l_t^{i*}(\eta)| + \tag{ii}$$

$$|\sum_{t=0}^{T} w_t^*(\eta^n) - \sum_{t=0}^{T} w_t^*(\eta)| \tag{iii}$$

$$+ \sum_{t=T}^{\infty} p_t^*(\eta^n) c_t^{i*}(\eta^n) + \sum_{t=T}^{\infty} p_t^*(\eta) c_t^{i*}(\eta)$$

$$+ \sum_{t=T}^{\infty} w_t^*(\eta^n) l_t^{i*}(\eta^n) + \sum_{t=T}^{\infty} w_t^*(\eta) l_t^{i*}(\eta)$$

$$+ \sum_{t=T}^{\infty} w_t^*(\eta^n) + \sum_{t=T}^{\infty} w_t^*(\eta) < \varepsilon$$

since, given T, the continuity of $p_t^*(\eta), w_t^*(\eta), c_t^{i*}(\eta), l_t^{i*}(\eta)$ implies that there exist N such that for any $n \geq N$, each term (i),(ii),(iii) is smaller than $\frac{\varepsilon}{9}$.

The similar arguments can show that

$$\vartheta^i r(\eta^n) k_0 - \alpha^i \pi^*(\eta^n) \rightarrow \vartheta^i r(\eta) k_0 - \alpha^i \pi^*(\eta).$$

Hence, $\phi_i(.)$ is a continuous function of η.

Let define $T : \Delta \rightarrow \Delta$, $T(\eta) = (T_1(\eta), T_2(\eta), ..., T_m(\eta))$ where $T_i(\eta)$ defined as

$$T_i(\eta) = \frac{\eta_i + \phi_i^{'}(\eta)}{1 + \sum_{i=1}^{m} \phi_i^{'}(\eta)}$$

with $\phi_i^{'}(\eta) = -\phi_i(\eta)$ if $\phi_i(\eta) < 0$, and $\phi_i^{'}(\eta) = 0$ if $\phi_i(\eta) \geq 0$. T is a continuous mapping from the simplex into itself. By the Brouwer fixed point theorem, there exists $\bar{\eta} \in \Delta$ such that $T(\bar{\eta}) = \bar{\eta}$. We have

$$\bar{\eta_i} = \frac{\overline{\lambda}_i + \phi_i^{'}(\bar{\eta})}{1 + \sum_{i=1}^{m} \phi_i^{'}(\bar{\eta})} \Leftrightarrow \bar{\eta_i} \sum_{i=1}^{m} \phi_i^{'}(\bar{\eta}) = \phi_i^{'}(\bar{\eta}) \tag{1}$$

If $\bar{\eta_i} = 0$, Proposition 4 (ii) implies that $\phi_i(\bar{\eta_i}) < 0$ and $\phi_i^{'}(\bar{\eta}) > 0$:a contradiction with (1). Thus, $\bar{\eta_i} > 0$, $\forall i$. If $\sum_{i=1}^{m} \Phi_i^{'}(\bar{\eta}) > 0$, then $\Phi_i^{'}(\bar{\eta}) > 0$, $\forall i$. From the definition of $\phi_i^{'}(\eta)$ this implies $\phi_i(\eta) < 0$, $\forall i$. But this contradicts Walras' Law which says $\sum_{i=1}^{m} \phi_i(\bar{\eta}) = 0$. Thus, $\sum_{i=1}^{m} \phi_i^{'}(\bar{\eta}) = 0$ which implies $\phi_i^{'}(\bar{\eta}) = 0$, $\forall i$. But in this case we have $\phi_i(\bar{\eta}) \geq 0$, $\forall i$. From Walras' Law we have $\phi_i(\bar{\eta}) = 0$, $\forall i$. ■

4 Conclusion

In this paper, we prove the existence of competitive equilibrium in an optimal growth model with heterogeneous agents and elastic labor supply. This paper is an extension of Le Van and Vailakis [2003] who studied the model without labor supply. It is also the completeness of the important issue about the existence of competitive equilibrium in the model of C. Le Van, M.H. Nguyen and Y. Vailakis [2007]. Following the Negishi approach, our strategy for tackling the question of existence relies on exploiting the link between Pareto-optima and competitive equilibria. The proof is based on the result of existence of Lagrange multipliers of the Pareto problem and their representation as a summable sequence. We show that there exists a Lagrange multiplier as a price system such that together with the Pareto-optimal solution they constitute a price equilibrium with transfers. These transfers depend on the individual weights involved in the social welfare function. An equilibrium exists provided that there is a set of welfare weights such that the corresponding transfers equal zero.

5 Appendix

Proof of Proposition 2

Proof: i) Let $k_0 > 0$ but assume that $k_1^* = 0$. Denote $L_t^* = m - \sum_{i\in I} l_t^{*i}$. Since

$$\sum_{i\in I} c_0^{*i} = f(k_0, L_0^*) > 0,$$

there exists some $i_1 \in I$ such that $c_0^{*i_1} > 0$. First, we claim that there exists p with $l_1^{*p} > 0$.

Assume the contrary that $l_1^{*i} = 0, \forall i \in I$. In this case, we prove that there exists p with $c_1^{*p} > 0$. Indeed, if $c_1^{*i} = 0\ \forall i \in I$ then $k_2^* = f(0, m)$. Choose $\varepsilon > 0$ such that $c_0^{*i_1} > \varepsilon + \varepsilon^2$. Let $\alpha = \frac{\varepsilon+1}{\beta_{i_1}}$ and $\gamma = \frac{\varepsilon+1}{\beta_{i_1}[c_0^{*i_1} - (\varepsilon+\varepsilon^2)]}$. Consider the alternative path $((\mathbf{c}^i, \mathbf{l}^i)_i, \mathbf{k})$ defined as follows:

$$\begin{array}{ll}
\text{i)} & c_0^{i_1} = c_0^{*i_1} - (\varepsilon + \varepsilon^2),\ c_0^i = c_0^{*i},\ \forall i \in I\backslash\{i_1\} \\
\text{ii)} & c_1^{i_1} = \alpha\varepsilon,\ c_1^i = 0,\ \forall i \in I\backslash\{i_1\} \\
\text{iii)} & l_0^i = l_0^{*i},\ \forall i \in I,\ l_1^{i_1} = \gamma\varepsilon,\ l_1^i = 0,\ \forall i \in I\backslash\{i_1\} \\
\text{iv)} & c_t^i = c_t^{*i} \text{ and } l_t^i = l_t^{*i},\ \forall i \in I, \forall t \geq 2 \\
\text{v)} & k_1 = \varepsilon,\ k_t = k_t^*,\ \forall t \geq 2.
\end{array}$$

Observe that

$$\begin{aligned} \sum_{i\in I} c_0^i + k_1 &= \\ c_0^{*i_1} - (\varepsilon+\varepsilon^2) + \sum_{i\in I\setminus\{i_1\}} c_0^{*i} + \varepsilon &\leq \sum_{i\in I} c_0^{*i} + k_1^* = f(k_0, L_0^*). \end{aligned}$$

Moreover,

$$\begin{aligned} & f(k_1, m - l_1^{i_1}) - k_2 - c_1^{i_1} \\ = \;& f(\varepsilon, m-\gamma\varepsilon) - f(0,m) - \alpha\varepsilon \\ \geq \;& \varepsilon[f_k(\varepsilon, m-\gamma\varepsilon) - f_L(\varepsilon, m-\gamma\varepsilon)\gamma - \alpha]. \end{aligned}$$

Due to the Inada conditions on F, the term inside the bracket is strictly positive for ε small enough. This proves feasibility of the alternative path.

Observe that as $\varepsilon \to 0$ both α and γ converge to a finite value. In addition, $\frac{\alpha}{\gamma} = c_0^{i_1}$. Define:

$$\begin{aligned} \Delta(\varepsilon) &= \sum_{i\in I} \eta_i \sum_{t=0}^{\infty} \beta_i^t u_i(c_t^i, l_t^i) - \sum_{i\in I} \eta_i \sum_{t=0}^{\infty} \beta_i^t u_i(c_t^{i*}, l_t^{i*}) \\ = \;& \eta_{i_1}[u_{i_1}(c_0^{i_1}, l_0^{i_1}) - u_{i_1}(c_0^{*i_1}, l_0^{*i_1})] + \eta_{i_1}\beta_{i_1}\left[u_{i_1}(c_1^{i_1}, l_1^{i_1}) - u_{i_1}(c_1^{*i_1}, l_1^{*i_1})\right]. \end{aligned}$$

The concavity of u^{i_1} implies that

$$\begin{aligned} \frac{\Delta(\varepsilon)}{\eta_{i_1}} &= \beta_{i_1}\left[u_{i_1}(c_1^{i_1}, l_1^{i_1}) - u_{i_1}(c_1^{*i_1}, l_1^{*i_1})\right] + [u_{i_1}(c_0^{i_1}, l_0^{i_1}) - u_{i_1}(c_0^{*i_1}, l_0^{*i_1})] \\ &\geq \beta_{i_1} u_{i_1}(\alpha\varepsilon, \gamma\varepsilon) - u_c^{i_1}(c_0^{i_1}, l_0^{i_1})(\varepsilon+\varepsilon^2). \end{aligned}$$

If $u_{cl}^{i_1} > 0$, then

$$\begin{aligned} \frac{\Delta_{i_1}(\varepsilon)}{\eta_{i_1}} &\geq \beta_{i_1} u_{i_1}\left(\gamma\varepsilon\frac{\alpha\varepsilon}{\gamma\varepsilon}, \gamma\varepsilon\right) - u_c^{i_1}(c_0^{i_1}, l_0^{i_1})(\varepsilon+\varepsilon^2) \\ &\geq \beta_{i_1} u_{i_1}\left(\frac{\alpha}{\gamma}, 1\right)\gamma\varepsilon - u_c^{i_1}(c_0^{i_1}, l_0^{i_1})(\varepsilon+\varepsilon^2) \\ &\geq \beta_{i_1} u_c^{i_1}\left(\frac{\alpha}{\gamma}, 1\right)\alpha\varepsilon - u_c^{i_1}(c_0^{i_1}, l_0^{i_1})(\varepsilon+\varepsilon^2) \\ &= \beta_{i_1} u_c^{i_1}\left(c_0^{i_1}, 1\right)\frac{\varepsilon^2+\varepsilon}{\beta_{i_1}} - u_c^{i_1}(c_0^{i_1}, l_0^{i_1})(\varepsilon+\varepsilon^2) \\ &= (\varepsilon^2+\varepsilon)[u_c^{i_1}\left(c_0^{i_1}, 1\right) - u_c^{i_1}(c_0^{i_1}, l_0^{i_1})] \geq 0. \end{aligned}$$

If $u_{cl}^{i_1} \leq 0$, then

$$\begin{aligned} \frac{\Delta(\varepsilon)}{\eta_{i_1}} &\geq \beta_{i_1} u_{i_1}(\alpha\varepsilon, \gamma\varepsilon) - u_c^{i_1}(c_0^{i_1}, l_0^{i_1})(\varepsilon+\varepsilon^2) \\ &\geq \beta_{i_1} u_c^{i_1}(\alpha\varepsilon, \gamma\varepsilon)\alpha\varepsilon - u_c^{i_1}(c_0^{i_1}, l_0^{i_1})(\varepsilon+\varepsilon^2) \\ &\geq (\varepsilon^2+\varepsilon)[u_c^{i_1}(\alpha\varepsilon, 1) - u_c^{i_1}(c_0^{i_1}, l_0^{i_1})]. \end{aligned}$$

Due to the Inada conditions on u_{i_1}, the term inside the bracket becomes non-negative for ε small enough. A contradiction.

Thus, there exists p such that $c_1^{*p} > 0$. For this p, we claim that $l_1^{*p} > 0$. Indeed, if this were false, define a feasible path as follows:

i) $l_1^p = \varepsilon,\ l_1^i = l_1^{*i},\ \forall i \neq p,$

ii) $c_1^p = c_1^{*p} + f(0, m-\varepsilon) - f(0,m), c_1^i = c_1^{*i}\ \forall i \neq p,$

iii) $c_t^i = c_t^{*i},\ l_t^i = l_t^{*i}, \forall i, \forall t \neq 1, k_t = k_t^*\ \forall t.$

Define:

$$\Delta_p(\varepsilon) = \sum_{i\in I}\eta_i\sum_{t=0}^{\infty}\beta_i^t u_i(c_t^i, l_t^i) - \sum_{i\in I}\eta_i\sum_{t=0}^{\infty}\beta_i^t u_i(c_t^{*i}, l_t^{*i})$$
$$= \eta_p\beta_p\left[u_p(c_1^p,\varepsilon) - u^p(c_1^{*p},0)\right] \geq \eta_p\beta_p u_c^p(c_1^p,\varepsilon)(f(\varepsilon,L_1^*) - f(0,L_1^*))$$
$$+\eta_p\beta_p u_l^p(c_1^p,\varepsilon)\varepsilon \geq \eta_p\beta_p[-u_c^p(c_1^p,\varepsilon)(f_L(0,m-\varepsilon) + u_l^p(c_1^p,\varepsilon)]\varepsilon \quad .$$

As $\varepsilon \to 0$, $u_l^p(c_1^p,\varepsilon) \to +\infty$ while $-u_c^p(c_1^p,\varepsilon)(f_L(0,m-\varepsilon) < +\infty$. Hence, for $\varepsilon > 0$ small enough, $\Delta_p(\varepsilon) > 0$: a contradiction. Thus, $l_1^{*p} > 0$.

Now, we consider the alternative feasible path $((\mathbf{c}_i, \mathbf{l}_i)_i, \mathbf{k})$ defined as follows:

i) $c_0^p = c_0^{*p} - \varepsilon,\ c_1^p = c_1^{*p} + f(\varepsilon, L_1^*) - f(0, L_1^*),\ c_t^p = c_t^{*p},\ \forall t \geq 2,$

ii) $c_t^i = c_t^{*i}\ \forall i \neq p, \forall t$ and $l_t^i = l_t^{*i},\ \forall i, \forall t$

iii) $k_1 = \varepsilon,\ k_t = k_t^*,\ \forall t \geq 2.$

Define:

$$\Delta_\varepsilon = \sum_{i\in I}\eta_i\sum_{t=0}^{\infty}\beta_i^t u_i(c_t^i, l_t^i) - \sum_{i\in I}\eta_i\sum_{t=0}^{\infty}\beta_i^t u_i(c_t^{i*}, l_t^{i*})$$

The concavity of u_p and f implies that

$$\begin{aligned}\frac{\Delta_p(\varepsilon)}{\eta_p} &= u_p(c_0^p, l_0^p) - u_p(c_0^{*p}, l_0^{*p}) + \beta_p\left[u_p(c_1^p, l_1^p) - u_p(c_1^{*p}, l_1^p)\right] \\ &\geq [-u_c^p(c_0^p, l_0^p) + \beta_p u_c^p(c_1^p, l_1^p) f_k(\varepsilon, L_1^*)]\varepsilon.\end{aligned}$$

As $\varepsilon \to 0$, $\beta_p u_c^p(c_1^p, l_1^p) f_k(\varepsilon, L_1^*) \to +\infty$ while $u_c^p(c_0^p, l_0^p) \to u_c^p(c_0^{*p}, l_0^{*p}) < +\infty$. Hence, for $\varepsilon > 0$ small enough, $\Delta_p(\varepsilon) > 0$: a contradiction. It follows that $k_1^* > 0$. Working by induction we can show that $k_t^* > 0$ for any t.

ii) It follows from proposition 10 in C. Le Van, M.H Nguyen and Y. Vailakis [2007] that there exists $\gamma > 0$ such that $k_t^* > \gamma\ \forall t$. Suppose that there exist an optimal paths $(\mathbf{c}^*, \mathbf{l}^*, \mathbf{k}^*)$ with $c_0^{1*} = 0$,we can choose a feasible paths from this optimal paths where we just repalce $c_0^{1^*}, k_t^*$ with $c_0^1 = \varepsilon_0 > 0$, $k_t = k_t^* - \varepsilon_t$ in which $\{\varepsilon_t\}$ is an increasing sequence bounded from above by γ (for example, $\varepsilon_t = \gamma - \frac{1}{t+n}, n > 0$) such that $\sum_{i\in I} c_t^{*i} + k_{t+1}^* - \varepsilon_{t+1} \leq f(k_t^* - \varepsilon_t, L_t^*)$. This feasible path create a new greater value than optimal value which leads to a contradiction. Thus $c_t^{*i} > 0$ for all t. It follows from (7) that $l_t^{*i} > 0$. (Otherwise, λ_t^1 would not belong to l_+^1). ■

Proof of Lemma 1

Proof: Consider $\lambda(\eta) = (\lambda^{\mathbf{1}}, \lambda^{\mathbf{2i}}, \lambda^{\mathbf{3}}, \lambda^{\mathbf{4i}}, \lambda^{\mathbf{5i}})$ of Proposition 1. Conditions (7),(8),(9) in Proposition 1 show that $\forall i = 1...m$, $\partial u_i(c_t^{i*}, l_t^{i*})$ and $\partial F(k_t^*, L_t^*)$ are nonempty. Moreover,$\forall t, \forall i = 1...m$, there exists $u_c^i(c_t^{i*}, l_t^{i*}) \in \partial_1 u_i(c_t^{i*}, l_t^{i*})$, $u_l^i(c_t^{i*}, l_t^{i*}) \in \partial_2 u_i(c_t^{i*}, l_t^{i*})$, $f_k(k_t^*, L_t^*) \in \partial_1 f(k_t^*, L_t^*)$and $f_L(k_t^*, L_t^*) \in \partial_2 f(k_t^*, L_t^*)$ such that

$$\eta_i \beta_i^t u_c^i(c_t^{i*}, l_t^{i*}) - \lambda_t^1 + \lambda_t^{2i} = 0, \forall i = 1...m \tag{10}$$

$$\eta_i \beta_i^t u_l^i(c_t^{i*}, l_t^{i*}) - \lambda_t^1 f_L(k_t^*, L_t^*) + \lambda_t^{4i} - \lambda_t^{5i} = 0, \forall i = 1...m \tag{11}$$

$$\lambda_t^1 f_k(k_t^*, L_t^*) + \lambda_t^3 - \lambda_{t-1}^1 = 0 \tag{12}$$

We have

$$+\infty > \sum_{t=0}^{\infty} \beta_i^t u_i(c_t^{i*}, l_t^{i*}) - \sum_{t=0}^{\infty} \beta_i^t u_i(0,0) \geq$$

$$\sum_{t=0}^{\infty} \beta_i^t u_c^i(c_t^{i*}, l_t^{i*}) c_t^{i*} + \sum_{t=0}^{\infty} \beta_i^t u_l^i(c_t^{i*}, l_t^{i*}) l_t^{i*}, \forall i = 1...m$$

which implies

$$\sum_{t=0}^{\infty} \beta_i^t u_l^i(c_t^{i*}, l_t^{i*}) l_t^{i*} < +\infty, \forall i = 1...m \tag{13}$$

and for any i,

$$+\infty > \sum_{t=0}^{\infty} \lambda_t^1 f(k_t^*, L_t^*) - \sum_{t=0}^{\infty} \lambda_t^1 f(0, L_t^* - L_t^{i*}) \geq$$

$$\sum_{t=0}^{\infty} \lambda_t^1 f_k(k_t^*, L_t^*) k_t^* + \sum_{t=0}^{\infty} \lambda_t^1 f_L(k_t^*, L_t^*) L_t^{i*}$$

which implies

$$\sum_{t=0}^{\infty} \lambda_t^1 f_L(k_t^*, L_t^*) L_t^{i*} < +\infty. \tag{14}$$

Given T, we multiply (11), for each i, by L_t^{i*} and sum up from 0 to T. We then obtain

$$\forall T, \ \sum_{t=0}^{T} \eta_i \beta_i^t u_l^i(c_t^{i*}, l_t^{i*}) L_t^{i*} = \sum_{t=0}^{T} \lambda_t^1 f_L(k_t^*, L_t^*) L_t^{i*} \tag{15}$$

$$- \sum_{t=0}^{T} \lambda_t^{4i} L_t^{i*} + \sum_{t=0}^{T} \lambda_t^{5i} L_t^{i*}, \forall i = 1...m$$

Observe that

$$0 \leq \sum_{t=0}^{\infty} \lambda_t^{5i} L_t^{i*} \leq \sum_{t=0}^{\infty} \lambda_t^{5i} < +\infty, \forall i = 1...m \tag{16}$$

$$0 \leq \sum_{t=0}^{\infty} \lambda_t^{4i} L_t^{i*} \leq \sum_{t=0}^{\infty} \lambda_t^{4i} < +\infty, \forall i = 1...m \tag{17}$$

Thus, since $L_t^{i*} = 1 - l_t^{i*}, \forall i = 1...m$, from (15), we get

$$\begin{aligned} \sum_{t=0}^{T} \eta_i \beta_i^t u_l^i(c_t^{i*}, l_t^{i*}) &= \sum_{t=0}^{T} \eta_i \beta_i^t u_l^i(c_t^{i*}, l_t^{i*}) l_t^{i*} + \sum_{t=0}^{T} \lambda_t^1 f_L(k_t^*, L_t^*) L_t^{i*} \\ &\quad + \sum_{t=0}^{T} \lambda_t^{5i} L_t^{i*} - \sum_{t=0}^{T} \lambda_t^{4i} L_t^{i*} \end{aligned}$$

Using (13),(14),(16),(17) and letting $T \to \infty$, we obtain

$$0 \leq \sum_{t=0}^{\infty} \eta_i \beta_i^t u_l^i(c_t^{i*}, l_t^{i*}) = \sum_{t=0}^{\infty} \eta_i \beta_i^t u_l^i(c_t^{i*}, l_t^{i*}) l_t^{i*} +$$

$$\sum_{t=0}^{\infty} \lambda_t^1 f_L(k_t^*, L_t^*) L_t^{i*} + \sum_{t=0}^{\infty} \lambda_t^{5i} L_t^{i*} - \sum_{t=0}^{\infty} \lambda_t^{4i} L_t^{i*} < +\infty$$

Consequently, from (11),

$$\sum_{t=0}^{\infty} w_t^* = \sum_{t=0}^{\infty} \lambda_t^1 f_L(k_t^*, L_t^*) < +\infty.$$

For all $i \in I$, Proposition 2 (ii) together with conditions (3), (10) imply that $p_t^* = \lambda_t^1 = \eta_i \beta_i^t u_c^i(c_t^{i*}, l_t^{i*}) > 0$. Inada condition on function F together with (8) imply that $L_t^* > 0$. Hence, by Proposition 2 (i), $w_t^*(\eta) = \lambda_t^1 f_L(k_t^*, L_t^*) > 0$. Therefore, p_t^*, w_t^* belong to $l_1^+ \backslash \{0\}$. This completes the proof. ∎

Proof of Theorem 2

Proof: i) From Proposition 1 and Lemma 1, we get

$$\mathbf{c}^* \in (l_+^\infty)^m, \mathbf{l}^* \in (l_+^\infty)^m, \mathbf{k}^* \in l_+^\infty, \mathbf{p}^* \in l_1^+ \backslash \{0\}, \mathbf{w}^* \in l_1^+ \backslash \{0\}, r > 0.$$

ii) We now show that $(\mathbf{c}^{i*}, \mathbf{l}^{i*})$ solves the consumer's problem. Let $(\mathbf{c}^i, \mathbf{l}^i)$ satisfies

$$\sum_{t=0}^{\infty} p_t^* c_t^i + \sum_{t=0}^{\infty} w_t^* l_t^i \leq \sum_{t=0}^{\infty} p_t^* c_t^{i*} + \sum_{t=0}^{\infty} w_t^* l_t^{i*}.$$

By the concavity of u_i, we have:

$$\sum_{t=0}^{\infty} \beta_i^t u_i(c_t^{i*}, l_t^{i*}) - \sum_{t=0}^{\infty} \beta_i^t u_i(c_t^i, l_t^i)$$

$$\geq \sum_{t=0}^{\infty} \beta_i^t u_c^i(c_t^{i*}, l_t^{i*})(c_t^{i*} - c_t^i) + \sum_{t=0}^{\infty} \beta_i^t u_l^i(c_t^{i*}, l_t^{i*}) \ (l_t^{i*} - l_t^i).$$

Combining (3),(6),(10),(11) yields that

$$\Delta \geq \sum_{t=0}^{\infty} \frac{(\lambda_t^1 - \lambda_t^{2i})}{\eta_i}(c_t^{i*} - c_t^i) + \sum_{t=0}^{\infty} \frac{(\lambda_t^1 f_L(k_t^*, L_t^*) - \lambda_t^{4i} + \lambda_t^{5i})}{\eta_i}(l_t^{i*} - l_t^i)$$

$$\geq \sum_{t=0}^{\infty} \frac{\lambda_t^1}{\eta_i}(c_t^{i*} - c_t^i) + \sum_{t=0}^{\infty} \frac{\lambda_t^1 f_L(k_t^*, L_t^*)}{\eta_i}(l_t^{i*} - l_t^i) + \sum_{t=0}^{\infty} \frac{\lambda_t^{5i}(1 - l_t^i)}{\eta_i}$$

$$\geq \sum_{t=0}^{\infty} \frac{\lambda_t^1}{\eta_i}(c_t^{i*} - c_t^i) + \sum_{t=0}^{\infty} \frac{\lambda_t^1 f_L(k_t^*, L_t^*)}{\eta_i}(l_t^{i*} - l_t^i)$$

$$= \sum_{t=0}^{\infty} \frac{p_t^*}{\eta_i}(c_t^{i*} - c_t^i) + \sum_{t=0}^{\infty} \frac{w_t^*}{\eta_i}(l_t^{i*} - l_t^i) \geq 0.$$

This means $(\mathbf{c}^{i*}, \mathbf{l}^{i*})$ solves the consumer's problem.

iii) We now show that $(\mathbf{k}^*, \mathbf{L}^*)$ is solution to the firm's problem. Since $p_t^* = \lambda_t^1$, $w_t^* = \lambda_t^1 f_L(k_t^*, L_t^*)$, we have

$$\pi^* = \sum_{t=0}^{\infty} \lambda_t^1 [f(k_t^*, L_t^*) - k_{t+1}^*] - \sum_{t=0}^{\infty} \lambda_t^1 f_L(k_t^*, L_t^*)\ L_t^* - rk_0.$$

Let :

$$\begin{aligned}
\Delta_T &= \sum_{t=0}^{T} \lambda_t^1 [f(k_t^*, L_t^*) - k_{t+1}^*] - \sum_{t=0}^{T} \lambda_t^1 f_L(k_t^*, L_t^*)\ L_t^* - rk_0 \\
&\quad - \left(\sum_{t=0}^{T} \lambda_t^1 [f(k_t, L_t) - k_{t+1}] - \sum_{t=0}^{T} \lambda_t^1 f_L(k_t^*, L_t^*) L_t - rk_0 \right)
\end{aligned}$$

By the concavity of f , we get

$$\begin{aligned}
\Delta_T &\geq \sum_{t=1}^{T} \lambda_t^1 f_k(k_t^*, L_t^*)(k_t^* - k_t) - \sum_{t=0}^{T} \lambda_t^1 (k_{t+1}^* - k_{t+1}) \\
&= [\lambda_1^1 f_k(k_1^*, L_1^*) - \lambda_0^1](k_1^* - k_1) + ... \\
&\quad + [\lambda_T^1 f_k(k_T^*, L_T^*) - \lambda_{T-1}^1](k_T^* - k_T) - \lambda_T^1 (k_{T+1}^* - k_{T+1}).
\end{aligned}$$

By (4) and (12), we have: $\forall t = 1, 2, ..., T$

$$[\lambda_t^1 f_k(k_t^*, L_t^*) - \lambda_{t-1}^1](k_t^* - k_t)$$
$$= -\lambda_t^3 (k_t^* - k_t) = \lambda_t^3 k_t \geq 0.$$

Thus,

$$\Delta_T \geq -\lambda_T^1 (k_{T+1}^* - k_{T+1}) = -\lambda_T^1 k_{T+1}^* + \lambda_T^1 k_{T+1} \geq -\lambda_T^1 k_{T+1}^*.$$

Since $\lambda^1 \in l_+^1$, $\sup_T k_{T+1}^* < +\infty$, we have

$$\lim_{T \to +\infty} \Delta_T \geq \lim_{T \to +\infty} -\lambda_T^1 k_{T+1}^* = 0.$$

We have proved that the sequences $(\mathbf{k}^*, \mathbf{L}^*)$ maximize the profit of the firm.

It is easy to see that market is clearing at the equilibrium. ■

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